Lectures on

Surgical Methods in Rigidity

F.T. Farrell

Tata Institute of Fundamental Research
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Contents

Introduction

This book originated from two courses given by the author at the Tata Institute of Fundamental Research during spring 1993. Lectures 1-14 are the first course which was originally meant to be an exposition of the recent topological rigidity result for non-positively curved manifolds due to L.E. Jones and the author. Since the intent was to make the proof of this result accessible to a wide audience, the author decided to include material on surgery theory and controlled topology prerequisite to understanding this proof. The first 14 lectures are consequently an introduction to rather than an exposition of the published work on topological rigidity. The last 6 lectures are the second course which concerns the question of smooth rigidity for non-positively curved manifolds. Lecture 15 gives a motivating example of an exotic expanding endomorphism constructed many years earlier by L.E. Jones and the author. Then Lectures 16-19 contains an exposition of some of the counterexamples to smooth rigidity found by L.E. Jones and the author. Not all of the technical details are given. But it is hoped that the references made to the literature will allow the curious reader to satisfy himself about them. The final lecture is a brief discussion (with references) to some more recent results on topological and smooth rigidity.

The author wishes to thank the faculty of the Tata Institute for their kind invitation to give these courses and for publishing the resulting lectures. He also thanks Shashidhar Upadhyay for his generous help both during the course and in preparing the lectures for publication. Finally, he wishes to thank Marge Pratt for the splendid typing of these lectures.

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Lecture 1. Borel's Conjecture

We start these lectures by defining a concept central to this course.

1.1. Definition A topological space is called aspherical if $\pi_n(X) = 0$ for $n \neq 1$. Here $\pi_n(X)$ denotes the n-th homotopy group of X.

If we assume X to be a CW-complex, then the above definition is equivalent to saying that the universal cover of X is contractible.

Let us recall a well-known result due to Hurewicz and then sketch its proof.

1.2. Theorem. *Let X and Y be aspherical CW-complexes and $\alpha : \pi_1(X) \to \pi_1(Y)$ be an isomorphism. Then α is induced by a homotopy equivalence $f : X \to Y$.*

Proof. We may assume (after some fuss) that the 1-skeleton of X, denoted by X^1, is a wedge of circles. Hence we get a map $f^1 : X^1 \to Y$ by sending the i-th circle a_i in X^1 to a loop in Y representing $\alpha([a_i])$ where $[a_i]$ is the homotopy class of a_i. This map extends over the 2-skeleton X^2 of X since the boundary of any 2-cell in X (thought of as an element in $\pi_1(X^1)$) maps to the trivial element of $\pi_1(Y)$ via $(f^1)_*$. We then easily extend the map over all of X by using the fact that Y is aspherical. Thus f is constructed satisfying $f_* = \alpha$. This map is clearly a weak homotopy equivalence since both X and Y are aspherical. And it is therefore a homotopy equivalence since both X and Y are CW-complexes. Q.E.D.

In view of the above theorem, one might ask whether any two aspherical CW-complexes with isomorphic fundamental groups are homeomorphic. The following examples show that this is not true.

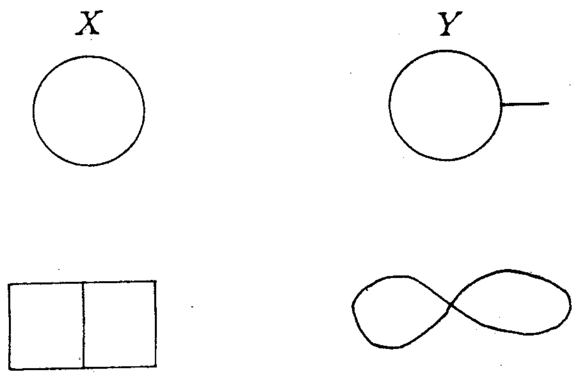

Figure 1

It is clear from these examples that one should require X and Y to have a fixed local homeomorphism type. But the answer to the above question is still no, even if we assume that the aspherical CW-complexes with isomorphic fundamental groups are also manifolds of the same dimension. A counterexample in this case is

$$X = S^1 \times \mathbb{R} \text{ and } Y = \text{Möbius band}.$$

But the following conjecture made by Borel about 1955 is still open.

1.3. Conjecture. *(Borel) If M and N are closed aspherical manifolds with isomorphic fundamental groups, then they are homeomorphic; in fact, the homeomorphism can be chosen to induce the given isomorphism.*

Remark. The following strengthenings of Borel's conjecture are both false as the next examples show.

(i) All closed aspherical manifolds support a smooth structure.

(ii) Any two closed smooth aspherical manifolds with isomorphic fundamental groups are diffeomorphic.

Let T^n denote the n-dimensional torus; i.e., $T^n = S^1 \times S^1 \times \ldots \times S^1$ (n-factors) where S^1 is the circle. Browder[13] constructed a smooth manifold which is homeomorphic but not diffeomorphic to T^7. This shows that (ii) is false. In fact, it follows from later results that T^n and the connected sum $T^n \# \Sigma^n$ are homeomorphic but not diffeomorphic if $n \geq 5$ and Σ^n is any exotic n-sphere. That is, Σ^n is a smooth manifold which is homeomorphic but not diffeomorphic to the standard n-dimensional sphere S^n, cf. [65]. On the other hand, M. Davis and J.C. Hausmann [23] constructed an example of a closed aspherical manifold which does not support any differentiable structure proving (i) to be false as well. Moreover, M. Davis and T. Januszkiewcz [24] gave an example of a closed aspherical manifold which can not be triangulated.

1.4. Examples of Aspherical Manifolds

(1) Any complete non-positively curved Riemannian manifold is aspherical. This follows from the Cartan-Hadamard Theorem. Special cases are:

(i) flat Riemannian manifolds,

(ii) hyperbolic manifolds,

(iii) locally symmetric space of non-compact type.

(2) If G is a virtually connected Lie group, K a maximal compact subgroup, and Γ a discrete torsion free subgroup of G, then the double coset space $\Gamma \backslash G/K$ is aspherical. In this case, G/K is diffeomorphic to $\mathbb{R}^n$ for some integer n. In the special case where G is virtually nilpotent and $\pi_1 G = 1$, the double coset space $\Gamma \backslash G/K$ is called an infranilmanifold.

But there are many other aspherical manifolds which are not of the types (1) or (2). For example, the closed smooth aspherical manifolds constructed by Davis [21] are not of these types since their universal covers are not even homeomorphic to $\mathbb{R}^n$. This is particularly surprising since the fundamental groups of these manifolds are relatively "tame". In fact, they are finite index subgroups of certain Coxeter groups. We will discuss his construction in Lecture 3.

Gromov [54] has more recently constructed many more examples of aspherical manifolds. Given any closed manifold N^n which is a polyhedron, he constructed a closed aspherical manifold M^n and a degree 1 map $f : M^n \to N^n$. The reason these manifolds M^n are aspherical is they are nonpositively curved complexes in the sense of Alexandroff.

Remark. Borel's conjecture implies Poincare's conjecture which says that any simply connected closed 3-manifold is homeomorphic to the unit sphere S^3 in $\mathbb{R}^4$. This is seen as follows. Let Σ^3 be a counterexample to Poincare's conjecture, and consider the connected sum $M = T^3 \# \Sigma^3$, where T^3 again denotes $S^1 \times S^1 \times S^1$. Van Kampen's theorem shows that T^3 and M^3 have isomorphic fundamental groups. And M is seen to be aspherical by applying the Hurewicz isomorphism theorem to the universal cover of $T^3 \# \Sigma^3$. Borel's conjecture is contradicted by showing that $T^3 \# \Sigma^3$ is not homeomorphic to T^3. For this we use the following two results.

1.5. Theorem. *(Schoenflies Theorem) Let $f : S^2 \to S^3$ be a bicollared embedding, then $f(S^2)$ bounds closed (topological) balls on both sides.*

1.6. Theorem. *(Alexander's Trick) Let $h : S^n \to S^n$ be any homeomorphism. Then h extends to a homeomorphism $\bar{h} : \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$, where $\mathbb{D}^{n+1}$ denotes the closed ball of radius 1 in $\mathbb{R}^{n+1}$ which bounds S^n.*

Now if $T^3 \# \Sigma^3$ were homeomorphic to T^3, then the universal cover of $T^3 \# \Sigma^3$ is homeomorphic to $\mathbb{R}^3$. Consequently, the Schoenflies theorem shows that $\Sigma^3 - \text{Int } \mathbb{D}^3$ is homeomorphic to $\mathbb{D}^3$. (This Int $\mathbb{D}^3$ is the interior of the 3-dimensional ball removed from Σ^3 in forming the connected sum with T^3.) Now applying Alexander's trick, we get Σ^3 is homeomorphic to S^3. It follows that $T^3 \# \Sigma^3$ is not homeomorphic to T^3.

Lecture 2. Generalized Borel Conjecture

2.1. Structure Sets

Let M be a closed manifold. We define $\mathcal{S}(M)$ to be the set of equivalence classes of pairs (N, f) where N is a closed manifold and $f : N \to M$ is a homotopy equivalence. And the equivalence relation is defined as follows:

$(N_1, f_1) \sim (N_2, f_2)$ if there is a homeomorphism $h : N_1 \to N_2$

such that $f_2 \circ h$ is homotopic to f_1.

The above definition is motivated by the definition of Teichmüller space T_g. In this classical situation, $M = M_g$ is a closed Riemann surface of genus $g \geq 2$. And T_g is the set of equivalence classes of pairs (N, f) where N is a Riemann surface of constant curvature -1 and $f : N \to M$ is a homotopy equivalence. The equivalence relation is the same as in $\mathcal{S}(M)$ except that homeomorphism is replaced by isometry.

For an aspherical finite polyhedron X, Hurewicz identified $\mathrm{Out}(\pi_1 X)$ with $\pi_0(\mathcal{E}(X))$, where $\mathrm{Out}(\pi_1 X)$ is the group of outer automorphisms of $\pi_1 X$ and $\mathcal{E}(X)$ is the space of self homotopy equivalences of X given the compact open topology. Using this identification, we get a natural action of $\mathrm{Out}(\pi_1 M_g)$ on T_g in the following way. If $\alpha \in \mathrm{Out}(\pi_1 M_g)$ and $[N, f] \in T_g$, then $\alpha[N, f] = [N, \tilde{\alpha} \circ f]$ where $\tilde{\alpha} : M_g \to M_g$ is any self homotopy equivalence inducing α. Moreover, $\mathrm{Out}(\pi_1 M_g)$ contains a subgroup Γ of finite index such that T_g/Γ is an aspherical manifold. Recall that T_g is diffeomorphic to $\mathbb{R}^{6g-6}$. The manifolds T_g/Γ are not compact; but can be compactified, cf. [56], [55]. These manifolds form another interesting class of aspherical manifolds apparently different from the other classes of examples mentioned in Lecture 1, cf. [55].

Remark. When M is aspherical, $\mathrm{Out}(\pi_1 M)$ also acts on $\mathcal{S}(M)$. The statement that $|\mathcal{S}(M)| = 1$ is equivalent to Borel's conjecture. While the statement that $\mathrm{Out}(\pi_1 M)$ acts transitively on $\mathcal{S}(M)$ is equivalent to the weaker statement that any closed aspherical manifold N with $\pi_1 N \simeq \pi_1 M$ is homeomorphic to M.

2.2. Variants of Structure Sets

(1) The smooth structure set is denoted by $\mathcal{S}^s(M)$. It consists of equivalence classes of objects (N, f) where now N is a smooth manifold, $f : N \to M$ is a homotopy equivalence and $h : N_1 \to N_2$ is required to be a diffeomorphism. Note that we do not require that M is a smooth manifold, hence $\mathcal{S}^s(M)$ could be empty. Likewise, the definitions of $\mathcal{S}(M)$ and $\mathcal{S}^s(M)$ make sense even if M is not a manifold. But for them to be possibly non-empty M must have the algebraic properties of a manifold; e.g., it must satisfy Poincare duality with arbitrary local coefficients. If this is so, then M is called a *Poincare duality space* and an interesting question is whether $\mathcal{S}(M)$ or $\mathcal{S}^s(M)$ is non-empty; i.e., does there exist a topological or smooth manifold N which is homotopically equivalent to M.

(2) It is also useful to define $\mathcal{S}(M, \partial M)$ where M is a compact manifold with boundary ∂M. Here, an object is again a pair (N, f) where N is a compact manifold with boundary ∂N and $f\colon (N, \partial N) \to (M, \partial M)$ is a homotopy equivalence such that the restricted map $f|_{\partial N}\colon \partial N \to \partial M$ is a homeomorphism. And $(N_1, f_1) \sim (N_2, f_2)$ if there is a homeomorphism $h : (N_1, \partial N_1) \to (N_2, \partial N_2)$ such that $f_2 \circ h$ is homotopic to f_1 rel ∂N_1; i.e., the homotopy between $f_2 \circ h$ and f_1 is constant on ∂N_1.

(3) The smooth structure set $\mathcal{S}^s(M, \partial M)$ is defined similarly.

Borel's conjecture can be generalized to the following statement.

2.3. Generalized Borel Conjecture. If M is a compact aspherical manifold with perhaps non-empty boundary, then $|\mathcal{S}(M, \partial M)| = 1$.

2.4. Examples of compact aspherical manifolds with boundary.

(1) Let G be a semi-simple Lie group and K a maximal compact subgroup. Suppose that Γ is a torsion free arithmetic subgroup of G for some algebraic group structure on G defined over $\mathbb{Q}$. Then M.S. Raghunathan [85] showed that the double coset space $M = \Gamma\backslash G/K$ is the interior of a compact manifold with boundary. Note that this shows that Γ is finitely generated and, in fact, of the type FL in the sense of Serre [88]. Later Borel and Serre [10] gave a second compactification $\bar{M}$ of M. In their compactification

the boundary of the universal cover of $\bar{M}$ is homotopically equivalent to a wedge of spheres S^m where m equals $\mathbb{Q}$ rank$(G)-1$. This has some useful consequences about the group cohomology of Γ. Namely, they deduce from this that the cohomological dimension of Γ is $\dim(G/K) - \mathbb{Q}$ rank(G) and that Γ is a duality group in the sense of Bieri and Eckmann [7]. It is a consequence of the vanishing results on Whitehead torsion (to be discussed later) that the two compactifications are diffeomorphic provided $\dim(G/K) \neq 3, 4, 5$.

(2) The compactification of T_g/Γ due to Harvey [56], mentioned earlier, is another example. Again, the boundary of its universal cover was shown by Harer [55] to be homotopically equivalent to a wedge of spheres. Harer showed, in this way, that Γ is a duality group and he also calculated its cohomological dimension.

A fundamental problem in topology is to calculate $|\mathcal{S}(M, \partial M)|$; i.e., the cardinality of the set $\mathcal{S}(M, \partial M)$. Surgery theory was developed to solve this problem. It essentially reduced the problem to calculating certain algebraically defined obstruction groups which are functors depending only on $\pi_1 M$ (when M is orientable). In particular, showing that

$$|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$$

for all sufficiently large integers n, yields a calculation of the obstruction groups for $\pi_1 M$ when M is aspherical. Hence the verification of the generalized Borel conjecture would make surgery theory an effective method for calculating $|\mathcal{S}(N, \partial N)|$ for any compact connected manifold (not necessary aspherical) with $\pi_1 N$ isomorphic to $\pi_1 M$, provided $\dim N \geq 5$.

We will still refer to the generalized Borel conjecture as Borel's conjecture because of a corollary to the following result of M. Davis [22].

2.5. Theorem. *(Davis) If K is a finite aspherical polyhedron, then there exists a closed aspherical manifold M such that K is a retract of M.*

2.6. Corollary. *If $|\mathcal{S}(M)| = 1$ for every closed aspherical manifold M, then $|\mathcal{S}(N, \partial N)| = 1$ for every compact aspherical manifold N with boundary ∂N.*

Remark. This theorem is implicit in Davis' paper [22] and is made explicit by Bizhong Hu in [63]. Hu shows that if K is non-positively curved in the sense of Alexandroff, then the manifold M of Davis' theorem can be constructed to also be non-positively curved in the sense of Alexandroff. If we consider a functor from topological spaces to groups, for instance the Whitehead group functor $X \to \mathrm{Wh}(\pi_1 X)$, then the following result is an immediate consequence of Davis' theorem. If such a functor vanishes on all closed aspherical manifolds, then it must also vanish on all finite aspherical complexes. (Hu uses this fact in his work on Whitehead groups.) An elaboration of this idea can be used to verify the above Corollary 2.6. We will discuss this in a later lecture.

Lecture 3. Davis' Construction

This lecture is devoted to sketching the proof of Theorem 2.5.

Step 1. We can assume that K is a compact smooth manifold with boundary. For example, embed K in $\mathbb{R}^n$ for some n. Then replace K with a regular neighborhood of it.

Step 2. Let K be a smooth manifold with boundary and T be a piecewise smooth triangulation of ∂K. Let V and E denote the set of vertices and edges of T, respectively. Associate to T the group Γ generated by the vertices $v \in V$ with the relations $v^2 = 1$ if $v \in V$ and $(uv)^2 = 1$ if $\{u, v\} \in E$. For each subset S of V, $\Gamma(S)$ denotes the subgroup of Γ generated by S. We identify a simplex σ of T with the subset of T consisting of the vertices of σ; consequently, $\Gamma(\sigma)$ is defined. For $v \in V$, let $D(v) = \text{Star}(v)$ be the closed dual cell of v in the first barycentric subdivision of T. For each simplex σ in T, its dual cell

$$D(\sigma) = \bigcap \{D(v)\}$$

where v varies over the vertices of σ. The dual cells give a regular cell complex structure to ∂K. To each $x \in \partial K$ associate the subgroup $\Gamma_x = \Gamma(\sigma)$ of Γ, where σ is the unique simplex such that $x \in \text{Interior } D(\sigma)$. If $x \in \text{Interior } K$, then define $\Gamma_x = \{1\}$. Define $\mathcal{M} = K \times \Gamma / \sim$, where $(x, \alpha) \sim (y, \beta)$ if and only if $x = y$ and $\alpha^{-1}\beta \in \Gamma_x$. There is an obvious action of Γ on $\mathcal{M}$; i.e., $g[x, h] = [x, gh]$ for $g \in \Gamma$ and $[x, h] \in \mathcal{M}$. (Compare J. Tits [95].) Note that the orbit space $\mathcal{M}/\Gamma = K$.

Step 3. This step is devoted to proving the following result.

3.1. Lemma. *Provided T is a first barycentric subdivision of a simplicial complex F, then $\Gamma(S)$ is a finite (Coxeter) group if and only if S is a simplex of T.*

Proof. If S is a simplex of T, then all the elements in $\Gamma(S)$ commute and hence are of order two. Therefore, $\Gamma(S) = (\mathbb{Z}/2)^{|S|}$. Conversely, if $\Gamma(S)$ is finite, then for each pair $v \neq w$ in S we must have $\{v, w\} \in E$; otherwise, $\Gamma(S)$ would map onto an infinite dihedral group. We will now use the assumption that T is a first barycentric subdivision of a simplicial complex F; i.e., the vertices V of T are the simplices of F. Note that V is partially ordered by the face relation on the simplices of K. The simplices of T are the totally ordered subsets of V; i.e., the subsets for which every pair of elements are comparable. But we have just seen that if $v \neq w \in S$, then $\{v, w\}$ is a simplex of T; hence v and w are comparable. This implies that S is a simplex of T.

Example. Let K be a 2-simplex and T be the 1-skeleton of K, then Γ is the finite group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But this does not contradict Lemma 3.1 since T is not the first barycentric subdivision of anything. Note in this example that $\mathcal{M}$ is the octahedron.

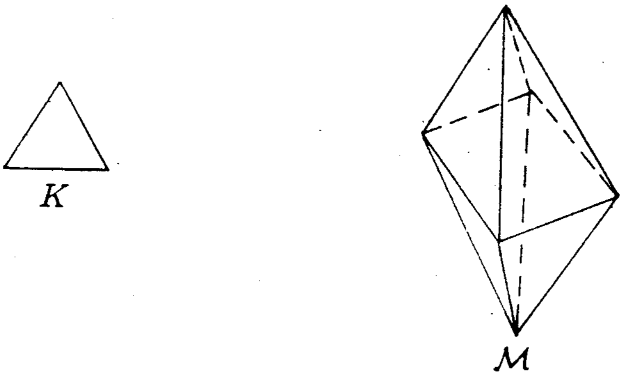

Figure 2

Step 4. We may assume during the remainder of the proof of Theorem 2.5 that T is the first barycentric subdivision of some simplicial complex K; e.g., replace T by its first barycentric subdivision. Hence, Lemma 3.1 is applicable to T. Since Γ is a Coxeter group, it has a faithful representation into $GL_{|V|}(\mathbb{R})$. Consequently, by a result of Selberg [87], Γ contains a normal subgroup Π of finite index such that Π is torsion free. In this case, Π can be taken to be the kernel of the obvious epimorphism $\Gamma \to \mathbb{Z}_2^{|V|}$ where $\mathbb{Z}_2^{|V|}$ is the abelianization of Γ. Using the fact (from Lemma 3.1) that Γ_x is a finite Coxeter group for each $x \in K$, Davis shows that $M = \mathcal{M}/\Pi$ is a closed manifold. Let $G = \Gamma/\Pi$, then G acts on M and $M/G = \mathcal{M}/\Gamma = K$. Now the maps $K \to M \to M/G = K$ demonstrate that K is a retract of M.

Step 5. Using the fact (from Lemma 3.1) that $\Gamma(S)$ finite implies S is a simplex of T, Davis shows that the group elements in Γ can be enumerated as $g_1, g_2, \ldots$ so that the following statement $(*)$ holds:

$$(*) \qquad \mathcal{M}_n \cap (K \times g_{n+1}) \text{ is contractible, where } \mathcal{M}_n = \bigcup\{K \times g_i \mid 1 \leq i \leq n\}.$$

The intersection in $(*)$ is, in fact, homeomorphic to $\mathbb{D}^{m-1}$ where $m = \dim K$. Consequently, if K is contractible, so is each $\mathcal{M}_n$; and if K is aspherical, so is each $\mathcal{M}_n$. It follows that $\mathcal{M}$ is contractible when K is contractible. Likewise, K aspherical implies $\mathcal{M}$ is aspherical which in turn implies M is aspherical. This completes our sketch of the proof of Theorem 2.5.

Remark 1. We see by $(*)$ that $\mathcal{M}_n$ is a manifold with boundary. Also, $\partial M_{n+1} = \partial \mathcal{M}_n \# \partial K$; therefore,

$$\pi_1(\partial \mathcal{M}_n) = \pi_1(\partial K) * \ldots * \pi_1(\partial K), \ n \text{ factors},$$

provided dim $K > 3$. Here $*$ denotes free product. This means that $\mathcal{M}$ is not simply connected at infinity if $\pi_1(\partial K) \neq \{1\}$. In particular, $\mathcal{M}$ is not homeomorphic to $\mathbb{R}^m$. Now it is well known that there exist K, for each $m \geq 4$, such that K is contractible but ∂K is not simply connected. In this way, Davis [21] gets his example of a closed aspherical manifold whose universal cover is not $\mathbb{R}^m$.

Remark 2. The example due to Davis and Hausmann [23] of a closed aspherical manifold which does not support a smooth structure is constructed by first finding a compact, piecewise linear, aspherical manifold K with non-empty boundary such that the interior of K does not support a smooth structure. (Davis' construction does not need that K is smooth; it only uses the triangulation T of ∂K.) Well known results from smoothing theory yields such a K. Consequently, $M = \mathcal{M}/\Pi$ does not support a smooth structure. If it did, then so would $\mathcal{M}$ and any open subset of $\mathcal{M}$; in particular, the interior of K.

Remark 3. The example of Davis and Januskiewicz [24] require new ideas employing Gromov's hyperbolization construction [54].

Lecture 4. Surgical method for analyzing $\mathcal{S}(M)$

The problem of showing $|\mathcal{S}(M)| = 1$ breaks into three geometric steps which we now describe.

Step 1. To show that $[N, f] \in \mathcal{S}(M)$ is *normally cobordant* to $[M, \mathrm{id}]$. This means that there exists a compact cobordism W and a map $F : (W, \partial W) \to (M \times [0,1], \partial)$ with the following properties.

(a) The boundary $\partial W = N \coprod M$. (Set $N = \partial^+ W$, $M = \partial^- W$.)

(b) The restriction map $F|_{\partial + W} : \partial^+ W \to M \times 1$ is equal to f and $F|_{\partial - W} : \partial^- W \to M \times 0$ is equal to the identity map.

(c) The map F is covered by an isomorphism $\bar{F} : N(W) \to \mathcal{E}$ where $N(W)$ is the stable normal bundle of W and $\mathcal{E}$ is a bundle over $M \times [0,1]$.

From now onwards we will denote a normal cobordism by the triple $(W, F, \simeq)$ where $\simeq$ denotes the isomorphism covering F.

Step 2. To show that the cobordism W in Step 1 can be chosen to be an h-cobordism; i.e., F is actually a homotopy equivalence.

Step 3. To show that the h-cobordism W of Step 2 has trivial Whitehead torsion and hence W is a product. (Here dim $M \geq 5$.)

