THE OBSTRUCTION TO FIBERING A MANIFOLD OVER A CIRCLE

F. Farrell, 1967

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The Obstruction to Fibering a Manifold over a Circle

By

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1967

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Summary

Let M^n be a closed connected smooth manifold of dimension $n \geq 6$. Let $f : M^n \longrightarrow S^1$ be a continuous map. Assume that $f_* : \pi_1(M) \longrightarrow \pi_1(S^1)$ is onto. Let G = kernel f_*. Then $\pi_1(M)$ is a semi-direct product of G and Z with respect to an automorphism α of G. Let X be the covering space of M corresponding to G. Let α_* denote the automorphism of $Wh(G)$ induced by α. Define $Wh(G,\alpha) = Wh(G)/$ image $(id-\alpha_*)$. Another abelian group $C(Z(G),\alpha)$ can also be defined via a Grothendieck construction. If X is dominated by a finite C.W. complex then an obstruction $c(f) \in C(Z(G),\alpha)$ is defined. If $c(f) = 0$ then a second obstruction $\tau(f) \in Wh(G,\alpha)$ is defined. We prove the following theorem.

<u>Theorem</u>. There exists a smooth fiber map $\bar{f} : M \longrightarrow S^1$ homotopic to f if and only if

1° X is dominated by a finite C.W. complex

2° $c(f) = 0$

3° $\tau(f) = 0$

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Introduction

Let M^n be a closed connected smooth manifold of dimension greater than five. Let $f : M \longrightarrow S^1$ be a continuous map. The purpose of this thesis is to give necessary and sufficient conditions for there to exist a smooth fiber map $\bar{f} : M \longrightarrow S^1$ such that $\bar{f}$ is homotopic to f. One condition is that $f_{\#}$ is not zero where $f_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1)$. This is seen by considering the homotopy sequence for a fiber space and remembering that the fiber should be a compact manifold. Now for convenience let us assume that $f_{\#}$ is onto. This corresponds to the fiber of $\bar{f}$ being connected. Let G denote the kernel of $f_{\#}$. Let X be the covering space of M corresponding to G. Then a second necessary condition is that X must be dominated by a finite C.W. complex. This is true since X is the homotopy type of the fiber of $\bar{f}$.

Let (N^{n-1},v) be a closed framed submanifold of M such that (N,v) represents f under the Pontryogin-Thom construction. Let M_N denote the compact manifold obtained by "cutting" M along N. Then ∂M_N consists of two copies of N which we denote by N' and N''. Then M_N is a cobordism from N' to N''. The pair (N^{n-1},v) is called a splitting of M with respect to f. Let s be an integer smaller than $n-2$ and larger than 1. Under the assumptions made thus far we can always find a splitting (N,v) such that

(M_N, N) has a handlebody decomposition consisting of only s and $s+1$ dimensional handles. The existence of a smooth fiber map $\overline{f}$ is equivalent to the existence of a splitting such that M_N is diffeomorphic to $N' \times [0,1]$. There are two obstructions to doing this. The first $c(f)$ with values in an abelian group $C(Z(G),\alpha)$ vanishes if and only if there exists a splitting (N,v) such that (M_N, N') is an h-cobordism. The second $\tau(f)$ which is defined if $c(f)$ vanishes has values in a quotient group of $Wh(G)$ and vanishes if and only if there exists a splitting (N,v) such that M_N is diffeomorphis to $N' \times [0,1]$. Put briefly $\overline{f}$ exists if and only if

1° X is dominated by a finite C.W. complex,

2° $c(f) = 0$, and

3° $\tau(f) = 0$.

For certain groups G both $C(Z(G),\alpha)$ and $Wh(G)$ vanish. For example if G is a free abelian group on k generators. When this is the case both conditions 2° and 3° drop and we are left with condition 1° which is a purely homotopy theoretic condition.

We also consider the case where M is a manifold with boundary where the boundary already fibers a circle and we wish to extend this fibration to the rest of M.

The result of this paper for $G = 0$ had already been obtained by W. Browder and J. Levine in [3]. I recommend reading their paper before reading this thesis. Finally I wish to remark that J. Stallings [16] has studied the case of when three dimensional manifolds fiber a circle. I know of no general results for manifolds of dimensions four or five.

In a similar fashion the groups $\widetilde{C}(R,\alpha)$ and $\widetilde{K}_0(R)$ are defined from the categories $\widetilde{\mathcal{C}}(R,\alpha)$ and $\mathcal{P}(R)$ respectively. The functor $\mathcal{F}$ induces a map $\overline{\mathcal{F}} : C(R,\alpha) \longrightarrow \widetilde{K}_0(R)$. There is a functor $\mathcal{J} : \mathcal{P}(R) \longrightarrow \mathcal{C}(R,\alpha)$ given by sending P to $(P,0)$. $\mathcal{J}$ induces a map $\overline{\mathcal{J}} : \widetilde{K}_0(R) \longrightarrow C(R,\alpha)$ $\overline{\mathcal{J}}$ is a splitting of $\overline{\mathcal{F}}$.

We can give an explicit construction of $C(R,\alpha)$ as equivalence classes of isomorphism classes of objects from $\mathcal{C}(R,\alpha)$ where the equivalence relation is generated by:

$1'^{o}$ $(P,f)\sim(P + F, f + 0)$ if F is free

$2'^{o}$ if $0 \longrightarrow (P_2,f_2) \longrightarrow (P_1,f_1) \longrightarrow (P_o,f_o) \longrightarrow 0$ is exact in $\mathcal{C}(R,\alpha)$ then $(P_1,f_1)\sim(P_2 \oplus P_o,f_2 \oplus f_o)$.

The group operation is defined by

$$\{(P,f)\} + \{(Q,g)\} = \{(P \oplus Q, f \oplus g)\}.$$

With this definition $C(R,\alpha)$ is an abelian group. The only difficult thing to verify is the existence of inverses. Once this is done it is easy to see that $C(R,\alpha)$ is universal with respect to properties 1^o and 2^o above. As one consequence this shows that the map $\sigma : \mathcal{C}(R,\alpha) \longrightarrow C(R,\alpha)$ is onto.

Now we will show that $C(R,\alpha)$ possesses inverses. We define a triangular object in $\mathcal{C}(R,\alpha)$ to be a pair (F,f) such that there exists a sequence $0 = F_o \subset F_1 \subset F_2 \subset \ldots \subset F_n = F$

where each F_i is a free finitely generated submodule of F, each F_i/F_{i-1} is also free, and $f(F_i) \subset F_{i-1}$.

Lemma 1^o. Let (K,k) be an object in $\mathcal{C}^*(R,\alpha)$. Let $0 = K_o \subset K_1 \subset \ldots \subset K_m = K$ be a filtration of K by finitely generated submodules such that $k(K_i) \subset K_{i-1}$. Then there exists an exact sequence in $C^*(R,\alpha)$ $0 \longrightarrow (L,\ell) \longrightarrow (F,f) \underset{p}{\longrightarrow} (K,k) \longrightarrow 0$ where (F,f) is triangular with respect to a filtration $0 = F_o \subset F_1 \subset \ldots \subset F_m = F$ (each F_i and F_i/F_{i-1} being free and $f(F_i) \subset F_{i-1}$) and such that $p(F_i) = K_i$.

Proof. The proof is by induction on n.

If $n = 1$ since K is finitely generated we can find a map $p : F \longrightarrow K \longrightarrow 0$ where F is a finitely generated free module; then $0 \longrightarrow (\ker p,0) \longrightarrow (F,0) \underset{p}{\longrightarrow} (K,0) \longrightarrow 0$ is the desired sequence. Assume that the lemma is true for $n-1$. Therefore there exists a map $p : (F_{n-1},f) \longrightarrow (K_{n-1},k) \longrightarrow 0$ satisfying the conclusion of Lemma 1^o. Since K_n/K_{n-1} is finitely generated there exists a finitely generated free module Q and a map $q : Q \longrightarrow K_n/K_{n-1} \longrightarrow 0$. This lifts to a map $\hat{q} : Q \longrightarrow K_n = K$. Let $F = F_{n-1} \oplus Q$ and extend the definition of p to F by the use of $\hat{q}$. Then $p : F \longrightarrow K \longrightarrow 0$. Consider the following diagram:

$$\begin{array}{ccc}
Q & \overset{\hat{f}}{\dashrightarrow} & F_{n-1} \\
p \downarrow & & \downarrow p \\
K & \underset{k}{\longrightarrow} & K_{n-1} \\
 & & \downarrow \\
 & & 0
\end{array}$$

Since Q is free and $k \circ p$ is α semi-linear it is easily seen that $k \circ p$ lifts to an α semi-linear map $\hat{f}$. Extend f from F_{n-1} to F by the use of $\hat{f}$. Then the pair (F,f) and the map p clearly satisfy the conclusion of Lemma 1^o.

Remark: In the above we let $L = \ker p$ and $\ell = f/\ker p$. Let $L_i = F_i \cap \ker p$, then $0 = L_o \subset L_1 \subset \ldots \subset L_n = L$ and $\ell(L_i) \subset L_{i-1}$. Also $0 \longrightarrow L_i/L_{i-1} \longrightarrow F_i/F_{i-1} \longrightarrow K_i/K_{i-1} \longrightarrow 0$.

The identity of the semi-group $C(R,\alpha)$ is $\{(0,0)\}$. Any triangular object is equivalent to $(0,0)$. This follows by induction on the length of the filtration by which the object (F,f) is said to be triangular. That is consider the sequence $0 \longrightarrow (F_{n-1}, f/F_{n-1}) \longrightarrow (F,f) \longrightarrow (F_n/F_{n-1}, 0) \longrightarrow 0$, then $(F,f) \sim (F_{n-1} \oplus F_n/F_{n-1}, f/F_{n-1} \oplus 0)$ by $2'^o$, $(F_{n-1} \oplus F_n/F_{n-1}, f/F_{n-1} \oplus 0) \sim (F_{n-1}, f/F_{n-1})$ by $1'^o$, and $(F_{n-1}, f/F_{n-1}) \sim (0,0)$ by the induction hypothesis.

If (P,g) is an object from $\mathcal{C}(R,\alpha)$ and g is nilpotent of order n, then let $K_i = \text{image } g^{n-i}$. Then the filtration $0 = K_o \subset K_1 \subset \ldots \subset K_n = P$ satisfies the hypothesis of Lemma 1^o. Hence there exists an exact sequence in $\mathcal{C}^*(R,\alpha)$ $0 \longrightarrow (L,\ell) \longrightarrow (F,f) \longrightarrow (P,g) \longrightarrow 0$. But since both (F,f) and (P,g) are in $\mathcal{C}(R,\alpha)$ this implies that (L,ℓ) is in $\mathcal{C}(R,\alpha)$. By $2'^o$ $(L \oplus P, \ell \oplus g) \sim (F,f)$ which is equivalent to $(0,0)$ since (F,f) is triangular. Hence $C(R,\alpha)$ is a group. We could give similar explicit constructions for the groups $\widetilde{C}(R,\alpha)$ and $\widetilde{K}_o(R)$. In particular the maps $\sigma : \widetilde{\mathcal{C}}(R,\alpha) \longrightarrow \widetilde{C}(R,\alpha)$ and $\sigma : \mathcal{P}(R) \longrightarrow \widetilde{K}_o(R)$ are both onto.

Next we show that $C(R,\alpha)$ splits as a direct sum of $\widetilde{C}(R,\alpha)$ and $\widetilde{K}_o(R,\alpha)$. Let $I : \widetilde{\mathcal{C}}(R,\alpha) \longrightarrow \mathcal{C}(R,\alpha)$ denote the inclusion functor. Then I induces a homomorphism $\overline{I} : \widetilde{C}(R,\alpha) \longrightarrow C(R,\alpha)$. It is easily seen that $\overline{\mathcal{J}} \circ \overline{I} = 0$. If (P,f) is an object in $\mathcal{C}(R,\alpha)$ then there exists Q in $\mathcal{P}(R)$ such that $P \oplus Q$ is free. Hence $\sigma(P,f) = \overline{I}\ \sigma(P \oplus Q, f \oplus 0) - \overline{J}\ \sigma(Q,0)$. Therefore image $\overline{I}$ equals kernel $\overline{\mathcal{J}}$. If to the object (P,f) in $\mathcal{C}(R,\alpha)$ we assign $\sigma(P \oplus Q, f \oplus 0) \in \widetilde{C}(R,\alpha)$, it is easily checked that this gives a well defined map of $\mathcal{C}(R,\alpha)$ to $\widetilde{C}(R,\alpha)$ which satisfies properties 1^o and 2^o and hence defines a map of $C(R,\alpha)$ to $\widetilde{C}(R,\alpha)$ which is seen to be a right inverse to $\overline{I}$. Hence $C(R,\alpha)$ splits naturally as a direct sum of $\widetilde{C}(R,\alpha)$ and $\widetilde{K}_o(R)$. A ring is said to be right regular if it is right Noetherian and every finitely generated right module has a projective resolution of finite length.

