MANIFOLDS WITH $\pi_1 = G \times_\alpha T$.

By F. T. FARRELL and W. C. HSIANG.[1]

1. Introduction. After the first author's thesis on fibring a manifold over a circle [11] and our joint paper on non-homeomorphic h-cobordant manifolds[2] [13], we realized that there is a very close relation between the structure of the Whitehead group $\mathrm{Wh}\, G \times_\alpha T$ of a semi-direct product $G \times_\alpha T$ of a finitely presented group G and the infinitely cyclic group T with the obstruction to splitting a homotopy equivalence $f: M \to M'$ along a codimension 1 submanifold $N' \subset M'$ where $\pi_1 M$ and $\pi_1 M'$ are isomorphic to $G \times_\alpha T$ (as usual, $\pi_1 M$ and $\pi_1 M'$ are identified under f_*), and the inclusion $N' \subset M'$ induces the inclusion $G \subset G \times_\alpha T$. By 'splitting f along N',' we mean that there is a codimension 1 submanifold $N \subset M$ and a homotopy equivalence of pairs $g: (M, N) \to (M', N')$ such that $g \mid M$ is homotopic to f. The result[3] is rather neat and the work was done during 1967. We announced both the algsbraic formula on $\mathrm{Wh}\, G \times_\alpha T$ and the geometric 'splitting theorem' in [14]. The details of the algebraic formula and some of its quick applications appeared in [12]. Due to our laziness, the details of the 'splitting theorem' itself were never formally published except that there were several copies of a rough version circulated among the friends. In this paper, we shall deliver this promise of [14]. The precise statement of this theorem will be given in § 2.

After the announcement of [14], several applications of the 'cplitting theorem' were obtained by other authors and ourselves. For example, Shaneson and Wall [31][36] proved a formula for the Wall group $L_{n+1}(G \times T)$ in terms of $L_n(G)$ and $L_n{}^h(G)$, and we verified the homotopy invariance of the higher indices of Novikov [25][15]. One can actually prove a formula for $L_{n+1}(G \times_\alpha T)$ using the full strength of the 'splitting theorem" and following the argument of Shaneson [31], Wall [36]. In particular, one should be able to obtain a classification theorem for manifolds of the homotopy

Received April 12, 1972.
Revised August 18, 1972.

[1] During the writing up of this paper, the second-named author is an NSF Senior Postdoctor and also supported in part by a grant from The Institute for Advanced Study.

[2] After Kirby-Siebenmann [19], this result is obsolete.

[3] See [10] for a generalization of the result.

type of a solvamanifold as a generalization of homotopy tori [16][17][36]. We shall not explore this possibility in this paper, but we shall give a proof of the homotopy invariance of the higher order indices. As we said at the beginning of the introduction that our motivation of this paper came from the intention to relate the algebraic structure of Wh $G \times_\alpha T$ with our theorem on non-homeomorphic h-cobodant manifolds, we shall give the geometric interpretation of the decomposition of Wh $G \times_\alpha T$ and recast the result on these non-homeomorphic h-cobordant manifolds. Even though this result is obsolete due to the recent developments [19][20], it seems to us that the geometric intepretation of the different pieces of Wh $G \times_\alpha T$ is still of considerable interest. We originally also had some results on 'projective class group,' but we shall leave them out because the recent understanding of topology made them out of date.

Finally, let us remark that the 'splitting theorem' is only valid for manifolds of the dimension $\geqq 6$. A weaker version of the theorem for 5-manfolds by the second-named author is given at the end as an apendix (cf. [37]).

2. Statement of the 'splitting theorem.' Let M' be finite Poincaré complex[4] of formal dimension m. Suppose that $\pi_1 M' = G \times_\alpha T$, a semi-direct product of a finitely presented group G and the infinitely cyclic group T with a preferred generator t such that the conjugation of G defined by tgt^{-1} is the automorphism α. Suppose that N' is a codimension 1 finite Poincaré subcomplex[5] of M' such that N' has a neighborhood homeomorphic to $N' \times (-\epsilon, \epsilon)$ for $\epsilon > 0$ in M' and the inclusion $N' \subset M'$ induces the π_1 inclusion $G \subset G \times_\alpha T$. Now, let M be a closed differentiable[6] manifold of dimension m. Suppose that $f\colon M \to M'$ is a homotopy equivalence. As always, we identify $\pi_1 M$ with $\pi_1 M'$ under f_*. We say that f is splittable along N' if there is a codimension 1 submanifold N of M and a homotopy equivalence of pairs

$$g\colon (M, N) \to (M', N') \tag{1}$$

such that $g \mid M$ is homotopic to f. Occasionally, we say that (g, N) is a splitting of (f, N'). In order to state our 'splitting theorem,' we recall the structure theaorem of Wh $G \times_\alpha T$ [12]. Let $\tilde{C}(\boldsymbol{Z}(G), \alpha)$, $\tilde{C}(\boldsymbol{Z}(G), \alpha^{-1})$ denote the Grothendieck groups of finitely generated free $\boldsymbol{Z}G$ modules with α-linear, respectively α^{-1}-linear nilpotent endomorphisms [11], [12]. Note that α induces automorphisms on Wh G and $\tilde{K}_0 G$ and let us denote these automor-

[4] See [36], see the definition of finite Poincaré complex.