We now put an equivalence relation "$\sim$" on the set of all normal cobordisms. We say $(W_1, F_1, \simeq_1) \sim (W_2, F_2, \simeq_2)$ if and only if there exists a triple $(\mathcal{W}, \mathcal{F}, \equiv)$ where $\mathcal{W}$ is a cobordism between W_1 and W_2 and $\mathcal{F} : \mathcal{W} \to M \times I \times I$ satisfying the following properties where $I = [0,1]$. The restriction $\mathcal{F}|_{W_1} : W_1 \to M \times 0 \times I$ is F_1, and $\mathcal{F}|_{W_2} : W_2 \to M \times 1 \times I$ is F_2. Also, $\mathcal{F}|_{\mathcal{W}^-} = \mathrm{id}_{M \times I \times 0}$ and $\mathcal{F}|_{\mathcal{W}^+}$ is a homotopy equivalence, where $\mathcal{W}^-$ and $\mathcal{W}^+$ are described in Figure 3.

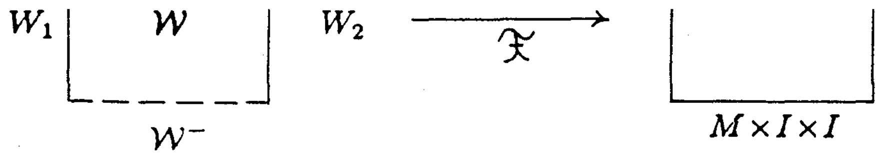

Figure 3

Also, $\equiv$ is an isomorphism of $\mathcal{N}(\mathcal{W})$ (the stable normal bundle) to some bundle $\mathcal{E}$ over $M \times I \times I$ which covers $\mathcal{F}$ and which restricts to $\simeq_1$ and $\simeq_2$ over W_1 and W_2, respectively. When $m = \dim M \geq 4$, the equivalence classes of normal cobordisms form a group which depends only on $\pi_1(M)$ and the first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}^2)$. When M is orientable, then $w_1(M) = 0$. (We will usually suppress mentioning $w_1(M)$.) The group of normal cobordisms modulo equivalence is denoted $L_{m+1}(\pi_1 M)$ where $m = \dim M$. (See [96] for a purely algebraic definition of the groups $L_n(\pi)$.) The results described above are proven in [96] and [67].

Remark. Although the groups $L_n(\pi_1 M)$ are always countable, they are sometimes not finitely generated. Cappell [16] for example showed that $L_{4n+2}(D_\infty)$ is not finitely generated (for all values of n) where D_∞ denotes the infinite dihedral group. On the other hand, Wall [97] has shown that $L_n(\Gamma)$ is finitely generated when Γ is a finite group. He also here calculated $L_n(\Gamma)$ for $|\Gamma|$ odd. And Wall [96] showed that $L_n(\Gamma) = L_{n+4}(\Gamma)$ for every Γ (not necessarily finite).

4.1. Definition. A normal cobordism $(W, F, \simeq)$ is called a *special normal cobordism* if $F|_{W^+}$ is a homeomorphism.

We note that a positive solution to Step 2 is equivalent to showing that the normal cobordism of Step 1 is $\sim$ to a special normal cobordism.

We can define a stronger equivalence relation $\sim_s$ on the set of special normal cobordisms by requiring the restriction $\mathcal{F}|_{\mathcal{W}+}$ of the earlier equivalence relation $\sim$ to be a homeomorphism. This set of special normal cobordisms modulo the equivalence relation $\sim_s$ is also an abelian group and it is naturally identified with $[M \times [0,1], \partial; G/\mathrm{Top}]$. Here, G/Top is an H-space and $[X, A; G/\mathrm{Top}]$ denotes the set of homotopy classes of maps $f : X \to G/\mathrm{Top}$ such that the restriction $f|_A = 1$; i.e., has constant value the homotopy identity element in G/Top.

Lecture 5. Surgery Exact Sequence

In the last lecture, we mentioned an H-space G/Top. We now note down some informations about G/Top without actually defining it:

$$\pi_n(G/\mathrm{Top}) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \mathbb{Z}, & \text{if } n \equiv 0 \bmod 4 \\ \mathbb{Z}_2, & \text{if } n \equiv 2 \bmod 4 \end{cases}$$

$$G/\mathrm{Top} \otimes \mathbb{Q} = \prod_{n=0}^{\infty} K(\mathbb{Q}, 4n)$$

$$G/\mathrm{Top} \otimes \mathbb{Z}_{(2)} = \prod_{n=0}^{\infty} K(\mathbb{Z}_{(2)}, 4n) \times \prod_{n=0}^{\infty} K(\mathbb{Z}_2, 4n+2)$$

$$G/\mathrm{Top} \otimes \mathbb{Z}_{\mathrm{odd}} = BO \otimes \mathbb{Z}_{\mathrm{odd}}.$$

(In these formulas, $\mathbb{Z}_{(2)} = \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \ldots]$ and $\mathbb{Z}_{\mathrm{odd}} = \mathbb{Z}[\frac{1}{2}]$.) It is a consequence of the second formula above that

$$[X, A; G/\mathrm{Top}] \otimes \mathbb{Q} = \bigoplus_{n=0}^{\infty} H^{4n}(X, A; \mathbb{Q})$$

for any pair of topological spaces (X, A).

Historical Remarks. D. Sullivan [93] determined the properties of G/PL in his works on the Hauptvermutung. His results combined with those of Kirby-Siebenmann [67] yield the results on G/Top mentioned above. The formulation of the surgery exact sequence given below is also due to work of Sullivan and Wall refining the earlier work of Browder [12] and Novikov [77]. The Kervaire-Milnor paper [65] on $\mathcal{S}^s(S^n)$ was the prototype for this method of studying $\mathcal{S}(M)$.

5.1. Definition. We next define a variant of $\mathcal{S}(M)$ denoted $\bar{\mathcal{S}}(M)$. The underlying set of $\bar{\mathcal{S}}(M)$ is same as that of $\mathcal{S}(M)$. But now $[N_1, f_1]$ is said to be equivalent to $[N_2, f_2]$ if there exist an h-cobordism W between N_1 and N_2 and a map $F : W \to M \times I$ such that $F\big|_{\partial^- W} = f_1$ and $F\big|_{\partial^+ W} = f_2$, where $\partial^- W = N_1$ and $\partial^+ W = N_2$.

Note that $\mathcal{S}(M) = \bar{\mathcal{S}}(M)$ when $\mathrm{Wh}(\pi_1(M)) = 0$ and $\dim M \geq 5$. A set $\bar{\mathcal{S}}(M, \partial M)$ can be defined similarly when M is a compact manifold with boundary. (The notation $\bar{\mathcal{S}}(M, \partial M)$ is sometimes abbreviated to $\bar{\mathcal{S}}(M, \partial)$ and likewise $[M, \partial M; G/\mathrm{Top}]$ to $[M, \partial; G/\mathrm{Top}]$.)

Surgery Exact Sequence. Let M^m be a compact connected manifold with non-empty boundary. For any non-negative integer n, we can form a new manifold $M^m \times \mathbb{D}^n$, where $\mathbb{D}^n$ is the closed n-ball. Then there is long exact sequence of pointed sets:

$$\cdots \xrightarrow{\pi} \bar{\mathcal{S}}(M \times \mathbb{D}^n, \partial) \xrightarrow{\omega} [M \times \mathbb{D}^n, \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_{m+n}(\pi_1 M) \longrightarrow \cdots$$

$$\longrightarrow \bar{\mathcal{S}}(m \times \mathbb{D}^1, \partial) \xrightarrow{\omega} [M \times \mathbb{D}^1, \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_{m+1}(\pi_1 M) \xrightarrow{\tau} \bar{\mathcal{S}}(M, \partial)$$

$$\xrightarrow{\omega} [M, \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_m(\pi_1 M).$$

The maps σ (when $n \geq 1$) and τ can be defined using the identifications mentioned earlier. Recall that $L_{m+n}(\pi_1 M) = L_{m+n}(\pi_1(M \times \mathbb{D}^{n-1}))$ is the set of equivalence classes of normal cobordisms on $M \times \mathbb{D}^{n-1}$ and that

$$[M \times \mathbb{D}^n, \partial; G/\mathrm{Top}] = [M \times \mathbb{D}^{n-1} \times [0,1], \partial; G/\mathrm{Top}]$$

consists of the equivalence classes of special normal cobordisms on $M \times \mathbb{D}^{n-1}$. Then, σ is the map which forgets the special structure; while, τ sends a normal cobordism W to its *top* $\partial^+ W$.

The maps ω, when $n \geq 1$, similarly have a natural geometric description. We illustrate this when $n = 1$. Let (W, F) represent an element x in $\bar{\mathcal{S}}(M \times [0,1], \partial)$. Then, W is an h-cobordism between $\partial^- W$ and $\partial^+ W$. Furthermore, the restrictions $F|_{\partial^- W} : \partial^- W \to M \times 0$ and $F|_{\partial^+ W} : \partial^+ W \to M \times 1$ are both homeomorphisms. If the first of these two homeomorphisms is id_M, then $(W, F, \simeq)$ is also a special normal cobordism and, considered as such, is $\omega(x)$. A bundle isomorphism $\simeq$ with domain $N(W)$ is determined since F is a homotopy equivalence. But it is easy to see that (W, F) is equivalent in $\bar{\mathcal{S}}(M \times [0,1], \partial)$ to an object (W', F') such that $\partial^- W' = M$ and $F'|_{\partial^- W'} = \mathrm{id}_M$. In this way, $\omega(x)$ is defined.

With this above description of the maps σ, τ and ω, it is not difficult to show that the surgery sequence (above the last few terms; i.e., above those near $L_m(\pi_1 M)$) is exact. The last few terms pose extra difficulty. But the following important periodicity result, due to Kirby-Siebenmann [67], allow us to avoid worrying about these difficulties when considering Borel's Conjecture.

5.2 Theorem. *(Kirby-Siebenmann) Let M be a compact, connected manifold with dim $M \geq 5$. If $\partial M \neq \emptyset$, then*

$$|\mathcal{S}(M, \partial)| = |\mathcal{S}(M \times \mathbb{D}^4, \partial)|.$$

If $\partial M = \emptyset$, then

$$|\mathcal{S}(M)| \leq |\mathcal{S}(M \times \mathbb{D}^4, \partial)|.$$

5.3. Definition. The map σ in the surgery sequence is called either the ***surgery map*** or the *assembly map*.

Remark 1. When $n \geq 1$, σ is a group homomorphism.

Remark 2. If $\partial M = \emptyset$, then the surgery sequence still exists and remains exact, provided we omit its last term.

Lecture 6. Condition (*)

Recall Alexander's Trick; namely, Theorem 1.6 from Lecture 1. This fundamental result is quite elementary. It has in fact the following one line proof.

Proof. Set $\bar{h}(tx) = th(x)$ where $x \in S^n$ and $t \in [0,1]$.

6.1. Addendum. *Even when $h : S^n \to S^n$ is not a homeomorphism, but only a continuous map, the above map $\bar{h} : \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ is a continuous extension of h.*

Remark. Note that $\bar{h}$ is rarely differentiable at 0 even if h is. Precisely stated, $\bar{h}$ is differentiable if and only if $h \in O(n+1)$; i.e., h is the restriction of an orthogonal linear transformation. There can, in fact, be no smooth analogue of Alexander's Trick as Milnor [72] proved by constructing examples of exotic 7-dimensional spheres.

Let us now use Alexander's Trick together with the topological version of Smale's h-cobordism theorem [90], due to Kirby-Siebenmann [67], to calculate $\mathcal{S}(\mathbb{D}^n, \partial)$ when $n \geq 5$.

6.2. Theorem. *When $n \geq 5$, $|\mathcal{S}(\mathbb{D}^n, \partial)| = 1$.*

Proof. Let $f : (N, \partial N) \to (\mathbb{D}^n, S^{n-1})$ represent an element in $\mathcal{S}(\mathbb{D}^n, \partial)$. Then, by definition, $f|_{\partial N} : \partial N \to S^{n-1}$ is a homeomorphism. We must show that f is homotopic rel ∂ to a homeomorphism. Let $\mathbb{B}^n$ be a (locally flatly) embedded closed n-ball in the interior of N. Then, $N - \text{Int } \mathbb{B}^n$ is an h-cobordism between $\partial\mathbb{B}^n$ and ∂N, and hence $N - \text{Int } \mathbb{B}^n$ is homeomorphic to $S^{n-1} \times [0,1]$ by the h-cobordism theorem. Using Alexander's Trick, we see that $N = (N - \text{Int } \mathbb{B}^n) \cup \mathbb{B}^n$ is homeomorphic to $\mathbb{D}^n$; i.e., $(N, \partial N) = (\mathbb{D}^n, S^{n-1})$. Let $\phi = f|_{S^{n-1}}$ and $\bar{\phi} : \mathbb{D}^n \to \mathbb{D}^n$ be the homeomorphism extending ϕ given by Alexander's Trick. Let $\psi = f \cup \bar{\phi} : S^n \to S^n$ denote the continuous map which is f on the northern hemisphere and $\bar{\phi}$ on the southern hemisphere. Then its continuous extension $\bar{\psi}$, given by Addendum 6.1, can be used to construct the desired homotopy rel ∂ between f and $\bar{\phi}$.

Remark 1. We will use Theorem 6.2, the surgery exact sequence and the Cartan-Hadamard theorem to show, in our next lecture, that the assembly map σ in the surgery exact sequence is a split monomorphism when M is a closed non-positively curved Riemannian manifold and $n \geq 1$.

Remark 2. Set $M = \mathbb{D}^5$ in the surgery exact sequence and observe that $M \times \mathbb{D}^n = \mathbb{D}^{n+5}$. Since $Wh(\pi_1 M) = Wh(1) = 0$, $\bar{S}(M \times \mathbb{D}^n, \partial) = S(M \times \mathbb{D}^n, \partial)$. These observations combined with Theorem 6.2 yield that the surgery map

$$\sigma : \pi_{n+5}(G/\text{Top}) \to L_{n+5}(1)$$

is an isomorphism for all $n > 0$. Now Kervaire-Milnor [65] calculated $L_m(1)$; namely,

$$L_m(1) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \bmod 4 \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \bmod 4 \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

This yields the following calculation of $\pi_m(G/\text{Top})$ valid for $m \geq 5$:

$$\pi_m(G/\text{Top}) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \bmod 4 \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \bmod 4 \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

The same result is shown to hold for $m < 5$ by special arguments. This is the method used by Sullivan in [92] to calculate $\pi_m(G/PL)$ and was the first step in his later determination of G/PL in [93]; cf. Historical Remarks in Lecture 5.

I now formulate a useful abstract property which is possessed by many aspherical manifolds M. This property was introduced in [30]. It will be shown in the next lecture how this property relates to the question of the split injectivity of the assembly map in the surgery exact sequence for M.

6.3. Definition. A closed manifold M^m satisfies *condition* $(*)$ provided there exists an action of $\pi_1 M^m$ on $\mathbb{D}^m$ with the following two properties.

1. The restriction of this action to $\mathrm{Int}(\mathbb{D}^m)$ is equivalent via a $(\pi_1 M)$-equivariant homeomorphism to the action of $\pi_1 M$ by deck transformations on the universal cover $\tilde{M}$ of M^m.
2. Given any compact subset K of $\mathrm{Int}(\mathbb{D}^m)$ and any $\epsilon > 0$, there exists a real number $\delta > 0$ such that the following is true for every $\gamma \in \pi_1 M$. If the distance between γK and $S^{m-1} = \mathbb{D}^m$ is less than δ, then the diameter of γK is less than ϵ.

Note that any manifold satisfying condition $(*)$ is obviously aspherical.

Remark. Every closed (connected) non-positively curved Riemannian manifold M satisfies condition $(*)$. This was shown in [30] by considering the geodesic ray compactification $\bar{M}$ of $\tilde{M}$ defined by Eberlein and O'Neill [26]. The compactification $\bar{M}$ of $\tilde{M}$ is homeomorphic to $\mathbb{D}^m$. The verification of property 2 in condition $(*)$ uses the well known fact that $\exp_x : T_x\tilde{M} \to \tilde{M}$ is weakly expanding; cf. [59, p. 172, Lemma 1].

Lecture 7. Splitting the Assembly Map

Between the notions of normal cobordism on a manifold M and special normal cobordism on M, there is an intermediate object called a *simple* normal cobordism. Recall that a normal cobordism on M is a triple $(W, F, \simeq)$ in which F is a map $F : W \to M \times I$ such that $F|_{\partial^+ W} : \partial^+ W \to M \times 1$ is a homotopy equivalence and $F|_{\partial^- W} : \partial^- W \to M \times 0$ is the identity map. If $F|_{\partial^+ W} : \partial^+ W \to M \times 1$ is a simple homotopy equivalence, we say that $(W, F, \simeq)$ is a *simple normal cobordism.* (Recall that $(W, F, \simeq)$ is a special normal cobordism if $F|_{\partial^+ W}$ is a homeomorphism.) There is an obvious equivalence relation on the set of simple normal cobordisms analogous to the equivalence relation on normal cobordisms and special normal cobordisms. Wall [96] showed that the equivalence classes of simple normal cobordisms form an abelian group, denoted by $L^s_{m+1}(\pi_1 M)$, which depends only on $\pi_1 M$ and on the first Stiefel-Whitney class $w_1(M)$. The forget structure maps define group homomorphisms

$$\bar{\sigma} : [M \times [0,1], \partial; G/\mathrm{Top}] \to L^s_{m+1}(\pi_1 M) \quad \text{and}$$

$$\eta : L^s_{m+1}(\pi_1 M) \to L_{m+1}(\pi_1 M).$$

And these homomorphisms factor the assembly map

$$\sigma : [M \times [0,1], \partial; G/\mathrm{Top}] \to L_{m+1}(\pi_1 M)$$

as $\sigma = \eta \circ \bar{\sigma}$. It is known that η is an isomorphism after tensoring with $\mathbb{Z}[\frac{1}{2}]$. This is a consequence of Rothenberg's exact sequence, cf. [96, p. 248]. Of course η is an isomorphism before tensoring with $Z[\frac{1}{2}]$ if $Wh(\pi_1 M) = 0$.

Recall condition $(*)$ defined in Lecture 6. Farrell and Hsiang [30] split the simple assembly map when M satisfies condition $(*)$. The precise statement is the following.

7.1. Theorem. *Let M^m be a closed manifold satisfying condition $(*)$ and let $\bar{\sigma} : [M \times \mathbb{D}^n, \partial; G/Top] \to L^s_{n+m}(\pi_1 M)$ be the simple assembly map. Then $\bar{\sigma}$ is a split monomorphism provided $n \geq 2$ and $n + m \geq 7$.*

Proof. In order not to obscure the main idea of the proof, we will assume that M is triangulated and that $n = 2$. We will construct a function

$$d : L^s_{m+2}(\pi_1 M) \to [M^m \times \mathbb{D}^2, \partial; G/\mathrm{Top}]$$

such that $d \circ \bar{\sigma} = \mathrm{id}$. This shows that $\bar{\sigma}$ is a monomorphism. We will not verify that d is a group homomorphism since this will not be important to our later applications of Theorem 7.1.

To construct d, we will use the geometric description of $\bar{\sigma}$ as the forget structure map from the equivalence classes of special normal cobordisms on $M \times [0,1]$ to the equivalence classes of simple normal cobordisms on $M \times [0,1]$. Hence, given a simple normal cobordism $(W, F, \simeq)$ on $M \times [0,1]$, I must give a method which produces a special normal cobordism $(W', F', \simeq')$ on $M \times [0,1]$ in such a way that if $(W, F, \simeq)$ is already special, then $(W', F', \simeq') = (W, F, \simeq)$.

Since M satisfies condition $(*)$, identify the deck transformation action of $\pi_1(M)$ on $\tilde{M}$ with the action of $\pi_1(M)$ on $\mathrm{Int}(\mathbb{D}^m)$ mentioned in the definition of condition $(*)$. The action of $\pi_1(M)$ on $\mathbb{D}^m$ naturally extends to an action of $\pi_1(M)$ on $\mathbb{D}^{m+1}$ as indicated in Figure 4.

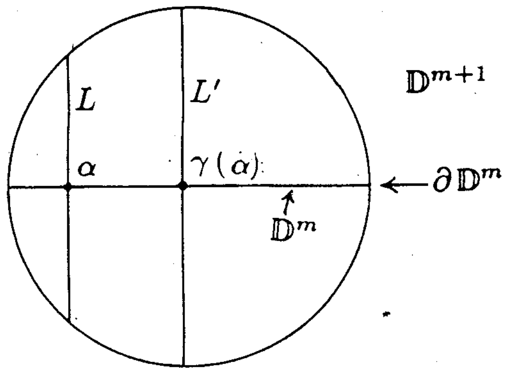

Figure 4

Here, the action of $\gamma \in \pi_1(M)$ linearly maps the vertical line segment L in $\mathbb{D}^{m+1}$ which meets $\mathbb{D}^m$ perpendicularly in the point x to the vertical line segment L' meeting $\mathbb{D}^m$ at $\gamma(x)$. Notice that the extended action satisfies the following analogues of properties 1 and 2 of condition $(*)$.

1.′ The restriction of the action of $\pi_1(M)$ to $\mathbb{D}^{m+1} - \partial\mathbb{D}^m$ is equivalent to the deck transformation action on the universal cover of $M \times [0,1]$.

2.′ Given $\epsilon > 0$ and a compact subset K of $\mathbb{D}^{m+1} - \partial\mathbb{D}^m$, there exists a number $\delta > 0$ such that the following is true for every $\gamma \in \pi_1(M)$. If the distance between γK and $\partial\mathbb{D}^m$ is less than δ, then $\operatorname{diam}(\gamma K) < \epsilon$.

Again identify the action of $\pi_1(M)$ on $\tilde{M} \times [0,1]$ with the extended action of $\pi_1(M)$ on $\mathbb{D}^{m+1} - \partial\mathbb{D}^m$. Now properties 1′ and 2′ have the following important consequence (**).

(**) Let $h : M \times I \to M \times I$ be any self-map with $h|_{M\times 0} = \mathrm{id}_{M\times 0}$. Then its unique lift $\tilde{h} : \tilde{M} \times I \to \tilde{M} \times I$, with $\tilde{h}|_{\tilde{M}\times 0} = \mathrm{id}_{\tilde{M}\times 0}$, uniquely extends to a self map $\bar{h} : \mathbb{D}^{m+1} \to \mathbb{D}^{m+1}$ by setting $\bar{h}|_{\partial\mathbb{D}^m} = \mathrm{id}_{\partial\mathbb{D}^m}$.

For the remainder of the proof, let Γ denote $\pi_1(M)$. Note that the extension $\bar{h}$ of (**) is a Γ-equivariant map. Since the universal cover $\tilde{M} \to M$ is a principal Γ-bundle, we can form the associated $\mathbb{D}^{m+1}$-bundle $\bar{E} \to M$ whose total space is $\tilde{M} \times_\Gamma \mathbb{D}^{m+1}$; i.e. $\bar{E}$ is the orbit space of the diagonal action of Γ.

The action of Γ on $\mathbb{D}^{m+1}$ leaves invariant the northern and southern hemispheres of $\partial\mathbb{D}^{m+1}$; which we denote by $\partial_+\mathbb{D}^{m+1}$ and $\partial_-\mathbb{D}^{m+1}$, respectively. It also leaves invariant the equator $\partial\mathbb{D}^m$. (See Figure 5.)

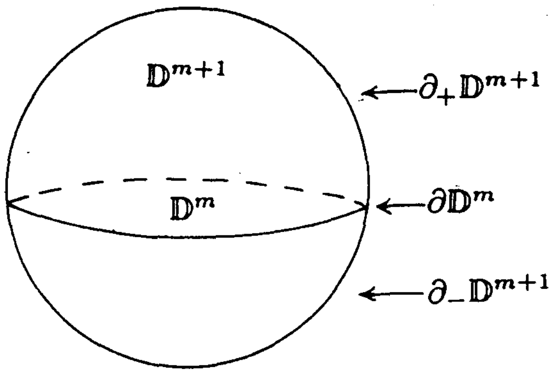

Figure 5

The associated $\partial_+\mathbb{D}^{m+1}$, $\partial_-\mathbb{D}^{m+1}$ and $\partial\mathbb{D}^m$-bundles to $\tilde{M} \to M$ are sub-bundles of

$\bar{E} \to M$. Their total spaces are denoted by $\partial_+\bar{E}$, $\partial_-\bar{E}$ and $\partial_0\bar{E}$, respectively.

Let $(W, F, \simeq)$ be a simple normal cobordism on $M \times I$. (See Figure 6.)

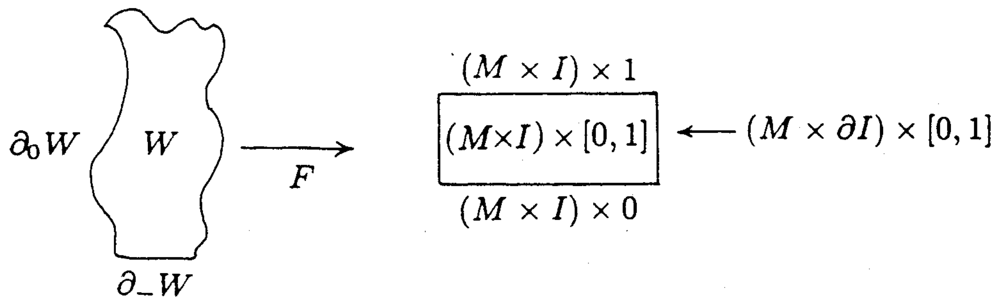

Figure 6

Recall $\partial W = \partial_- W \cup \partial_+ W \cup \partial_0 W$ and $F\big|_{\partial W} = F_- \cup F_+ \cup F_0$, where

$F_- : \partial_- W \to (M \times I) \times 0$ is the identity map;

$F_+ : \partial_+ W \to (M \times I) \times 1$ is a simple homotopy equivalence;

$F_0 : \partial_0 W \to (M \times \partial I) \times [0,1]$ is a homeomorphism.