Theorem 1^o: If R is right regular then $\widetilde{C}(R,\alpha) = 0$.

<u>Proof</u>: Let (P,g) be an object from $\widetilde{\mathcal{C}}(R,\alpha)$. Then g is nilpotent of some order n. Let K_i = image g^{n-i}, then $0 = K_o \subset K_1 \subset \ldots \subset K_n = P$ is a filtration of P satisfying the hypothesis of Lemma 1^o. Hence there exists an exact

sequence in $\mathcal{C}^*(R,\alpha)$ $0 \to (P_1,g_1) \to (F,f) \to (P,g) \to 0$. Since (F,f) and (P,g) are in $\tilde{\mathcal{C}}(R,\alpha)$, (P_1,g_1) is also in $\tilde{\mathcal{C}}(R,\alpha)$. Since (F,f) is triangular $\sigma(P_1,g_1) = -\sigma(P,g)$. But by the remark following Lemma 1^o we see that there exists a filtration of P_1, $0 = L_o \subset L_1 \subset \ldots \subset L_n = P_1$ such that $g(L_i) \subset L_{i-1}$ and $0 \to L_i/L_{i-1} \to F_i/F_{i-1} \to K_i/K_{i-1} \to 0$. Since R is Noetherian we see that L_i/L_{i-1} is finitely generated. If M is a right R module let $d(M)$ denote the length of a projective resolution of M of minimal length. Then $d(L_i/L_{i-1}) = \max(1, d(K_i/K_{i-1})-1)$. Since R is Noetherian each L_i is finitely generated. Hence (P_1,g_1) satisfies the hypothesis of Lemma 1^o with respect to $0 = L_o \subset L_1 \subset \ldots \subset L_n = P_1$. Let $m = \max_{i \leq i \leq m} d(K_i/K_{i-1})$. Then after m applications of Lemma 1^o we obtain an object (P_m,g_m) of $\tilde{\mathcal{C}}(R,\alpha)$ such that $\sigma(P_m,g_m) = (-1)^m \sigma(P,g)$ and a filtration $0 = S_o \subset S_1 \subset \ldots \subset S_n = P_m$ such that each S_i/S_{i-1} is an object from $\mathcal{P}(R)$. By application of 2^o we see that

$$\sigma(P_m,g_m) = \sum_{i=1}^{n} \sigma(S_i/S_{i-1},0) = \sigma\left(\bigoplus_{i=1}^{n}(S_i/S_{i-1}),0\right) = \sigma(P_m,0) = 0$$

since P_m is stably free. Hence $\tilde{C}(R,\alpha) = 0$.

Let γ be an anti-automorphism of R such that $\gamma \circ \gamma =$ identity and $\alpha \circ \gamma = \gamma \circ \alpha$. We proceed to define a duality functor $D : \mathcal{C}(R,\alpha) \to \mathcal{C}(R,\alpha^{-1})$. If $P \in \mathcal{P}(R)$

we denote by $\overline{\mathrm{Hom}}_R(P,R)$ the collection of anti-homomorphisms of P to R. ($\varphi \in \overline{\mathrm{Hom}}_R(P,R)$ if $\varphi(xr) = \gamma(r)\,\varphi(x)$ for all $x \in P$ and $r \in R$, see [2] page 119). Then $\overline{\mathrm{Hom}}_R(P,R) \in \mathcal{P}(R)$. Let us denote this object by $D(P)$. Then D becomes a contravariant exact functor from $\mathcal{P}(R)$ to $\mathcal{P}(R)$. Also $D \circ D$ is naturally equivalent to the identity functor. Let $(P,f) \in \mathcal{C}(R,\alpha)$. If $\varphi \in D(P)$ we define $f^*(\varphi) = \alpha^{-1} \circ \varphi \circ f$. Then $(D(P),f^*) \in \mathcal{C}(R,\alpha^{-1})$. Denote this object by $D(P,f)$. This defines a contravariant exact functor from $\mathcal{C}(R,\alpha)$ to $\mathcal{C}(R,\alpha^{-1})$. If D' is the analogous functor from $\mathcal{C}(R,\alpha^{-1})$ to $\mathcal{C}(R,\alpha)$ then $D' \circ D$ is naturally equivalent to the identity functor. Since D takes free modules to free modules and is an exact functor, one easily sees that D takes triangular object to triangular objects. Let $r \in R$ be an invertible element. Let $\mathcal{R}_r$ denote right multiplication by r. Then $\mathcal{R}_r$ induces a functor which is an isomorphism between $\mathcal{C}(R,\alpha)$ and $\mathcal{C}(R,I_r \circ \alpha)$ where I_r denotes the inner automorphism of R $s \longrightarrow r^{-1}sr$. The functor $\mathcal{R}_r$ sends (P,f) to $(P,\mathcal{R}_r \circ f)$. If 1 is the identity element of R, then -1 denotes its additive inverse. Sometimes we will be interested in the functor $\overline{D} = \mathcal{R}_{-1} \circ D$ instead of D. $\overline{D} : \mathcal{C}(R,\alpha) \longrightarrow \mathcal{C}(R,\alpha^{-1})$ enjoys all the properties which we listed above for the functor D.

We record for later use the following lemma:

Lemma 2°. Let $0 \longrightarrow P_2 \xrightarrow{i} P_1 \xrightarrow{p} P_o \longrightarrow 0$ and $0 \longrightarrow Q_2 \xrightarrow{i'} Q_1 \xrightarrow{p'} Q_o \longrightarrow 0$ be two exact sequences in $\mathcal{P}(R)$. Let $y_i : Q_i \longrightarrow P_i$ $i \neq 1$ be α semi-linear homomorphisms. Then there exists an α semi-linear homomorphism $y_1 = Q_1 \longrightarrow P_1$ such that $y_1 \circ i' = i \circ y_2$ and $p \circ y_1 = y_o \circ p'$.

Proof: Let c and c' be splittings of p and p' respectively. Then $P_1 = \text{image } i \oplus \text{image } c$ and $Q_1 = \text{image } i' \oplus \text{image } c'$. With these identifications y_o and y_2 clearly define y_1.

Chapter 2°. Preliminary Geometric Preparations.

Let M^n be a compact connected C^∞ manifold of dimension ≥ 6. Let B denote the boundary of M. Assume that B is connected (or empty). Suppose that $f : B \longrightarrow S^1$ is a smooth fiber map (see [3]). Let $\hat{f}$ be a continuous map of M into S^1 such that $\hat{f}/B$ is homotopic to f. Suppose $\hat{f}_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1)$ is onto. Our problem is to determine when there exists a smooth fiber map $\overline{f} : M \longrightarrow S^1$ such that $\overline{f}/B = f$ and $\overline{f}$ is homotopic to $\hat{f}$. In order to avoid obscuring the main ideas in our discussion we postpone to an appendix consideration of the cases where B is disconnected or $\hat{f}_{\#}$ is not onto.

<u>Note</u>. The homotopy classes of maps of a space X into S^1 correspond in a one to one fashion to elements of $H^1(X)$. If $a \in H^1(B)$ corresponds to f then the homotopy classes of extensions of f to M correspond to the elements $b \in H^1(M)$ such that $i^*(b) = a$, where i denotes the inclusion map of B into M.

A pair (N,v) will be called a splitting of M if N is a compact $n-1$ dimensional submanifold of M such that $\partial N \subseteq \partial M$ (N should meet ∂M transversely) and v is a framing for the normal bundle of N such that under the Pontryagin-Thom construction (N,v) represents $\hat{f}$. We also require that ∂N is a fiber of B (that is $\partial N = f^{-1}(x)$ for some $x \in S^1$). When no confusion results we denote a splitting merely by N. W is an elementary cobordism of dimension i from splitting (N,v) to splitting (N',v') if W is an n dimensional compact submanifold (with corner, see [4] part I) of M such that $\partial W = N \cup f^{-1}(I) \cup N'$, where I is an arc in S^1 with endpoints x and y. $N \cap N' = \phi$, $N \cap f^{-1}(I) = f^{-1}(x)$, $N \cap f^{-1}(I) = f^{-1}(y)$ and the vector field v should point into W while v' should point out of W. Also W should be diffeomorphic to $N \times [0,1] \cup_{\varphi} D^i \times D^{n-1}$, where φ is a diffeomorphism of $S^{i-1} \times D^{n-i}$ into the interior of $N \times 1$. Note that W has corners at $f^{-1}(x)$ and $f^{-1}(y)$, while $N \times [0,1] \cup_{\varphi} D^i \times D^{n-i}$ has corners at $\partial N \times 0$, $\partial N \times 1$, and $S^{i-1} \times S^{n-i-1}$. We smooth these corners as in [4] and the diffeomorphism is then between smoothed manifolds. Since Wall in [4] claims that smoothings are unique this is well defined. We will avoid further mention of smoothing corners. In the literature the passage from N to N' via W is referred to as exchanging an i-dimensional handle from one side of N to the other.

We now embark on a program of improving splittings. If we start with an arbitrary N (splittings always exist by the Pontryagin-Thom construction) we can pass to a new splitting N', where N' is connected, by a sequence of elementary cobordisms of dimension 1. This is done explicitly in [3] on page 157. Here we use the fact that $\hat{f}_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1)$ is onto. This is equivalent to the element $b \in H^1(M)$ which corresponds to $\hat{f}$ not being expressible as $m\,b'$ where $b' \in H^1(M)$ and m is an integer larger than 1.

Let $G = \ker \hat{f}_{\#}$, $f_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1)$. Since G is a normal subgroup, we will generally omit considerations of base point. Let X be the covering space of M corresponding to G. A necessary condition for a smooth fiber map $\bar{f}$ homotopic to $\hat{f}$ to exist is that X be dominated by a finite C.W. complex. Under this single assumption we will see how much improvement of splittings we can effect. We have N connected. We wish next to obtain the situation where $i_{\#} : \pi_1(N) \longrightarrow \pi_1(M)$ is a monomorphism with image G (i denotes the inclusion of N into M). First we need the following algebraic lemma. Let A, B, C, and D be groups, $f : A \longrightarrow B$ and $g : C \longrightarrow B$ homomorphisms such that kernel g is D and g is onto. Assume that $A, C,$ and D are finitely generated. Let $A \circ C$ denote the free product of A and C and $h : A \circ C \longrightarrow B$ the homomorphism induced by f and g. Then h is onto and its kernel is finitely generated.

For a proof of this see [5] page 4.

Since X is dominated by a finite C.W. complex we have that $\pi_1(x) \cong G$ is finitely presented (see [6] Lemma 1.3). Let F be a free group on m generators $a_1, \ldots, a_m$ and g a homomorphism of F into G whose kernel K is finitely generated. Let x be a point inside a tubular nbd of -

N but not N. Let $L_1, \ldots L_m$ be circles embedded in M such that each L_i meets N transversally and $L_i \cap L_j = x$ if $i \neq j$. We pick L_i so that L_i represents $g(a_i)$. Since under the Pontryogin-Thom construction (N, v) represents $\hat{f}$ and since $\hat{f}_{\#}(g(a_i)) = 0$, we see that the intersection number of L_i with N is 0. Hence after exchanging a finite number of handles of dimension one we can obtain the situation where L_i does not meet N. We do this for each i, finally obtaining the situation where the bouquet of circles $L_1, \ldots, L_m$ is disjoint from N. Next by exchanging one dimensional handles $W_1, \ldots, W_m$ such that the core of W_i is 'homotopic' to L_i we obtain a connected splitting N' such that by van Kampen's theorem $\pi_1(N') \cong \pi_1(N) \circ F$ and the inclusion map $i_{\#} : \pi_1(N') \longrightarrow G \subseteq \pi_1(M)$ is induced from $i_{\#} : \pi_1(N) \longrightarrow G$ and $g : F \longrightarrow G \longrightarrow 0$. Hence by the algebraic lemma above the kernel of $i'_{\#}$ is finitely generated. Now by exchanging a finite sequence of 2 dimensional handles (see [7] proof of Lemma 3.1) we obtain a connected splitting N'' such that $i'_{\#} : \pi_1(N'') \longrightarrow \pi_1(M)$ is a monomorphism whose image is G.