[5] Cf. [18].

[6] Everything works in PL or Top category.

phisms by α again. Let $I(\alpha)$ denote the subgroup of $\mathrm{Wh}\, G$ generated by $\{x-\alpha(x) \mid x \in \mathrm{Wh}\, G\}$ and let $(\tilde{K}_0 G)^{\alpha}$ be the subgroup of $\tilde{K}_0 G$ consisting of the elements invariant under α. Then, we have

(2) $$\mathrm{Wh}\, G \times_{\alpha} T = X \oplus \tilde{C}(\mathbf{Z}(G), \alpha) \oplus \tilde{C}(\mathbf{Z}(G), \alpha^{-1})$$

where X fits into the following short exact sequence:

(3) $$0 \to \mathrm{Wh}\, G/I(\alpha) \to X \to (\tilde{K}_0 G)^{\alpha} \to 0.$$

In fact, if we put $C(\mathbf{Z}(G), \alpha) = \tilde{C}(\mathbf{Z}(G), \alpha) \oplus \tilde{K}_0 G$, then it follows from the 'five term exact sequence' for the α-twist localization [12][3] that there is a projection map

(4) $$p \colon \mathrm{Wh}\, G \times_{\alpha} T \to C(\mathbf{Z}(G), \alpha),$$

and the projection $X \to (\tilde{K}_0 G)^{\alpha}$ of (3) is induced from p. The inclusion $\mathrm{Wh}\, G/I(\alpha) \to X$ is induced from the inclusion $G \subset G \times_{\alpha} T$ as one expects. Suppose that $w \colon \pi = G \times_{\alpha} T \to \{\pm 1\}$ is a homomorphism. It induces an involution $\mathbf{Z}(G) \to \mathbf{Z}(G)$ which is also denoted by w defined by

$$w(\sum_g n(g) g) = \sum_g w(g) n(g) g^{-1}.$$

w induces a conjugation '—' on $\mathrm{Wh}\, G \times_{\alpha} T$. Under theconjugation '—,' the short exact sequence (3) is invariant and the factors $\tilde{C}(\mathbf{Z}(G), \alpha)$ and $\tilde{C}((G), \alpha^{-1})$ of (2) are interchanged.

We are now ready to state the 'splitting theorem.'

THEOREM 2.1. *Let (M', N') be a pair of finite Poincaré complexes and let $f \colon M \to M'$ be a homotopy equivalence from a manifold M to M' as given above. Suppose that* $\dim M = \dim M' = m \geqq 6$. *Let $\tau(f) \in \mathrm{Wh}\, G \times_{\alpha} T$ be the torsion of f and let $0(f) = p\tau(f) \in C(\mathbf{Z}(G, \alpha)$ be the image of $\tau(f)$ under the projection p of* (4). *Then the obstruction to splitting f along N' is $0(f)$.*

Let us briefly indicate the program for the proof of Theorem 2.1. We first make f transversely regular with respect to N' such that the restriction map $f \mid N \colon N = f^{-1}(N') \to N'$ induces an isomorphism on π_1. Then, we embark a program for improving the connectivity of $f \mid N$. For this purpose, we consider the ∞-cycle covering spaces Y_M, $Y_{M'}$ corresponding to the normal subgroup $G = \pi_1(N) \subset G \times_{\alpha} T$ and the induced map $Y_M \to Y_{M'}$. Suitably lifting N, N' into $Y_M, Y_{M'}$ respectively, we have a map of triads

$$(Y_M; A_N, B_N) \to (Y_{M'}; A_{N'}, B_{N'})$$

where A_N, B_N and $A_{N'}$, $B_{N'}$ are gotten from Y_M, $Y_{M'}$ by dividing through N, N' respectively. Suppose that the homology kernels $K_i(A_N, N; \mathbf{Z}G) = 0$,

$K_i(B_N, N; \mathbf{Z}G) = 0$ for $i < k \leqq \frac{m-1}{2}$. Then, $K_k(A_N, N; \mathbf{Z}G)$, $K_k(B_N, N; \mathbf{Z}G)$ are finitely generated $\mathbf{Z}G$ modules and we may represent the generators by immersed handles on N. If $k < \frac{m-1}{2}$, we may represent these handles by embeddings and we may suitably exchange them to the other side to eliminate the kernels. The connectivity will be improved by 1. However, when we come to the middle dimension, we meet difficulties. It turns out that the obstructions to eliminating these difficulties are precisely described as stated in the theorem.