Consequently, $\partial_+ W$ is an s-cobordism and hence a cylinder. We can therefore identify $\partial_+ W$ with $M \times I = M \times [0,1]$ in such a way that $F_+ : M \times I \to M \times I$ is a homotopy equivalence; $F_+\big|_{M\times 0} = \mathrm{id}_{M\times 0}$ and $F_+\big|_{M\times 1}$ is a self homeomorphism.

Lecture 8. Novikov's Conjecture

The proof of Theorem 7.1 is completed in this lecture. After that, I formulate a conjecture due to Novikov which is related to the splitting of the assembly map.

Let $(W, F, \simeq)$ be a simple normal cobordism over $M \times I$ where dim $M \geq 5$. Recall from lecture 7 that we have identified $\partial_+ W$ with $M \times I$ so that $h|_{M\times 0} = \mathrm{id}_M$ and $h|_{M\times 1}$ is a homeomorphism, where h is an abbreviation for F_+. Let $\tilde{h} : \tilde{M} \times I \to \tilde{M} \times I$ denote the unique lift satisfying $\tilde{h}|_{\tilde{M}\times 0} = \mathrm{id}_{\tilde{M}\times 0}$. And let $\bar{h} : \mathbb{D}^{m+1} \to \mathbb{D}^{m+1}$ be the unique extension of $\tilde{h}$ which exists because M satisfies condition $(*)$ and hence condition $(**)$ is satisfied. (See Lecture 7.) Note the following properties of $\bar{h}$.

0. $\bar{h}$ is Γ-equivariant. (Recall $\Gamma = \pi_1 M$.)
1. $\bar{h}|_{\partial\mathbb{D}^{m+1}}$ is a self homeomorphism of $\partial\mathbb{D}^{m+1}$
2. $\bar{h}|_{\partial_-\mathbb{D}^{m+1}} = \mathrm{id}_{\partial_-\mathbb{D}^{m+1}}$.
3. $\bar{h}^{-1}(\partial\mathbb{D}^m) = \partial\mathbb{D}^m$.

Recall, also from Lecture 7, that $q : \bar{E} \to M$ is the $\mathbb{D}^{m+1}$-bundle associated to the universal cover $\tilde{M} \to M$. Let $E = \bar{E} - \partial_0\bar{E}$; it is identified with the total space of the $\tilde{M} \times [0,1]$-bundle associated to $\tilde{M} \to M$. (Remember the Γ-spaces $\tilde{M} \times [0,1]$ and $\mathbb{D}^{m+1} - \partial\mathbb{D}^m$ were identified in Lecture 7.) So property 0 yields that $\mathrm{id}_{\tilde{M}} \times \bar{h}$ induces a self bundle map of $\bar{E} \to M$ covering id_M. Denote this map by $k : \bar{E} \to \bar{E}$. Fix a triangulation of M. The following properties $1'$, $2'$, $3'$ of k are consequences of the corresponding properties 1, 2, 3 of $\bar{h}$. And property $0'$ is obvious since k is a bundle map covering id_M.

$0'$. $k(q^{-1}(\Delta)) \subset q^{-1}(\Delta)$ for each closed simplex Δ of the fixed triangulation of M.

$1'$. $k\big|_{\partial \bar{E}}$ is a self homeomorphism of $\partial \bar{E}$.

$2'$. $k\big|_{\partial_- \bar{E}} = \mathrm{id}_{\partial_- \bar{E}}$.

$3'$. $k^{-1}(\partial_0 \bar{E}) = \partial_0 \bar{E}$.

8.1 Lemma. *There is a homotopy k_t $(0 \leq t \leq 1)$ of k such that $k_0 = k$, k_1 is a homeomorphism, and each map k_t satisfies properties $0'$, $1'$, $2'$, $3'$.*

Proof. The construction of k_t proceeds by induction over the skeleta of M via a standard obstruction theory argument. Note the following. If Δ is an n-simplex in M, then $q^{-1}(\Delta)$ is homeomorphic to $\mathbb{D}^{n+m+1}$. Hence, the obstructions encountered in extending the homotopy from over the $(n-1)$-skeleton to over the n-skeleton lie in $\mathcal{S}(\mathbb{D}^{n+m+1}, \partial)$. But these all vanish because of Theorem 6.2; i.e., $|\mathcal{S}(\mathbb{D}^s, \partial)| = 1$ when $s \geq 5$. Q.E.D.

Let us now continue with the proof of Theorem 7.1. Consider the universal cover $\tilde{W}$ of W and let $\tilde{F} : \tilde{W} \to (\tilde{M} \times I) \times [0,1]$ be the lift of F such that $\tilde{F}_- = \mathrm{id}_{\tilde{M} \times I}$. (Recall $F_- = \mathrm{id}_{M \times I}$). Since $\tilde{W}$ is a Γ-space, we can let $V^{2m+2} \to M$ be the $\tilde{W}$-bundle associated to the universal cover $\tilde{M} \to M$. Note that $E \times I \to M$ is the $(\tilde{M} \times I) \times I$-bundle associated to $\tilde{M} \to M$. Hence $\mathrm{id}_{\tilde{M}} \times \tilde{F}$ induces a bundle map $G : V \to E \times I$ covering id_M. Let $\partial_+ \tilde{W}$, $\partial_- \tilde{W}$, $\partial_0 \tilde{W}$ be the parts of $\tilde{W}$ lying over $\partial_+ W$, $\partial_- W$, $\partial_0 W$, respectively. These are Γ-invariant subspaces of $\partial \tilde{W}$ and $\partial \tilde{W} = \partial_+ \tilde{W} \cup \partial_0 \tilde{W} \cup \partial_- \tilde{W}$. Denote the total spaces of associated subbundles by $\partial_+ V$, $\partial_0 V$, $\partial_- V$, respectively. We can identify $G\big|_{\partial_- V} : \partial_- V \to E \times 0$ with id_E, and $G\big|_{\partial_+ V} : \partial_+ V \to E \times 1$ with $k\big|_E$. Also observe that $G\big|_{\partial_0 V} : \partial_0 V \to \partial E \times I$ is a homeomorphism. We may assume, after a special homotopy using Lemma 8.1, that $G\big|_{\partial_+ V}$ is a homeomorphism and hence

(1) $$G\big|_{\partial V} : \partial V \to \partial(E \times I) \text{ is a homeomorphism.}$$

Identify M with the submanifold of $\tilde{M} \times_\Gamma \tilde{M}$ which is the image of diagonal in $\tilde{M} \times \tilde{M}$ under the quotient map. (We think of this submanifold as the "zero-section" to the bundle $\tilde{M} \times_\Gamma \tilde{M} \to M$.) We also, in this way, identify $M \times I^2$ with a submanifold of $E \times I$ using that $E \times I = (\tilde{M} \times_\Gamma \tilde{M}) \times I^2$. Note that

$$\partial(M \times I^2) \subset \partial(E \times I). \tag{2}$$

Because of facts (1) and (2), we can make G transverse to $M \times I^2$ rel ∂. Then, set $\mathcal{W}^{m+2} = G^{-1}(M \times I^2)$ and let $\mathcal{F} : \mathcal{W}^{m+2} \to M \times I^2$ be $G|_{\mathcal{W}^{m+2}}$. Note that $F|_{\partial\mathcal{W}} : \partial\mathcal{W} \to \partial(M \times I^2)$ is a homeomorphism which over $(M \times I) \times 0$ is $\mathrm{id}_{M \times I}$. The isomorphism $\simeq$ also naturally induces an isomorphism $\cong$ of the stable normal bundle of $\mathcal{W}$ to a bundle over $M \times I^2$. The triple $(\mathcal{W}, \mathcal{F}, \cong)$ is therefore a special normal cobordism over $M \times I$. One checks that the correspondence

$$(W, F, \simeq) \mapsto (\mathcal{W}, \mathcal{F}, \cong)$$

sends equivalent simple normal cobordisms to equivalent special normal cobordisms and hence defines a function

$$d : L^s_{m+2}(\pi_1 M) \to [M \times I^2, \partial; G/\mathrm{Top}].$$

(This checking uses a relative version of Lemma 8.1.)

Now suppose that $(W, F, \cong)$ is a special normal cobordism. Then, in constructing $(\mathcal{W}, \mathcal{F}, \cong)$, it is unnecessary to use Lemma 8.1. Hence $\mathcal{W}$ is the "graph of F". To be more precise, let $\tilde{f} : \tilde{W} \to \tilde{M}$ denote the composition of $\tilde{F} : \tilde{W} \to \tilde{M} \times I^2$ with projection onto the first factor of $\tilde{M} \times I^2$. Then $\mathcal{W} = G^{-1}(M \times I^2)$ is the image of *graph* $(\tilde{f}) \subset \tilde{M} \times \tilde{W}$ under the quotient map to $V = \tilde{M} \times_\Gamma \tilde{W}$. Since the map $x \to (\tilde{f}(x), x)$ is Γ-equivariant, it determines a cross-section $\hat{f}$ to the bundle $V \to W$ and $\hat{f} : W \to \mathcal{W}$ is a homeomorphism such that $\mathcal{F} \circ \hat{f} = F$. In this way, we see that $(W, F, \simeq)$ and $(\mathcal{W}, \mathcal{F}, \cong)$ are isomorphic, hence equivalent, special normal cobordisms. Therefore, $d \circ \sigma = \mathrm{id}$. Q.E.D.

Remark 1. Lemma 8.1 is the key to the proof of Theorem 7.1. Note we only needed a proper homotopy rel ∂ of $k|_E$ to a homeomorphism. But the obstructions encountered in extending such a homotopy from the $(n-1)$-skeleton of M to the n-skeleton lie in $\pi_{n+1}(G/\mathrm{Top})$ and these groups are $\mathbb{Z}$, $\mathbb{Z}_2$ or 0 depending on the congruence class of n mod 4. But condition $(*)$ enabled us to convert the problem to one over $\bar{E}$ where the obstructions were automatically 0, since $|\mathcal{S}(\mathbb{D}^n, \partial)| = 1$ for all $n \geq 5$.

Remark 2. The simple assembly map $\bar{\sigma} : [M^m \times \mathbb{D}^n, \partial; G/\mathrm{Top}] \to L^s_{n+m}(\pi_1 M^m)$ is also a split injection when $M^m = \Gamma\backslash G/K$, where G is a virtually connected Lie group, K is a maximal compact subgroup of G and Γ is a co-compact, discrete, torsion-free subgroup of G. This is proven by Farrell and Hsiang in [33], [34]. Here it is not known, in general, that M^m satisfies condition $(*)$. (It does when G is a semi-simple linear Lie group, since M^m then supports a non-positively curved Riemannian metric.) A weaker condition than condition $(*)$ does hold, and hence a stronger fact is needed than that $|\mathcal{S}(\mathbb{D}^n, \partial)| = 1$ for $n \geq 5$. The needed fact is that $|\mathcal{S}(N^n \times \mathbb{D}^i, \partial| = 1$ for every closed infranilmanifold N^n, provided $n + i \geq 5$. This fact was proven by Farrell and Hsiang in [32].

Theorem 7.1 has the following geometric consequence which is also proven in [30].

8.2. Corollary. *Let $f : N \to M$ be a homotopy equivalence between closed manifolds such that M supports a non-positively curved Riemannian metric. Then, $f \times id : N \times \mathbb{R}^3 \to M \times \mathbb{R}^3$ is properly homotopic to a homeomorphism.*

This corollary is a consequence of the surgery exact sequence together with another fundamental result in surgery theory; namely, the π-π theorem, due to C.T.C. Wall. (See [30] for details.)

Remark 3. It was conceivable, when [30] was written, that every closed aspherical manifold M^m satisfies condition $(*)$. But Davis [21] showed this is not so. His examples where the universal cover $\tilde{M}^m$ is not homeomorphic to $\mathbb{R}^m$ contradict property 1 of condition $(*)$. (See Remark 1 in Lecture 3.)

On the other hand, $M^m \times S^1$ satisfies property 1 of condition $(*)$ whenever $\tilde{M}^m$ is homeomorphic to $\mathbb{R}^m$. This is seen as follows. Let $\mathbb{Z}$ denote the additive group of integers. Its natural action by translations on $\mathbb{R}$ extends to an action on $[-\infty, +\infty)$ where each group element fixes $-\infty$. We hence have a product action of $\pi_1(M \times S^1) = \pi_1(M) \times \mathbb{Z}$ on $\tilde{M} \times [-\infty, +\infty) = \mathbb{R}^m \times [0, +\infty)$ which extends to its one point compactification $\mathbb{D}^{m+1}$. If we let this be the action posited in Definition 6.3, then it satisfies property 1 of condition $(*)$; but, not property 2.

Note that the universal cover X of $M^m \times S^1$ is homeomorphic to $\mathbb{R}^{m+1}$ when $m \geq 5$. This fact is due to Newman [51], since X is contractible and simply connected at infinity. Consequently, $M^m \times S^1 \times S^1 \times S^1$ satisfies property 1 of condition $(*)$ whenever M^m is a closed aspherical manifold with $m \geq 4$. Also, if one examines the proof in [30] of Corollary 8.2, it is seen that this result remains true when the condition "M is non-positively curved" is replaced by the weaker condition "$M \times S^1 \times S^1 \times S^1$ satisfies condition $(*)$."

There is the following question apropos Remark 3.

8.3. Question. Let M^m be a closed manifold such that $\pi_1 M^m$ is virtually solvable. Suppose M^m satisfies condition $(*)$. Does this imply that $\pi_1 M$ is virtually abelian?

This question is motivated by Yau's result [98] that such an M cannot support a non-positively curved Riemannian metric which is not flat.

We end this lecture with a description of a conjecture due to Novikov and its relation to splitting the assembly map. There are two sets of rational characteristic classes associated to a manifold M; namely, its rational Pontryagin classes $p_i(M)$ and its L-genera $L_i(M)$; both of which are elements of $H^{4i}(M, \mathbb{Q})$ and are defined for all integers $i \geq 0$. They contain essentially the same information about M since the L-genera are polynomials in the Pontryagin classes and vice versa. Novikov [78] proved the fundamental fact that the rational Pontryagin classes are topological invariants; i.e., if $f : M \to N$ is a homeomorphism between manifolds, then $f^*(p_i(N)) = p_i(M)$. This is, of course, equivalent to the analogous statement for L-genera.

Associate to any map $f : M \to N$ elements $L_i(f) \in H^{4i}(M, \mathbb{Q})$ defined by

$$L_i(f) = L_i(M) - f^*(L_i(N)).$$

Then Novikov's theorem is equivalent to saying $L_i(f) = 0$, for all $i \geq 0$, when f is a homeomorphism. Now it is easy to construct examples where this vanishing fails when f is merely a homotopy equivalence. But Novikov conjectured a partial vanishing result which we proceed to formulate.

8.4. Definition. Given a group Γ, let $H^i_\Gamma(M, \mathbb{Q})$ denote the subset of $H^i(M, \mathbb{Q})$ consisting of all elements of the form $\bar{\phi}^*(x)$ where $x \in H^i(\Gamma, \mathbb{Q})$ and $\phi : \pi_1 M \to \Gamma$ is a group homomorphism. Here, $\bar{\phi} : M \to K(\Gamma, 1)$ is the continuous map induced by ϕ.

8.5. Conjecture. (Novikov [79]) Let $f : M^m \to N^m$ be a homotopy equivalence between closed (connected) orientable manifolds. Then the cup products $L_i(f) \cup x$ vanish for all i and every $x \in H^{m-4i}_\Gamma(M, \mathbb{Q})$, where Γ is an arbitrary group.

If we fix a group Γ but allow f, M and N to vary, then the above assertion is called Novikov's conjecture for the group Γ.

Remark 4. Suppose $H^*(M, \mathbb{Q}) = H^*_\Gamma(M, \mathbb{Q})$. This happens, for example, when M is aspherical and $\pi_1 M = \Gamma$. With this assumption, the rational Pontryagin classes of M are invariants of homotopy equivalence provided Novikov's conjecture for Γ is true. This assertion follows from Poincare duality.

Wall [97, pp. 263-267], expanding on ideas of Novikov [79], gives the following relationship between Novikov's conjecture and the assembly map.

8.6. Theorem. *Let M^m be a compact, orientable, aspherical manifold with $\pi_1 M^m = \Gamma$. Then, Novikov's conjecture for Γ is true if and only if the (rational) assembly maps*

$$\bar{\sigma}_n : [M^m \times \mathbb{D}^n, \partial; G/Top] \otimes \mathbb{Q} \to L^s_{n+m}(\pi_1 M^m) \otimes \mathbb{Q}$$

are monomorphisms for all integers n satisfying both $n \geq 2$ and $n + m \geq 7$.

Remark 5. Hence, Theorem 7.1 implies that Novikov's conjecture for Γ is true when $\Gamma = \pi_1 M$ and M is a closed (connected) non-positively curved Riemannian manifold.

However, this result was proven much earlier and via a different technique in Mishchenko's seminal paper [73].

Remark 6. Although much work has been done verifying Novikov's conjecture for a very large class of groups Γ, it remains open and is still an active area of research. See Kasparov's paper [64] for a description of the state of the conjecture as of 1988. Additional important work on it has been done since that date.

Lecture 9. Geometric groups

Connell and Hollingsworth [20] introduced the notion of geometric group in the hope of proving the topological invariance of Whitehead torsion. Although they did not succeed at that time, their concept was revived by F. Quinn [84] who showed that it is a useful framework in which to prove "control theorems" in topology.

A *geometric* group G on a metric space X is a finite sequence $x_1, \ldots, x_n$ of points in X; G also denotes the free abelian group equipped with the ordered basis $x_1, \ldots, x_n$. (More precisely, the basis is $(1, x_1), (2, x_2), \ldots, (n, x_n)$; i.e., if $i \neq j$ but $x_i = x_j$, we will consider x_i and x_j to be distinct elements in G.)

Remark. One can similarly define the concept of geometric R-module over X where R is a fixed ring. This notion is particularly useful when $R = \mathbb{Z}\Pi$ where Π is a discrete group. But, for the present, we will stick to the case $\Pi = 1$; i.e., $R = \mathbb{Z}$.

Let G_1 and G_2 be two geometric groups with bases $\{x_i\}$ and $\{y_j\}$, respectively, and $f : G_1 \to G_2$ be a homomorphism. It determines a set valued function $C(f)$ defined by

$$C(f)(x_i) = \{y_i \mid y_i \text{ has nonzero coefficient in } f(x_i)\}.$$

The *diameter* of f, denoted diam f, is defined to be

$$\sup_i \textit{diameter}\,(\{x_i\} \cup C(f)(x_i)).$$

9.1. Definition. A homomorphism $f : G \to G$ is a δ-endomorphism if diam $f \leq \delta$. And f is a δ-automorphism if f is invertible and both f and f^{-1} are δ-endomorphisms.

Assume from now on that X is compact, locally contractible and arc-connected; e.g., X could be either a finite (connected) simplicial complex or a compact (connected) manifold. Fix a pair of positive real numbers $\delta_1 > \delta_0 > 0$ such that the following metric connectedness conditions are satisfied.

1. Any two points $x, y \in X$ with $d(x, y) \leq \delta_0$ are connected by an arc of diameter $\leq \delta_1$.

2. Any closed curve of diameter $\leq 2\delta_1$ is null homotopic.

Let $\Gamma = \pi_1(X, *)$ and $M_n(\mathbb{Z}\Gamma)$ be the ring of all $n \times n$ matrices with entries in $\mathbb{Z}\Gamma$. Given any δ_0-endomorphism $f : G \to G$ of a geometric group on X, there is associated an element $\hat{f} \in M_n(\mathbb{Z}\Gamma)$ defined as follows. Let $x_1, x_2, \dots, x_n \in X$ be the basis of G. The construction of $\hat{f}$ depends on a choice of paths $\alpha_1, \dots, \alpha_n$ in X such that α_i connects $*$ to x_i. Let $f_{ij} \in \mathbb{Z}$ denote the coefficient of x_i in the expression of $f(x_j)$; i.e.,

$$f(x_j) = \sum_i f_{ij} x_i.$$

For each pair of indices i, j such that $d(x_i, x_j) \leq \delta_0$, pick a path γ_{ij} connecting x_j to x_i and such that diameter $\gamma_{ij} \leq \delta_1$. This is possible because of condition 1. (The construction of $\hat{f}$ will be independent of this choice.) We now define the i, j entry of $\hat{f}$ by the formula

$$\hat{f}_{ij} = \begin{cases} f_{ij}\alpha_i^{-1} * \gamma_{ij} * \alpha_j & \text{where } f_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_i^{-1} * \gamma_{ij} * \alpha_j$ is the homotopy class of the closed loop gotten by concatenating the paths α_j, α_{ij}, α_i^{-1}. Condition 2 shows that $\hat{f}$ is independent of the choice γ_{ij}. We now examine its dependence on $\{\alpha_i\}$. Suppose new paths $\{\beta_i\}$ are chosen and let $\gamma_i \in \pi_1(X, *)$ be the homotopy class represented by the closed loop $\beta_i^{-1} * \alpha_i$. Let D denote the diagonal $n \times n$ matrix defined by $D_{ii} = \gamma_i$. A consideration of Figure 7 yields the first result below.

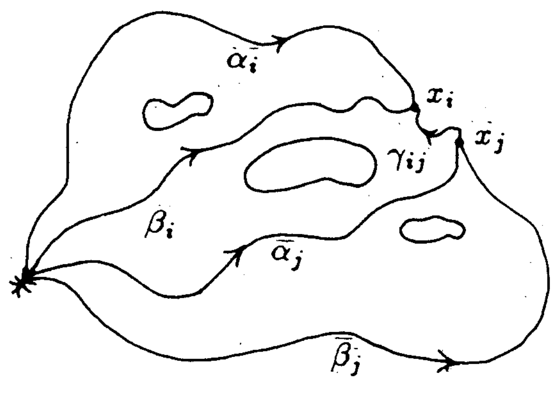

Figure 7

9.2. Lemma. *If we choose new paths β_i as above, then the matrix associated to f changes from $\hat{f}$ to $D\hat{f}D^{-1}$.*

9.3. Lemma. *Suppose $f, g : G \to G$ are two $\delta_0/3$-endomorphisms of the geometric group G. Then their composition $f \circ g$ is a δ_0-endomorphism of G and $\widehat{f \circ g} = \hat{f}\hat{g}$.*

Proof. It is clear that $f \circ g$ is a δ_0-endomorphism. Fix a choice of paths α_i and γ_{ij} as before. Because of conditions 1 and 2, the concatenation $\gamma_{ik} * \gamma_{kj}$ is homotopic to γ_{ij}. Consider the following calculation

$$\begin{aligned}
(\hat{f}\hat{g})_{ij} &= \sum_k \hat{f}_{ik}\hat{g}_{kj} \\
&= \sum_k f_{ik}(\alpha_i^{-1} * \gamma_{ik} * \alpha_k) g_{kj}(\alpha_k^{-1} * \gamma_{kj} * \alpha_j) \\
&= \sum_k f_{ik} g_{kj}(\alpha_i^{-1} * \gamma_{ik} * \gamma_{kj} * \alpha_j) \\
&= \left(\sum_k f_{ik} g_{kj}\right) \alpha_i^{-1} * \gamma_{ij} * \alpha_j \\
&= (fg)_{ij}\alpha_i^{-1} * \gamma_{ij} * \alpha_j \\
&= (\widehat{f \circ g})_{ij}. \quad \text{Q.E.D.}
\end{aligned}$$

Remark. If $f_1, f_2, \ldots, f_n$ are endomorphisms of G such that diam $f_i \leq a$ for each i, then $\operatorname{diam}(f_n \circ \ldots \circ f_1) \leq 2na$.

9.4. Corollary. *If $f : G \to G$ is a $\delta_0/3$-automorphism, then $\hat{f} \in GL_n(\mathbb{Z}\Gamma)$ and determines a well defined element in $Wh(\Gamma)$, which we also denote by $\hat{f}$.*

Proof. Apply Lemma 9.3 with $g = f^{-1}$. Then,

$$\widehat{\mathrm{id}} = \widehat{f \circ g} = \hat{f} \circ \hat{g} \quad \text{and}$$

$$\widehat{\mathrm{id}} = \widehat{g \circ f} = \hat{g} \circ \hat{f}.$$

But clearly $\widehat{\mathrm{id}} = \mathrm{id}$. Therefore, $\hat{f}$ is invertible. And its value in $Wh(\Gamma)$ is independent of the choice of the paths α_i, because of Lemma 9.2.

Lecture 10. Connell-Hollingsworth Conjecture

This lecture continues the discussion of geometric groups started in Lecture 9.

10.1. Definition. An automorphism $f : G \to G$ is *ϵ-blocked* if there exists a partition $\mathcal{P}$ of the basis $\{x_1, \ldots, x_n\}$ for the geometric group G such that, for each set $S \in \mathcal{P}$, we have

1. diam $S \leq \epsilon$ and
2. $f(\bar{S}) \subset \bar{S}$, where $\bar{S}$ denotes the subgroup of G generated by S.

10.2. Lemma. *If f is $\delta_0/3$-blocked, then $\hat{f}$ represents 0 in $Wh(\Gamma)$.*

Proof. Pick a base point $*_S$ for each partition set $S \in \mathcal{P}$ from among the elements $x_j \in S$, and pick paths α_S connecting $*$ to $*_S$. For each $x_i \in S$ pick a path $\bar{\alpha}_i$ connecting $*_S$ to x_i such that diam $\bar{\alpha}_i \leq \delta_1$. Let $\alpha_i = \bar{\alpha}_i * \alpha_S$. Then the loops $\alpha_i^{-1} * \gamma_{ij} * \alpha_j$, such that $f_{ij} \neq 0$, are all inessential as is seen from Figure 8.