With N thus improved, choose a lifting $\hat{N}$ of N to X. Then $\hat{N}$ divides X into two connected components which we denote A and B (see [3] sec. 3.1) The framing v also lifts to a framing $\hat{v}$. Let B denote the component into which $\hat{v}$ points. When no confusion can result we will use N and v to denote $\hat{N}$ and $\hat{v}$. Let T denote the generator of the group of covering transformations of X such that $A \subset T(A)$. Let $\tilde{X}$ denote the universal covering space of X and $p : \tilde{X} \longrightarrow X$ the covering map. Denote by $\tilde{A}$, $\tilde{B}$ and $\tilde{N}$ $p^{-1}(A)$, $p^{-1}(B)$ and $p^{-1}(N)$ respectively. Since the inclusion map of N into X induces an isomorphism on fundamental groups we see that $\tilde{N}$ is connected. and simply connected. Since $N \subset A \subset X$ we see that the inclusion of A into X induces an epimorphism on fundamental groups. Hence $\tilde{A}$ is connected. Likewise $\tilde{B}$ is connected. By van Kampen's theorem applied to $\tilde{N}$, $\tilde{A}$, $\tilde{B}$ and $\tilde{X}$ we see that $\tilde{A}$ and $\tilde{B}$ are both simply connected. Hence the inclusion maps of A and B into X are isomorphisms on fundamental groups.

Consider the groups $H_i(\tilde{X},\tilde{A};Z)$. Identify G with the group of covering transformations of $\tilde{X}$. Then $H_i(\tilde{X},\tilde{A};Z)$ becomes a right $Z(G)$ module because of the action of G on $\tilde{X}$. We denote these modules by $H_i(X,A;Z(G))$. Since X and N are dominated by finite C.W. complexes; and the inclusions of A and B into X induce isomorphisms on fundamental groups; and A,B,N and X are connected we have that A and B are both dominated by finite C.W. complexes (see [2] Complement 6.6). It also follows that $H_0(X,A; Z(G))$ and $H_1(X,A;Z(G))$ are both zero. Let i be the first integer such that $H_i(X,A;Z(G)) \neq 0$. By excision $H_i(X,A;Z(G)) = H_i(B,N;Z(G))$. Let N_{i-1} denote the i-1 skeleton of N (in some triangulation of N). Consider the homology exact sequence for the triple $N_{i-1} \subset N \subset B$. We see that $H_j(B,N_{i-1};Z(G)) = H_j(B,N;Z(G)) = 0$ for $j \leq i-1$ and that $H_i(B,N;Z(G))$ is a quotient module of $H_i(B,N_{i-1};Z(G))$. By Theorem A of [6] we have that $H_i(B,N_{i-1};Z(G))$ is finitely generated. Hence $H_i(X,A;Z(G))$ is finitely generated.

Consider $W = \overline{T(A)-A}$. This is a connected manifold with boundary. $\partial W = N \cup \partial_\circ W \cup T(N)$ where $\partial_\circ W$ is diffeomorphic to $F \times [0,1]$ (F denotes the fiber of $f : \partial M \longrightarrow S$, we assume the diffeomorphism chosen so that $F \times t$ corresponds to the inverse image of a fiber of ∂M under covering projection $\vec{p} : \partial X \longrightarrow \partial M$.) Identify $\partial_\circ W$ with $F \times [0,1]$ under this diffeomorphism. Then $N \cap \partial_\circ W = F \times 0 = \partial N$ and $\partial T(N) = T(N) \cap \partial_\circ W = F \times 1$. Notice that W has a corner at $F \times 0$ and at $F \times 1$. By considering the four spaces $T(N)$, B, W and $\overline{X - T(A)}$ where $T(N) = W \cap \overline{X - T(A)}$ and $B = W \cup \overline{X - T(A)}$, we can show that the inclusion map of $T(N)$ into W induces an isomorphism on fundamental groups. To do this we use an argument analogous to that used in showing that the inclusion of N into B induces an isomorphism on fundamental groups. Likewise the inclusion of N into W induces an isomorphism on fundamental groups. Therefore $H_1(W,N;Z(G)) = 0$. Consider this omology exact sequence for the triple $A \subset T(A) \subset X$. In particular $H_2(X,A;Z(G)) \xrightarrow{j_*} H_2(X,T(A);Z(G)) \xrightarrow{\partial} H_1(T(A),A;Z(G))$. By excision $H_1(T(A),A;Z(G)) \cong H_1(W,N;Z(G)) = 0$. Hence j_* is onto. The collection of modules $\{H_i(X,T^m(A);Z(G))\}$ for i fixed form a directed system whose maps are induced by the nclusions of $(X,T^m(A))$ into $(X,T^{m'}(A))$ for $m' \geq m$. The direct limit of this system is $H_i(X,X;Z(G)) = 0$ (see [3] section2.6). For $i = 2$ the two facts j_* onto and $H_2(X,A;Z(G))$ finitely generated yield $H_2(X,A;Z(G)) = 0$.

We shall call a splitting N s-connected if N is connected, $i_\#$ is a monomorphism onto G (i denoting the inclusion of N into M), and $H_j(X,A;Z(G)) = 0$ for $j \leq s$. In our program for improving splittings we have shown that 2-connected splittings can always be found. Next we show that n-3 connected splittings can always be found.

$\tilde{X}$ can also be considered as the universal covering space of M. As such we can identify $\pi_1(M)$ with the group of covering transformations of $\tilde{X}$. Let $t \in \pi_1(M)$ such that $\hat{f}_\#(t)$ is the generator of $\pi_1(S^1)$ determined by the orientation of S^1 used in setting up the Pontryagin-Thom correspondence. Under our identification $t : \tilde{X} \longrightarrow \tilde{X}$ covers $T : X \longrightarrow X$. t is not uniquely defined; but for the remainder of this paper our choice of t will be fixed. Let A be a subset of X such that $A \subset T(A)$. Let $j : (\tilde{X},\tilde{A}) \longrightarrow (\tilde{X},t(\tilde{A}))$ denote the inclusion map where if S is a subset of X $\widetilde{S}$ denotes $p^{-1}(S)$ (note that $t(\widetilde{A}) = \widetilde{T(A)}$). $t_*^{-1} \circ j_*$ induces an endomorphism of $H_i(\widetilde{X},\widetilde{A})$ for each i. $g \longrightarrow tgt^{-1}$ is an automorphism of G which induces an automorphism of $Z(G)$ which we denote by α. Then one sees easily that $t_*^{-1} \circ j_*$ is an α semi-linear endomorphism of $H_i(X,A;Z(G))$. Hence this module with this endomorphism can be considered as an object in $\mathcal{C}^*(R,\alpha)$. Let B be a second subset of X such that $B \subset T(B)$ and $A \subset B$. Let $j : (\widetilde{B},\widetilde{A}) \longrightarrow (t(\widetilde{B}),t(\widetilde{A}))$ be the inclusion map. Then $t_*^{-1} \circ j_*$ is an α semi-linear endomorphism of $H_i(B,A;Z(G))$, and hence this pair defines an object in $\mathcal{C}^*(R,\alpha)$. Consider the exact sequence of homology for the triple $\widetilde{A} \subset \widetilde{B} \subset \widetilde{X}$. By a straightforward verification one shows that this is an exact sequence in the category $\mathcal{C}^*(R,\alpha)$. If C is a third subset of X such that $C \subset T(C)$ and $B \subset C$, then the long exact sequence of homology for the triple $\widetilde{A} \subset \widetilde{B} \subset \widetilde{C}$ is an exact sequence in $\mathcal{C}^*(R,\alpha)$. Denote by j' the inclusion map of $(\widetilde{X},t^{-1}(\widetilde{A}))$ into $(\widetilde{X},\widetilde{A})$. Then one verifies easily that $t_*^{-1} \circ j_* = j'_* \circ t_*^{-1}$. Denote by j_m the inclusion map of $(\widetilde{X},\widetilde{A})$ into $(\widetilde{X},t^m(\widetilde{A}))$. Then by repeated application of the above equality one obtains $(t_*^{-1} \circ j_*)^m = t_*^{-m} \circ j_{m*}$. Suppose N is an s-1-connected splitting of M, then $t_*^{-1} \circ j_*$ is a nilpotent endomorphism of $H_s(X,A;Z(G))$. This is a consequence of three facts:

1° $H_s(X,A;Z(G))$ is finitely generated,

2° direct limit $\{H_s(X,T^r(A);Z(G))\}$ equals 0, and

3° $(t_*^{-1} \circ j_*)^m = t_*^{-m} \circ j_{m*}$.

Denote $t_*^{-1} \circ j_*$ by φ. Then $\varphi^m = 0$ for some integer m. Denote $H_s(X,A;Z(G))$ by K. Let K_i = image φ^{m-i}. Then $0 = K_0 \subset K_1 \subset \ldots \subset K_m = K$ is a filtration of K by finitely generated submodules such that $\varphi(K_i) \subset K_{i-1}$. Consider the homology exact sequence for $\widetilde{A} \subset \widetilde{T(A)} \subset \widetilde{X}$. We see that $H_i(T(A),A;Z(G)) = 0$ for $i < s-1$. By excision this module is isomorphic to $H_i(W,N;Z(G))$. By the theory of relative cobordisms (see [4] Part IV the remarks proceeding Theorem 6.2 and [8] Theorem 6.1) $W = W_1 \cup W_2 \cup \ldots \cup W_m$ where each W_i is an elementary cobordism (i.e. its image under $\overline{p} : X \longrightarrow M$ is an elementary cobordism) and $s-1 \leq \dim W_i \leq n-2$.

If $x \in H_s(W,N;Z(G))$ then it is possible to arrange things so that $\dim W_1 = s$ and there exists a generator $\overline{x}$ of $H_s(W_1,N;Z(G))$ such that $i'_*(\overline{x}) = x$ where i' denote the inclusion of $(\widetilde{W}_1,\widetilde{N})$ into $(\widetilde{W},\widetilde{N})$ (see [2] proof of the Fundamental Lemma 4.8 for the details of this fact). Here we need that $s \leq n-3$. Let $\hat{x}$ be one of a fixed finite collection of generators for K_1. By considering the homology sequence for $A \subset T(A) \subset X$ we see that there exists an $x \in H_s(T(A),A;Z(G))$ such that $i_*(x) = \hat{x}$. This follows since $\varphi(x) \in K_0 = 0$ and t_*^{-1} is an isomorphism, hence $j_*(\hat{x}) = 0$. Pick W_1 as above. Then $\overline{p}(W_1)$ is an elementary cobordism between N and a splitting N'. Consider the homology sequence for the triple $A \subset A' \subset X$ (where the particular lifting of N' is determined by W_1). We see that $H_i(X,A';Z(G)) = 0$ for $i < s$. Hence N' is also an s-1-connected splitting. Let K' denote $H_s(X,A';Z(G))$. Then in dimension s the sequence becomes:

$$Z(G) \xrightarrow[i_* \circ i_*']{} K \xrightarrow[j_*']{} K' \longrightarrow 0 .$$

Let φ' denote $t_*^{-1} \circ j_* : K' \longrightarrow K'$ and $K_i' = \text{image } \varphi'^{m-i}$. Since j_*' is a map in $\mathcal{C}(R,\alpha)$, $(\varphi')^m = 0$. Also j_*' induces a map $K_i \longrightarrow K_i' \longrightarrow 0$ for each i. For $i = 1$ this becomes $0 \longrightarrow (\hat{x}) \longrightarrow K_1 \longrightarrow K_1' \longrightarrow 0$. Hence K_1' is generated by one fewer elements than K_1. Therefore after repeating this process a finite number of times we obtain an s-connected splitting. Hence we see that we can obtain n-3 connected splittings.