The proof of Theorem 2.1 occupies § 3 to § 5.

Now, suppose that g, N exist, i.e., f is splittable with respect to N' and (g, N) is a splitting. Cut M along N and M' along N' and we form M_N and $M'_{N'}$ respectively. It is easy to check that M_N is a smooth manifold with boundary after we smooth out the corners and there is a quotient map

$$q: M_N \to M. \tag{5}$$

In fact, $q^{-1}(N)$ is a disjoint union of two smooth manifolds with boundary, N_+ and N_- such that

(6)
(A) $q_\pm = q \mid N_\pm: N_\pm \to N$ are diffeomorphisms,
(B) M_N is a cobordism between N_+ and N_-,
(C) $q \mid (M_N - (N_+ \cup N_-))$ is a diffeomorphism of $(M_N - (N_+ \cup N_-))$ to $(M - N)$.

Similarly, we can show that $M'_{N'}$ is a finite Poincaré complex with boundary and there is a quotient map

$$q': M'_{N'} \to M' \tag{7}$$

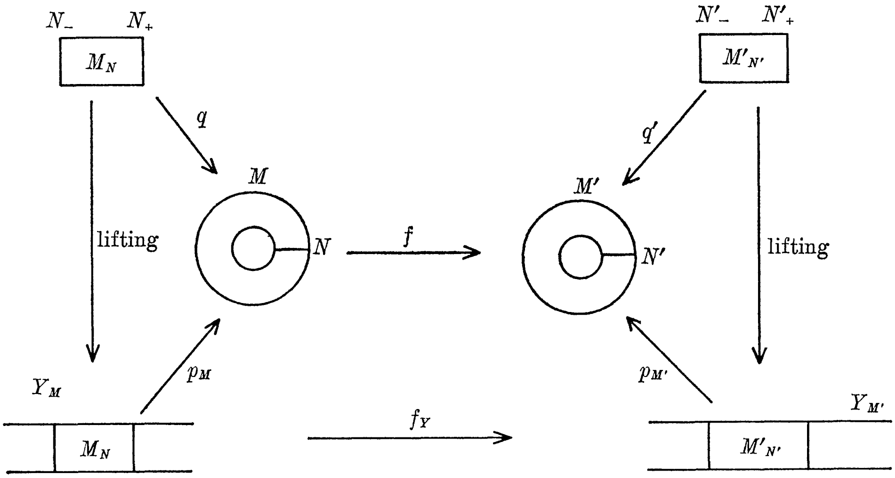
such that $q_0^{-1}(N')$ is a disjoint union of two finite Poincaré complexes with boundary, N'_+ and N'_- satisfying the conditions:

(8)
(A) $(q')_\pm. = q \mid N'_\pm: N'_\pm \to N$ are homeomorphisms,
(B) $q \mid (M'_{N'} - (N'_+ \cup N'_-))$ is a homeomorphism of

$$(M'_{N'} - (N'_+ \cup N'_-)) \text{ to } (M' - N').$$

Let $p_M: Y_M \to M$, $p_{M'}: Y_{M'} \to M'$ be the covering projections where Y_M, $Y_{M'}$ are the ∞-cyclic covering spaces of M and M' respectively corresponding to the normal subgroup $G \subset G \times_\alpha T$. We can lift M_N and $M'_{N'}$ to Y_M and $Y_{M'}$ respectively and find a covering map $f_Y: Y_M \to Y_{M'}$ $p_{M'} f_Y = f p_M$ and f_Y sends M_N onto $M'_{N'}$. (Note that the lifting is not unique!)

FIGURE 1



Let $h\colon M_N \to M'_{N'}$, $h'\colon N_- \to N'_-$ be the induced maps. It is easy to see that h, h' are homotopy equivalence, and hence the torsions $\tau(h)$, $\tau(h')$ are well-defined. If we identify $\pi_1 M_N$ and $\pi_1 N_-$ with G, $\tau(h)$ and $\tau(h')$ lie in $\mathrm{Wh}\, G$. Let us consider $\tau = j_*(\tau(h) - \tau(h'))$ in $\mathrm{Wh}\, G \times_\alpha T$ where j_* is induced by the inclusion $j\colon G \to G \times_\alpha T$.

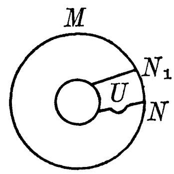
THEOREM 2.2. (A) *If the splitting exists and* τ *is defined as above, then* $\tau = \tau(f)$. *In particular,* τ *depends only on* f, *and we shall call it the torsion of the splitting.*

(B) *If we consider* $\tau(f)$ *as the coset of* $\mathrm{Wh}\, G$ *with respect to* $I(\alpha_*)$, *then for every element* x *of* $\mathrm{Wh}\, G$ *in the coset, we can find a splitting* (N, g) *and maps* $h\colon M_N \to M'_{N'}$, $h'\colon N_- \to N'_-$ *induced by suitable liftings such that* $x = \tau(h) - \tau(h')$.