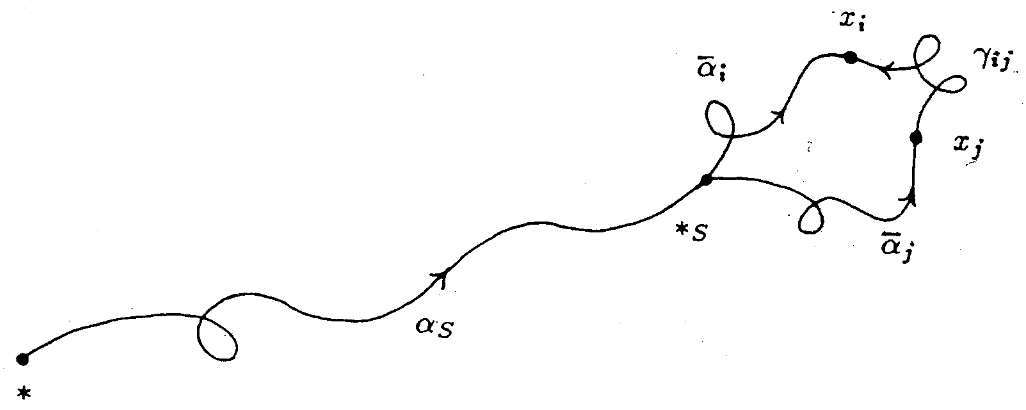

Figure 8

Therefore, $\hat{f} = f \in GL_n(\mathbb{Z})$. Hence, its value in $Wh(\Gamma)$ is in the image of $Wh(1) = 0$. Q.E.D.

Remark. Note that the entries of $\hat{f}$, where f is any $\delta_0/3$-automorphism, are monomials from $\mathbb{Z}\Gamma$. The following is an example of a monomial matrix which represents a non-zero element in $Wh\Gamma$. In this example, Γ is the cyclic group of order 5 generated by x.

Consider the unit $1+x-x^{-3}$ in $Units(\mathbb{Z}\Gamma)$. It represents a non-zero element in $Wh\Gamma$ since it is *not* a monomial. But $1+x-x^{-3}$ is equivalent to the following 3×3 matrix

$$\begin{pmatrix} 1+x-x^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by stabilization. This matrix is in turn equivalent to

$$A=\begin{pmatrix} 1 & x & x^3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

by elementary column operations. Hence A represents a non-zero element in $Wh\Gamma$ and the entries in A are all monomial. It appears likely that every element in $Wh\Gamma$ is represented by a monomial matrix for an arbitrary group Γ.

10.3. Conjecture (Connell-Hollingsworth [20]) Let X be an n-dimensional finite simplicial complex. Given $\epsilon>0$, there exists $\delta>0$ such that the following is true. Every δ-automorphism $f: G\to G$ of a geometric group on X can be factored

$$f=f_1\circ f_2\cdots\circ f_{n+1}$$

as the composite of $n+1$ ϵ-blocked automorphisms.

10.4. Lemma. *Assume Conjecture 10.3 is true and that X is a finite simplicial complex. Then there exists $\delta>0$ such that $\hat{f}$ represents 0 in $Wh\Gamma$ for any δ-automorphism $f: G\to G$ of a geometric group G on X.*

Proof. Let $n=\dim X$ and $\epsilon=\delta_0/6n$. Then δ is the number posited in Conjecture 10.3 relative to ϵ. (We may assume that $\delta\le\delta_0/3$.) Express f as the composition

$$f=f_1\circ f_2\cdots\circ f_{n+1}$$

where each f_i is ϵ-blocked. Note that each f_i is a $\delta_0/6n$-automorphism of G. Applying Lemma 9.3 and the remark following it, we obtain

(1) $$\hat{f}=\hat{f}_1\hat{f}_2\ldots\hat{f}_{n+1}.$$

But Lemma 10.2 states that each $\hat{f}_i$ represents 0 in $Wh\Gamma$. And equation (1) implies that the element in $Wh\Gamma$ represented by $\hat{f}$ is the sum of the elements represented by $\hat{f}_i$. Q.E.D.

It is useful to extend the construction $\hat{f}$ and the results about it to the case of δ-isomorphisms.

10.5. Definition. A homomorphism $f : G_1 \to G_2$ between two geometric groups on X is a δ-homomorphism if diam $f \leq \delta$. It is a δ-isomorphism if both f and f^{-1} are δ-homomorphisms.

If f is a δ_0-homomorphism, then it determines an $m \times n$ matrix $\hat{f}$ with entries in $\mathbb{Z}\Gamma$ where $m =$ rank G_1, $n =$ rank G_2. This is done by the natural analogue of the procedure given in Lecture 9 for the special case $G_1 = G_2$. But now the construction depends on choices of paths α_i from $*$ to x_i and β_j from $*$ to y_j, where $x_1, x_2, \ldots, x_n$ is the basis for G_1 and $y_1, y_2, \ldots, y_m$ is the basis for G_2. If we change the choice of these paths, then the following analogue of Lemma 9.2 is true.

10.6. Lemma. *When changes in the choices of the paths α_i, β_j are made, $\hat{f}$ changes to $\mathcal{D}\hat{f}D$ where both $\mathcal{D}$ and D are square diagonal matrices whose diagonal entries are elements in Γ.*

The following analogues of Lemma 9.3 and its Corollary 9.4 are also true.

10.7. Lemma. *Suppose $g : G_1 \to G_2$ and $f : G_2 \to G_3$ are both $\delta_0/3$-homomorphisms, then $f \circ g$ is a δ_0-homomorphism and $\widehat{f \circ g} = \hat{f}\hat{g}$.*

10.8. Corollary. *If $f : G_1 \to G_2$ is a $\delta_0/3$-isomorphism, then $\hat{f} \in GL_n(\mathbb{Z}\Gamma)$, where $n = rank\ G_1 = rank\ G_2$ and $\hat{f}$ determines a well defined element in $Wh\Gamma$, which is also denoted $\hat{f}$.*

10.9. Definition. A geometric isomorphism of geometric groups is an isomorphism induced by a bijection of their bases.

10.10. Lemma. *Let $f : G_1 \to G_2$ be a geometric $\delta_0/3$-isomorphism, then $\hat{f}$ represents 0 in $Wh\Gamma$.*

Proof. Note that $\hat{f}$ can be factored as a product of a diagonal matrix with diagonal entries in Γ and a permutation matrix. But both these represent 0 in $Wh\Gamma$. Q.E.D.

10.11. Lemma. *If two geometric groups G_1 and G_2 are isomorphic by a δ-homomorphism, then they are geometrically δ-isomorphic.*

Proof. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be the bases for the geometric groups G_1 and G_2. They are also the bases for the corresponding *geometric $\mathbb{Q}$-vector spaces* $G_1 \otimes \mathbb{Q}$ and $G_2 \otimes \mathbb{Q}$. We will show, by induction on n, that if $G_1 \otimes \mathbb{Q}$ and $G_2 \otimes \mathbb{Q}$ are isomorphic via a δ-homomorphism f, then G_1 and G_2 are geometrically δ-isomorphic. The lemma then follows since a δ-homomorphism which induces an isomorphism of G_1 to G_2 induces an isomorphism of $G_1 \otimes \mathbb{Q}$ to $G_2 \otimes \mathbb{Q}$. Let the $n \times n$ matrix $F = (f_{ij})$ be defined by

$$f(x_j) = \sum_i f_{ij} y_i. \tag{1}$$

Consider the expansion of $\det(F)$ using the last row

$$\det(F) = \sum_j (-1)^{n+j} f_{nj} \det(F^{nj}) \tag{2}$$

where F^{nj} denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the last row and j-th column from F. Since $\det(F) \neq 0$, equation (2) shows that there exists an index j such that both

$$f_{nj} \neq 0 \quad \text{and} \quad \det(F^{nj}) \neq 0. \tag{3}$$

Now, let $\bar{G}_1$ and $\bar{G}_2$ denote the geometric groups with bases $x_1, \ldots, \hat{x}_j, \ldots, x_n$ and $y_1, y_2, \ldots, y_{n-1}$, respectively. Using the second assertion in (3), we see that $\bar{G}_1 \otimes \mathbb{Q}$ and $\bar{G}_2 \otimes \mathbb{Q}$ are isomorphic via a δ-homomorphism. Hence our inductive assumption yields a geometric δ-isomorphism $\bar{g} : \bar{G}_1 \to \bar{G}_2$. We extend $\bar{g}$ to geometric isomorphism

$g : G_1 \to G_2$ by requiring $g(x_j) = y_n$. The first assertion in (3) now shows that g is a δ-isomorphism. Q.E.D.

Remark. A geometric isomorphism which is a δ-isomorphism is a δ-isomorphism. But, an isomorphism which is a δ-homomorphism need not be a δ-isomorphism in general. To construct an example, consider the matrix identity

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & \cdot & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} .$$

We can now prove the following analogue of Lemma 10.4.

10.12. Lemma. *Assume that the Conjecture 10.3 is true and that X is a finite simplicial complex. Then there exists $\delta > 0$ such that $\hat{f}$ represents 0 in $Wh\Gamma$ for any δ-isomorphism $f : G_1 \to G_2$ of geometric groups on X.*

Proof. Denote the number δ whose existence is posited in Lemma 10.4 by δ_2. (Recall $\delta_2 \le \delta_0/3$.) Then set $\delta = \delta_2/3$. Let $g : G_2 \to G_1$ be the geometric δ-isomorphism given by Lemma 10.11. Then the composite $g \circ f : G_1 \to G_1$ is a δ_2-isomorphism. Hence, $\widehat{g \circ f}$ represents 0 in $Wh\Gamma$, by Lemma 10.4. But $\widehat{g \circ f} = \hat{g}\hat{f}$, by Lemma 10.7, and $\hat{g}$ represents 0 in $Wh\Gamma$, by Lemma 10.10. Consequently, $\hat{f}$ also represents 0. Q.E.D.

Remark. The results so far described are due to Connell and Hollingsworth [20]. After proving these, Connell and Hollingsworth then proceed to show that the topological invariance of Whitehead torsion, first proven by Chapman [18], would be a consequence of Conjecture 10.3. It is also implicit in their paper that the controlled (thin) h-cobordism, eventually proved by Ferry [57], would also be a consequence of Conjecture 10.3. Our next lecture will be devoted to formulating this theorem and showing how Conjecture 10.3 implies it.

Lecture 11. Thin h-Cobordism Theorem

Recall that an h-cobordism W^{m+1} with base a closed manifold M^m is a compact manifold with boundary such that $\partial W = M \amalg N$ (disjoint union) and both M and N are deformation retracts of W. If $r_t : W \to W$, where $t \in [0,1]$, is a deformation retraction onto M, then the *tracks* of r_t are the following family of curves $\{\alpha_x \mid x \in W\}$ in M defined by

$$\alpha_x(t) = r_1(r_t(x)), \quad t \in [0,1].$$

The h-cobordism W is said to be ϵ-*controlled* if there exists a deformation retraction r_t onto M such that each of its tracks has diameter $\leq \epsilon$. (Here, some metric $d(\ ,\ )$ on M is fixed.) The next result is called the Controlled (or Thin) h-Cobordism Theorem.

11.1. Theorem. *(Ferry [51]) Let M^m be a closed, connected, smooth manifold with $m \geq 5$ and equipped with a fixed metric $d(\ ,\ )$. Then, there exists a number $\epsilon > 0$ such that every ϵ-controlled h-cobordism with base M is a cylinder; i.e., is homeomorphic to* $M \times [0,1]$.

11.2. Proposition. *The Controlled h-Cobordism Theorem is implied by the Connell-Hollingsworth Conjecture.*

The remainder of this lecture is devoted to proving Proposition 11.2. The proof given here is implicit in the Connell-Hollingsworth paper [20].

Assume Conjecture 10.3 is true and let W be an h-cobordism with base M. Smoothing theory, as developed in [67], shows that W has a smooth manifold structure inducing the given structure on M. Hence, by the s-Cobordism Theorem, W is a cylinder if its Whitehead torsion $\tau(W, M)$ is 0 in $Wh\Gamma$, where $\Gamma = \pi_1 M$. Because of Lemma 10.12, there is a number $\delta > 0$ such that $\hat{f} = 0$ in $Wh\Gamma$ for any δ-isomorphism $f : G_1 \to G_2$ between geometric groups on M. We next show that, if we set $\epsilon = \delta/(16m + 32)$ in the statement of Theorem 11.1, then there exists a δ-isomorphism $f : G_{\text{odd}} \to G_{\text{even}}$ such that

$\hat{f} = \tau(W, M)$ as elements in $Wh\Gamma$. (Here, f depends on W.) This proves Proposition 11.2, once f is constructed.

Let K be a triangulation of the pair (W, M) by small simplices; i.e., we require

$$\text{diameter } r_1(\Delta) \leq \epsilon$$

for each simplex Δ in K. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\tau_1, \tau_2, \ldots, \tau_n$ be, respectively, orderings of the odd and even dimensional open simplices of K in $W - M$. (Since W is an h-cobordism, the number of odd and even dimensional simplices is same.) Pick points $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ in M such that $x_i \in r_1(\sigma_i)$ and $y_j \in r_1(\tau_j)$. Then, G_{odd} and G_{even} are the geometric groups with bases $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, respectively.

Let (C_i, d_i) be the integral simplicial chain complex for (W, M). There is an obvious identification

(0) $$G_{\text{even}} = \bigoplus_{i \geq 0} C_{2i} \quad \text{and} \quad G_{\text{odd}} = \bigoplus_{i \geq 0} C_{2i+1}.$$

Under this identification, the differentials d_i determine a pair of homomorphisms

$$D_{\text{odd}} : G_{\text{odd}} \to G_{\text{even}}, \quad D_{\text{odd}} = \bigoplus_{i \geq 0} d_{2i+1};$$

$$D_{\text{even}} : G_{\text{even}} \to G_{\text{odd}}, \quad D_{\text{even}} = \bigoplus_{i \geq 1} d_{2i}.$$

Using the fact that the simplices of K are small, it is seen that both D_{odd} and D_{even} are ϵ-homomorphisms.

The deformation retraction r_t determines a chain contraction $c_i : C_i \to C_{i+1}$. Under the above identification, this contraction defines a pair of homomorphisms

$$\Sigma_{\text{odd}} : G_{\text{odd}} \to G_{\text{even}}, \quad \Sigma_{\text{odd}} = \bigoplus_{i \geq 0} c_{2i+1};$$

$$\Sigma_{\text{even}} : G_{\text{even}} \to G_{\text{odd}}, \quad \Sigma_{\text{even}} = \bigoplus_{i \geq 0} c_{2i}.$$

Using that the tracks of r_t and the simplices of K both have diameter $\leq \epsilon$, it is seen that both Σ_{odd} and Σ_{even} are 3ϵ-homomorphisms.

Let $f : G_{\mathrm{odd}} \to G_{\mathrm{even}}$ be the sum $D_{\mathrm{odd}} + \Sigma_{\mathrm{odd}}$. Then, f is a 4ϵ-homomorphism. Likewise, $g : G_{\mathrm{even}} \to G_{\mathrm{odd}}$ is also a 4ϵ-homomorphism, where $g = D_{\mathrm{even}} + \Sigma_{\mathrm{even}}$. The following calculation is a consequence of the fact that (C_i, c_i) is a chain contraction for (C_i, d_i).

$$
\begin{aligned}
f \circ g &= (D_{\mathrm{odd}} + \Sigma_{\mathrm{odd}}) \circ (D_{\mathrm{even}} + \Sigma_{\mathrm{even}}) \\
&= D_{\mathrm{odd}} D_{\mathrm{even}} + \Sigma_{\mathrm{odd}} \Sigma_{\mathrm{even}} + \Sigma_{\mathrm{odd}} D_{\mathrm{even}} + D_{\mathrm{odd}} \Sigma_{\mathrm{even}} \qquad (1) \\
&= 0 + \Sigma_{\mathrm{odd}} \Sigma_{\mathrm{even}} + \mathrm{id}_{(G_{\mathrm{even}})}.
\end{aligned}
$$

Since $\Sigma_{\mathrm{odd}} \Sigma_{\mathrm{even}}$ raises degree by 2, it is nilpotent. Consequently,

$$
f^{-1} = g \circ (1 + \alpha + \alpha^2 + \ldots + \alpha^s) \qquad (2)
$$

where $s = [(m+1)/2]$ and $\alpha = -\Sigma_{\mathrm{odd}} \Sigma_{\mathrm{even}}$. (Recall that $m + 1 = \dim W$.) Using the Remark after Lemma 9.3 together with equation (2), we conclude that f^{-1} is a $16(m+2)\epsilon$-homomorphism; i.e., it is a δ-homomorphism. Consequently, f is a δ-isomorphism and hence $\hat{f}$ represents 0 in $Wh\Gamma$, because of Lemma 10.12 and the assumption that Conjecture 10.3 is true.

It is a pleasant exercise, using 11.3 below, to show that the matrices $\hat{d}_i$ represent the differentials in the simplicial chain complex $H_i(\tilde{W}, \tilde{M})$ in terms of a natural $\mathbb{Z}\Gamma$ basis given by lifts of the simplices in K. (Here, $\tilde{W}$ denotes the universal cover of W. Also, $d_i : C_i \to C_{i-1}$ is regarded as a δ-homomorphism between geometric submodules of G_{even} and G_{odd} using the identifications given in (0).) Likewise, consider $\sigma_i : C_i \to C_{i+1}$. Using Lemma 10.7 on the equation

$$
d_{i+1} \sigma_i + \sigma_{i-1} d_i = \mathrm{id},
$$

we obtain the matrix equation

$$
\hat{d}_{i+1} \hat{\sigma}_i + \hat{\sigma}_{i-1} \hat{d}_i = I.
$$

Hence, the matrices $\hat{\sigma}_i$ give a chain contraction of the $\mathbb{Z}\Gamma$-chain complex $(H_i(\tilde{W}, \tilde{M}), \hat{d}_i)$. Therefore, the Whitehead torsion $\tau(W, M)$ of this chain complex is represented by

$$
\hat{D}_{\mathrm{odd}} + \hat{\Sigma}_{\mathrm{odd}} = \hat{f}.
$$

The s-Cobordism Theorem now implies that W is a cylinder since f represents 0 in $Wh\Gamma$. Q.E.D.

11.3. Second method of constructing $\hat{f}$. We end this lecture by giving an alternative method for constructing $\hat{f}$, where $f : G_1 \to G_2$ is a δ-homomorphism of geometric groups on a compact Riemannian manifold X. Let $p : \tilde{X} \to X$ denote the universal covering space of X, and identify $\Gamma = \pi_1(X, *)$ with the group of all its deck transformations by picking a base point $\tilde{*} \in \tilde{X}$ with $p(\tilde{*}) = *$. Each G_i induces a *geometric* $\mathbb{Z}\Gamma$*-module* p^*G_i on $\tilde{X}$ as follows. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ be respectively the bases of G_1 and G_2, and α_i, β_j be a choice of paths connecting $*$ to x_i, y_j. Let $\tilde{\alpha}_i$, $\tilde{\beta}_j$ be the lifts of α_i, β_j to $\tilde{X}$ starting at $*$, and let $\tilde{x}_i$, $\tilde{y}_j$ be the endpoints of $\tilde{\alpha}_i$, $\tilde{\beta}_j$, respectively. Then, p^*G_1 and p^*G_2 are the free $\mathbb{Z}\Gamma$-modules with bases $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ and $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m$, respectively. Note that p^*G_1 is a, perhaps infinitely generated, "geometric group" on $\tilde{X}$ with basis $\{\gamma\tilde{x}_i \mid 1 \leq i \leq n, \gamma \in \pi_1(X)\}$. A similar remark holds for p^*G_2.

Observe the following consequences of the fact that p is a local isometry when δ is picked to be smaller than the injectivity radius of X.

Observation. Let $\tilde{x} \in \tilde{X}$, $y \in X$ be any pair of points such that $d(p(\tilde{x}), y) \leq \delta$. Then, there is a unique point $\hat{y} \in \tilde{X}$, with $d(\tilde{x}, \hat{y}) \leq \delta$, solving the equation $p(\hat{y}) = y$.

Recall that the δ-homomorphism f defines an integral matrix (a_{ij}) via the system of equations

$$f(x_j) = \sum_{i=1}^{n} a_{ij} y_i.$$

Since $a_{ij} \neq 0$ implies $d(x_j, y_i) \leq \delta$, we can apply the observation to the pair $\tilde{x}_j$, y_i. This yields a point $\hat{y}_{ij}$ with $p(\hat{y}_{ij}) = y_i$. And, there is a unique group element $\gamma_{ij} \in \Gamma$ such that $\gamma_{ij}\tilde{y}_i = \hat{y}_{ij}$, since also $p(\tilde{y}_i) = y_i$. It can be seen, in this way, that $\hat{f}$ is the $\mathbb{Z}\Gamma$-matrix whose entries are $\hat{f}_{ij} = a_{ij}\gamma_{ij}$.

Lecture 12. Quinn's Theorem

Quinn [84] used Kirby's torus trick [66] to verify the stable version of the Connell-Hollingsworth conjecture. He then gave an alternate proof of Ferry's thin h-cobordism theorem by the method, due to Connell-Hollingsworth, given in the last lecture. Namely, he proved the following result.

12.1. Theorem. *Let M^m be a closed (connected) smooth manifold (equipped with a compatible metric $d(\ ,\ )$) and let $\epsilon > 0$ be a real number. Then there exists a real number $\delta > 0$ such that any δ-automorphism f of a geometric group G on M is stably the composition of $m+1$ ϵ-blocked automorphisms. That is, there exists a second geometric group G_0 such that the automorphism*

$$f \oplus id_{G_0} : G \oplus G_0 \to G \oplus G_0$$

can be factored as the composition of $m+1$ ϵ-blocked automorphisms of $G \oplus G_0$.

This lecture and the next are devoted to proving Theorem 12.1. We start by introducing some notation. Let $A \subset M$ be a subset and G be a geometric group on M. Then $G\big|_A$ is the geometric subgroup on M generated by the basis elements of G in A. Note that $G = G\big|_A \oplus G\big|_{M-A}$. Let i_A and p_A be the inclusion $i_A : G\big|_A \to G$ and projection $p_A : G \to G\big|_A$, respectively.

12.2. Definition. An endomorphism $f : G \to G$ is a δ-automorphism over A if there exists an endomorphism $g : G \to G$ satisfying the following.

1. Both f and g are δ-endomorphisms.
2. $p_A \circ f \circ g \circ i_A = \mathrm{id}_{G|_A} = p_A \circ g \circ f \circ i_A$.

If $A \subset M$, let $A^\delta = \{x \in M \mid d(x, A) \leq \delta\}$.

The next result follows directly from the definitions.

12.3. Lemma. *Let* $A^{\delta} \subset B$, $B^{\delta} \subset C$ *and* $C \subset M$. *If* $f : G \to G$ *is a* δ*-automorphism of a geometric group on* M. *Then* $f \circ p_B : G_C \to G_C$ *is a* δ*-automorphism over* A *where* $G_C = G|_C$ *is considered as a geometric group on* C.

The key step in the proof of Theorem 12.1 is the following lemma which Quinn proves by a variant of Kirby's torus trick.

12.4. Lemma. *Fix an integer* $n \geq 2$. *Given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that the following is true for any geometric group* G *on* $\mathbb{D}^n$ *and any* δ'*-automorphism* $f : G \to G$ *over* $(2/3)\mathbb{D}^n$, *where* $\delta' \leq \delta$. *There exists a second geometric group* H *on* $\mathbb{D}^n - \frac{1}{2}\mathbb{D}^n$ *and a* ϵ*-automorphism* $\bar{f} : G \oplus H \to G \oplus H$ *with*

$$\bar{f}\big|_{\frac{2}{5}\mathbb{D}^n} = f\big|_{\frac{2}{5}\mathbb{D}^n}.$$

12.5. Remark. Let $\mathcal{G} = G \oplus H$. If $\epsilon + \delta' \leq \frac{1}{15}$, then $\bar{f}^{-1} \circ (f \oplus \mathrm{id}_H) : \mathcal{G} \to \mathcal{G}$ is blocked relative to the decomposition

$$\mathcal{G} = \mathcal{G}\big|_{\frac{1}{3}\mathbb{D}^n} \oplus \mathcal{G}\big|_{\mathbb{D}^n - \frac{1}{3}\mathbb{D}^n}$$

and $\bar{f}^{-1} \circ (f \oplus \mathrm{id}_H)$ restricted to the 1st block is the identity map.