Chapter 3°. The definition of the obstruction $c(\hat{f}) \in C(Z(G),\alpha)$.

Let $r : S^1 \longrightarrow S^1$ denote a fixed diffeomorphism of degree -1 such that $r \circ r = id$, for example reflection thru the x-axis. If (N,v) is a splitting of M with respect to $\hat{f}$ then $(N,-v)$ is a splitting with respect to $r \circ \hat{f}$. A splitting (N,v) is said to be s-bi-connected if (N,v) is s-connected and (N,v) is $(n-s)-1$ connected, that is if $H_i(X,A;Z(G)) = 0$ for $i \leq s$ and $H_i(X,B;Z(G)) = 0$ for $i \leq (n-s)-1$. By handlebody theory one sees that this is equivalent to $W = \overline{T(A) - A}$ having a handle decomposition consisting of only $s + 1$ and s dimensional handles (see [8] Theorem 6.1). Hence we see that (B,N) is the homotopy type of a pair (K,N) where K is a C.W. complex obtained from N by attaching s and $s + 1$ dimensional cells. Therefore the chain groups for $(\tilde{K},\tilde{N})$ vanish except in dimensions s and $s + 1$. Look at the following exact sequence:

$0 \longrightarrow H_{s+1}(K,N;Z(G)) \longrightarrow C_{s+1}(K,N;Z(G)) \xrightarrow{\partial} C_s(K,N;Z(G)) \longrightarrow H_s(K,N;Z(G)) \longrightarrow 0$. $H_{s+1}(K,N;Z(G)) \cong H_{s+1}(X,A;Z(G))$, $H_s(K,N;Z(G)) \cong H_s(X,A;Z(G)) = 0$, and both $C_{s+1}(K,N;Z(G))$ and $C_s(K,N;Z(G))$ are free $Z(G)$ modules. Hence $H_{s+1}(X,A;Z(G))$ is a projective $Z(G)$ module. Therefore the pair $(H_{s+1}(X,A;Z(G)),\ t_*^{-1} \circ j_*)$ is an object in the category $\mathcal{C}(R,\alpha)$. Denote this object by $c(N,v)$. We will eventually define an obstruction $c(\hat{f}) \in C(R,\alpha)$ by $c(\hat{f}) = (-1)^{s+1}\sigma(c(N,v))$. But first we must show that this is independent of the splitting (N,v).

Define $\theta : G \longrightarrow Z_2$ (cyclic group of order 2) as follows: If $g : \tilde{X} \longrightarrow \tilde{X}$ is orientation preserving $\theta(g) = 0$, if it is orientation reversing $\theta(g) = 1$. Define an anti-automorphism γ of $Z(G)$ by $\gamma(g) = (-1)^{\theta(g)}g^{-1}$. $\alpha(\gamma(g)) = (-1)^{\theta(g)}tg^{-1}t^{-1} = (-1)^{\theta(tgt^{-1})}(tgt^{-1})^{-1} = \gamma(\alpha(g))$. Therefore γ commutes with α.

Let $\mathcal{D} : \mathcal{C}(R,\alpha) \longrightarrow \mathcal{C}(R,\alpha^{-1})$ be the functor defined as follows:

1° if t is orientation preserving let $\mathcal{D} = D$ (see Chapter 1°)

2° if t is orientation reversing let $\mathcal{D} = \overline{D}$.

Lemma 1°. If (N,v) is an $(s-1)$-bi-connected splitting of M with respect to $\hat{f}$ then $\mathcal{D}\, c(N,v) \cong c(N,-v)$ where $(N,-v)$ is an $(n-s)$-bi-connected splitting of M with respect to $r \circ \hat{f}$.

Proof: Let $c(N,v) = (P,\varphi)$ and $C(N,-v) = (Q,\psi)$. Then there exists an integer m such that $\varphi^m = \psi^m = 0$. Consider the homology sequence for the triple $A \subset T^m(A) \subset X$. Since $\varphi^m = 0$ we see that $\overline{j}_* = 0$ where $\overline{j}$: $(X,A) \longrightarrow (X,T^m(A))$. Therefore $(P,\varphi) \cong (H_s(T^m(A),A;Z(G)), t_*^{-1} \circ j_*)$. Likewise $(Q,\psi) \cong (H_{n-s}(B,T^m(B);Z(G)), t_* \circ j_*')$. Let $W = \overline{T^m(A) - A}$.

If R is a ring and β an automorphism of R, we denote by $\mathcal{C}^{**}(R,\beta)$ the category whose objects are pairs (K,Y) where K is a left R module and Y is a β semi-linear endomorphism of K.

Step 1°. $(H^s(W,N;Z(G)), j^* \circ t^{-1*}) = (\mathrm{Hom}_{Z(G)}(P,Z(G)), \mathrm{Hom}(\varphi,\alpha^{-1}))$ as objects in $\mathcal{C}^{**}(Z(G),\alpha^{-1})$. By $H^s(W,N;Z(G))$ we mean $H^s_{comp}(\tilde{W},\tilde{N};Z)$, that is cohomology with compact supports.

Demonstration of Step 1°:

Since (W,N) has a handlebody decomposition consisting of only s and $s-1$ dimensional handles, (P,φ) is isomorphic to the kernel of ∂ in the following free based $Z(G)$ chain complex:

$$(C_s, \varphi_s) \xrightarrow{\partial} (C_{s-1}, \varphi_{s-1})$$

where both C_s and C_{s-1} are finitely generated and ∂ is a map in $\mathcal{C}^*(Z(G),\alpha)$. The cokernel of ∂ is also isomorphic to (P,φ) (look again at the homology sequence for the triple $A \subset T^m(A) \subset X$ and use fact that $\bar{j}_* = 0$). But P is projective. Hence $\ker \partial$ is a direct summand of C_s. Look at the chain complex

$$* \quad (\mathrm{Hom}_{Z(G)}(C_s,Z(G)), \mathrm{Hom}(\varphi_s,\alpha^{-1})) \xleftarrow{\mathrm{Hom}(\partial,\mathrm{id})} (\mathrm{Hom}_{Z(G)}(C_{s-1},Z(G)), \mathrm{Hom}(\varphi_{s-1},\alpha^{-1})).$$

The cokernel of this complex is isomorphic to

$(\mathrm{Hom}_{Z(G)}(P, \varphi), \mathrm{Hom}(\varphi, \alpha^{-1}))$. But $(H^s_{comp}(\tilde{W}, \tilde{N}; Z), j^* \circ t^{-1*})$ is isomorphic to the cokernel of the complex:

$$** \quad (\mathrm{Hom}_Z^{comp}(C_s, \underline{Z}), \mathrm{Hom}(\varphi_s, id)) \xleftarrow{\mathrm{Hom}(\partial, id)}$$

$$(\mathrm{Hom}_Z^{comp}(C_{s-1}, Z), \mathrm{Hom}(\varphi_{s-1}, id))$$

where $\mathrm{Hom}_Z^{comp}(C_s, Z)$ denotes the Z homomorphisms which vanish all but a finite number of the distinguished basis elements of C_s (that is the Z basis for C_s determined by the distinguished $Z(G)$ basis for C_s by the action of G). But the complexes * and ** are isomorphic (see [9] page 223). An explicit isomorphism is the following. Let $h : Z(G) \longrightarrow Z$ be the Z linear map determined by $h(g) = 0$ if $g \neq 1$ $g \in G$ and $h(1) = 1$. Then the map of $\mathrm{Hom}_{Z(G)}(C_i, Z(G))$ into $\mathrm{Hom}_Z^{comp}(C_i, Z)$ $(i = s$ or $s-1)$ given by sending y to $h \circ y$ is the isomorphism.

Step 2°. $\partial W = N \cup \partial N \times I \cup T^m(N)$. Let $N' = \partial N \times I \cup T^m(N)$. Let u represent the fundamental homology class (perhaps represented by an infinite chain) for $(\tilde{W}, \partial W)$ (this amounts to choosing an orientation for $\tilde{W}$). Now Poincaré duality implies that $\cap u : H^s_{comp}(\tilde{W}, \tilde{N}; Z) \longrightarrow H_{n-s}(\tilde{W}, \tilde{N}'; Z)$ is an isomorphism as Z modules (see [9] page 225). For $g \in G$ consider the following commutative diagram:

$$\begin{array}{ccc} H^s_{comp}(\tilde{W},\tilde{N};Z) & \xrightarrow{\cap u} & H_{n-s}(\tilde{W},\tilde{N}';Z) \\ \downarrow g^{-1*} & & \downarrow g_* \\ H^s_{comp}(\tilde{W},\tilde{N};Z) & \xrightarrow{\cap g_*(u)} & H_{n-s}(\tilde{W},\tilde{N}';Z) \end{array}$$

If g is orientation preserving then $g_*(u) = u$. If g reverses orientation then $g_*(u) = -u$. Hence if we change $H^s_{comp}(\tilde{W},\tilde{N};Z)$ into a right $Z(G)$ module by use of the anti-automorphism γ then $\cap u$ becomes an isomorphism between right $Z(G)$ modules.

Using the naturality properties of cap product one can also obtain the following diagram:

$$\begin{array}{ccc} H^s_{comp}(\tilde{W},\tilde{N};Z) & \xrightarrow{\cap u} & H_{n-s}(\tilde{W},\tilde{N}';Z) \\ \downarrow j^*\circ t^{-1*} & & \downarrow j'_*\circ t_* \\ H^s_{comp}(\tilde{W},\tilde{N}:Z) & \xrightarrow{\cap u} & H_{n-s}(\tilde{W},\tilde{N}';Z) \end{array}$$

which commutes if t is orientation preserving and skew commutes if t reverses orientation. Also as noted before $j'_*\circ t_* = t_*\circ j'_*$. Hence putting these facts together with Step 1^o we obtain that

$$\mathcal{D}(P, \varphi) = (H_{n-s}(W,N';Z(G)), \ t_*\circ j'_*) .$$

By considering the homology exact sequence for the triple $T^m(N) \subset N' \subset W$ and remembering that $N' = T^m(N) \cup \partial(T^m(N)) \times [0,1]$ we see that $(H_{n-s}(W,N';Z(G)), t_*\circ j'_*) = (H_{n-s}(W,T^m(N);Z(G)), t_*\circ j'_*)$. But this completes the proof of Lemma 1^o. (For properties of cap products see [10], [11] and [9] page 225.)

If W is an elementary cobordism from a splitting (N,v) to a splitting (N',v') where v points into W and v' points out of W then N is called the left side of W while N' is called the right side of W.

Lemma 2°. If (N,v) and (N',v') are two splittings of M with respect to $\hat{f}$ and N' is connected then there exists a sequence of elementary cobordisms $W_1,W_2,\ldots,W_m$ such that the right side of W_i is the same as the left side of W_{i+1}, the left side of W_1 is N while the right side of W_m is N'.

An ordered elementary cobordism is one in which we designate a front and back side. (For example front equals right side, back equals left side.) An ordered chain from N to N' is a sequence of ordered elementary cobordisms $W_1,W_2,\ldots,W_m$ such that the front side of W_1 is N while the back side of W_m is N' and the front side of W_{i+1} is the same as the back side of W_i. The first stage in the proof of Lemma 2° will be to demonstrate the existence of an ordered chain from N to N'.

Step 1°. Let M^n be a closed oriented manifold. Let (N,v) and (N',v') be $n-1$ dimensional submanifolds which intersect transversely and represent the same element of $H^1(M;Z)$ under the Pontryagin-Thom construction. Then there exists a compact submanifold W^n of M such that $\overset{o}{W}$ is a component of $M - (N \cup N')$; $\partial W = \partial_+ W \cup \partial_- W$ where $\partial_+ W$ is a compact submanifold of N while $\partial_- W$ is a compact submanifold N'; and v points into W while $-v'$ points out of W. Note that W has a corner at $\partial_+ W \cap \partial_- W$. We proceed now to give a demonstration of Step 1°.