Proof. (A) Let Z, Z' be the mapping cylinders of h and h' respectively. Denote the acyclic complex of the pair $(Z, M_N \cup Z')$ with coefficients in $\mathbf{Z}(G)$ by $\mathbf{C}$. Let W be the mapping cylinder of f_Y. We see that $\mathbf{C} \otimes_{Z(G)} \mathbf{Z}(G \times_\alpha T)$ is the chain complex of the pair (W, Y_M) with coefficients in $\mathbf{Z}(G)$. In fact, we can identify the action of T on $\mathbf{C} \otimes_{Z(G)} \mathbf{Z}(G \times_\alpha T)$ (from the right) with the action on the chain complex induced by the covering transformation, when we consider W as the ∞-cyclic covering space of the mapping cylinder V of f. Therefore, we may identify $\mathbf{C} \otimes_{Z(G)} \mathbf{Z}(G \times_\alpha T)$ with the chain complex of (V, M) with the coefficients in $\mathbf{Z}(G \times_\alpha T)$. Since the torsion $j_*(\tau(h) - \tau(h'))$ is exactly equal to the torsion of the complex $\mathbf{C} \otimes_{Z(G)} \mathbf{Z}(G \times_\alpha T)$ in $\mathrm{Wh}\, G \times_\alpha T$, we see that it is equal to $\tau(f)$.

(B) Recall that I is the subgroup of Wh G consisting of the elements of the type $u - \alpha_* u$ for $u \in \text{Wh}\, G$. Let (N_1, g_1) be a fixed splitting and $h_1: M_{N^1} \to M'_{N'}$, $h': (N_1)_- \to N'_-$ be fixed maps constructed as above. Denote $\tau(h_1) - \tau(h'_1)$ by $\tau_1 \in \text{Wh}\, G$. For $u \in \text{Wh}\, G$, we choose a small h-cobordism U on N_1 inside a tubular neighborhood of N_1 in M with $\tau(U, N_1) = u$.

FIGURE 2



There is a deformation retract $r: U \to N_1$. When we choose the right side of the tubular neighborhood of N_1 to put the h-cobordism, we have an embedding $U \subset M_{N_1}$. When we identify M_{N_1} with its lifting to Y_M, the covering transformation t corresponding to the preferred generator of T induces a diffeomorphism of $t(N_1)_-$ with $(N_1)_+$. Let us consider the submanifold $V = M_{N_1} \cup_{t(N_1)_-} t(U)$ of Y_M. Set $W = \overline{M_{N_1} - U}$. It is clear that $\overline{V - U} = W \cup_{t(N^1)_-} tW$ is a lifting of M_N where M_N is the manifold gotten from M by cutting along N as we defined before. The deformation retract $r: U \to N_1$ induces a homotopy equivalence $l_1: W \to M_{N_1}$ relative to $(N_1)_+ = t(N_1)_-$ and a deformation retract $r_1: tU \to t(N_1)_- = N_+$. Piecing them together, we have a homotopy equivalence $l: M_N \to M_{N_1}$ such that $q_1 l = l' q$ where $q: M_N \to M$, $q_1: M_{N_1} \to M$ are the quotient map as defined before and $l': M \to M$ is the induced homotopy equivalence. Set $g = g_1 l'$. We see that (N, g) is again a splitting of f. Let h, h' be the maps of M_N and N_- corresponding to the lifting of the above. Let us compute $\tau(h) - \tau(h')$. It follows from Lemma 7.8 of [22] that $\tau(h) = \tau(h_1) + \tau(l)$. Since l is gotten from l_1 and r_1 by piecing them together, we have $\tau(l) = \tau(l_1) + \tau(r_1)$. Note that $\tau(l_1) = -\tau(U, N_1) = (-1)^m \bar{u}$. Using the automorphism on $\tau_1(N_1)_-$ under the covering transformation, we have $\tau(r_1) = \alpha_* u$. Putting these we have

$$\tau(h) = \tau(h_1) + \alpha_* u + (-1)^m \bar{u}. \tag{9}$$

We also have

$$\begin{aligned} \tau(h') &= \tau(r_1(N_-) + \tau(h'_1) \\ &= u + (-1)^m \bar{u} + \tau(h'_1). \end{aligned} \tag{10}$$

By (9) and (10), we have

$$(11) \qquad \tau(h) - \tau(h') = \alpha_* u - u + \tau(h_1) - \tau(h'_1)$$

and the theorem is proved.