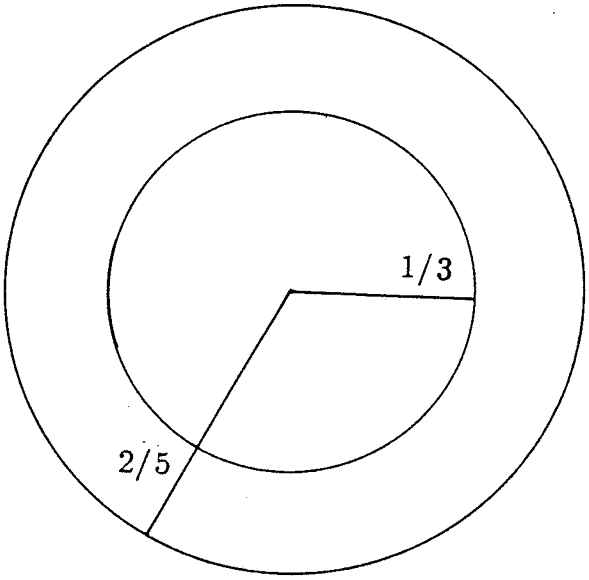

Figure 9

Lemma 12.4 will be proved in the next lecture after we use it in this lecture to prove Theorem 12.1. We start by giving a first approximation to our proof and then discuss the modifications needed to give a complete proof. Fix a handle body decomposition of M where each handle has diameter $\leq \epsilon/4$. Such a handlebody can be constructed as dual to a sufficiently fine triangulation K of M. See Figure 10.

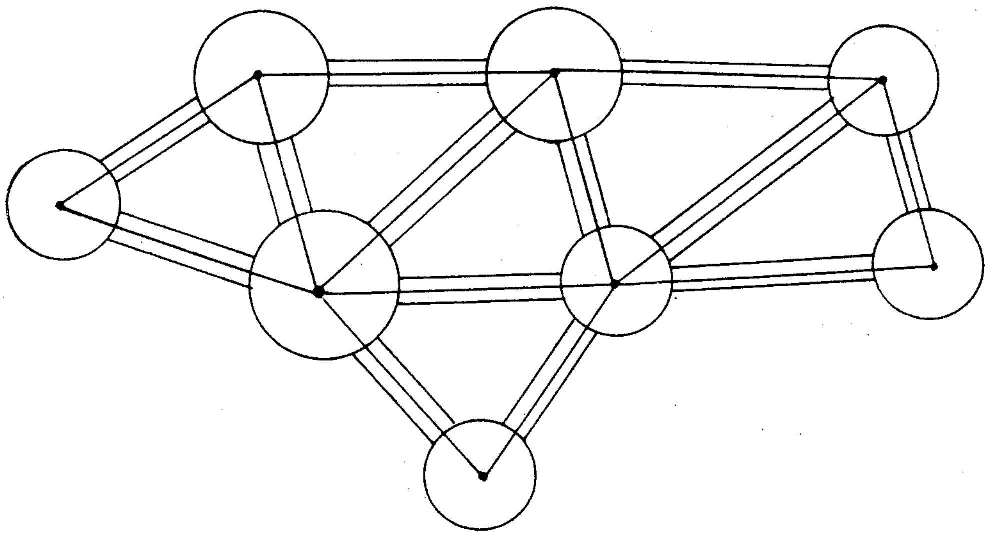

Figure 10

Let M_j = union of all the closed i-handles where $i \leq j$. We can arrange, for each j-handle, an embedding

$$h : (\mathbb{D}^j \times \mathbb{D}^{m-j}, S^{j-1} \times \mathbb{D}^{m-j}) \to (M - \text{Int } M_{j-1}, \partial M_{j-1})$$

so that the following conditions are satisfied.

1. Diameter (image h) $\leq \epsilon/2$.

2. These embeddings have disjoint images.

3. The j-handle $= h(\mathbb{D}^j \times \frac{1}{3}\mathbb{D}^{m-j})$.

See Figure 11 for $j = 1$.

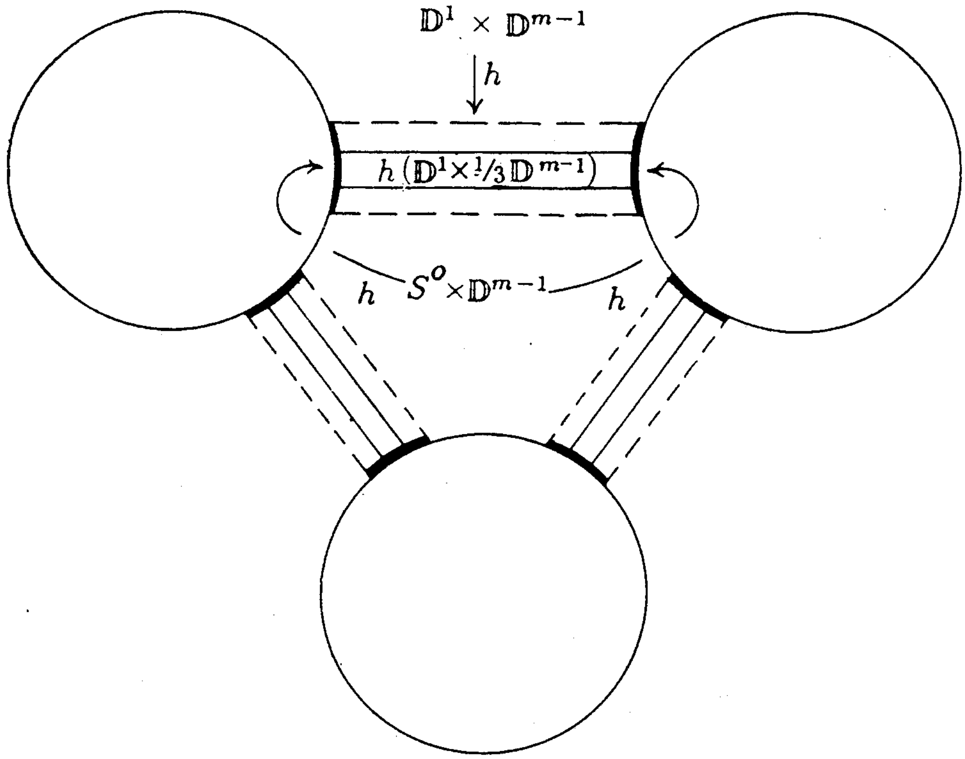

Figure 11

Let $f : G \to G$ be a δ_0-automorphism of a geometric group G on M. We proceed to verify the conclusion of 12.1 for f, provided $\delta = \delta_0$ is sufficiently small. (How small will become evident from the proof and depends on the handlebody structure just chosen.) First apply Lemma 12.4 and Remark 12.5 to each 0-handle independently. Hence there exists a geometric group G_0, whose basis is in $M - M_0$, such that

$$f_0 = f \oplus \mathrm{id} : \mathcal{G}_0 = G \oplus G_0 \to \mathcal{G}_0$$

is the composite map $\phi \circ \psi$ where ϕ is ϵ-blocked and ψ is an ϵ_0-automorphism of $\mathcal{G}_0$ which is blocked relative to

$$\mathcal{G}_0\big|_{M_0} \oplus \mathcal{G}_0\big|_{M-M_0}$$

and ψ restricted to $\mathcal{G}_0\big|_{M_0}$ is id. Let ψ_1 denote the restriction of ψ to $\mathcal{G}_0\big|_{M-M_0}$. Then it suffices to show that ψ_1 is stably the composition of m ϵ-blocked automorphisms over $M - \mathrm{Int}\, M_0$.

For each 1-handle, ψ_1 restricted to $\mathbb{D}^1 \times \frac{2}{3}\mathbb{D}^{m-1}$ (identifying via the embedding h) is a δ_1-automorphism over $\mathbb{D}^1 \times \frac{1}{2}\mathbb{D}^{m-1}$. Projecting the basis for the geometric group

$\mathcal{G}_0|_{\mathbb{D}^1\times\mathbb{D}^{m-1}}$ to $\mathbb{D}^{m-1}$ gives a δ_1-automorphism over $\frac{1}{2}\mathbb{D}^{m-1}$. Applying Lemma 12.4 and Remark 12.5 once again over the transverse core (i.e., $\mathbb{D}^{m-1}$) of each 1-handle yields a geometric group G_1 whose base is in $M - M_1$ and the following factorization. Let

$$\mathcal{G}_1 = \mathcal{G}_0|_{M-M_0} \oplus G_1 \quad \text{and} \quad f_1 = \psi_1 \oplus \mathrm{id}_{G_1} : \mathcal{G}_1 \to \mathcal{G}_1,$$

then f_1 is factored as the composite map $\phi_1 \circ \psi_1$ where ϕ_1 is ϵ-blocked and ψ_1 is an ϵ_1-automorphism of $\mathcal{G}_1$ which is blocked relative to $\mathcal{G}_1|_{M_1} \oplus \mathcal{G}_1|_{M-M_1}$ and ψ_1 restricted to $\mathcal{G}_1|_{M_1}$ is id. The number ϵ_1 is determined by

$$\max\{\mathrm{diam}\, h(\mathbb{D}^1 \times pt)\}$$

and by $\bar{\epsilon}_1$ which is the number 'ϵ' used in this application of Lemma 12.4.

If we can continue this argument inductively to the 2-handles, 3-handles, ..., $(m-1)$-handles and m-handles, then we will have proven Theorem 12.1. But there are two weak points which must be faced. First, if the diameters of the cores of the 1-handles are too large, i.e., the numbers diam $h(\mathbb{D}^1 \times pt)$, then ψ_1 may not be sufficiently controlled to continue the argument to the 2-handles. (And the same problem must be faced in going from the 2-handles to the 3-handles, etc.) Quinn overcomes this difficulty by carefully subdividing the original handlebody so that the new cores of the handles $h(\mathbb{D}^j \times \mathbb{D}^{m-j})$ all have "very small" diameter. This turns out not to be too difficult to do once one has correct bookkeeping; i.e., once the notion of very small is made precise. See [84, p. 330] for details.

The second difficult occurs in proceeding over the $(m-1)$-handles since their transverse cores are $\mathbb{D}^1$ and Lemma 12.4 is false for $\mathbb{D}^1$. (A little thought shows that the final step of proceeding over m-handles presents no difficulty.) There are three ways around this. Quinn's method is by introducing the notion of flux for a δ-automorphism of $\mathbb{D}^1$ over $\frac{1}{2}\mathbb{D}^1$. See [84, §8] for details. On the other hand, the core of $M - \mathrm{Int}(M_{m-2})$ is a 1-complex and Connell-Hollingsworth [20] proved their Conjecture 10.3 for geometric groups on 1-complexes. Hence their result can also be used to complete the argument.

The third method uses that $\pi_1(M - M_{m-2})$ is a free group. It combines this fact with a vanishing result due to Stallings [91]; namely, $Wh(\Gamma) = 0$ when Γ is a free group. Now the main argument, valid through the $(m-2)$-handles, shows that f stably factors as the composition of $m-1$ ϵ-blocked automorphisms $f_0, f_1, \ldots, f_{m-2}$ together with an ϵ-automorphism ϕ which is blocked relative to M_{m-2} and $M - M_{m-2}$ and such that $\phi|_{M_{m-2}} = \text{id}$. Consequently, $\hat{\phi}$ represents 0 in $Wh(\pi_1 M)$ since it is in the image of $Wh(\pi_1(M - M_{m-2})) = 0$. And Lemma 9.3 yields the equation

$$\hat{f} = \hat{f}_0 + \ldots + \hat{f}_{m-2} + \hat{\phi}$$

in $Wh(\pi_1 M)$, provided $\epsilon < \bar{\delta}_0/6m$ where $\bar{\delta}_0$ denotes the number "δ_0" fixed in Lecture 9. Furthermore, under this proviso each $\hat{f}_i = 0$ by Lemma 10.2. Therefore, $\hat{f} = 0$. Although this third method does not prove Theorem 12.1, it does establish the two applications discussed in the Lectures 10 and 11; namely, the topological invariance of Whitehead torsion and the thin h-cobordism theorem.

Lecture 13. Torus Trick

This lecture is devoted to giving Quinn's proof from [84] of Lemma 12.4. The key to his argument is constructing a geometric group version of Kirby's torus trick [66]. To do this, Quinn makes crucial use of the Bass-Heller-Swan Theorem [5] which states that $Wh\Gamma = 0$ when Γ is a free abelian group.

Identify the n-torus T^n with the quotient group $\mathbb{R}^n/(3\mathbb{Z})^n$. Then, $\mathbb{D}^n \subset T^n$. See Figure 12.

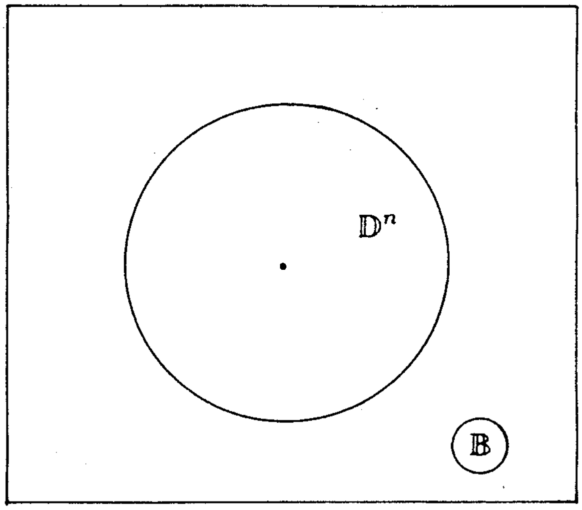

Figure 12

Let $\mathbb{B}$ be a second small closed n-dimensional round ball which is disjoint from $\mathbb{D}^n$ in T^n. And let

$$j : T^n - \operatorname{Int}\left(\frac{1}{2}\mathbb{B}\right) \to \frac{2}{3}\mathbb{D}^n$$

be a smooth immersion such that $j\big|_{\frac{1}{2}\mathbb{D}^n} = \mathrm{id}$. Let $0\mathbb{B}$ denote the center of $\mathbb{B}$. More generally, let

$$r\mathbb{B} = \{x \in \mathbb{B} \mid d(x, 0\mathbb{B}) \leq r\} \text{ and}$$

$$r\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x| \leq r\}$$

where $0 \leq r \leq 1$. Such an immersion exists because of the Smale-Hirsch immersion theorem and the fact that $T^n - 0\mathbb{B}$ is parallelizable. An explicit such immersion, when $n = 2$, can be constructed by considering Figure 13.

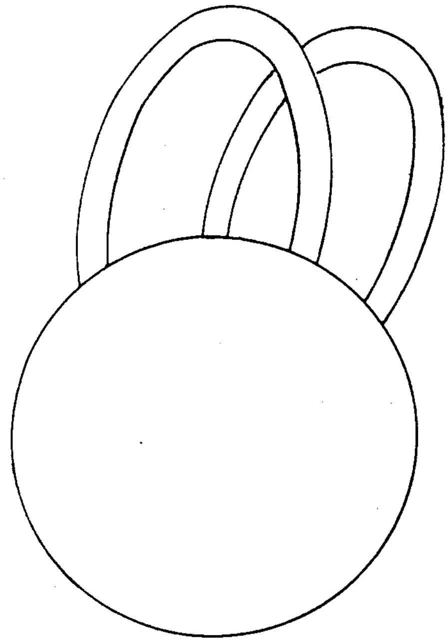

Figure 13

Given a geometric group G with basis $x_1, \dots, x_m$ on $\mathbb{D}^n$, define a geometric group j^*G on $T^n - \text{Int}(\frac{1}{2}\mathbb{B})$ with basis $\bigcup_{i=1}^{m} \{j^{-1}(x_i)\}$. The order of the basis elements in j^*G is ambiguously determined; but this is unimportant. Fix a positive real number γ satisfying the following condition:

$$\text{(0)} \qquad \text{If } d(x,y) \leq 2\gamma \text{ and } j(x) = j(y), \text{ then } x = y.$$

Such numbers exist since j is an immersion with compact domain. We also choose γ to be smaller than the number ϵ given in the hypotheses of Lemma 12.4. The following assertion is also a consequence of the fact that j is an immersion. There exists a number $\delta > 0$ such that, for each $a \in T^n - (\gamma + \frac{1}{2})\mathbb{B}$ and each $b \in \mathbb{D}^n$ with $d(j(a), b) \leq \delta$, the equation $j(x) = b$ has a unique solution $x \in T^n - \frac{1}{2}\mathbb{B}$ subject to the constraint $d(x,a) \leq \gamma$. This property allows us to associate to any δ-homomorphism $f : G \to G$ a γ-homomorphism $\bar{f} : j^*G \to j^*G$ such that $\bar{j} \circ \bar{f}$ and $f \circ \bar{j}$ are equal when restricted to $T^n - (\gamma + \frac{1}{2})\mathbb{B}$. Here, $\bar{j} : j^*G \to G$ is the canonical homomorphism induced by the natural basis correspondence. Furthermore, $\bar{f}$ restricted to $T^n - (\gamma + \frac{1}{2})\mathbb{B}$ is uniquely determined. (Note the similarity between this construction and the second construction of $\hat{f}$ given in 11.3.)

This uniqueness property shows that if f is a δ-automorphism over $\frac{2}{3}\mathbb{D}^n$, then $\bar{f}$ is a γ-automorphism over $T^n - (2\gamma + \frac{1}{2})\mathbb{B}$. We assume from now on that f is the δ-automorphism given in the statement of the Lemma 12.4. Let $\mathcal{G}_0 = j^*G|_{T^n - \frac{2}{3}\mathbb{B}}$ and $\mathcal{G}_1 = j^*G|_{\frac{2}{3}\mathbb{B}}$. Hence, $j^*G = \mathcal{G}_0 \oplus \mathcal{G}_1$. Using that $\bar{f}$ is a γ-automorphism over $T^n - (2\gamma + \frac{1}{2})\mathbb{B}$, it is seen that $\bar{f}(\mathcal{G}_0)$ is also a direct summand of j^*G. Stated more precisely

$$\mathcal{G} = \bar{f}(\mathcal{G}_0) \oplus \mathcal{G}_3$$

where $\mathcal{G}_3$ is a subgroup of $j^*G|_{\frac{3}{4}\mathbb{B}}$. But the Fundamental Theorem of Finitely Generated Abelian Groups implies that $\mathcal{G}_1 \simeq \mathcal{G}_3$. Using these facts, we can construct an automorphism $\bar{f}_1 : j^*G \to j^*G$ such that

$$\bar{f}_1|_{T^n - \frac{2}{3}\mathbb{B}} = \bar{f} \quad \text{and}$$
$$\bar{f}_1(\mathcal{G}_1) \subset j^*G|_{\frac{3}{4}\mathbb{B}}.$$

It is easy to construct a self-diffeomorphism $g : T^n \to T^n$ with the following properties:

1. $g|_{T^n - \mathbb{B}} = \text{id}$.
2. diam $g(\frac{3}{4}\mathbb{B}) \leq \gamma$.
3. $|dg(v)| \leq 4|v|$ for each vector v tangent to T^n.

A new geometric group G_1 on T^n is constructed by applying g to the basis elements of j^*G. This process also yields a new automorphism $f_1 : G_1 \to G_1$ such that

1. f_1 is a 4γ-automorphism, and
2. $f_1|_{\frac{1}{2}\mathbb{D}} = f$.

Assuming $4\gamma \leq 1$, we can form the square matrix $\hat{f}_1$ with entries from $\mathbb{Z}(\pi_1 T^n)$. If we use its second construction from 11.3 to do this, then $\hat{f}_1$ can be regarded as an automorphism of the "geometric group" p^*G_1 over $\mathbb{R}^n$, where $p : \mathbb{R}^n \to T^n = \mathbb{R}^n/(3\mathbb{Z})^n$

is the canonical projection. Considered this way, p^*G_1 is a free but not finitely generated $\mathbb{Z}$-module. It is however a free and finitely generated $\mathbb{Z}(\pi_1 T^n)$-module.

Quinn now uses the Fundamental Theorem of Algebraic K-theory, due to Bass-Heller-Swan [5]; this results states that

$$Wh(\pi_1 T^n) = 0.$$

It allows him to factor $\hat{f}$ as a certain product

$$E_k E_{k-1} \cdots E_1 E_0 \tag{1}$$

after stabilizing $\hat{f}$ outside the orbit of $\frac{1}{2}\mathbb{D}^n$ under the action of $\pi_1(T^n)$. The matrix E_0, in this factorization, is *strongly diagonal*; i.e., only its first (diagonal) entry is possibly not equal 1 and that entry is either α or $-\alpha$ for some $\alpha \in \pi_1(T^n)$. The matrices E_i (for $i > 0$) *are strongly elementary*; i.e., the non-zero off diagonal entry of E_i is $a_i\alpha_i$ where $a_i \in \mathbb{Z}$ and $\alpha_i \in \pi_1(T^n)$. Let G_2 be a geometric group over $T^n - \frac{1}{2}\mathbb{D}^n$ such that $p^*(G_1 \oplus G_2)$ is the stabilization in the factorization (1).

Now fix a very large positive real number s. How large will be presently evident. We proceed next to modify each of the automorphisms

$$E_i : p^*(G_1 \oplus G_2) \to p^*(G_1 \oplus G_2)$$

where $i = 0, 1, \ldots, k$. We change E_i to a new automorphism $\mathcal{E}_i$ satisfying the following properties:

1. $\mathcal{E}_i\big|_{s\mathbb{D}^n} = E_i\big|_{s\mathbb{D}^n}$;
2. $\mathcal{E}_i\big|_{\mathbb{R}^n - 2s\mathbb{D}^n} = \text{id}$;
3. $\text{diam } \mathcal{E}_i = \text{diam } \mathcal{E}_i^{-1} \leq \text{diam } E_i$.

The $\mathcal{E}_i$ are automorphisms of $\hat{\mathcal{G}} = p^*(G_1 \oplus G_2) \oplus G_3$ where G_3 is a (finitely generated) geometric group on $\mathbb{R}^n - s\mathbb{D}^n$. Also each $\mathcal{E}_i$, where $i \geq 1$, is blocked relative to the subgroups $p^*(G_1 \oplus G_2)$ and G_3 so that $\mathcal{E}_i\big|_{G_3} = \text{id}$. Note that each E_i $(i \geq 1)$ is blocked via a partition of the basis of $p^*(G_1 \oplus G_2)$ into subsets containing either 1 or 2 elements. And the matrix representing E_i on the 2 element partitions is

$$\begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$$

Also each 2 element member of the partition relative to E_i has the same diameter. Then $\mathcal{E}_i$ is determined by changing the matrix

$$\begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on each 2 element member of this partition which meets $\mathbb{R}^n - 2s\mathbb{D}^n$.

Note that E_0 is blocked relative to a partition $\mathcal{P}$ consisting of singletons and infinite sets S_t where each S_t lies on a straight line L_t in $\mathbb{R}^n$. The set S_t divides L_t into intervals of constant length l, independent of the index t. Furthermore, E_0 maps each point in S_t one unit l in the same direction. For each L_t that intersects $s\mathbb{D}^n$, draw a smooth curve in $2s\mathbb{D}^n - s\mathbb{D}^n$ connecting the first exit point on L_t to the first entrance point. (Note this is impossible to do when $n = 1$.) See Figure 14.

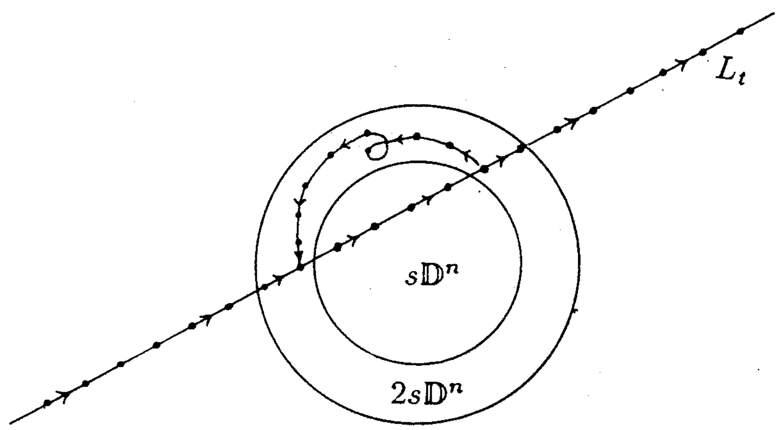

Figure 14

Introduce new basis elements along these curves so that the distance between successive elements is $\leq l$. These new points are the basis for G_3. Define $\mathcal{E}_0$ to circulate around the new cycles just constructed as indicated in Figure 14, and to be the identity off these cycles.

It is clear that the automorphisms $\mathcal{E}_i$ thus constructed satisfy conditions 1, 2 and 3 posited above. Furthermore, when s is sufficiently large, each $\mathcal{E}_i$ is blocked relative to the decomposition

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}\big|_{3s\mathbb{D}^n} \oplus \hat{\mathcal{G}}\big|_{\mathbb{R}^n - 3s\mathbb{D}^n}$$

and is the identity map on the second factor.

It is easy to construct a diffeomorphism $h : 3s\mathbb{D}^n \to \mathbb{D}^n$ having the following three properties:

1. $h\big|_{\frac{2}{3}\mathbb{D}^n} = \mathrm{id}$;
2. $|dh(v)| \leq |v|$ for each vector v tangent to $3s\mathbb{D}^n$;
3. $|dh(v)| \leq \frac{1}{s}|v|$ for each vector v tangent to $3s\mathbb{D}^n - s\mathbb{D}^n$.

A new geometric group $\hat{\mathcal{G}}_1$ on $\mathbb{D}^n$ is constructed by applying h to the basis elements of $\hat{\mathcal{G}}\big|_{3s\mathbb{D}^n}$. And conjugating $\mathcal{E}_k\mathcal{E}_{k-1}\cdots\mathcal{E}_1\mathcal{E}_0\big|_{3s\mathbb{D}^n}$ with the geometric automorphism induced by h yields an automorphism $\mathcal{E} : \hat{\mathcal{G}}_1 \to \hat{\mathcal{G}}_1$. Let the geometric group H posited in Lemma 12.4 be $\hat{\mathcal{G}}_1\big|_{\mathbb{D}^n - \frac{1}{2}\mathbb{D}^n}$, and notice that

$$G \oplus H = G\big|_{\mathbb{D}^n - \frac{1}{2}\mathbb{D}^n} \oplus \hat{\mathcal{G}}_1.$$

Set $\bar{f} = \mathrm{id} \oplus \mathcal{E}$ relative to this decomposition. Then $\bar{f}\big|_{\frac{2}{5}\mathbb{D}^n} = f\big|_{\frac{2}{5}\mathbb{D}^n}$ and $\bar{f}$ is a 4γ-automorphism provided s is chosen to be sufficiently large. If we pick $4\gamma \leq \epsilon$ and s sufficiently large, then this proves Lemma 12.4.