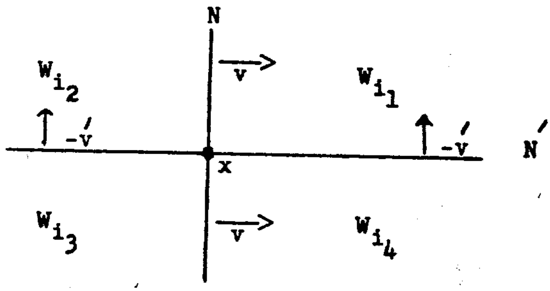
The framing v together with the orientation of M determines an orientation for N, that is an element $u_N \in H_{n-1}(N;Z)$. Likewise v' determines an element $u_{N'} \in H_{n-1}(N';Z)$. Denote also by u_N and $u_{N'}$ the images of these elements in $H_{n-1}(N \cup N';Z)$. Consider the homology sequence for $N \cup N' \subset M$:

$$H_n(M, N \cup N'; Z) \xrightarrow{\partial} H_{n-1}(N \cup N'; Z) \xrightarrow{i_*} H_{n-1}(M;Z).$$

Since (N,v) and (N',v') represent the same element of $H^1(M;Z)$ under the Pontryagin-Thom construction we have that $i_*(u_N) = i_*(u_{N'})$.

Let W_i denote the components of $M - (N \cup N')$. There are only a finite number of these since N and N' meet transversely. Therefore $H_n(M, N \cup N'; Z) = \sum_i H_n(M, M-W_i; Z)$. Let u_{W_i} denote the generator of $H_n(M, M-W_i; Z)$ determined by the orientation of M. Then there exist integers n_i such that $\partial(\sum_i n_i u_{W_i}) = u_N - u_{N'}$. Since there are only a finite number of n_i there is a maximum n_i. If we let $W = \overline{W}_i$ it is easily checked that W satisfies the conditions asserted.

Let us check more closely through the situation when $x \in W \cap N \cap N'$. There exists a nbd. U of x and coordinate functions $x_1, \ldots, x_n$ on U such that $x_i(x) = 0$, $N \cap U = \{x | x_1(x) = 0\}$ and $N' \cup U = \{x | x_2(x) = 0\}$. Then N and N' divide U into four regions which are contained in components $W_{i_1}, W_{i_2}, W_{i_3}, W_{i_4}$ of $M - N \cup N'$ (see picture).



Clearly $n_{i_1} > n_{i_2} > n_{i_3}$ and $n_{i_1} > n_{i_4} > n_{i_3}$. Hence $W = \overline{W}_{i_1}$ and W_{i_1} is distinct from W_{i_2}, W_{i_3}, and W_{i_4}. Hence W satisfies the condition to be a manifold with boundary at x (x being the corner of W). Also we see that if $N \cap N' \neq \phi$ then $N \cup N'$ divides M into at least three components. Everything proven so far for orientable manifolds M is also true for non-orientable manifolds M. Proofs can be given by considering the orientation covering $\widetilde{M}$ of M and making the obvious observations.

Step 2^o. Let (N,v) and (N',v') be two splittings of M^n such that N and N' meet transversely and $\partial M \cap N$ is disjoint from $\partial M \cap N'$. Also assume that N' is connected and that $N \cap N' \neq \phi$. Then there exists a compact submanifold W^n of M such that $\partial W = \partial_+ W \cup \partial_- W$ where $\partial_+ W$ is a compact submanifold of N while $\partial_- W$ is a compact submanifold of N', $\overset{o}{W}$ is a component of $M-(N \cup N')$, and either v points into W while v' points out of W or vice versa.

Let (N'_*, v'_*) be a connected splitting of M such that $N'_* \cap \partial M = N \cap \partial M$ and $N' \cap N'_* = \phi$. Such a splitting can be obtained by using half of the boundary of a tubular nbd of N' in M which extends the product tubular nbd. of $\partial N'$ in ∂M half of whose boundary is $N \cap \partial M$. Now consider the double of M which we denote by $M \cup M$. Applying Step 1^o to the two framed submanifolds $(N \cup N', v \cup v')$ and $(N'_* \cup N, v'_* \cup v)$ we can prove Step 2^o. That these two framed submanifolds represent the same element in $H^1(M \cup M; Z)$ can be seen by using the reduced Mayer-Vietoris sequence. This is the only place where we need that ∂M is connected.

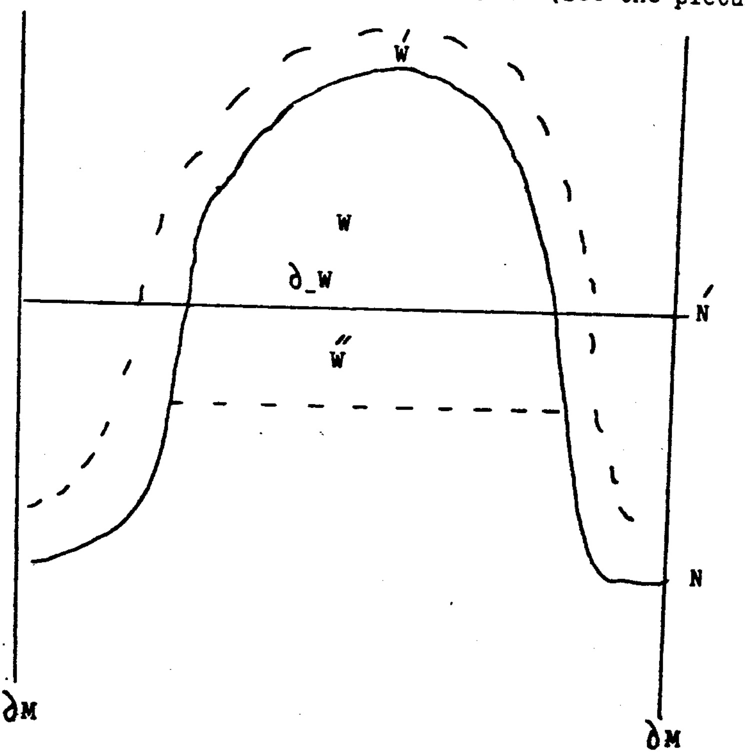
If we drop the assumption that $N \cap N' \neq \phi$ then there are two further possibilities namely:

Case 1^o. A compact submanifold W exists as above with the additional condition that $\partial_- W = \phi$.

Case 2^o. A compact submanifold W exists such that

$\partial W = \partial_+ W \cup \partial N' \times [0,1] \cup \partial_- W$ where $\mathring{W}$ is a component of $\mathring{M} - (N \cup N')$ while $\partial_+ W$ and $\partial_- W$ are as above. Here $\partial_- W = N'$.

Step 3°. We proceed now to construct an ordered chain of elementary cobordisms associated with the compact submanifold W of Step 2°. First we construct a thickened $W = W' \cup W \cup W''$ where W' is diffeomorphic to $N \times [0,1]$ and meets W in N; and W'' is diffeomorphic to $\partial_- W \times [0,1]$ and meets $W' \cup W$ in $\partial_- W \cup \partial\partial_- W \times [0,1]$. We also require that $W' \cap \partial M = \partial N \times [0,1]$ where each $\partial N \times t$ is a fiber of ∂M. (see the picture below.)



W' can be constructed by taking half of a sufficiently narrow tubular nbd. of N (the half determined by v or $-v$ depending on which points out of W). The tubular nbd. can be picked so as to have the desired property at the boundary of M. Suppose $\partial_+W \cap \partial_-W \neq \phi$. Let U be an open set containing $\partial_+W \cap \partial_-W$. There exists a Riemannian metric on M such that $N \cap U$ is a totally geodesic submanifold of M (that is if $x \in N \cap U$ and α is a geodesic such that $\alpha(o) = x$ and $\frac{d\alpha}{dt}\big|_{t=0}$ is tangent to N then there exists an $\varepsilon > 0$ such that $\alpha : [-\varepsilon,\varepsilon] \longrightarrow N$). For a proof of this fact see [12] page 74. We construct W'' as half of a tubular nbd. of ∂_-W relative to the exponential map determined by this metric and the vector field v' or $-v'$ depending on which points out of W. Note that we may assume that for $x \in \partial_+W \cap \partial_-W$ $v'(x)$ is tangent to N. If $\partial_+W \cap \partial_-W = \phi$ construct W'' similar to the way that W' was constructed.

Clearly W' since it is a product can be expressed as the union of a complementary pair of elementary cobordisms. Also $W' \cup W \cup W''$ can be expressed as the union of elementary cobordisms. Hence we obtain an ordered chain from N to a new splitting $\hat{N}$. $\hat{N}$ has the following properties: 1) if $\partial_+W \cap \partial_-W \neq \phi$ then the number of components of $\hat{N} \cap N'$ is smaller than the number of components of $N \cap N'$. 2) if Case 1° of Step 2° applies to W then the number of components of $\hat{N} \cap N'$ is the same as the number of components of $N \cap N'$. But the number of components of $\hat{N}$ is smaller than the number of components of N. Case 2^o of Step 2^o can only occur if $N \cap N' = \phi$. Hence after a finite number of applications of Step 3^o we obtain an ordered chain from our original splitting N to a new splitting $\hat{N}$ such that $N' \cap \hat{N} = \phi$. To be able to apply Step 3^o one must first make N transverse to N' by an ordered chain. But this is easily done. If we continue to apply Step 3^o we know that Case 2^o must eventually occur. Applying Step 3^o now we obtain a splitting $\hat{N}$ such that one of the components of $\hat{N}$ is N'. (Here we use a modified version of Step 3^o where in the construction of the thickened $\overline{W}$ we omit W''.) Hence $\hat{N} = N' \cup N''$ where (N'', v'') represents the zero element of $H^1(M;Z)$ under the Pontryagin-Thom construction. Also $\partial N'' = \phi$. Hence (N'', v'') represents the zero element in $H^1(M \cup M;Z)$. Applying Step 1^o we see that there exists a compact submanifold W of M such that $\partial W \subset N''$, $W \cap N' = \phi$ and for all $x \in \partial W$ either $v''(x)$ points into W or $-v''(x)$ points into W. Hence by applying Step 3^o to W (where $\partial_- W = \phi$) we reduce the number of components of N''. Hence eventually we eliminate N'' altogether. In this way we have constructed an ordered chain from (N,v) to (N',v').

A properly ordered elementary cobordism is one whose front side is the same as its left side. In order to prcve Lemma 2^o we must find an ordered chain from (N,v) to (N',v') in which each elementary cobordism is properly ordered. In the ordered chain just constructed let W be an elementary cobordism with front side (N_1,v_1) and back side (N_2,v_2) such that v_1 points out of W. We now show how to replace W by an ordered chain from (N_1,v_1) to (N_2,v_2) such that each elementary cobordism in it is properly ordered. To do this let $W' = \overline{M - W}$. Then $\partial W' = N_1 \cup \partial N_1 \times I \cup N_2$. v_1 points into W' while v_2 points out of W'. We can express W' as the union elementary cobordisms and this gives the desired ordered chain. This completes the proof of Lemma 2^o.

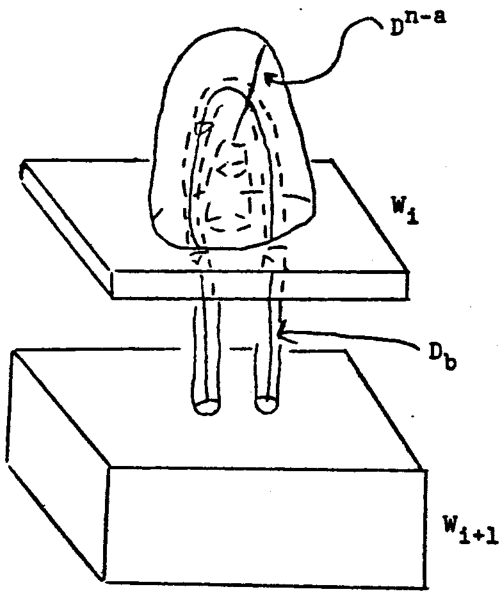
Corollary: If (N,v) and (N',v') are as in Lemma 2^o then there exists a splitting (N'',v'') and a diffeomorphism f of M homotopic to the identity map such that f maps (N',v') to (N'',v'') and an ordered chain $W_1,W_2,\ldots W_m$ from (N,v) to (N'',v'') such that each W_i is properly ordered and $\dim W_i \leq \dim W_{i+1}$.