3. Exchanging handles below the middle dimension. Starting from this section, we proceed to prove Theorem 2.1. Let us first make $f\colon M \to M'$ transversely regular with respect to N'. (Following [6], we make (M', N') a manifold pair with N' as a codimensional 1 submanifold of M' and we may speak of transversality.) We shall call this new map and all the later maps gotten by modifications and homotopies by g. Following an argument of [29, p. 15], we can make $N = g^{-1}(N')$ connected by modifying N. By exchanging a finite number of 2-handles, we can make g in such a way that the inclusion $N \subset M$ induces a monomorphism and $\pi_1 N$ may be identified as the subgroup G of $\pi_1 M = G \times_\alpha T$. This is explicitly done in Chapter II of [29] and Chapter III of [11]. Then $g_N = g \mid N\colon N \to N'$ induces an isomorphism on π_1.

We now embark on a program for improving the connectivity of g_N. In the present section, we shall show that we can always make g satisfy the condition

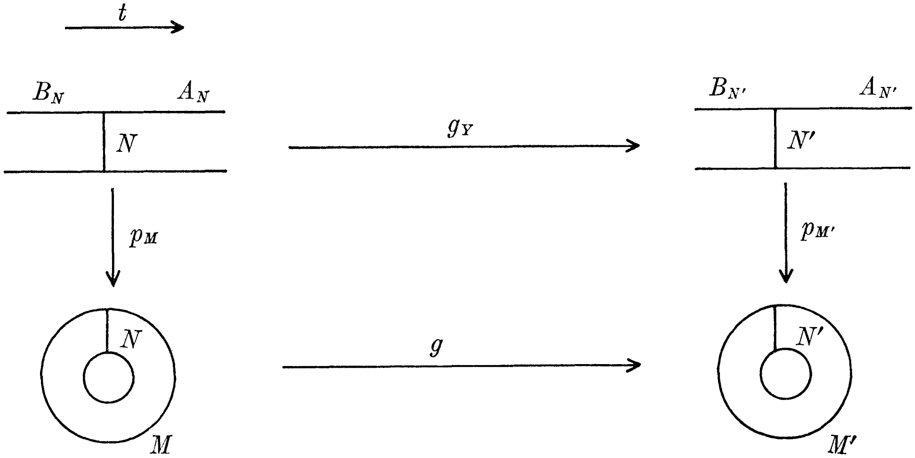
$$(12) \qquad \pi_i(g) = 0 \text{ for } i < m/2.$$

Let $p_M\colon Y_M \to M$, $p_{M'}\colon Y_{M'} \to M'$ be the covering projection of § 2. Consider the following commutative diagram

$$(13) \qquad \begin{array}{ccc} Y_M & \xrightarrow{\;g_Y\;} & Y_{M'} \\ \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_{M'}} \\ M & \xrightarrow{\;g\;} & M' \end{array}$$

where g_Y covers g (not unique, of course!). Since $\pi_1 N = \pi_1 N' = G$, we may life N, N' into Y_M and $Y_{M'}$ respectively such that g_Y sends the lifted N onto the lifted N'. N divides Y_M into A_N and B_N, and similarly N' divides $Y_{M'}$ into $A_{N'}$ and $B_{N'}$ such that $t(A_N) \subset A_N$, $t(A_{N'}) \subset A_{N'}$, $t^{-1}(B_N) \subset B_N$ and $t^{-1}(B_{N'}) \subset B_{N'}$. Let $g_Y \mid A_N = g_A$ and $g_Y \mid B_N = g_B$. We have $g_A\colon A_N \to A_{N'}$ and $g_B\colon B_N \to B_{N'}$. By an abuse of notations, we shall also denote the map of the triad $(Y_M; A_N, B_N) \to (Y_{M'}; A_{N'}, B_{N'})$ by g_Y, the induced maps $(A_N, N) \to (A_{N'}, N')$ by g_A and $(B_N, N) \to (B_{N'}, N')$ by g_B.

FIGURE 3



Let us consider the homology (cohomology with compact support) with the coefficients in $R = \mathbf{Z}(G)$ of various spaces. By [6][7][9][23], these maps induce split epimorphisms on homology groups[7] and split monomorphisms on cohomology groups with compact support [35][36]. We use K_i and K^i to denote the kernel and cokernel of the i-th cohomology respectively.

LEMMA 3.1. (A) $K_i(N) = K_i(A_N) \oplus K_i(B_N)$,
(B) $K_{i-1}(A_N) = K_i(B_N, N)$,
(C) $K_{i-1}(B_N) = K_i(A_N, N)$.

Similarly, we have the corresponding statement for cohomology kernels. (Note that the coefficient group is $R = \mathbf{Z}(G)$ which is always suppressed. In fact, it is valid for any R-module B.)

Proof. Let us prove the homology part of (A) and (B). The other parts follow similarly.