Lecture 14. Non-positively curved manifolds

This lecture is devoted to motivating the proof of the following topological rigidity result due to Farrell and Jones [43].

14.1. Theorem. *Let M^m be a closed non-positively curved Riemannian manifold. Then* $|\mathcal{S}(M^m \times \mathbb{D}^n, \partial)| = 1$ *when* $m + n \geq 5$.

Remark. This is a partial verification of Borel's Conjecture 1.3 and its generalization, Conjecture 2.3. The special cases of 14.1 when M is Riemannian flat or real hyperbolic were proven earlier by Farrell and Hsiang in [32] and by Farrell and Jones in [38], respectively. The assertion of 14.1 is also true when M^m is a closed infrasolvmanifold. This was proven in [37] by Farrell and Jones extending the result for infranilmanifolds proven by Farrell and Hsiang in [32]. Yau showed in [98] that a closed infrasolvmanifold M^m supports a non-positively curved Riemannian metric only when $\pi_1(M)$ is virtually abelian; hence, neither class of manifolds contains the other.

Theorem 14.1 is proven by the surgical method for analyzing $\mathcal{S}(M^m \times \mathbb{D}^n, \partial)$ described in Lecture 4. Recall this is a three step method. Step 1 is an immediate consequence of Theorem 7.1 in Lecture 7 since closed (connected) non-positively curved Riemannian manifolds satisfy condition $(*)$. (See the Remark at the end of Lecture 6.) And Step 3 is a direct consequence of the following result proven by Farrell and Jones in [42].

14.2. Theorem. *Let M be a closed (connected) non-positively curved Riemannian manifold. Then* $Wh(\pi_1 M) = 0$.

Remark. The special cases of 14.2 when M is Riemannian flat or real hyperbolic were proven earlier by Farrell and Hsiang in [29] and by Farrell and Jones in [36], respectively. Farrell and Hsiang also showed in [31] that $Wh(\pi_1 M) = 0$ when M is a closed infrasolvmanifold.

Theorem 14.2 was discussed for the case when M is real hyperbolic in a separate course of lectures given by Professor Raghunathan. We also refer the reader to the expositions of the Riemannian flat and real hyperbolic cases of Theorem 14.2 given in Chapters 2 and 3 of the book [40]. Lectures 9-13 above give the background material prerequisite for the foliated control theorem used in proving Theorem 14.2.

We hence only discuss Step 2 in this lecture. This is the most complicated step and the last to be solved. We make simplifying assumptions in order to make the discussion as transparent as possible; e.g., we assume throughout that M^m is orientable and $n = 0$. Refer now back to Lecture 5 and note that

$$L^s_{m+n}(\pi_1 M^m) = L_{m+n}(\pi_1 M^m) \text{ and}$$

$$\bar{\mathcal{S}}(M \times \mathbb{D}^n, \partial) = \mathcal{S}(M \times \mathbb{D}^n, \partial)$$

since $Wh(\pi_1 M) = 0$. These facts together with the periodicity of the surgery exact sequence (and Theorem 7.1) yield the following short exact sequence of pointed sets

$$0 \to [M^m \times I, \partial; G/\mathrm{Top}] \xrightarrow{\sigma} L_{m+1}(\pi_1 M^m) \to \mathcal{S}(M^m) \to 0.$$

Hence it remains to show that σ is an epimorphism. The argument accomplishing this is modeled after the one used to solve Step 3 in [36], [42]; i.e., the method used to show that the only h-cobordism with base M is the cylinder.

The s-cobordism theorem was used in that argument. It's surgery analogue is the algebraic classification of normal cobordisms over M due to Wall [96]. We refer back to Lecture 4 and expand on the discussion given there. Given a group Γ, Wall algebraically defined a sequence of abelian groups $L_n(\Gamma)$ with $L_{n+4}(\Gamma) = L_n(\Gamma)$ for all $n \in \mathbb{Z}$. He then showed that there is a natural bijection between the equivalence classes of normal cobordisms W over $M^m \times I^{n-1}$ and $L_{m+n}(\pi_1 M^m)$ with the trivial normal cobordism corresponding to 0. Denote this correspondence by

$$W \mapsto \omega(W) \in L_{m+n}(\pi_1 M^m).$$

He also proved the following product formula. Let N^{4k} be a simply connected closed oriented manifold and $\mathcal{W} = (W, f, \simeq)$ be a normal cobordism over $M^m \times I^{n-1}$. Form a new normal cobordism $\mathcal{W} \times N$ over $M^m \times I^{n-1} \times N^{4k}$ by producting $\mathcal{W}$ with N; i.e.,

$$\mathcal{W} \times N = (W \times N, f \times \mathrm{id}, \simeq \times \mathrm{id}).$$

Then,

$$\omega(\mathcal{W} \times N) = \mathrm{Index}(N)\omega(\mathcal{W})$$

under the (algebraic) identification

$$L_{m+n+4k}(\pi_1(M \times N)) = L_{m+n}(\pi_1 M).$$

This product formula has following geometric consequence.

14.3. Proposition. *Let K^{4k} be a closed oriented simply connected manifold with Index $(K) = 1$. Let $f : N \to M$ be a homotopy equivalence where N is also a closed manifold. If*

$$f \times id : N \times K \to M \times K$$

is homotopic to a homeomorphism, then f is also homotopic to a homeomorphism.

The complex projective plane $\mathbb{C}P^2$ is the natural candidate for K when applying 14.3. It is important for this purpose to have the following alternate description of $\mathbb{C}P^2$. Let C_2 denote the cyclic group of order 2. It has a natural action on $S^n \times S^n$ determined by the involution

$$(x, y) \mapsto (y, x)$$

where $x, y \in S^n$. Denote the orbit space of this action by F_n; i.e.,

$$F_n = S^n \times S^n / C_2.$$

14.4. Lemma. $\mathbb{C}P^2 = F_2$.

Proof. Let $\mathfrak{sl}_2(\mathbb{C})$ be the set of all 2×2 matrices with complex number entries and trace zero. Since $\mathfrak{sl}_2(\mathbb{C})$ is a 3-dimensional $\mathbb{C}$-vector space, $\mathbb{C}P^2$ can be identified as the set of all equivalence classes $[A]$ of non-zero matrices $A \in \mathfrak{sl}_2(\mathbb{C})$ where A is equivalent to B if and only if $A = zB$ for some $z \in \mathbb{C}$. The characteristic polynomial of $A \in \mathfrak{sl}_2(\mathbb{C})$ is $\lambda^2 + \det A$. Consequently, A has two distinct 1-dimensional eigenspaces if $\det A \neq 0$, and a single 1-dimensional eigenspace if $\det A = 0$ and $A \neq 0$. Also, A and zA have the same eigenspaces provided $z \neq 0$. These eigenspaces correspond to points in S^2 under the identification $S^2 = \mathbb{C}P^1$. The assignment

$$[A] \mapsto \text{ the eigenspaces of } A$$

determines a homeomorphism of $\mathbb{C}P^2$ to F_2.

Remark. Theorem 14.1 was first proved in the case where M^m is a hyperbolic 3-dimensional manifold by making use of Lemma 14.4. It was then realized that the general result for m odd could be proven using F_{m-1} once one could handle the technical complications arising from the fact that F_k is not a manifold when $k > 2$. The following result is used in overcoming these complications. It shows that F_k is "very close" to being a manifold of index equal to 1 when k is even.

14.5. Lemma. *Let n be an even positive integer. Then F_n has the following properties.*

1. *F_n is an orientable $2n$-dimensional $\mathbb{Z}[\frac{1}{2}]$-homology manifold.*
2. *F_n is simply connected.*
3. $$H_i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0,\ n,\ 2n \\ \mathbb{Z}_2 & \text{if } n < i < 2n \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$
4. $$H^i(F_n) = \begin{cases} \mathbb{Z} & \text{if } i = 0,\ n,\ 2n \\ \mathbb{Z}_2 & \text{if } n + 2 < i < 2n \text{ and } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

5. *The cup product pairing*

$$H^n(F_n) \otimes H^n(F_n) \to H^{2n}(F_n)$$

is unimodular and its signature is either 1 or -1.

Proof. There is a natural stratification of F_n consisting of two strata B and T. The bottom stratum B consists of all (agreeing) unordered pairs $\langle u, v\rangle$ where $u = v$; while the top stratum T consists of all (disagreeing) pairs $\langle u, v\rangle$ where $u \neq v$. Note that B can be identified with S^n. Also real projective n-space $\mathbb{R}P^n$ can be identified with the set of all unordered pairs $\langle u, -u\rangle$ in F_n. It is seen that F_n is the union of "tubular neighborhoods" of S^n and $\mathbb{R}P^n$ intersecting in their boundaries. The first tubular neighborhood is a bundle over S^n with fiber the cone on $\mathbb{R}P^{n-1}$. The second tubular neighborhood is a bundle over $\mathbb{R}P^n$ with fiber $\mathbb{D}^n$. Furthermore, they intersect in the total space of the $\mathbb{R}P^{n-1}$-bundle associated to the tangent bundle of S^n. This description of F_n can be used to verify 14.5.

Caveat. The fundamental class of B represents *twice* a generator of $H_n(F_n)$. On the other hand, if we fix a point $y_0 \in S^n$, then the map $x \mapsto \langle x, y_0\rangle$ is an embedding of S^n in F_n which represents a generator of $H_n(F_n)$.

Let $f : N \to M$ represent an element in $\mathcal{S}(M)$. Then $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ represents an element in $\mathcal{S}(M \times S^1)$. This defines a map $\mathcal{S}(M) \mapsto \mathcal{S}(M \times S^1)$. It is seen that this map is monic by using Theorem 14.2. Hence it suffices to show that $f \times \mathrm{id}$ is homotopic to a homeomorphism in order to prove Theorem 14.1. Note that $M \times S^1$ is also non-positively curved. One consequence of this discussion is that we may assume that $m = \dim M$ is odd when proving 14.1.

We now formulate a variant of Proposition 14.3 which is used in showing that $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ is homotopic to a homeomorphism. There is a bundle $p : \mathcal{F}M \to M \times S^1$ whose fiber over a point $(x, \theta) \in M \times S^1$ consists of all unordered pairs of unit length vectors $\langle u, v\rangle$ tangent to $M \times S^1$ at (x, θ) satisfying the following two constraints.

1. If $u \neq v$, then both u and v are tangent to the level surface $M \times \theta$.
2. If $u = v$, then the projection $\bar{u}$ of u onto $T_\theta S^1$ points in the counterclockwise direction (or is 0).

The total space $\mathcal{F}M$ is stratified with three strata:

$$\mathbb{B} = \{\langle u, u\rangle \mid \bar{u} = 0\},$$

$$\mathbf{A} = \{\langle u, u\rangle \mid \bar{u} \neq 0\},$$

$$\mathbb{T} = \{\langle u, v\rangle \mid u \neq v\}.$$

Note that $\mathbb{B}$ is the bottom stratum and $\mathcal{F}M - \mathbb{B}$ is the union of the two open sets $\mathbf{A}$ and $\mathbb{T}$. The restriction of p to each stratum is a sub-bundle. Let $\mathcal{F}_x$, B_x, A_x and T_x denote the fibers of these bundles over $x \in M \times S^1$; i.e.,

$$\mathcal{F}_x = p^{-1}(x),$$

$$B_x = p^{-1}(x) \cap \mathbb{B},$$

$$A_x = p^{-1}(x) \cap \mathbf{A},$$

$$T_x = p^{-1}(x) \cap \mathbb{T}.$$

Note that $B_x = S^{m-1}$, $A_x = \mathbb{D}^m$, $T_x \cup B_x = F_{m-1}$ and the bundle $p : \mathbb{B} \to M \times S^1$ is the pullback of the tangent unit sphere bundle of M under the projection $M \times S^1 \to M$ onto the first factor.

The space F_{m-1} will play the role of the index one manifold K in our variant of Proposition 14.3. Since it is unfortunately not a manifold when $m > 3$, we need to introduce the auxiliary fibers A_x. Hence the total fiber is homeomorphic to $F_{m-1} \cup \mathbb{D}^m$ where the subspace B in F_{m-1} is identified with $S^{m-1} = \partial\mathbb{D}^m$. Let $\mathcal{F}_f \to N \times S^1$ denote the pullback of $\mathcal{F}M \to M \times S^1$ along $f \times \mathrm{id} : N \times S^1 \to M \times S^1$ and let $\hat{f} : \mathcal{F}_f \to \mathcal{F}M$ be the induced bundle map. Note that the stratification of $\mathcal{F}M$ induces one on $\mathcal{F}_f$ and that $\hat{f}$ preserves strata. We say that $\hat{f}$ is *admissibly homotopic to a split map* provided there exists a homotopy h_t, $t \in [0,1]$, with $h_0 = \hat{f}$ and satisfying the following four conditions.

1. Each h_t is strata preserving.
2. Over some closed "tubular neighborhood" $\mathcal{N}_0$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$, each h_t is a bundle map; in particular, h_t maps fibers homeomorphically to fibers.
3. There is a larger closed "tubular neighborhood" $\mathcal{N}_1$ of $\mathbb{B}$ in $\mathbb{B} \cup \mathbb{T}$ such that h_1 is a homeomorphism over $\mathbb{B} \cup \mathbb{T} - \operatorname{Int} \mathcal{N}_1$ and over $\mathbb{B} \cup \mathbf{A}$.
4. Let $\rho : \mathcal{N}_1 \to M \times S^1$ denote the composition of the two bundle projections $\mathcal{N}_1 \to \mathbb{B}$ and $\mathbb{B} \to M \times S^1$. Then there is a triangulation K for $M \times S^1$ such that h_1 is transverse to $\rho^{-1}(\sigma)$ for each simplex σ of K. Furthermore, $h_1 : h_1^{-1}(\rho^{-1}(\sigma)) \to \rho^{-1}(\sigma)$ is a homotopy equivalence.

The variant of Proposition 14.3 needed to prove Theorem 14.1 is the following result.

14.6. Proposition. *The map $f : N \to M$ is homotopic to a homeomorphism provided $\hat{f} : \mathcal{F}_1 \to \mathcal{F}M$ is admissibly homotopic to a split map.*

Proposition 14.6 is the surgery theory part of the proof of Theorem 14.1 and is proven in [38, Theorem 4.3]. The geometry of M (in particular, its non-positive curvature) is used to show that the hypothesis of 14.6 is satisfied; i.e., that $\hat{f}$ is admissibly homotopic to a split map. This is done in [43, Proposition 0.4]. The key step there is the construction of a "focal transfer" which improves on the "asymptotic transfer" used in [38] to prove 14.1 in the special case when M is real hyperbolic. The focal transfer is used to gain control; after this, applications of both foliated and ordinary control theorems are made to complete the argument. The reader is referred to [43] and [38] for details. Also, [40, Chapter 5] contains a detailed sketch of the proof of Theorem 14.1 for the case when M is real hyperbolic.

Lecture 15. Expanding Endomorphisms

We will be concerned for the remainder of these lectures with constructing exotic examples of certain natural structures. We start this program by constructing in this lecture exotic examples of one of the simplest types of dynamical systems; namely, of expanding endomorphisms.

15.1. Definition. Let M be a closed smooth manifold. A self-map $f : M \to M$ is said to be an expanding endomorphism provided M supports a Riemannian metric such that $|df(v)| > |v|$ for every non-zero vector v tangent to M.

15.2. Question. What closed smooth manifolds support expanding endomorphisms?

The question is answered up to topological classification as follows by results due to Shub [89], Franks [52] and Gromov [53].

15.3. Theorem. *If a closed smooth manifold M supports an expanding endomorphism, then M is homeomorphic to an infranilmanifold.*

Recall that infranilmanifolds were defined in 1.4. Shub showed that the universal cover $\tilde{M}$ of M is diffeomorphic to $\mathbb{R}^m$ where $m = \dim M$. Then Franks showed that $\pi_1 M$ has polynomial growth and that M is homeomorphic to an infranilmanifold provided $\pi_1 M$ is virtually solvable. Gromov completed the proof of 15.3 by showing that a group of polynomial growth must be virtually nilpotent. Gromov's result was motivated by Hirsch's paper [60] where it is shown that the solution to Hilbert's fifth problem is related to 15.3. Hirsch also implicitly poised Question 15.2 in his Remark 1; i.e., whether the word "homeomorphism" can be replaced by "diffeomorphism" in 15.3. But Farrell and Jones showed in [35] that this is not the case; namely, they proved the following result.

15.4. Theorem. *Let T^n be the n-torus $(n > 4)$ and Σ^n an arbitrary homotopy sphere, then the connected sum $T^n \# \Sigma^n$ admits an expanding endomorphism.*

When Σ^n is not the standard sphere, Wall [96, §15A] showed that $T^n \# \Sigma^n$ is not diffeomorphic to T^n. This fact combined with the classical rigidity results of Bieberbach

[6] and Malcev [70] yields that $T^n \# \Sigma^n$ is *not* diffeomorphic to any infranilmanifold. More details of this argument will be given in lecture 16. The remainder of this lecture is devoted to constructing the expanding endomorphism $f : T^n \# \Sigma^n \to T^n \# \Sigma^n$ posited in 15.4.

Let θ_n denote the Kervaire-Milnor group of homotopy n-spheres ($n \geq 5$). We note that $\theta_n = \mathcal{S}^s(S^n)$ as sets and the abelian group structure on θ_n is given by the connected sum operation of oriented manifolds; cf. [65]. Kervaire and Milnor prove that θ_n is a finite group and that it is a non-trivial group for infinitely many n. They also calculate its order $|\theta_n|$ for small values of n. This calculation is given by the following table:

n	5	6	7	8	9	10	11	12	13	14	15
$\lvert\theta_n\rvert$	1	1	28	2	8	6	992	1	3	2	16,256

Let s denote the order of $[\Sigma^n]$ in θ_n and set $t = ms + 1$ where m is a *large* positive integer. Let $\bar{f} : T^n \to T^n$ be multiplication by t. (Recall T^n is an abelian group.) It is an expanding endomorphism. The exotic expanding endomorphism f is topologically conjugate to $\bar{f}$ and is constructed as the composite of the 4 maps illustrated in the diagram below; i.e., $f = P \circ E \circ D \circ F$. (In the illustration, $s = 2$.)

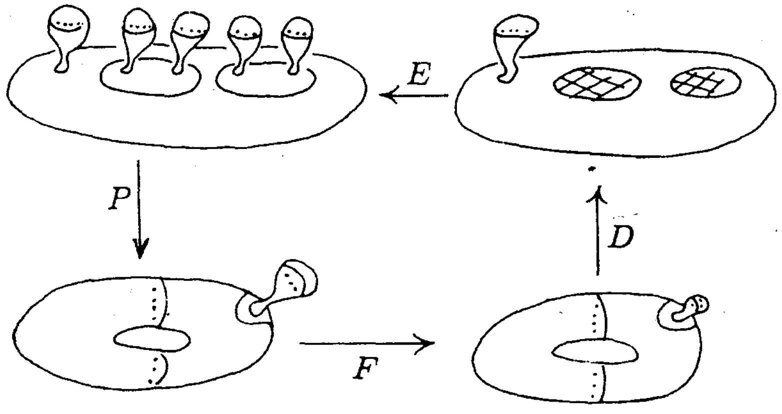

Figure 15

Here P denotes the covering projection in the standard t^n-sheeted covering space of $T^n \# \Sigma^n$; i.e., the one such that image $(P_\#)$ is the subgroup of $\pi_1(T^n \# \Sigma^n)$ consisting of all elements divisible by t^n. Note that P is a local isometry. The map D is the diffeomorphism which is uniform dilation by t. It can of course be made as expanding as we want by choosing m large enough.

The map E is a diffeomorphism with bounded distortion independent of m. It is constructed by carefully grouping the t^n-connected summand of Σ^n occurring in its range into sets of s-each with 1-summand left over. And then independently "canceling" each group of s-summands using the fact that $[\Sigma^n]$ has order s. Each canceled group is illustrated by the shaded disc in the domain of E.

Finally, the diffeomorphism F is the identity map outside a closed n-ball $\mathbb{B}^n$ contained in T^n. The domain and range of F are both connected sums of T^n with Σ^n and $1/t\ \Sigma^n$, respectively. Both connected summings occur inside of $\mathbb{B}^n$. Here, $1/t\ \Sigma^n$ is the same smooth manifold as Σ^n; but its Riemannian metric is dilated by $1/t$. The diffeomorphism F is constructed "inside of $\mathbb{B}^n$" by using the "commutator" $1/t\ \Phi t \Phi^{-1}$ illustrated below.

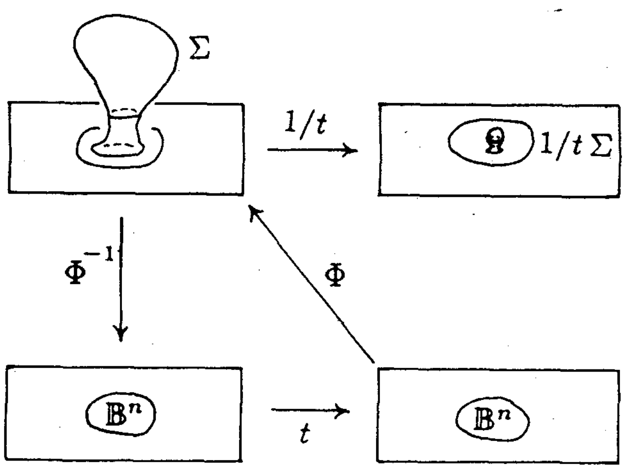

Figure 16

Here $\Phi : \mathbb{R}^n \to \mathbb{R}^n \# \Sigma^n$ is the diffeomorphism defined as follows. Note first that

$$\mathbb{R}^n \# \Sigma^n = \mathbb{B}^n \amalg_\phi (\mathbb{R}^n - \text{Int } \mathbb{B}^n)$$

where $\phi : S^{n-1} \to S^{n-1}$ is a self-diffeomorphism and the identification $\sim$ above is given in polar co-ordinates by

$$(x, r) \sim (\phi(x), r) \quad \text{where } r = 1.$$

(We can take $\mathbb{B}^n$ to be the ball of radius 1 centered at the origin of $\mathbb{R}^n$.) Then Φ is defined in terms of polar co-ordinates (x, r) where $x \in S^{n-1}$ and $r \in [0, +\infty)$ by the formula

$$\Phi(x, r) = \begin{cases} (x, r) & \text{if } r < 1 \\ \\ (\phi(x), r) & \text{if } r \geq 1. \end{cases}$$

It is easily shown that the "commutator"

$$y \mapsto 1/t \ \Phi(t\Phi^{-1}(y))$$

is the identity map outside of $\mathbb{B}^n$ and also that its derivative has bounded distortion independent of our choice of m. Consequently, f is an expanding endomorphism of $T^n \# \Sigma^n$ provided m is chosen to be sufficiently large.

Lecture 16. Exotic Smoothings

This lecture is concerned with the problem of detecting non-diffeomorphic smooth structures on the same topological manifold M. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are equivalent provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is *topologically concordant* (a.k.a. topologically pseudoisotopic) to f_1; i.e., there exists a homeomorphism $F : N_1 \times [0,1] \to M$ such that

$$F\big|_{N_1 \times 0} = f_1 \quad \text{and} \quad F\big|_{N_2 \times 0} = f_2.$$

Note that F is *not* required to be level preserving; i.e., $F(N_1 \times t)$ needn't be contained in $M \times t$. (If F is additionally level preserving, then it is a *topological isotopy.*) The set of all such equivalence classes is denoted $\mathcal{C}(M)$ and an equivalence class is called a smooth structure on M.

The key to analyzing $\mathcal{C}(M)$ is the following result due to Kirby and Siebenmann [67, p. 194].

16.1. Theorem. *There exists a connected H-space Top/O such that there is a bijection between $\mathcal{C}(M)$ and $[M; Top/O]$ for any smooth manifold M with dim $M \geq 5$. Furthermore, the equivalence class of (M, id_M) corresponds to the homotopy class of the constant map under this bijection.*

Assume $n \geq 5$ and recall that θ_n is $\mathcal{S}^s(S^n)$. Consider the obvious forgetting information map

$$\mathcal{C}(S^n) \to \mathcal{S}^s(S^n).$$

This map is a bijection which can be seen by using the fact that both $|\mathcal{S}(S^n)| = 1$ and $|\mathcal{S}(S^n \times [0,1], \partial)| = 1$. We can hence identify $\mathcal{C}(S^n)$ with the set of all equivalence classes of oriented n-dimensional homotopy spheres Σ. Here two oriented homotopy spheres Σ_1 and Σ_2 are equivalent provided they are orientation preservingly diffeomorphic. Also the abelian group structure on θ_n given by connected sum agrees with the one given by Theorem 16.1 via the identification $\mathcal{C}(S^n) = \pi_n(\mathrm{Top}/O)$.