Proof: Let $W_1,W_2,\ldots,W_m$ be the ordered chain from (N,v) to (N',v') whose existence is given by Lemma 2^o. Suppose $\dim W_{i+1} < \dim W_i$. Let N_{i-1} denote the left side of W_i while N_i and N_{i+1} denote the left and right sides of W_{i+1}.

Case 1^o. If $W_{i+1} \cap W_i = N_i$ let $W = W_i \cup W_{i+1}$. By rearrangement of handles we have that $W = W_i' \cup W_{i+1}'$ where $\dim W_i' = \dim W_{i+1}$ and $\dim W_{i+1}' = \dim W_i$. Then the ordered chain $W_1,W_2,\ldots W_{i-1},W_i',W_{i+1}',W_{i+2},\ldots,W_m$ would be one step closer to the desired chain than $W_1,W_2,\ldots,W_m$.

Case 2°. There exists a diffeomorphism g of M which is homotopic to the identity map such that g leaves a nbd. of N_i fixed, g is fiber preserving when restricted to ∂M and $g(W_{i+1}) \cap W_i = N_i$. Then let $W'_s = g(W_s)$ for $s > i$ and $W'_s = W_s$ for $s \leq i$. This gives an ordered chain from (N,v) to $(g(N'),dgv')$ in which Case 1° applies to W'_i, W'_{i+1}. Hence by repeated applications of Case 2° and Case 1° we obtain a proof of the corollary.

The diffeomorphism g of Case 2° can be constructed by observing the following facts. W_i is diffeomorphic to $N_{i-1} \times [0,1] \cup D^a \times D^{n-a}$ where 'a' is the dimension of W_i. Let D^a denote the core of the handle W_i and D^{n-a} the disc transverse to the core. Let D^b denote the core of the handle W_{i+1} where $\dim W_{i+1}$ equals b. When D^b is isotoped slightly so as to be transverse to D^{n-a} we have that D^b must miss D^{n-a} since $b < a$. Hence we can isotope D^b out of $D^a \times D^{n-a}$ without crossing $D^a \times S^{n-a-1}$. With this fact the construction of g is easy. See [8] page 6 for a similar construction. Also see picture on following page.


D^n-a
W_i
D_b
W_i+1

We proceed now to show that we have a well defined obstruction $c(\hat{f}) \in C(R,\alpha)$. Let (N,v) be an s-bi-connected splitting of M. First we must show that $\sigma(c(N,v))$ is independent of the lifting of N to X. This reduces to showing that

$$\sigma(H_{s+1}(X,A;Z(G)),\ t_*^{-1} \circ j_*) = \sigma(H_{s+1}(X,T(A);Z(G)), t_*^{-1} \circ j_*).$$

Let $W = \overline{T(A) - A}$. Then $W = W_s \cup W_{s+1}$ where W_s is a cobordism consisting of only 's' dimensional handles and W_{s+1} consists of 's+1' dimensional handles. Let $A' = A \cup W_s$. Consider the homology sequence for the triple $A \subset A' \subset X$. It reduces to:

$$0 \longrightarrow H_{s+1}(X,A;Z(G)) \longrightarrow H_{s+1}(X,A';Z(G)) \longrightarrow H_s(A',A;Z(G)) \longrightarrow 0.$$

Since $A' \subset T(A)$ $t_*^{-1} \circ j_*$ is then zero endomorphism on the free $Z(G)$ module $H_s(A',A;Z(G))$. Hence

$$\sigma(H_{s+1}(X,A;Z(G)),\ t_*^{-1} \circ j_*) = \sigma(H_{s+1}(X,A';Z(G)),\ t_*^{-1} \circ j_*).$$

Similarly $\sigma(H_{s+1}(X,A';Z(G)),\ t_*^{-1} \circ j_*) =$ $\sigma(H_{s+1}(X,T(A);Z(G)),\ t_*^{-1} \circ j_*)$. Hence $\sigma(c(N,v))$ is independent of the lifting of N to X.

Suppose $g : M \longrightarrow M$ is a diffeomorphism homotopic to the identity map and that (N',v') is the image of (N,v) under this diffeomorphism. We proceed to show that $\sigma(c(N,v)) = \sigma(c(N',v'))$. Regarding $\tilde{X}$ as a principal $\pi(M)$ bundle over M we have that $id : \tilde{X} \longrightarrow \tilde{X}$ is a bundle map covering $id : M \longrightarrow M$. Since id is homotopic to g, we have a lifting $\tilde{g}$ of g to $\tilde{X}$ which is homotopic to id through bundle maps. Such a lifting is unique. From its construction one easily sees that $\tilde{g}$ commutes with the action of $\pi_1(M)$ on $\tilde{X}$. Hence $\tilde{g}$ defines a diffeomorphism $\hat{g} : X \longrightarrow X$. Let $\hat{N}$ be a lifting of N to X and $g(\hat{N})$ the corresponding lifting of N' to X. $\tilde{g} : (\tilde{X},\tilde{A}) \longrightarrow (\tilde{X},\tilde{A}')$ induce an isomorphism $\tilde{g}_* : H_{s+1}(\tilde{X},\tilde{A};Z) \longrightarrow H_{s+1}(\tilde{X},\tilde{A}';Z)$. Since $\tilde{g}$ commutes with the action of $\pi_1(M)$ we see that $\tilde{g}_*$ is an isomorphism between $(H_{s+1}(X,A;Z(G)),\ t_*^{-1}\circ j_*)$ and $(H_{s+1}(X,A';Z(G)),\ t_*^{-1}\circ j_*)$ as objects from $\mathcal{C}(Z(G),\alpha)$. Hence $\sigma(c(N,v)) = \sigma(c(N',v'))$.

Let (N,v) and (N',v') be two s-bi-connected splittings of M. We wish to show that $\sigma(c(N,v)) = \sigma(c(N',v'))$. By the above remarks and the corollary to Lemma 2° we may assume that there exists an ordered chain $W_1,W_2,\ldots W_m$ of properly ordered elementary cobordisms from (N,v) to (N',v') such that $\dim W_i \leq \dim W_{i+1}$. Let $\hat{N}$ be a lifting of N to X. Lifting the W_i to X we can obtain a sequence A_i of submanifolds of X such that $A_i \subset A_{i+1} \subset T(A_i)$ and $\overline{A_{i+1} - A_i}$ is diffeomorphic to W_i under the covering projection. Then $\overline{\partial A_m - \partial X}$ is a lifting of N' to X. Let $A_0' \subset A_1' \subset \ldots \subset A_n'$ be a subcollection of the A_i's such that $\overline{A_{i+1}' - A_i}$ consists of only 'i+1' dimensional handles in the handlebody decomposition induced by the W_s's. One shows easily that $(H_{s+1}(A'_{s+1},A_s';Z(G)),\ t_*^{-1}\circ j_*)$ is a triangular object in

$\mathcal{C}(Z(G),\alpha)$. (Use the fact that $A_{i+1} \subset T(A_i)$.) By considering the homology sequence for triple $A \subset A' \subset X$ (where $A' = A_n'$) we see that $H_i(A',A;Z(G)) = 0$ except when 'i' equals 's' or 's+1'. By considering the sequence for $A \subset A_i' \subset A'$ where $i \leq s$ we see that $H_j(A_i',A;Z(G)) = 0$ unless $j = i$. Now consider the sequence for $A \subset A_{i-1}' \subset A_i'$ where $i \leq s$. By the above remarks it reduces to the following short exact sequence of objects from $\mathcal{C}^*(R,\alpha)$:

$$0 \longrightarrow H_i(A_i',A;Z(G)) \longrightarrow H_i(A_i',A_{i-1}';Z(G)) \longrightarrow H_{i-1}(A_{i-1}',A;Z(G)) \longrightarrow 0.$$

If by induction we assume that $H_{i-1}(A_{i-1}',A;Z(G))$ is a projective $Z(G)$ module such that $\sigma(H_{i-1}(A_{i-1}',A;Z(G)),\ t_*^{-1} \circ j_*) = 0$ (the case $i-1 = 0$ was proven above) we see that $(H_i(A_i',A;Z(G)),\ t_*^{-1} \circ j_*)$ is an object in $\mathcal{C}(Z(G),\alpha)$ and that $\sigma(H_i(A_i',A;Z(G)),\ t_*^{-1} \circ j_*) = 0$. In particular we see that $\sigma(H_s(A_s',A;Z(G)),\ t_*^{-1} \circ j_*) = 0$. Now consider the sequence for $A \subset A_s' \subset X$. It reduces to the following short exact sequence of objects from $\mathcal{C}^*(Z(G),\alpha)$:

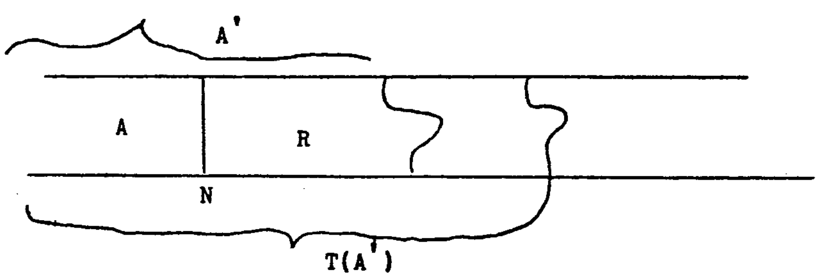
$$0 \longrightarrow H_{s+1}(X,A;Z(G)) \longrightarrow H_{s+1}(X,A_s';Z(G)) \longrightarrow H_s(A_s',A;Z(G)) \longrightarrow 0.$$

Hence $(H_{s+1}(X,A_s';Z(G)),\ t_*^{-1} \circ j_*)$ is an object from $\mathcal{C}(Z(G),\alpha)$ and $\sigma(H_{s+1}(X,A_s';Z(G)),\ t_*^{-1} \circ j_*) = \sigma(c(N,v))$. $\partial A_s' - \partial X$ is the lifting of an s-bi-connected splitting which we denote (N'',v''). Hence we have shown that $\sigma(c(N,v)) = \sigma(c(N'',v''))$. By analogous arguments we can show that $\sigma(c(N',-v')) = \sigma(c(N'',-v''))$. But by Lemma 1° $\mathcal{D}(c(N',v')) \cong c(N',-v')$ and $\mathcal{D}(c(N'',v'')) \cong c(N'',-v'')$. By the discussion of the functor $\mathcal{D} : \mathcal{C}(Z(G),\alpha) \longrightarrow \mathcal{C}(Z(G),\alpha^{-1})$ given in Chapter 1° we see that $\mathcal{D}$ induces an isomorphism of $C(Z(G),\alpha)$ onto $C(Z(G),\alpha^{-1})$. Hence $\sigma(c(N',v')) = \sigma(c(N'',v''))$. Hence $\sigma(c(N,v) = \sigma(c(N',v'))$. Therefore if we define $c(\hat{f}) = (-1)^{s+1} \ \sigma(c(N,v))$ we see that this obstruction is well defined for a fixed 's'. We note that Lemma 1° was not used in an essential fashion here since we could have shown that $\sigma(c(N',v')) = \sigma(c(N'',v''))$ by continuing the line of argument used in showing that $\sigma(c(N,v)) = \sigma(c(N'',v''))$. In order to show that $c(\hat{f})$ is independent of s we need to find an s-bi-connected splitting (N,v) and a $(s+1)$-bi-connected splitting (N',v') such that $\sigma(c(N,v)) = -\sigma(c(N,v))$. This will follow from the following lemma.

Lemma 3°. Let (N,v) be an s-bi-connected splitting of M where $2 \leq s \leq n-4$. Let $0 \longrightarrow (P_2,\varphi_2) \longrightarrow (P_1,\varphi_1) \longrightarrow (P_o,\varphi_o) \longrightarrow 0$ be an exact sequence in $\mathcal{C}(Z(G),\alpha)$. Suppose that (P_1,φ_1) is a triangular object in $\mathcal{C}(Z(G),\alpha)$ and that $c(N,v) = (P_o,\varphi_o)$. Then there exists an $(s+1)$-bi-connected splitting (N',v') such that $c(N',v') \cong (P_2,\varphi_2)$. Of course $c(N,v)$ and $c(N',v')$ are defined relative to particular liftings of N and N' to X.