(A) We have the following maps of exact sequences:

$$
(14)\qquad
\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \| & & & & & \\
\to & K_{i+1}(Y_M) & \to & K_i(N) & \to & K_i(A_N) \oplus K_i(B_N) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & H_{i+1}(Y_M) & \to & H_i(N) & \to & H_i(A_N) \oplus H_i(B_N) & \to \\
 & \downarrow\cong & & \downarrow & & \downarrow & \\
\to & H_{i+1}(Y_{M'}) & \to & H_i(N') & \to & H_i(A_{N'}) \oplus H_i(B_{N'}) & \to
\end{array}
$$

[7] If $w_1(M) \neq 0$, we should consider the twisted homology of [36], but we shall suppress the notation.

such that each column is 'naturally' split exact [6][23]. $K_i(Y_M) = 0$ for all i implies that $K_i(N) = K_i(A_N) \oplus K_i(B_N)$.

(B) Let us consider the following maps of exact sequences:

$$
(15)\qquad
\begin{array}{ccccccccc}
\to K_i(Y_M) & \to & K_i(Y_M, A_N) & \cong & K_i(B_N, N) & \to & K_{i-1}(A_N) & \to \\
\downarrow & & \downarrow & & & & \downarrow & \\
\to H_i(Y_M) & \to & H_i(Y_M, A_N) & \cong & H_i(B_N, N) & \to & H_{i-1}(A_N) & \to \\
\downarrow & & \downarrow & & & & \downarrow & \\
\to H_i(Y_{M'}) & \to & H_i(Y_{M'}, A_{N'}) & \cong & H_i(B_{N'}, N') & \to & H_{i-1}(A_{N'}) & \to
\end{array}
$$

The assertion follows from a similar argument as that of (A).

Lemma 3.2. *If g_N is k-connected, then $K_k(N)$, $K_k(A_N)$, $K_k(B_N)$, $K_{k+1}(A_N, N)$ and $K_{k+1}(B_N, N)$ are all finitely generated R-modules.*

Proof. By Lemma 3.1, it suffices to show that $K_k(N)$ is finitely generated. Let $H_i(g_N)$ be the homology of the mapping cone of g_N (with the coefficients in R, of course!). We have the following exact sequence of R-modules.

$$(16)\qquad \to H_{i+1}(g_N) \to H_i(N) \to H_i(N') \to .$$

(We suppress the coefficient group as always!) Since $g_N\colon N \to N'$ is of degree 1, $H_{i+1}(g_N) \cong K_i(N)$. Because $\pi_i(g_N) = H_i(g_N) = 0$ for $i \leqq k$, it follows from [35] that $H_{k+1}(g_N)$ is finitely generated.

Let us now consider the inclusion $i\colon (Y_M, B_N) \to Y_M, tB_N)$. We have the α^{-1}-linear endomorphism $t_*^{-1}\colon H_i(A_N, N) \to H_i(A_N, N)$ defined by the following commutative diagram:

$$
(17)\qquad
\begin{array}{ccccc}
H_i(Y_M, B_N) & \xrightarrow{i_*} & H_i(Y_M, tB_N) & \xrightarrow{t_*^{-1}} & H_i(Y_M, B_N) \\
\cong \big\downarrow \text{excision} & & & & \cong \big\downarrow \text{excision} \\
H_i(A_M, N) & & \xrightarrow{\quad ``t_*^{-1}" \quad} & & H_i(A_N, N).
\end{array}
$$

Similarly, we have α-linear endomorphism $t_*\colon H_i(B_N, N) \to H_i(B_N, N)$. These endomorphisms induce endomorphisms on $K_i(A_N, N)$ and $K_i(B_N, N)$.

Lemma 3.3. *The α^{-1}-linear endomorphisms (resp. α-linear endomorphism) t_*^{-1} (resp. t_*) of $K_{k+1}(A_N, N)$ (resp. $K_{k+1}(B_N, N)$) is nilpotent, of g is k-connected.*

Proof. Let us prove the assertion for t_*^{-1} and leave the case for t_* to the reader. By Lemma 3.2, $K_{k+1}(A_N, N)$ is finitely generated. Let $a_1, \cdots, a_s$

be a set of generators. It suffices to show that there is a large integer $l > 0$ such that $(t_*^{-1})^l a_i = 0$ for $i = 1, \cdots, s$. Let us prove it for a_1. Let c_1 be a cycle representing a_1. We can find an integer $l > 0$ such that the support of c_1 lies in $\overline{A_N - t^l A_N}$. Therefore a_1 is in the kernel of

$$(18) \qquad (t_*^{-1})^l\colon\ H_{k+1}(A_N, N) \cong H_{k+1}(Y_M, B_N) \to H_{k+1}(Y_M\ t^l B_N) \xrightarrow{(t_*^{-1})^l} H_{k+1}(Y_M, B_N) \cong H_{k+1}(A_N, N).$$

Hence $K_{k+1}(A_N, N)$ is nilpotent under t_*^{-1}.