We can more generally show that the natural map $\mathcal{C}(M) \to \mathcal{S}^s(M)$ is a bijection for any closed smooth manifold M such that both $\mathcal{S}(M)$ and $\mathcal{S}(M \times [0,1], \partial)$ have cardinality 1 (and dim $M \geq 5$). To do this, one notices that $|\mathcal{S}(M \times [0,1], \partial)| = 1$ implies that any self-diffeomorphism of M which is homotopic to id_M is, in fact, topologically pseudo-isotopic to id_M. Combining this observation with Theorem 14.1, we see, in particular, that the natural map $\mathcal{C}(M^m) \to \mathcal{S}^s(M^m)$ is a bijection for every closed non-positively curved manifold M^m (with $m \geq 5$).

Recall now from Lecture 2 how $\pi_0\mathcal{E}(M^m)$ acts on $\mathcal{S}^s(M^m)$. One sees immediately from this description how Mostow's Rigidity Theorem, cf. [75], implies that the concordance class of (M^m, id_{M^m}) is a fixed point of this action whenever M^m is a non-positively curved locally symmetric space such that its universal cover has neither a one nor a two dimensional metric factor. This is, in particular, the case when M^m is negatively curved. And Bieberbach's Rigidity Theorem, cf. [6], shows that this is also the case when M^m is a flat Riemannian manifold. Stringing the above remarks together yields the following consequence of 16.1.

16.2. Corollary. *Let M^m be a closed Riemannian manifold (with $m \geq 5$) which is a locally symmetric space whose sectional curvatures are either identically zero or all negative. Let (N^m, f) be a smoothing of M^m. If N^m is diffeomorphic to M^m, then (N^m, f) and (M^m, id_{M^m}) represent the same element in $\mathcal{C}(M^m)$; i.e., they are topologically concordant.*

We will next apply 16.2 to the problem of determining when connected sum with a homotopy sphere Σ changes the differential structure on a smooth oriented (connected) manifold M. Start by noting that the homeomorphism $M\#\Sigma$ to M which is the inclusion map outside of Σ is well defined up to topological concordance. We will denote the class in $\mathcal{C}(M)$ of $M\#\Sigma$ equipped with this homeomorphism by $[M\#\Sigma]$. (Note that $[M^m\#S^m]$

is the class of (M^m, id_{M^m}).) Let $f_M : M^m \to S^m$ be a degree-one map and note that f_M is well-defined up to homotopy. Composition with f_M defines a homeomorphism

$$f_M^* : [S^m, \mathrm{Top}/O] \to [M^m, \mathrm{Top}/O].$$

And in terms of the identifications

$$\theta_m = [S^m, \mathrm{Top}/O] \text{ and } \mathcal{C}(M^m) = [M^m, \mathrm{Top}/O]$$

given by 16.1, f_M^* becomes $[\Sigma^m] \mapsto [M^m \# \Sigma^m]$.

Recall that a smooth manifold is *stably parallelizable* (a.k.a. a π-*manifold*) if its tangent bundle is stably trivial. We need the following result due to Browder [13] and Brumfiel [14].

16.3. Lemma. *Assume that M^m is an oriented closed (connected) smooth manifold which is stably parallelizable and that $m \geq 5$. Then $f_M^* : \theta_m \to \mathcal{C}(M^m)$ is monic.*

Proof. Since $X \mapsto [X, \mathrm{Top}/O]$ is a homotopy functor on the category of topological spaces, 16.3 would follow immediately if $f_M : M^m \to S^m$ is homotopically split. That is, if there exists a map $g : S^m \to M^m$ such that $f_M \circ g$ is homotopic to id_{S^m}. Unfortunately, f_M is only homotopically split when M is a homotopy sphere. But we can use the fact that M is stably parallelizable to always stably split f_M up to homotopy; i.e., to show that the $(m+1)$-fold suspension

$$\Sigma^{m+1}(f_M) : \Sigma^{m+1} M^m \to S^{2m+1}$$

of f_M is homotopically split. This is done as follows. Note first that $M^m \times \mathbb{D}^{m+1}$ can be identified with a codimension-0 smooth submanifold of S^{2m+1} by using the Whitney embedding theorem together with the fact that M is stably parallelizable. Let $*$ be a base point in M. Then dual to the inclusion

$$\mathbf{M}^m \times \mathbb{D}^{m+1} \subseteq S^{2m+1}$$

is a quotient map $\phi : S^{2m+1} \to \Sigma^{m+1}M^m$ realizing the $(m+1)$-fold reduced suspension $\Sigma^{m+1}M^m$ of M^m as a quotient space of S^{2m+1}. Namely, ϕ collapses everything outside of $M^m \times \text{Int}(\mathbb{D}^{m+1})$ together with $* \times \mathbb{D}^{m+1}$ to the base point of $\Sigma^{m+1}M^m$, and is a bijection between the remaining points. And it is easy to see that the composition $\Sigma^{m+1}(f_M) \circ \phi$ is homotopic to id_{S^m}; i.e., $\Sigma^{m+1}(f_M)$ is homotopically split.

But this is enough to show that f_M^* is monic since Top/O is an ∞-loop space [8]; in particular, there exists a topological space Y such that $\Omega^{m+1}(Y) = \text{Top}/O$. This fact is used to identify the functor

$$X \mapsto [X, \text{Top}/O] = [X, \Omega^{m+1}(Y)]$$

with the functor $X \mapsto [\Sigma^{m+1}X, Y]$. Consequently,

$$f_M^* : [S^m, \text{Top}/O] \to [M^m, \text{Top}/O]$$

is identified with

$$(\Sigma^{m+1}(f_M))^* : [S^{2m+1}, Y] \to [\Sigma^{m+1}M^m, Y].$$

But, this last homomorphism is monic since $\Sigma^{m+1}(f_M)$ is homotopically split. Q.E.D.

A homotopy m-sphere is called *exotic* if it is not diffeomorphic to S^m. The following result is an immediate consequence of 16.2 and 16.3; it will be used to construct exotic smoothings of some symmetric spaces.

16.4. Corollary. *Let M^m be a closed, oriented (connected) stably parallelizable Riemannian locally symmetric space (with $m \geq 5$) whose sectional curvatures are either identically zero or all negative (e.g., M^m could be an m-torus). Let Σ^m be an exotic homotopy sphere, then $M^m \# \Sigma^m$ is not diffeomorphic to M^m.*

Remark. Recall we asserted in Lecture 15 that $T^n \# \Sigma^n$ is *not* diffeomorphic to any infranilmanifold when Σ^n is exotic (and $n \geq 5$). We now give a more complete argument for this fact. Corollary 16.4 shows that $T^n \# \Sigma^n$ is *not* diffeomorphic to T^n. Also Malcev's

Rigidity Theorem, cf. [70], shows that any closed infranilmanifold with abelian fundamental group must be Riemannian flat. And finally Bieberbach's Rigidity Theorem shows that any such manifold is diffeomorphic to a torus. Q.E.D.

Lecture 17. Smooth Rigidity Problem

Recall we remarked in Lecture 1 that the obvious smooth analogue of Borel's Conjecture 1.3 is false. Namely, Browder had shown in [13] that it is false even in the basic case where M is an n-torus. In fact, it was shown in Lecture 16 that T^n and $T^n \# \Sigma^n$ $(n \geq 5)$ are not diffeomorphic when Σ^n is an exotic sphere; although they are clearly homeomorphic.

But when it is assumed that both M and N in Conjecture 1.3 are non-positively curved Riemannian manifolds, then smooth rigidity frequently happens. The most fundamental instance of this is an immediate consequence of Mostow's Strong Rigidity Theorem; cf. [75]. Namely, Mostow showed that any isomorphism between fundamental groups is, in fact, induced by an isometry if M and N satisfy some more geometric constraints and provided we are allowed to change the metric on M by scaling it on each irreducible metric factor of its universal cover. Adequate extra constraints are that both manifolds be locally symmetric spaces and that the universal cover of M does not have a 1 or 2 dimensional metric factor. Mostow's result led Lawson and Yau [99, p. 673, Problem 12] to pose the problem of whether smooth rigidity always holds when both M and N are negatively curved; in particular, does $\pi_1 M \simeq \pi_1 N$ imply that M and N are diffeomorphic? Farrell and Jones showed in [39] that this is *not* always true even when M is a real hyperbolic manifold. This lecture is devoted to constructing such a counterexample to smooth rigidity. It is loosely motivated by the construction used to prove Theorem 15.4.

The manifold N^m in this counterexample will be $M^m \# \Sigma^m$ where Σ^m is an exotic sphere and M^m is a stably parallelizable, real hyperbolic manifold (with $m \geq 5$) and having sufficiently large injectivity radius; i.e., every closed geodesic in M^m must be sufficiently long. To implement this program, we need to know that such manifolds M^m exist. To show this, we will use the following three results: the first due to Sullivan [94], the second to Borel [9] and the third to Malcev [71]; cf. [68].

17.1. Theorem. *(Sullivan) Every closed real hyperbolic manifold has a stably parallelizable finite sheeted cover.*

17.2. Theorem. *(Borel) There exists closed real hyperbolic manifolds in every dimension* $m \geq 2$*, as well as closed complex hyperbolic manifolds in every even (real) dimension.*

17.3. Theorem. *(Malcev) Let* M *be a closed Riemannian manifold which is either real or complex hyperbolic, then* $\pi_1 M$ *is residually finite; i.e., the intersection of all its subgroups of finite index contains only the trivial element.*

The combination of Theorem 17.1 with Theorem 17.2 clearly yields the existence, in every dimension $m \geq 2$, of a m-dimensional, closed and stably parallelizable real hyperbolic manifold $\mathcal{M}^m$. Furthermore, any finite sheeted cover of $\mathcal{M}^m$ will have these same properties. And Theorem 17.3 can be used, as follows, to show that there exist finite sheeted covers of $\mathcal{M}^m$ of arbitrarily large injectivity radius r. Since there are only a finite number of closed geodesics in $\mathcal{M}^m$ with length less than $2r+1$, we can use 17.3 to find a normal subgroup Γ with finite index in $\pi_1(\mathcal{M}^m)$ such that every closed geodesic in $\mathcal{M}^m$ which represents the free homotopy class of an element in $\Gamma - \{e\}$ must have length at least $2r+1$. Then the injectivity radius of the finite sheeted cover of $\mathcal{M}^m$ which corresponds to Γ is at least r. This demonstrates the existence of the manifolds M^m needed for our counterexample.

It remains to put a negatively curved Riemannian metric on $M^m \# \Sigma^m$. We use the following result to do this.

17.4. Lemma. *Given* $m \geq 5$ *and* $\epsilon > 0$*, there exists a real number* $\alpha > 0$ *such that the following is true. Let* M^m *be a* m*-dimensional, oriented, closed, real hyperbolic manifold whose injectivity radius is bigger than* α*, and let* Σ^m *be an exotic sphere. Then* $M^m \# \Sigma^m$ *supports a Riemannian metric whose sectional curvatures all lie in the open interval* $(-1-\epsilon, -1+\epsilon)$.

Before proving 17.4, let us precisely describe the counterexamples from [39] that it immediately yields.

17.5. Theorem. *The following statement is true in each dimension $m \geq 5$ such that there exists an exotic m-dimensional sphere (e.g., $m = 4k-1$ where $k \geq 2$ and $k \in \mathbb{Z}$). Given $\epsilon > 0$, there exist two m-dimensional, closed, negatively curved Riemannian manifolds M^m and N^m such that*

1. *M^m is real hyperbolic;*
2. *the sectional curvatures of N^n are all contained in the interval $(-1-\epsilon, -1+\epsilon)$;*
3. *M^m is not diffeomorphic to N^m;*
4. *M^m is homeomorphic to N^m.*

The remainder of this lecture is devoted to proving 17.4. Each exotic sphere Σ^m arises by taking 2 disjoint copies of the closed m-ball $\mathbb{D}^m$ and identifying their boundaries S^{m-1} by a self-diffeomorphism $f : S^{m-1} \to S^{m-1}$. (Note that this construction yields S^m when $f = \text{id}_{S^{m-1}}$.) Since there are only a finite number of exotic spheres in each dimension $m \geq 5$, it suffices to consider a single Σ^m and thus fix a single diffeomorphism $f : S^{m-1} \to S^{m-1}$.

The connected sum $M^m \# \Sigma^m$ is likewise constructed from the disjoint union

$$(M^m - \text{Int}(\mathbb{D}^m)) \amalg \mathbb{D}^m$$

by identifying the boundaries of its two components using f. Let us make this construction more explicit. Fix a point $x \in M^m$ and look at the exponential map

$$\exp : T_x M^m \to M^m.$$

The closed ball in $T_x M^m$ of radius α and center 0, denoted by $\alpha\mathbb{D}^m$, is smoothly embedded via exp into M^m since the injectivity radius of M^m is greater than α. Dilate f by α to define a self-diffeomorphism f_α of $\partial(\alpha\mathbb{D}^m)$; i.e., set

$$f_\alpha(\alpha x) = \alpha f(x) \quad \text{for all} \quad x \in S^{m-1} = \partial\mathbb{D}^m$$

Then $M^m \# \Sigma^m$ is

$$(0) \qquad (M^m - \mathrm{Int}(\alpha\mathbb{D}^m)) \amalg_{f_\alpha} (\alpha\mathbb{D}^m)$$

where $\amalg_{f_\alpha}$ means to glue together the boundaries of the two components in the disjoint union $\amalg$ using f_α.

We put a Riemannian metric $B_\alpha(\ ,\ )$ on M^m in terms of this decomposition (0) as follows. Restricted to both $M^m - \mathrm{Int}(\alpha\mathbb{D}^m)$ and $\frac{\alpha}{2}\mathbb{D}^m$, $B_\alpha(\ ,\ )$ is the real hyperbolic metric. Then we interpolate to define $B_\alpha(\ ,\ )$ on the rest of M^m; namely, on $\alpha\mathbb{D}^m - \mathrm{Int}(\frac{\alpha}{2}\mathbb{D}^m)$ which we denote by A_α. To do this interpolation, put a Riemannian metric $\langle\ ,\ \rangle$ on $S^{m-1} \times [\frac{1}{2}, 1]$ with the following properties (1-4).

(1) The manifolds $S^{m-1} \times t$ and $x \times [\frac{1}{2}, 1]$ intersect perpendicularly at (x, t) for each $(x, t) \in S^{m-1} \times [\frac{1}{2}, 1]$.

(2) The map $t \mapsto (x, t)$ is an isometry between $[\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1] \times x$ for each $x \in S^{m-1}$

(3) The map $x \mapsto (x, \frac{1}{2})$ is an isometry from S^{m-1} to $S^{m-1} \times \frac{1}{2}$.

(4) The map $x \mapsto (f(x), 1)$ is an isometry from S^{m-1} to $S^{m-1} \times 1$.

It is easy to construct such a metric $\langle\ ,\ \rangle$. Then we "warp" $\langle\ ,\ \rangle$ by $\sinh^2(\alpha t)$ to do the interpolation. That is, let ξ and η be the distributions tangent, respectively, to the first and second factors in the product structure $S^{m-1} \times [\frac{1}{2}, 1]$. And define $B_\alpha(\ ,\ )$ on A_α by

$$(5) \qquad B_\alpha(u, v) = \begin{cases} \sinh^2(\alpha t)\langle u, v\rangle & \text{if } u, v \in \xi \\ \alpha^2\langle u, v\rangle & \text{if } u, v \in \eta \\ 0 & \text{if } u \in \xi \text{ and } v \in \eta. \end{cases}$$

Remark. In this formula (5), $S^{m-1} \times [\frac{1}{2}, 1]$ is identified with A_α via multiplication by α on the $[\frac{1}{2}, 1]$ factor; i.e., we used multiplication by α to shift $\langle\ ,\ \rangle$ to a Riemannian metric $\langle\ ,\ \rangle_\alpha$ on A_α and then "warped" this metric by $\sinh^2(t)$.

It is easily seen that these definitions fit together to give a well-defined Riemannian metric on all of $M^m \# \Sigma^m$. Note also that the Riemannian metric $B_\alpha(\ ,\ )$ defined on

$S^{m-1} \times [\frac{1}{2}, 1]$ by (5) is independent of M^m. Hence to complete the proof of Lemma 17.4, it suffices to verify the following statement.

17.6. Lemma. *The sectional curvatures of the Riemannian metric $B_\alpha(\ ,\ )$ defined by formula (5), converge uniformly to -1 as $\alpha \to \infty$.*

The proof of 17.6 uses that sectional curvatures are computable in terms of the first and second order partial derivatives of the first fundamental form together with the form itself. The following is a precise statement of what is used.

17.7. Theorem. *Given a positive integer m, there exists a polynomial $p(\ )$ such that the following is true. Let $g_{ij}(y)$, where $y \in \mathbb{R}^m$ and $1 \le i, j \le m$, be any smooth Riemannian metric on $\mathbb{R}^m$ satisfying*

$$g_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and let $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ be any pair of vectors in $\mathbb{R}^m$ satisfying

$$\Sigma_{i=1}^m (u_i)^2 = 1, \quad \Sigma_{i=1}^m (v_i)^2 = 1, \quad \Sigma_{i=1}^m (u_i v_i) = 0.$$

Then the sectional curvature of this Riemannian metric at 0 in the direction of the tangent plane spanned by u and v is the polynomial $p(\ )$ evaluated at

$$\{u_i\}_{i=1}^m, \quad \{v_i\}_{i=1}^m, \quad \left\{\frac{\partial g_{ij}}{\partial x_k}\right\}_{i,j,k=1}^m, \quad \left\{\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right\}_{i,j,k,l=1}^m$$

Remark. This fundamental result is a direct consequence of Cartan's second structural equation combined with the Koszul formula description of the Levi-Civita connection; cf. [59, §§5.3 and 6.2].

We proceed to sketch the proof of 17.6. (See [39] for details.) Given $(x, t) \in S^m \times [\frac{1}{2}, 1]$, one first constructs local co-ordinates $y_1, y_2, \ldots, y_{m-1}, \bar{t}$ about (x, t) sending (x, t) to $0 \in \mathbb{R}^m$ and such that the matrix entries $g_{ij}(y_1, y_2, \ldots, y_{m-1}, \bar{t})$ of the first fundamental form satisfy

$$g_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \neq j \end{cases}$$

and their partial derivatives $D(g_{ij})$ evaluated at 0 have the following limiting values V (uniformly in (x, t)) as $\alpha \to +\infty$:

$$V = \begin{cases} 0 & \text{if } D = \dfrac{\partial}{\partial y_k}, \dfrac{\partial^2}{\partial y_k \partial \bar{t}} \text{ or } \dfrac{\partial^2}{\partial y_k \partial y_l} \\ \\ 2 & \text{if } D = \dfrac{\partial}{\partial \bar{t}} \\ \\ 4 & \text{if } D = \dfrac{\partial^2}{\partial \bar{t} \partial \bar{t}}. \end{cases} \tag{6}$$

Recall next the "cusp description" of real hyperbolic m-dimensional space $\mathbb{H}^m$ is given by the warped product $\mathbb{R}^{m-1} \times_{e^t} \mathbb{R}$ of the Euclidean spaces $\mathbb{R}^{m-1}$ and $\mathbb{R}$ in terms of the warping function e^t on $\mathbb{R}$. (See [82, pp. 204-211] for the definition and basic properties of warped products. Note that we've reversed above the normal order of base and fibre.) Let (h_{ij}) be the first fundamental form of $\mathbb{H}^m$ in terms of the canonical co-ordinates $(y_1, y_2, \ldots, y_{m-1}, \bar{t})$ on $\mathbb{R}^{m-1} \times_{e^{\bar{t}}} \mathbb{R}$. Then it is easily seen that

$$h_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \neq j. \end{cases}$$

and that the values V of the partial derivatives $D(h_{ij})$ evaluated at 0 are given by formula (6). But the sectional curvatures of $\mathbb{H}^m$ are identically -1. Hence an elementary continuity argument based on Theorem 17.7 shows that the sectional curvatures of $B_\alpha(\ ,\ )$ approach the value -1 uniformly as $\alpha \to +\infty$. Q.E.D.

Lecture 18. Complex hyperbolic manifolds

This lecture starts the discussion of the counterexamples constructed by Farrell and Jones in [46] to the smooth rigidity problem when M^m is a complex hyperbolic manifold. (The discussion is completed in the following lecture.) I start by identifying complex hyperbolic n-space $\mathbb{CH}^n$ with an open subset of complex projective n-space $\mathbb{C}P^n$. Recall that $\mathbb{C}P^n$ is the space whose points consist of all complex lines containing 0 in $\mathbb{C}^{n+1}$. Fix the following non-degenerate indefinite Hermitian form $b(\ ,\ )$ on $\mathbb{C}^{n+1}$ defined by

$$b(x,y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n - x_{n+1}\bar{y}_{n+1}$$

where the subscript denotes the co-ordinate of the vector referred to. Then

$$\mathbb{CH}^n = \{L \in \mathbb{C}P^n \mid \text{the restriction } b\big|_L \text{ is negative definite}\}.$$

It is relatively easy to see that the thus defined $\mathbb{CH}^n$ is an open subset of $\mathbb{C}P^n$ and is biholomorphically equivalent to $\mathbb{C}^n$. The group of all isometries of $b(\ ,\ )$ with determinant equal to 1 is $SU(n,1)$. This Lie group acts transitively on $\mathbb{CH}^n$ and its isotropy subgroup at the complex line spanned by $(0,\ldots,0,1)$ is $S(U(n)\times U(1))$ which is a maximal compact subgroup of $SU(n,1)$. There is a Riemannian metric $\langle\ ,\ \rangle$ on $\mathbb{CH}^n$ such that $SU(n,1)$ acts via isometries of it. And $\langle\ ,\ \rangle$ is unique if we require, as we now do, that its maximal sectional curvature is -1. This is the canonical Riemannian metric on $\mathbb{CH}^n$. Let $A(\mathbb{CH}^n)$ denote the subgroup of $\mathrm{Iso}(\mathbb{CH}^n)$ consisting of all holomorphic isometries; it has index 2 in $\mathrm{Iso}(\mathbb{CH}^n)$. The homomorphism $SU(n,1) \to \mathrm{Iso}(\mathbb{CH}^n)$ has image $A(\mathbb{CH}^n)$ and its kernel is the center of $SU(n,1)$ which is a finite group and is identified as

$$\{\omega I \mid \omega \in \mathbb{C} \text{ and } \omega^{n+1} = 1\}$$

where I denotes the identity matrix.

We now recall the basic curvature properties of $\mathbb{CH}^n$ starting with the fact that all of its sectional curvatures lie in the closed interval $[-4,-1]$. Since $\mathbb{CH}^n$ is a complex manifold, given a tangent vector v, we can form iv. Then the sectional curvature in the direction of the $\mathbb{R}$-plane P spanned by $\{v, iv\}$ is -4. On the other hand if $u \perp P$, then the sectional curvature in the direction of the $\mathbb{R}$-plane Q spanned by $\{u, v\}$ is -1. This is dual in some specific sense to the situation for $\mathbb{C}P^n$; in particular, $\mathbb{C}P^n$ has a canonical Riemannian metric whose sectional curvatures all lie in $[1,4]$. And, if P and Q are as above, then the sectional curvature of P is 4 while that for Q is 1; cf. [82, pp. 321-329].

A *complex n-dimensional hyperbolic manifold* is a orbit space $\mathbb{CH}^n/\Gamma$ where Γ is a discrete, torsion-free subgroup of $A(\mathbb{CH}^n)$. (The real dimension of $\mathbb{CH}^n/\Gamma$ is, of course, $2n$.) Such a group Γ is said to be *regular* provided it splits back to $SU(n,1)$; i.e., if there exists a subgroup $\bar{\Gamma}$ of $SU(n,1)$ mapping isomorphically onto Γ under $SU(n,1) \twoheadrightarrow A(\mathbb{CH}^n)$. It is an easy consequence of Theorem 17.3 that Γ contains a regular subgroup of finite index when $\mathbb{CH}^n/\Gamma$ is compact.

Since complex hyperbolic manifolds are negatively curved locally symmetric spaces, Mostow's Strong Rigidity theorem yields isometric rigidity in the special case of Conjecture 1.3 where both M and N are complex hyperbolic of $\mathbb{C}$-dim $\neq 1$. Hernández [58] and Yau and Zheng [100] independently extended this result to the situation where N is assumed only to be a Riemannian manifold whose sectional curvatures are all contained in $[-4,-1]$. (But M is still assumed to be complex hyperbolic of $\mathbb{C}$-dim $\neq 1$.) I now state precisely the nature of the counterexamples to smooth rigidity constructed in [46] in the case where M is a complex hyperbolic manifold.

18.1. Theorem. *Given any positive numbers $n \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$, there exists a pair of closed negatively curved Riemannian manifolds M and N having the following properties:*

1. *M is a complex $4n+1$ dimensional hyperbolic manifold.*
2. *The sectional curvatures of N are all in the interval $[-4-\epsilon, -1+\epsilon]$.*
3. *The manifolds M and N are homeomorphic but not diffeomorphic.*

The manifold N in 18.1 is $M \# \Sigma$ where M and Σ are, respectively, an appropriately chosen complex $4n+1$ dimensional hyperbolic manifold and a $8n+2$ dimensional exotic sphere. (Recall that a complex manifold is canonically oriented.) The choices must be made so that properties 2 and 3 of 18.1 hold. I show in this lecture how to choose M and Σ so that property 3 holds. And the extra conditions necessary to guarantee that property 2 also holds will be discussed in the next lecture. Recall that M and $M\#\Sigma$ are always homeomorphic since dim $\Sigma > 4$. Hence we need only choose M and Σ so that M and $M\#\Sigma$ are not diffeomorphic in order to satisfy property 3. Letting $[M]$ denote the concordance class of (M, id_M), Corollary 16.2 shows that it is sufficient to choose M and Σ so that $[M\#\Sigma] \neq [M]$ in $\mathcal{C}(M)$; i.e., so that $f_M^*([\Sigma]) \neq 0$. It would be convenient at this point to be able to use Lemma 16.3; but unfortunately this can't be done since a closed complex m-dimensional manifold M is *never* stably parallelizable when $m > 1$; in fact, its first Pontryagin class is never zero.