Proof: Let $p : P_1 \longrightarrow H_{s+1}(X,A;Z(G))$ denote the map from P_1 to P_o composed with the given isomorphism from P_o to $H_{s+1}(X,A;Z(G))$. Since (P_1,φ_1) is triangular there exists a filtration $o = F_o \subset F_1 \subset \ldots \subset F_m = P_1$ such that $\varphi_1(F_{i+1}) \subset F_i$, each F_i is free and $Q_i = F_i/F_{i-1}$ is free on one generator. We prove Lemma 3^o by induction where the 'i-th' inductive statement is as follows: After exchanging i 's+1' dimensional handles from B to A we can find a monomorphism $\rho : H_{s+1}(R,N,Z(G)) \longrightarrow P_1$ where R is the cobordism formed from the union of the i exchanged elementary cobordisms. The image of ρ is F_i, ρ is a map in $\mathcal{C}(Z(G),\alpha)$, and $p \circ \rho = i_*$ where i is the inclusion of (R,N) into (X,A). By exchanging i 's+1' dimensional handles from B to A we mean lifting a chain of properly ordered 's+1' dimensional elementary cobordisms to X so that the lifted chain starts at the given lifting of (N,v) to X.

The '0'th statement is clearly true. Let us assume the 'i'th statement and try to prove the 'i+1'st statement. Pick $e \in F_{i+1}$ such that $F_i \oplus [e] = F_{i+1}$. Let $b = p(e)$. Denote the region obtained by exchanging the first i handles by R and let $A' = A \cup R$.



Since $\varphi_1(e) \in F_i$ $\varphi_1(e)$ is in the image of ρ. Let c be the element of $H_{s+1}(R,N)$ such that $\rho(c) = \varphi_1(e)$ (for the rest of this proof we will always be dealing with homology with local coefficients in $Z(G)$.

Some notation

The following are inclusion maps:

$i : (A',A) \longrightarrow (X,A)$

$i' : (T(A'),A) \longrightarrow (X,A)$

$i'' : (A',A) \longrightarrow (T(A'),A)$

$k : (T(A'),A) \longrightarrow (T(A'),T(A))$

$k' : (A',T^{-1}(A)) \longrightarrow (A',A)$

$k'' : (T(A'),A) \longrightarrow (T(A'),A')$

∂ denotes the boundary map in homology associated to the triple $A \subset A' \subset X$.

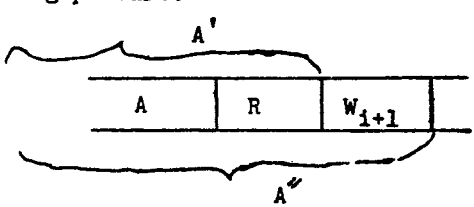
∂' denotes the boundary map associated to the triple $A \subset T(A') \subset X$.

∂'' is the boundary map associated to $T(A) \subset T(A') \subset X$.

Step 1°. We wish to find an element $x \in H_{s+1}(T(A'),A)$ such that $i'_*(x) = b$ and $t^{-1}_* k_*(x) = c$. One can show "easily" that there exists an $x' \in H_{s+1}(T(A'),A)$ such that $i'_*(x') = b$. It is also easy to see that $i_*(c) = t^{-1}_* j_*(b) = i'_*(t_* j_*(x))$ (remember that i'_* is a map in $\mathcal{C}(Z(G),\alpha)$). $i = i' \circ i''$ and $t^{-1} \circ j = j \circ t^{-1} = i'' \circ k' \circ t^{-1} = i'' \circ t^{-1} \circ k$. Therefore $i' \circ t^{-1} \circ j = i \circ t^{-1} \circ k$ and hence $i_*(c) = i_*(t^{-1}_* \circ k_*(x))$. Thus

there exists $Z \in H_{s+1}(X,A')$ such that $\partial Z = c - t_*^{-1}\circ k_*(x)$. Let $Z' = t_*(Z) \in H_{s+1}(X,T(A'))$. Let $x = \partial'(Z') + x'$. Then ince $i_*\circ\partial' = 0$ we see that $i'_*(x) = b$. But $k_*\circ\partial' = \partial''$ ence $t_*^{-1}k_*\partial'(Z') = t_*^{-1}\partial''(Z')$. Since $t^{-1} : (\tilde{X},\widetilde{T(A')},\widetilde{T(A)}) \longrightarrow (\tilde{X},\tilde{A}',\tilde{A})$ we have that $t_*^{-1}\circ\partial'' = \partial\circ t_*$. Hence $t_*^{-1}\partial''(Z') = \partial t_*^{-1}(Z') = t_*^{-1}t_*(Z) = \partial Z = c - t_*^{-1}k_*(x)$. Therefore
$t_*^{-1}k_*(x) = t_*^{-1}k_*\partial'(Z) + t_*^{-1}k_*(x) = t_*^{-1}\partial''(Z') + t_*^{-1}k_*(x')$
$= c - t_*^{-1}k_*(x') + t_*^{-1}k_*(x') = c$.

Step 2°. Let $y' = k''_*(x)$, hence $y' \in H_{s+1}(T(A'),A')$. Let $W_1,W_2,\dots,W_i$ be the chain of properly ordered elementary cobordisms constructed so far. Let (N_i,v_i) denote the right side of W_i. Then by considering the triple $A \subset A' \subset X$ we see that (N_i,v_i) is an s-connected splitting of M. By applying the final construction of Chapter 2° to the element y' we obtain an elementary cobordism W_{i+1} of dimension $s+1$ with left side (N_i,v_i) such that a generator of $H_{s+1}(W_{i+1},N_i)$ maps onto y' under the inclusion map of (W_{i+1},N_i) into $(T(A'),A')$. Let y'' denote this generator. (Here W_{i+1} also is used to denote the lifting of W_{i+1} to X determined by the lifting N_i to X). Let $A'' = A' \cup W_{i+1}$. Consider the following picture:



Some more notation

The following are inclusion maps:

$I : (A'',A) \longrightarrow (X,A)$

$I' : (A'',A) \longrightarrow (T(A'),A)$

$I'' : (A'',A') \longrightarrow (T(A'),A')$

$K : (A'',A) \longrightarrow (A'',A')$

$L : (A',A) \longrightarrow (A'',A)$

$L' : (T(A'),T(A)) \longrightarrow (T(A''),T(A))$

The inclusion map of the triple (A'',A',A) into $(T(A'),A',A)$ induces the following diagram in homology.

$$
\begin{array}{ccccccc}
 & \overset{L_*}{\nearrow} & H_{s+1}(A\ ,A) & \xrightarrow{K_*} & H_{s+1}(A'',A') & \searrow & \\
H_{s+1}(A',A) & & \downarrow I'_* & & \downarrow I''_* & & 0 \\
 & \underset{i''_*}{\searrow} & H_{s+1}(T(A'),A) & \xrightarrow[k''_*]{} & H_{s+1}(T(A'),A') & \nearrow &
\end{array}
$$

Using the facts that $I''_*(y'') = y$ and $k''_*(x) = y'$ we find after chasing the above diagram an element $y \in H_{s+1}(A'',A)$ such that $K_*(y) = y''$ and $I'_*(y) = x$. Define $\rho : H_{s+1}(A'',A) \longrightarrow P_1$ by $\rho'(y) = e$ and $\rho' \circ L_* = \rho$. This is well defined since $H_{s+1}(A'',A) \cong \text{image } L_* \oplus \{y\}$, and L_* is a monomorphism. $p\rho'(y) = p(e) = b = i'_*(x) = i'_*I'_*(y) = I_*(y)$. Also if $u \in H_{s+1}(A'',A)$ such that $u = L_*(v)$ for some $v \in H_{s+1}(A',A)$ then $p\rho'(u) = p\rho' L_*(v) = p\rho(v) = i_*(v) = I_*L_*(v) = I_*(u)$.

Hence $p \circ \rho' = I_*$. $\varphi_1 \rho'(u) = \varphi_1 \rho(v) = \rho t_*^{-1} j_*(v) = \rho' L_* t_*^{-1} j_*(v)$ $= \rho' \circ (t_*^{-1} j_*) \circ L_*(v) = \rho'(t_*^{-1} j_*(u))$. (Remember from Chapter 2° that L_* is a map in $\mathcal{C}^*(Z(G),\alpha)$ and hence $L_* \circ (t_*^{-1} j_*) = (t_*^{-1} j_*) \circ L_*$.) $\varphi_1 \rho'(y) = \varphi_1(e) = \rho(b) = \rho' L_*(b) =$ $\rho L_*(t_*^{-1} k_*(x))$ (by Step 1°) $= \rho' L_* t_*^{-1} k_*(I'_*(y)) =$ $\rho' t_*^{-1}(L_* k_* I'_*(y))$. But $L' \circ k \circ I' = j$. Hence $\varphi_1 \rho'(y) = \rho'(t_*^{-1} j_*(y))$. Therefore ρ' is a map in $\mathcal{C}(Z(G),\alpha)$. Hence we have completed the proof of the inductive statement. Consider the inductive statement for $i = m$. Also consider the homology exact sequence for the triple $A \subset A' \subset X$. We obtain the following diagram:

$$\begin{array}{ccccccc} & & & & & & P_1 \\ & & & & & \overset{\rho}{\underset{\approx}{\nearrow}} & \downarrow P \\ 0 \longrightarrow & H_{s+2}(X,A') & \longrightarrow & H_{s+1}(A',A) & \longrightarrow & & H_{s+1}(X,A) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Since all the maps in this diagram are maps in $\mathcal{C}(Z(G),\alpha)$ we see that $(H_{s+2}(X,A';Z(G)), t_*^{-1} \circ j_*) = (P_2, \varphi_2)$. One also shows easily that (N_m, v_m) is an $(s+1)$-bi-connected splitting. Hence we have proven Lemma 3°.

Let (N,v) be an s-bi-connected splitting of M where $2 \leq s \leq n-4$. Let (P, φ) be an object of $\tilde{\mathcal{C}}(Z(G),\alpha)$ such that $\sigma(P,\varphi) = \sigma(c(N,v))$. We now will show that there exists an s-bi-connected splitting (N',v') such that

$c(N',v') \cong (P,\varphi)$. (Remember that $c(N,v)$ and $c(N',v')$ are defined with respect to particular liftings of N and N' to X.) To do this we need only show that the equivalence relations $1'^{o}$ and $2'^{o}$ of Chapter 1^{o} can be realized.

First let us consider relation $1'^{o}$. Let $0 \to (P_2,\varphi_2) \to (P_1,\varphi_1) \to (P_o,\varphi_o) \to 0$ be an exact sequence in $\mathcal{C}(Z(G),\alpha)$. Suppose that (P_1,φ_1) is triangular and that $c(N,v) \cong (P_o,\varphi_o)$. Then by Lemma 3^{o} there exists an s+1-bi-connected splitting (N_1,v_1) of M such that $c(N_1,v_1) \cong (P_2,\varphi_2)$. By Lemma 1^{o} $c(N_1,-v_1) \cong \mathcal{D}(P_2,\varphi_2)$. But since $\mathcal{D}$ is an exact contravariant functor $0 \leftarrow \mathcal{D}(P_2,\varphi_2) \leftarrow \mathcal{D}(P_1,\varphi_1) \leftarrow \mathcal{D}(P_o,\varphi_o) \leftarrow 0$ is exact in $\mathcal{C}(Z(G),\alpha^{-1})$. Also as remarked in Chapter 1^{o} $\mathcal{D}(P_1,\varphi_1)$ is triangular. Let F be a finitely generated free $Z(G)$ module. Let $g : \mathcal{D}(P_o,\varphi_o) \to \mathcal{D}(P_1,\varphi_1)$ denote the map in the exact sequence above. Then

$$0 \to \mathcal{D}(P_o,\varphi_o) \oplus (F,0) \xrightarrow{g \oplus id} \mathcal{D}(P_1,\varphi_1) \oplus (F,0) \to \mathcal{D}(P_2,\varphi_2) \to 0$$

is exact and $\mathcal{D}(P_1,\varphi_1) \oplus (F,0)$ is triangular. Hence there exists a splitting (N_2,v_2) with respect to $r \circ \hat{f}$ such that $C(N_2,v_2) \cong \mathcal{D}(P_o,\varphi_o) \oplus (F,0)$. By Lemma 2^{o} again $C(N_2,-v_2) \cong \mathcal{D}'\mathcal{D}(P_o,\varphi_o) \oplus \mathcal{D}'(F,0)$. But $\mathcal{D}' \circ \mathcal{D}$ is naturally equivalent to the identity functor and $\mathcal{D}'(F) \cong F$. Hence $C(N_2,-v_2) \cong (P_o,\varphi_o) \oplus (F,0)$. Thus relation $1'^{o}$ going in one direction can be realized. The opposite direction can be realized in a similar fashion.

