April 4, 1978

Dear David,

I've been thinking about fibering over a circle; in particular, how to construct a closed manifold M and ~~2~~ two elements $f, g \in H^1(M, \mathbb{Z})$ such that M fibers in the direction of f ~~g~~ but ~~not~~ does not fiber in the direction of g, although the ∞-cyclic cover of M corresponding to g has the homotopy type of a finite complex. I thought in 1968 that I had constructed such an example (at least ~~when~~ if only M is a finite C.W. complex). I've been unable to reconstruct it so perhaps I never had it. If you ~~still want~~ need such an example, I'll try some more to construct one. (Though it is possible none exist.)

While thinking about this question, I've tentatively ~~verified the following~~ ruled out manifolds ~~of~~ M of the form $N \times S^1$ ~~Lemma 1~~ as possibilities.

Lemma 1. If K is a finite C.W. complex and $f \in H^1(\cancel{K}^{K\times S^1}, \mathbb{Z})$ is indivisible such that the so-cyclic "covering space" of $K \times S^1$ corresponding to f has the homotopy type of a finite complex then the fibering obstruction $\hat{\tau}(f) = 0$.

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... my talk in ~~at~~ the Nice Congress (vol. II) for the definition of $\tau(f)$.)

This ~~result~~ lemma is motivated by your result that the set ~~of~~ of directions in which M fibre open. Note ~~that~~ there is the following corollary.

Theorem. If K and L are finite complexes, then any homotopy equivalence $K \times S^1 \to L \times S^1$ is simple; i.e. has zero Whitehead torsion.

This generalizes the result of ~~Mather and~~ Kwun–Szczarba ~~(it implies)~~ that

f is simple when it has the form $\varphi \times id$.

To ~~prove~~ this Theorem and lemma!, ~~[illegible]~~ I ~~also~~ ~~use~~ need two ~~[illegible]~~ results which I proved in 1968 but never ~~wrote~~ published.

<u>Lemma 2. Let A and B be finite</u> CW <u>complexes</u>, ~~and~~ $f : A \to B$ <u>be a homotopy equivalence and</u> $\varphi \in H^1(B, \mathbb{Z})$ ~~so~~ <u>such that the</u> ∞-<u>cyclic cover corresponding to</u> φ <u>has the homotopy type of a finite complex</u>, <u>then</u> $\tau(\varphi) - f_* \tau(\varphi f) = \tau(f)$.

(Here, I'm abusing notation, ~~and~~ using τ to denote both Whitehead and fibering torsion.)

Lemma 3. Let K be a finite CW complex and $f, g \in H^1(K, \mathbb{Z})$ such that the covering space corresponding jointly to f and g ~~jointly~~ (i.e., the total space of the ~~pull~~ pull back of $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ via $x \to (f(x), g(x))$) is dominated by a finite complex, then $\tau(f) = \tau(g)$.

(I've not gone back yet and verified Lemma 3 in detail as I think I did in 1968. It's proof is funny since the cover is ~~not~~ only dominated by a finite complex.)

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To deduce the Theorem (from lemmas 1 and 2) ~~from lemmas 1 and 2~~ in lemma 2,

let $B = L \times S^1$, $A = K \times S^1$ and φ be

projection onto S^1 (given by the cartesian product

structure), ~~then by lemma 2~~ hence it suffices to

show $\tau(\varphi f) = 0$; but this follows from lemma 1

<u>Proof of lemma 1.</u> Let p and q be the

projections onto K and S^1, respectively,

determined ~~given~~ by the product structure, then

~~$f = p^*(g) + n$~~ $f = p^*([g]) + n[q]$

where $[g] \in H^1(K, \mathbb{Z})$ $(g : K \to S^1)$.

<u>Case 1.</u> $(n=0)$ When $n=0$, let

$\bar{K}$ be the ∞-cyclic cover of K determined by g, Then $\mathbb{X} \times S^1$ is the cover ~~determined by~~ $\mathbb{X} \times \mathbb{R}$ corresponding to ξ and $\mathbb{X}$ the cover jointly determined by ξ and $[q]$.

By the hypotheses of lemma 1, $\mathbb{X} \times S^1$ is the homotopy type of a finite complex; hence, $\mathbb{X} \times \mathbb{R}$ is dominated by a finite complex ($\mathbb{X}$ is a retract of $\mathbb{X} \times S^1$). Therefore, (by lemma 3) $\tau(\xi) = \tau([q]) = 0$.

<u>Case 2</u>. $(n \neq 0)$ let $r : S^1 \to S^1$ be the map $r(z) = z^n$. This is a covering space with fiber $\mathbb{Z}_n$ (the cyclic group of order n). Form the covering space $\hat{r} : \tilde{K} \to K$ which is the pullback

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of $r : S^1 \to S^1$ via q. (See the diagram below.)

(1)
$$\begin{tikzcd} \tilde{K} \arrow[r, "\hat{q}"] \arrow[d, "\hat{r}"'] & S^1 \arrow[d, "r"] \\ K \arrow[r, "q"'] & S^1 \end{tikzcd}$$

Since $\hat{r} : \tilde{K} \to K$ is a principal Z_4-bundle we can form the associated principal S^1 bundle

(2) $$S^1 \to \tilde{K} \times_{Z_4} S^1 \to K$$

which is the pullback of

(3) $$S^1 \to S^1 \times_{Z_4} S^1 \to S^1$$

via q. But, (3) has a cross section (obstruction theory); hence (2) has cross section

Therefore (2) is trivial (its a principle bundle i.e., $\tilde{K} \times_{\mathbb{Z}_4} S^1 = K \times S^1$.

On the other hand, projecting $\tilde{K} \times_{\mathbb{Z}_4} S^1$ onto the second factor, determines a fibration

(4) $$\tilde{K} \to \tilde{K} \times_{\mathbb{Z}_4} S^1 \to S^1/\mathbb{Z}_4 = S^1;$$

i.e., a ~~fibration~~ fibration of $K \times S^1$ over S^1. The direction of this fibration can be checked to be ξ. This completes the proof of Lemma 1.

Best regards,
Tom Farrell