This last fact is a result of the close relationship between the tangent bundle TM of M and that of its positively curved dual symmetric space $\mathbb{C}P^m$. In fact, the following result was proven in [46; §3].

18.2. Lemma. *Let $\mathcal{M}$ be any closed complex m-dimensional hyperbolic manifold. Then there exists a finite sheeted cover M of $\mathcal{M}$ and a map $f : M \to \mathbb{C}P^m$ such that the pullback bundle $f^*(T\mathbb{C}P^m)$ and TM are stably equivalent complex vector bundles.*

Remark. This result has recently been generalized by Boris Okun in [80]. He obtains a similar relationship between the tangent bundles of any finite volume locally symmetric space of non-compact type and that of its dual symmetric space of compact type. Both 18.2 and [80] depend on a deep result about flat complex vector bundles due to Deligne and Sullivan [25]. Their result was also used by Sullivan in [94] to prove Theorem 17.1. The observation made above that $\mathcal{M}$ has a finite sheeted cover $\mathbb{C}\mathbb{H}^m/\Gamma$ where Γ is regular is needed to apply [25] in proving 18.2.

Now Lemma 18.2 can be used to prove the following useful analogue of 16.3.

18.3. Corollary. *Let $\mathcal{M}$ be any closed complex m-dimensional hyperbolic manifold. Then there exists a finite sheeted cover $\mathcal{M}_0$ of $\mathcal{M}$ such that the following is true for every finite sheeted cover M of $\mathcal{M}_0$. The group homomorphism $f^*_{\mathbb{C}P^m} : \theta_{2m} \to \mathcal{C}(\mathbb{C}P^m)$ factors through $f^*_M : \theta_{2m} \to \mathcal{C}(M)$.*

The posited factor homomorphism $\eta : \mathcal{C}(M) \to \mathcal{C}(\mathbb{C}P^m)$ is constructed geometrically as follows. Note first that 18.2 implies $M \times \mathbb{D}^{2m+1}$ embeds as a codimension-0 submanifold in $\mathrm{Int}(\mathbb{C}P^m \times \mathbb{D}^{2m+1})$. This embedding determines a "dual map"

$$\phi : \Sigma^{2m+1}\mathbb{C}P^m \to \Sigma^{2m+1}M$$

satisfying $\Sigma^{2m+1}(f_{\mathbb{C}P^m})$ and the composite $\Sigma^{2m+1}(f_M) \circ \phi$ are homotopic. The homomorphism η is then induced by ϕ via the construction given at the end of the proof of 16.3. See [46, p. 70] for details.

Recall that 17.2 yields a closed complex m-dimensional hyperbolic manifold $\mathcal{M}$ in every $\mathbb{C}$-dimension m. Hence 18.3 reduces the problem of satisfying property 3 of 18.1 to that of showing the homomorphism

$$f^*_{\mathbb{C}P^{4n+1}} : \theta_{8n+2} \to \mathcal{C}(\mathbb{C}P^{4n+1})$$

is non-zero for every $n \in \mathbb{Z}^+$. This is a classical (albeit hard) type of algebraic topology problem. Results of Adams [1], Adams and Walker [2], and Brumfiel [15] are used in [46] to give a positive solution to it; i.e., $f^*_{\mathbb{C}P^{4n+1}}$ is *never* the zero homomorphism. See again [46] for details.

Lecture 19. Berger spheres

The proof of Theorem 18.1 is completed in this lecture by constructing a negatively curved Riemannian metric on $M\#\Sigma$ satisfying property 2 of 18.1 when the injectivity radius of M is large enough; how large depends on how small the given real number ϵ is. This construction depends on the following result that puts a negatively curved Riemannian metric on $\mathbb{R}^{2m}$ which agrees with $\mathbb{CH}^m$ near ∞, with $\mathbb{H}^{2m}$ near 0, and whose sectional curvatures are ϵ-pinched close to $[-4,-1]$. Here is the precise statement.

19.1. Lemma. *Given a positive integer m, there exists a family $b_\gamma(\ ,\ )$ of complete Riemannian metrics on $\mathbb{R}^{2m}$ which is parameterized by the real numbers $\gamma \geq e$ and has the following three properties.*

1. *The sectional curvatures of $b_\gamma(\ ,\ )$ are all contained in $[-4-\epsilon(\gamma), -1+\epsilon(\gamma)]$ where $\epsilon(\gamma)$ is a $\mathbb{R}^+$ valued function such that $\lim_{\gamma\to+\infty} \epsilon(\gamma) = 0$.*
2. *The ball of radius γ about 0 in $(\mathbb{R}^{2m}, b_\gamma)$ is isometric to a ball of radius γ in $\mathbb{H}^{2m}$.*
3. *There is a diffeomorphism f from $(\mathbb{R}^{2m}, b_\gamma)$ to $\mathbb{CH}^m$ which maps the complement of the ball of radius γ^2 centered at 0 isometrically to the complement of the ball of radius γ^2 centered at $f(0)$.*

Before constructing these metric $b_\gamma(\ ,\ )$, let us use them to complete the proof of 18.1. Start by using the last paragraph of Lecture 18 to select an exotic $8n+2$ dimensional sphere Σ such that $f^*_{\mathbb{C}P^{4n+1}}([\Sigma]) \neq 0$. Then fix a positive real number γ such that $\epsilon(\gamma) < \epsilon_0$ where $\epsilon_0 = \min(\epsilon, \frac{1}{2})$ and let M be a closed complex $4n+1$ dimensional hyperbolic manifold such that $M\#\Sigma$ is not diffeomorphic to M and so that the injectivity radius of M is bigger than γ^2+1. Such a manifold M exists because of 17.2, 17.3, 18.3 and 16.2. Let $\mathbb{D}_1$ be a ball of radius γ^2+1 in $\mathbb{CH}^{4n+1}$ centered at $f(0)$ where f is the diffeomorphism given by property 3 of 19.1. Isometrically identify $\mathbb{D}_1$ with a codimension-0 submanifold of M. Now change the metric on $\mathbb{D}_1$ to $b_\gamma(\ ,\ )$ using f. Let $\mathbb{D}_2$ be the ball in $(\mathbb{R}^{2n+1}, b_\gamma)$ with center 0 and radius γ. Perform the connected summing of M with Σ inside of $f(\mathbb{D}_2)$. And additionally require that $\gamma > \alpha + 1$ where α is the real number given in Lemma 17.4 which depends on $8n+2$ and ϵ_0. Then the argument proving 17.4 also shows how to put a Riemannian metric on $M\#\Sigma$, keeping the already constructed Riemannian metric on $M - f(\mathbb{D}_2)$, so that property 2 of 18.1 is satisfied. Setting $N = M\#\Sigma$, Theorem 18.1 is proven.

The remainder of this lecture is devoted to constructing the Riemannian metrics $b_\gamma(\ ,\ )$ posited in 19.1. This is done by finding a "nice" family of Riemannian metrics $c_t(\ ,\ )$ on S^{2m-1} parameterized by $t \in (0, +\infty)$ such that (S^{2m-1}, c_t) is isometric to the boundary of the ball of radius t in $\mathbb{H}^{2m}$ for $t \le \gamma$ and, respectively, in $\mathbb{CH}^m$ for $t \ge \gamma^2$. Use vector space scalar multiplication to identify $S^{2m-1} \times (0, +\infty)$ with $\mathbb{R}^{2m}$. Then $b_\gamma(\ ,\ )$ is defined so that the induced metric on $S^{2m-1} \times t$ is $c_t(\ ,\ )$, the induced metric on $x \times (0, +\infty)$ comes from the canonical one on $\mathbb{R}$, and $S^{2m-1} \times t$ is perpendicular to $x \times (0, +\infty)$ for each $(x, t) \in S^{2m-1} \times (0, +\infty)$. This outlines the construction. We now furnish details.

Let S^{2m-1} be the sphere of unit radius in $\mathbb{C}^m$ relative to the standard positive definite Hermitian form

$$u \cdot v = u_1\bar{v}_1 + u_2\bar{v}_2 + \cdots + u_m\bar{v}_m.$$

There is a natural free action of the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

on S^{2m-1} whose orbits fiber S^{2m-1} over $\mathbb{C}P^{m-1}$. This equips S^{2m-1} with complementary distributions η_1, η_2 where the Whitney sum $\eta_1 \oplus \eta_2$ equals the tangent bundle TS^{2m-1} of S^{2m-1}. The 1-dimensional distribution η_1 is tangent to the orbits of the S^1-action while the $(2m-2)$-dimensional distribution η_2 consists of all tangent vectors perpendicular to η_1 relative to the inner product real $(u \cdot v)$; i.e., the real part of the Hermitian form $u \cdot v$. Note that η_2 is a $\mathbb{C}$-vector bundle.

Fix an ordered pair of positive real numbers a, b and define a new Riemannian metric $\langle\ ,\ \rangle$ on S^{2m-1} by requiring the following equations be valid when $v \in \eta_1$ and $u \in \eta_2$;

$$\langle u, u\rangle = a^2 u \cdot u,$$

$$\langle u, v\rangle = b^2 v \cdot v, \text{ and}$$

$$\langle u, v\rangle = 0.$$

The Riemannian manifold S^{2m-1} equipped with $\langle\ ,\ \rangle$ is called a Berger sphere and is denoted by $S^{2m-1}_{a,b}$.

Berger spheres are relevant to constructing the family of Riemannian manifolds (S^{2m-1}, c_t) since the distance sphere of radius t in $\mathbb{CH}^m$ and $\mathbb{H}^{2m}$, respectively, is $S_{a,b}$ where $a = \sinh(t)$, $b = \sinh(t)\cosh(t)$ in the case of $\mathbb{CH}^m$ and $a = b = \sinh(t)$ in the case of $\mathbb{H}^{2m}$. It is also interesting to note that the distance spheres of radius t in the positively curved dual symmetric spaces to $\mathbb{CH}^m$ and $\mathbb{H}^{2m}$; namely, in $\mathbb{C}P^m$ and S^{2m}, are also the Berger spheres $S_{a,b}$ where $a = \sin(t)$, $b = \sin(t)\cos(t)$ in the case of $\mathbb{C}P^m$, $t \in (0, \frac{\pi}{2})$, and where $a = b = \sin(t)$ in the case of S^{2m}, $t \in (0, \pi)$.

To prove that the metrics $b_\gamma(\ ,\ )$ constructed by the above outline using Berger spheres for (S^{2m-1}, c_t) satisfy property 1 of 19.1, it is necessary to know the sectional curvatures of Berger spheres. Fortunately, these are well known; cf. [11] and [19]. One method for calculating them is the following. Let $U(m)$ denote the unitary group; i.e., the isometry group of the Hermitian form $u \cdot v$. It acts transitively on S^{2m-1}. On the other hand, it is also contained in $\mathrm{Iso}(S^{2m-1}_{a,b})$; i.e., $\langle\ ,\ \rangle$ is a $U(m)$-invariant Riemannian metric on S^{2m-1}. Therefore O'Neill's Riemannian submersion formula [81] can be used to give the following calculation of the sectional curvature $K(P)$ of a plane P tangent to $S^{2m-1}_{a,b}$. (See [46, §2] for the details of how this calculation is done.) Pick an orthonormal basis $\{u, \cos\theta v + \sin\theta w\}$ for P where $u, v \in \eta_2$ and $w \in \eta_1$. (All measurements relative to the calculation of $K(P)$ are with respect to $\langle\ ,\ \rangle_{ab}$.) Note that θ is the angle between η_2 and P. And let ω denote the angle between v and iu, then

$$K(P) = \frac{b^2}{a^4}\sin^2\theta + \left(\frac{1}{a^2} + \frac{3(a^2-b^2)}{a^4}\cos^2\omega\right)\cos^2\theta.$$

Berger's interest in the Riemannian manifold $S_{a,b}$ can be explained from this formula by setting $a = 1$ so that it specializes to

$$K(P) = b^2 \sin^2 \theta + (1 + 3(1 - b^2)\cos^2 \omega)\cos^2 \theta.$$

It is then immediately seen that $\{S_{1,b} \mid b < 1\}$ is a set of positively curved, simply connected, Riemannian manifolds whose sectional curvatures are all bounded above by 5 but which contains manifolds of arbitrarily small injectivity radius. In fact, this family of Riemannian manifolds more and more resembles $\mathbb{C}P^{m-1}$ as $b \to 0$. Berger called attention to this interesting phenomenon when $m = 2$. It is also interesting to note that these examples of Berger are essentially the distance spheres of radius t in $\mathbb{C}P^m$ as $t \to \frac{\pi}{2}$; more precisely, these distance spheres are $S_{a,b}$ where $a = \sin t$, $b = \sin t \cos t$.

On the other hand, we are interested in the distance sphere of radius t in $\mathbb{C}\mathbb{H}^m$ which is $S_{a,b}$ where $a = \sinh(t)$, $b = \sinh(t)\cosh(t)$, and that of radius t in $\mathbb{H}^{2m}$ which is $S_{a,a}$. We also need to interpolate between these two; i.e., to consider $S_{a,b}$ where $a = \sinh(t)$ and $a \leq b \leq \sinh(t)\cosh(t)$. Now the above curvature formula in this situation yields the following fact.

19.2. Proposition. *Assume that $a = \sinh(t)$ and $a \leq b \leq \sinh(t)\cosh(t)$. Then all of the sectional curvatures of $S_{a,b}$ lie in the interval $[-3, \coth^2(t)]$.*

I now construct the Riemannian manifolds $(S^{2m-1}, c_t(\ ,\ ))$ used to define $b_\gamma(\ ,\ )$. To do this, fix a smooth function $\psi : \mathbb{R} \to [0,1]$ which has the following properties.

1. $\psi(t) \geq 0$ for all $t \in \mathbb{R}$;
2. $\psi^{-1}(0) = (-\infty, 1]$;
3. $\psi^{-1}(1) = [2, +\infty)$.

Then use ψ to define a family of smooth functions ϕ_γ parameterized by all real numbers $\gamma \geq e$ where $\phi_\gamma : (0, +\infty) \to [0, +\infty)$. These functions are defined by the following formula:

$$\phi_\gamma(t) = \psi\left(\frac{\ln t}{\ln \gamma}\right) t \quad \text{where} \quad t \in (0, +\infty).$$

Then $(S^{2m-1}, c_t(\ ,\ ))$ is the Berger sphere $S_{a,b}$ where $a = \sinh(t)$ and $b = \sinh(t)\cosh(\phi_\gamma(t))$. It is easy to see from this definition that $b_\gamma(\ ,\ )$ satisfies properties 2 and 3 of 19.1. However, the curvature calculations needed to verify property 1 are quite complicated. These calculations use 19.2; but, more importantly, they use the method for obtaining 19.2 via O'Neill's Riemannian submersion formula. This method is used in a more elaborate way in verifying property 1. See [46, §2] for details.

Lecture 20. Final Remarks

This final lecture contains a potpourri of some of the more recent results related to the topics discussed in my previous lectures.

I start by mentioning a result due to Ontaneda [83] giving counterexamples to "PL-rigidity" for closed negatively curved manifolds.

20.1. Theorem. *(Ontaneda) Given $\epsilon > 0$, there exist a pair of 6-dimensional, closed, negatively curved Riemannian manifolds M and N with the following four properties.*

1. *M is real hyperbolic.*
2. *The sectional curvatures of N are all contained in the interval $(-1-\epsilon, -1+\epsilon)$.*
3. *M and N are homeomorphic.*
4. *But, M and N are not piecewise linearly homeomorphic.*

Remark. Since M and N are smooth manifolds, they can be piecewise smoothly triangulated. Property 4 of Theorem 20.1 means that the underlying simplicial complex of any such triangulation of M must be different from that of any such triangulation of N. In particular, M and N are *not* diffeomorphic. On the other hand, the counterexamples to smooth-rigidity for negatively curved manifolds given by Theorems 17.4 and 18.1 *are* piecewise linearly homeomorphic.

Ontaneda's construction builds on the ideas used in making the counterexamples given by Theorem 17.4; but employs the Kirby-Siebenmann obstruction to PL-equivalence, which lies in $H^3(M, \mathbb{Z}_2)$, instead of exotic spheres. The paper of Millson and Raghunathan [71] provides the real hyperbolic manifolds M with rich enough cohomology structure to carry out the argument.

There are also counterexamples to smooth rigidity for finite volume but non-compact negatively curved Riemannian manifolds. These were constructed by Farrell and Jones in [44]. Here is employed a different technique to change the differential structure than that used to prove Theorems 17.4 and 18.1. A new technique is necessary since connected summing with an exotic sphere *never* changes the smooth structure on a non-compact, connected manifold M^m, where $m \geq 5$. The technique used in [44] is a type of "Dehn surgery" along a closed geodesic in M^m using an $m-1$ dimensional exotic sphere.

There are also examples constructed by Farrell and Jones in [47] of complete real hyperbolic manifolds M with finite volume where some exotic smoothing of M *cannot* support a complete, finite volume, pinched negatively curved Riemannian metric. However, it is an open question whether any such examples exist where M is closed. See [3] for a related result.

These counterexamples taken together with the discussions in Lectures 17-19 motivate a search for extra geometric conditions which will yield smooth or PL-rigidity when both M and N are non-positively curved. This problem was addressed by Farrell and Jones in [49]. I now describe the result obtained in [49]; but, make the added assumption that both M and N are negatively curved to sharpen the discussion. We also assume that M and N are closed with dim $M \geq 5$ and $\alpha : \pi_1(M) \to \pi_1(N)$ is an isomorphism. Let $\tilde{M}(\infty)$, $\tilde{N}(\infty)$ be the Eberlein-O'Neill visibility spheres of the universal covers $\tilde{M}$, $\tilde{N}$ of M and N, respectively; cf. [26]. The fundamental groups $\pi_1(M)$ and $\pi_1(N)$ act naturally on $\tilde{M}(\infty)$ and $\tilde{N}(\infty)$, respectively. And Mostow showed (implicitly; cf. [75]) there is a unique α-equivariant homeomorphism $\alpha_\infty : \tilde{M}(\infty) \to \tilde{N}(\infty)$. There is also a natural map

$$F : S\tilde{M} \to \tilde{M}(\infty)$$

defined by $F(v) = \gamma_v(+\infty)$. Here, $S\tilde{M} \to \tilde{M}$ denotes the tangent sphere bundle of $\tilde{M}$; γ_v is the unique geodesic in $\tilde{M}$ satisfying $\dot{\gamma}_v(0) = v$ and $\gamma_v(+\infty)$ is the asymptoty class containing the geodesic ray $\{\gamma_v(t) \mid t \geq 0\}$. (Recall from [26] that the asymptoty classes of geodesic rays in $\tilde{M}$ are the points of $\tilde{M}(\infty)$.) The map F restricted to any fiber of $S\tilde{M} \to \tilde{M}$ is a homeomorphism onto $\tilde{M}(\infty)$. And the visibility sphere $\tilde{M}(\infty)$ is said to be *naturally* C^1 provided it has a C^1-manifold structure such that F is a C^1-map and when restricted to each fiber of $S\tilde{M} \to \tilde{M}$ is a C^1-diffeomorphism.

Remarks. If $\tilde{M}(\infty)$ is naturally C^1, then this C^1 structure is unique and the action of $\pi_1(M)$ on $\tilde{M}(\infty)$ is via C^1-diffeomorphisms. Furthermore, $\tilde{M}(\infty)$ is naturally C^1 when M is *strictly* $\frac{1}{4}$*-pinched*; i.e., when there exists a positive real number b such that all the sectional curvatures of M lie in the open interval $(-b, -b/4)$. This second comment is a consequence of the fundamental result of Hirsch and Pugh [61].

The rigidity result in [49] can now be stated as follows.

20.2 Theorem. *Assume, in addition to the above assumption, that both* $\tilde{M}(\infty)$ *and* $\tilde{N}(\infty)$ *are naturally* C^1 *and that* α_∞ *is a* C^1*-diffeomorphism. Then,* α *is induced by a piecewise linear homeomorphism. In fact, there is a smooth diffeomorphism*

$$f : M\#s\Sigma \to N$$

inducing α*. Here,* Σ *is a homotopy sphere and* s *denotes the Euler characteristic of* M*.*

Remark. If dim M is odd, then $s = 0$. And hence $f : M \to N$ is a diffeomorphism; i.e., smooth-rigidity holds in this case.

Unfortunately, the condition in 20.2 that α_∞ is a C^1-diffeomorphism is quite strong. In particular, it is *not* a necessary condition for smooth rigidity. Mostow showed in his original work on strong rigidity [74] that if α_∞ is a C^1-diffeomorphism and M and N are both real hyperbolic manifolds, then α is induced by an isometry; even when dim $M = 2$! In fact, we do not know an example where the conclusions of 20.2 cannot be replaced by the stronger statement "α is induced by an isometry after multiplying the metric on M by a suitable constant." Hopefully, weaker conditions "at ∞" will be found which imply smooth (or PL)-rigidity.

Let M be a closed and connected Riemannian manifold. Then there is a natural sequence of groups and homomorphisms

$$\text{Iso}(M) \to \text{Diff}(M) \to \text{Top}(M) \to \text{Out}(\pi_1 M);$$

consisting of all self-isometries, diffeomorphisms, homeomorphisms of M and outer automorphisms of $\pi_1(M)$, respectively. An immediate consequence of Mostow's Strong Rigidity Theorem; cf. [75], is that the composition of these homomorphisms maps $\mathrm{Iso}(M)$ onto $\mathrm{Out}(\pi_1 M)$ when M is a non-positively curved locally symmetric space whose universal cover has no 1 or 2 dimensional metric factor. Likewise, it is an immediate consequence of Theorem 14.1 that $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$ is an epimorphism when M is non-positively curved and $\dim M \neq 3, 4$. On the other hand, the examples of Theorem 17.5 were used by Farrell and Jones in [41] to show that the homomorphism $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1 M)$ is *not*, in general, an epimorphism under these same assumptions. However, it had been hoped that this map was always epimorphic; cf. [86]. One reason for this optimism was the following fundamental result due to Eells and Sampson [28].

20.3. Theorem. *Let $f : M \to N$ be a homotopy equivalence where N is a closed non-positively curved Riemannian manifold. Then f is homotopic to a harmonic map.*

Remark. A harmonic map is a smooth map which is a critical point of the energy functional. And the energy of $f : M \to N$ is essentially the integral over all $v \in SM$ of $\frac{1}{2}|df(v)|^2$.

It was hoped that every harmonic homotopy equivalence between closed non-positively curved Riemannian manifolds was a diffeomorphism, or at least a homeomorphism; cf. [27, Problems 5.4 and 5.5]. Theorem 17.5 showed that the diffeomorphism conclusion is false, in general. But the homeomorphism conclusion is still an open problem. The paper [50] of Farrell and Jones is an attempt to address this problem. Among other things, there are constructed in [50] examples of harmonic homotopy equivalences $f : M \to N$ which are *not* homeomorphisms even though N is negatively curved. But in these examples it is unknown if M can also be non-positively curved. The examples are based on a topological result due to Hatcher and Igusa, cf. [57, §4].

I finish these lectures with some additional comments about Whitehead groups. Recall Theorem 14.2 showed that $\mathrm{Wh}(\Gamma) = 0$ for a large class of torsion-free groups Γ; namely, for

$\Gamma = \pi_1(M)$ where M is a closed (connected) non-positively curved Riemannian manifold. Much is also known about $\mathrm{Wh}(F)$ where F is a finite group, cf. [4]; in particular, it is finitely generated and its rank is $r - q$, where r is the number of irreducible real representations of F and q is the number of irreducible rational representations. On the other hand, there are many examples due to M.P. Murthy (cf. [4]) of finitely generated abelian groups Γ such that $\mathrm{Wh}(\Gamma)$ is *not* finitely generated. Recently, Farrell and Jones in [45] have given a method for "computing" $\mathrm{Wh}(\Gamma)$ in terms of $\mathrm{Wh}(S)$, $\tilde{K}_0(\mathbb{Z}S)$ and $K_{-n}(\mathbb{Z}S)$ $(n \geq 1)$, where S varies over the class of all virtually cyclic subgroups of Γ. This method is valid for any subgroup Γ of a uniform lattice in a Lie group G (where G has only finitely many connected components). Their method is additionally conjectured in [45] to be valid for *all* groups Γ. A similar conjecture is also made in [45] for a method of calculating the surgery L-groups of integral group rings.

Remarks. A group is *virtually cyclic* if it contains a cyclic subgroup with finite index; e.g., finite groups, the infinite cyclic group, and the infinite dihedral group are virtually cyclic. The *lower K-groups* of a ring R, denoted by $K_{-n}(R)$ where n is any integer ≥ 1, were defined by Bass in [4].

W.-C. Hsiang conjectured in [62] that $K_{-n}(\mathbb{Z}\Gamma) = 0$ for every group Γ and every integer $n \geq 2$. Farrell and Jones in [48] verified Hsiang's conjecture for any subgroup Γ of a uniform lattice in a Lie group with finitely many connected components. They did this by directly verifying Hsiang's conjecture for all infinite virtually cyclic groups and then applying the main result of [45]. Carter in [17] had previously verified Hsiang's conjecture for all finite groups.

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