Next we consider relation $2'^{o}$. Let $0 \to (P_2, \varphi_2) \to (P_1, \varphi_1) \to (P_o, \varphi_o) \to 0$ and $0 \to (Q_2, \psi_2) \to (Q_1, \psi_1) \to (Q_o, \psi_o) \to 0$ be two exact sequences in $\mathcal{C}(Z(G),\alpha)$. Let $Y_o, Y_o' : Q_o \to P_o$ be two α-semi-linear maps. Define $g_o : P_o \oplus Q_o \to P_o \oplus Q_o$ by $g_o(x,y) = (\varphi_o(x) \oplus Y_o(y), \psi_o(y))$. Define $g_o' : P_o \oplus Q_o \to P_o \oplus Q_o$ by $g_o'(x,y) = (\varphi_o(x) \oplus Y_o'(y), \psi_o(y))$. Then applying Lemma 2^o of Chapter 1^o twice with $Y_2 = 0 = Y_2'$ we obtain the following two exact sequences in $\mathcal{C}(Z(G),\alpha)$:

$$0 \to (P_2 \oplus Q_2, \varphi_2 \oplus \psi_2) \to (P_1 \oplus Q_1, g_1) \to (P_o \oplus Q_o, g_o) \to 0$$

$$0 \to (P_2 \oplus Q_2, \varphi_2 \oplus \psi_2) \to (P_1 \oplus Q_1, g_1') \to (P_o \oplus Q_o, g_o') \to 0.$$

Assume that (P_1, φ_1) and (Q_1, ψ_1) are triangular. Then $(P_1 \oplus Q_1, g_1)$ and $(P_1 \oplus Q_1, g_1')$ are triangular. Suppose that $c(N,v) \cong (P_1 \oplus Q_1, g_1)$. Then by Lemma 3^o there exists an s+1-bi-connected splitting (N_1, v_1) such that $c(N_1, v_1) \cong (P_2 \oplus Q_2, \varphi_2 + \psi_2)$. By Lemma 2^o $c(N_1, -v_1) \cong \mathcal{D}(P_2 \oplus Q_2, \varphi_2 \oplus \psi_2)$. But since $\mathcal{D}$ is contravariant and exact

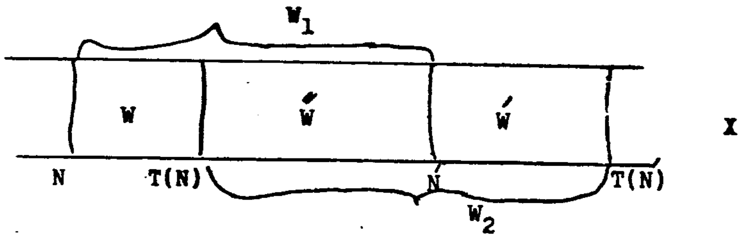
$$0 \leftarrow \mathcal{D}(P_2 \oplus Q_2, \varphi_2 \oplus \psi_2) \leftarrow \mathcal{D}(P_1 \oplus Q_1, g_1') \leftarrow \mathcal{D}(P_o \oplus Q_o, g_o') \leftarrow 0$$

is exact in $\mathcal{C}(Z(G),\alpha^{-1})$. By Lemma 3^o there exists an n-(s+1)-bi-connected splitting (N_2, v_2) of M with respect to $r \circ \hat{f}$ such that $c(N_2, v_2) \cong \mathcal{D}(P_o \oplus Q_o, g_o')$. Applying Lemma 2^o again we see that $(N_2, -v_2)$ is an s-bi-connected splitting of M with respect to $\hat{f}$ such that $c(N_2, -v) \cong (P_o \oplus Q_o, g_o')$. Hence relation $2'^{o}$ can be realized.

Hence we have shown that there exists a splitting (N,ν) with respect to $\hat{f}$ such that $\overline{T(A) - A}$ is an h-cobordism iff $c(\hat{f}) = 0$. (If we delete an open tubular nbd. of N from M then the resulting manifold is diffeomorphic to $\overline{T(A) - A}$. see [3] Section 2^{o}.)

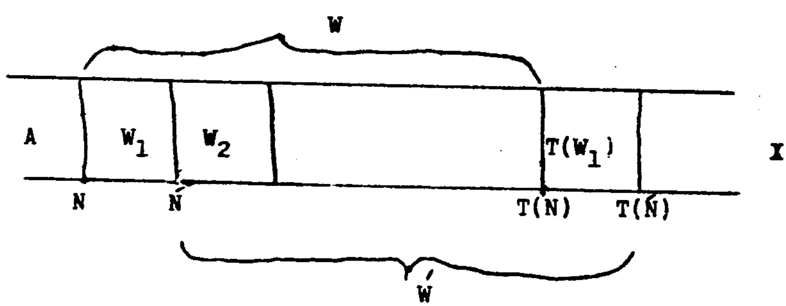
Chapter 4°. The definition of the obstruction $\tau(\hat{f})$.

If $c(\hat{f}) = 0$ then there exists a splitting (N,v) such that $W = \overline{T(A) - A}$ is a relative h-cobordism. $\partial W = N \cup \partial_c(W) \cup T(N)$ where we have a given diffeomorphism of $\partial_c W$ onto $\partial N \times [0,1]$. This diffeomorphism can be extended to a diffeomorphism of W onto $N \times [0,1]$ if and only if $\tau(W,N)$ is the zero element of $Wh(G)$. If this happens then there exists a differentiable fiber map $\overline{f} : M \longrightarrow S^1$ such that $\overline{f}$ is homotopic to $\hat{f}$ and $f/\partial M = f$. But it is possible that $\tau(W,N) \neq 0$ although there exists another splitting (N',v') such that $\tau(W',N') = 0$. We proceed to measure this ambiguity. Let $\alpha_* : Wh(G) \longrightarrow Wh(G)$ be the automorphism induced by $\alpha : Z(G) \longrightarrow Z(G)$. Let $Wh_\alpha(G) = Wh(G)/\{x-\alpha_*(x) | x \in Wh(G)\}$. If (N,v) is as above we define $\tau(\hat{f}) \in Wh_\alpha(G)$ to be the image of $\tau(W,N)$ in $Wh_\alpha(G)$. We proceed to show that $\tau(\hat{f})$ is well defined. Since $T : (W,N) \longrightarrow (T(W),T(N))$ is a diffeomorphism we see easily that $\tau(T(W),T(N)) = \alpha_*^{-1}(\tau(W,N))$. Therefore if p denotes the quotient map of $Wh(G)$ onto $Wh_\alpha(G)$ we see that $p(\tau(W,N))$ is independent of the lifting of N to X. Let (N',v') be a second splitting such that (W',N') is also a relative h-cobordism. We may assume that $T(A) \subset A'$. Let $W = \overline{A' - T(A)}$. Then one sees easily that $(W'',T(N))$ is a relative h-cobordism. Let $W_1 = W \cup W''$ and $W_2 = W'' \cup W'$ (see picture on next page.)



Since $T : (W_1, N) \longrightarrow (W_2, T(N))$ is a diffeomorphism we see that $\tau(W_2, T(N)) = \alpha_*^{-1}\tau(W_1, N)$. But $\tau(W_1, T(N)) = \tau(W, N) + \tau(W'', T(N))$ while $\tau(W_2, T(N)) = \tau(W'', T(N)) + \tau(W', N')$. Putting these three equations together we see that $p(\tau(W, N)) = p(\tau(W', N'))$. Hence $\tau(\hat{f})$ is well defined.

Let $x \in p^{-1}\tau(\hat{f})$, then $x = \tau(W, N) + y - \alpha_*(y)$ for some $y \in Wh(G)$. By a result due to Stallings (see [13] page 398) there exists a cobordism (W_1, N) such that $\tau(W_1, N) = \alpha_*(y)$. Let N' be the right side of W_1. Then there exists a second cobordism (W_2, N') such that $\tau(W_2, N') = -\alpha_*(y)$. We can identify half of a narrow tubular nbd. of N with $W_1 \cup W_2$ (see picture below.)



Let $A' = A \cup W_1$ and $W = \overline{T(A') - A'}$. Then $\tau(W_1 \cup W', N) = \tau(W \cup T(W_1), N), \tau(W_1 \cup W', N) = \alpha_*(y) + \tau(W', N')$, and $\tau(W \cup T(W_1), N) = \tau(W, N) + \alpha_*^{-1} \alpha_*(y)$. Hence $\tau(W', N') = x$, and (N', v') is a splitting relative to $\hat{f}$. Hence we obtain our theorem namely:

Theorem: $\hat{f}$ is homotopic to a smooth fiber map $\bar{f}$ such $\bar{f}/\partial M$ is the given fiber map f if and only if

1° X is dominated by a finite C.W. complex.

2° $C(\hat{f}) = 0$

3° $\tau(f) = 0$

Appendix

If the boundary of M is disconnected then the statement of the fibering theorem must be modified. Namely we must consider smooth fiber maps $\overline{f}$ which are homotopic to $\hat{f}$ relative to the boundary. (If ∂M is connected $\overline{f}$ homotopic to $\hat{f}$ implies that $\overline{f}$ is homotopic to $\hat{f}$ leaving the boundary fixed.) The problem occurs in the proof of Lemma 2° of Chapter 2°. But if we restrict ourselves to maps homotopic to $\hat{f}$ relative to ∂M everything goes through as before. The major technical annoyance is to give the proper interpretation of the Pontryagin-Thom correspondence in this context. This is accomplished by working with pairs (g,h) where $g : M \longrightarrow S^1$ and $h : \partial M \;\; I \longrightarrow S^1$ such that $h/\partial M \times 0 = f$ and $h/\partial M \times 1 = g/\partial M$. A homotopy from (g,h) to (g',h') is a pair (G,H) such that G is a homotopy from g to g' and $H : \partial M \times I \times I \longrightarrow S^1$ such that $H/\partial M \times I \times 0 = h$, $H/\partial M \times I \times 1 = h'$, $H/\partial M \times 1 \times I = G/\partial M$ and $H(x,0,t) = f(x)$ for $x \in \partial M$ and $t \in I$.

Next we consider the situation when $\hat{f} : \pi_1(M) \longrightarrow \pi_1(S^1)$ is not onto. This corresponds to the case where the fiber is disconnected. For convenience assume that $\partial M = \phi$. Regard $\hat{f}$ as an element of $H^1(M,Z)$. Let $\hat{f} = m\,\hat{\hat{f}}$ where $\hat{\hat{f}}$ is indivisible. If $m = 0$ there cannot exist a smooth fiber map homotopic to $\hat{f}$. $\hat{\hat{f}}_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1) \longrightarrow 0.$

If $m \neq 0$ then there exists a smooth fiber map homotopic to $\hat{f}$ if and only if there exists a smooth fiber map homotopic to $\hat{\hat{f}}$. The case when $\partial M \neq \phi$ can be treated similarly.

Let G be a finitely presented group. Let $x \in \tilde{C}(Z(G), id)$ then there exists a closed manifold M and a map $\hat{f} : M \longrightarrow S^1$ such that $\ker \hat{f}_{\#} \cong G$, $\hat{f}_{\#}$ is onto, $\bar{X}$ is the homotopy type of a finite C.W. complex and $c(\hat{f}) = x$.

By results of Bass and Murthy if G is a finitely generated abelian group of rank ≥ 2 whose torsion subgroup is not cyclic of square free order then $\tilde{C}(Z(G), id)$ is not finitely generated. Also Bass and Murthy show that $C(R, id)$ fits into the following exact sequence:

$K_1(R[t]) \longrightarrow K_1(R[t,t^{-1}]) \longrightarrow C(R,id) \longrightarrow K_o(R[t]) \longrightarrow K_o(R[t,t^{-1}])$.

There is also a product formula for $c(\hat{f})$ analogous to that discovered by Kwun and Szczarba for Whitehead torsion (see [15].)

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