Suppose that $(t_*^{-1})^l = 0$ on $K_{k+1}(A_N, N)$. We have the following filtration:

$$(19) \qquad K_{k+1}(A_N, N) \supset t_*^{-1} K_{k+1}(A_N, N) \supset \cdots \supset (t_*^{-1})^{l-1} K_{k+1}(A_N, N) \supset 0 = (t_*^{-1})^l K_{k+1}(A_N, N).$$

Let H denote the Hurewicz homomorphism and let l be the following composite map:

$$(20) \qquad l\colon \pi_{k+1}\overline{(A_N - tA_N, N)} \xrightarrow{H} H_{k+1}\overline{(A_N - tA_N, N)} \to H_{k+1}(A_N, N)$$

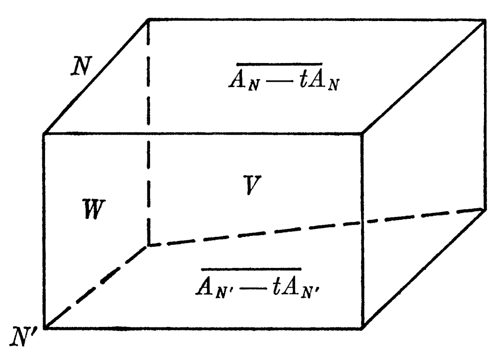
where the unlabelled map is induced by inclusion.

Let us consider the mapping cone (V, W) of map

$$\overline{(A_N - tA_N, N)} \to \overline{(A_{N'} - tA_{N'}, N')}.$$

We have a triad $(V; \overline{A_N - tA_N}, W)$.

FIGURE 4



LEMMA 3.4. *Suppose that g_N is k-connected for $k > 1$, and $(t_*^{-1})^l = 0$ on $K_{k+1}(A_N, N) = 0$. Then, the image of the following composite homomorphism*

$$(21)\qquad \pi_{k+2}(V; A_N - tA_N, W) \xrightarrow{\ \partial\ } \pi_{k+1}(A_N - tA_N, N) \xrightarrow{\ l\ } H_{k+1}(A_N, N)$$

contains $(t_*^{-1})^{l-1}(K_{k+1}(A_N, N))$.

Proof. Let us consider the following commutative diagram

$$(22)\qquad \begin{array}{ccccc}
K_{k+1}(\overline{A_N - tA_N}, N) & \longrightarrow & K_{k+1}(A_N, N) & & \xrightarrow{\ t_*^{-1}\ } \\
\downarrow & & \downarrow & & \\
\rightarrow H_{k+1}(\overline{A_N - tA_N}, N) & \longrightarrow & H_{k+1}(A_N, N) & \cong H_{k+1}(Y_M, B_N) & \xrightarrow{\ t_*^{-1}\circ i_*\ } \\
\downarrow & & \downarrow & & \\
H_{k+1}(\overline{A_{N'} - tA_{N'}}, N') & \rightarrow & H_{k+1}(A_{N'}, N') & \cong H_{k+1}(Y_{M'}, B_{N'}) & \xrightarrow{\ t_*^{-1}\circ i_*\ }
\end{array}$$

$$\begin{array}{ccccc}
K_{k+1}(A_N, N) & & & & \\
\downarrow & & & & \\
H_{k+1}(A_N, N) & \cong H_{k+1}(Y_M, B_N) & \rightarrow & H_k(\overline{A_N - tA_N}, N) & \rightarrow \\
\downarrow & & & \downarrow & \\
H_{k+1}(A_{N'}, N') & \cong H_{k+1}(Y_{M'}, B_{N'}) & \rightarrow & H_k(\overline{A_{N'} - tA_{N'}}, N') & \rightarrow
\end{array}$$

It is an easy ' diagram chasing of (22) that the image of $K_{k+1}(\overline{A_N - tA_N}, N)$ in $H_{k+1}(\overline{A_N - tA_N}, N)$ sontains $\operatorname{Ker} t_*^{-1}$. Since $(t_*^{-1})(t_*^{-1})^{l-1} = 0$, it contains $(t_*^{-1})^{l}{}_{+1}K_{k+1}(A_N, N)$. Denote the map $\overline{(A_N - tA_N, N)} \rightarrow \overline{(A_{N'} - tA_{N'}, N')}$ induced from g by g_1. Since $K_i(A_N, N) = 0$ for $i \leqq k$ $H_j(g_1) \cong K_{j-1}(A_N, N) = 0$ for $j \leqq k+1$ and g_1 is $(k+1)$-connected. By [35], the triad $(V; \overline{A_N - tA_N}, W)$ is $(k+1)$-connected, and the homomorphism

$$(23)\qquad \pi_{k+1}(\overline{A_N - tA_N}, N) \rightarrow \pi_{k+1}(\overline{A_N - tA_N} \cup W, W)$$

is an epimorphism. The following commutative diagram

$$(24)\qquad \begin{array}{ccc}
\pi_{k+2}(V; \overline{A_N - tA_N}, W) & \xrightarrow{\ \partial\ } & \pi_{k+1}(\overline{A_N - tA_N}, N) \\
\Big\downarrow H & & \Big\downarrow H \\
H_{k+2}(g_1) & \xrightarrow{\ \partial\ } & H_{k+1}(\overline{A_N - tA_N}, N) \\
& & \Big\downarrow (g_1)_* \\
& & H_{k+1}(\overline{A_{N'} - tA_{N'}}, N')
\end{array}$$

shows that the image of the maps in (21) contains $\operatorname{Ker}(g_1)_*$. This completes the proof of the lemma.

The following lemma is the main result of this action, and it is the first step toward the proof of Theorem 2.1.

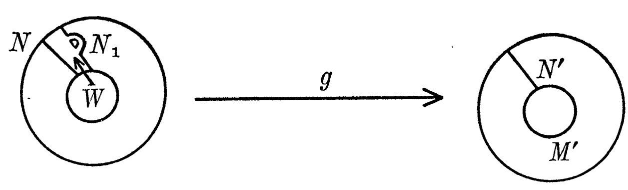
LEMMA 3.5. *Under the hypothesis of Theorem* 2.1, *there is a map* $g: M \to M'$ *such that* g *is homotopic to* f *and there is a codimension* 1 *submanifold* $N \subset M$ *with* $g_N = g \mid N: N \to N'$ *which is* k*-connected for* $k < m/2$.

Proof. Assume that we have done the preliminary steps of the beginning of the section and keep all the notations of the above. Recall that we may assume that g_N induces an π_1-isomorphism. Let us assume that we have proved our assertions for $1 \leqq i - 1 < m/2$. If $i \geqq m/2$. If $i \geqq m/2$, the lemma is proved. So, we have to show the assertion for $i < m/2$. By Lemma 3.1, it suffices to modify g such that $K_i(A_N, N) = 0$ and $K_i(B_N, N) = 0$ for the new map. By Lemma 3.2 and Lemma 3.3, $K_i(A_N, N)$ is a finitely generated R-module with an α^{-1}-linear nilpotent endomorphism t_*^{-1}, and hence we have the filtration (19) with $i = k + 1$. Clearly, $(t_*^{-1})^{l-1} K_i(A_N, N)$ is again finitely generated and let $a_1, \cdots, a_s$ be a set of generators. By Lemma 3.4, there are $c_1, \cdots, c_s$ in $\pi_{i+1}(V; \overline{A_N - tA_N}, W)$ whose images under the composite homomorphism (21) are $a_1, \cdots, a_s$. Set $\partial c_1 = b_1, \cdots, \partial c_s = b_s \in \pi_i(\overline{A_N - tA_N}, N)$. Since $i < m/2$, we can represent them by disjoint embeddings

$$b_j: (\boldsymbol{D}^i, \boldsymbol{S}^{i-1}) \to (A_N, N) \tag{25}$$

$j = i, \cdots, s$. These embeddings project down to embeddings of $(\boldsymbol{D}^i, \boldsymbol{S}^{i-1})$ into (M, N). Following [11, Chapter III], we exhange the handles on N with the above $\boldsymbol{D}^i$'s as the central discs from A_N to B_N. Using Lemma 3.4, we have a new map $M \to M'$ homotopic to g which is again denoted by g such $g(^W) \subset N'$ where W is the elementary cobordism on N with $\boldsymbol{D}^j$ $(j = 1, \cdots, s)$ as the central discs of the added handles. The other boundary of W is denoted by N_1 for the moment.

FIGURE 5



Let A_{N_1}, B_{N_1} denote the sets corresponding to A_N, B_N for N_1. We have epimorphisms

$$\begin{array}{ccc} K_i(A_N,N) & \longrightarrow & K_i(A_{N_1},N_1)\to 0 \\ \text{excision}\downarrow\cong & & \text{excision}\downarrow\cong \\ K_i(Y_M,B_N) & \longrightarrow & K_i(Y_M,B_{N_1})\to 0 \end{array}$$

induced by inclusions, and these epimorphisms respect the α^{-1} endomorphism t_*^{-1}. It follows from the construction that the kernel of the epimorphism $K_i(A_N,N)\to K_i(A_{N_1})$ cntains $(t_*^{-1})^{l-1}K_i(A_N,N)$. Hence $(t_*^{-1})^{l-1}K_i(A_{N_1},N_1) = 0$. We may replace N by N_1 and repeat the argument. After a finite number of steps, $K_i(A_N,N)$ may be eliminated altogether. During the elimination procedure of $K_i(A_N,N)$, $K_i(B_N,N)$ is not affected provided that $i<m/2$. We can perform a similar procedue to elimnate $K_i(B_N,N)$. This completes the induction step and f is homotopic to a map g as described in the lemma.

4. The homological description of $0(f)$. In this section, we shall give a homological description of the obstruction $0(f)$ of Theorem 2.1, and we shall show that if f is splittable, then $0(f)$ must vanish.

LEMMA 4.1. *Under the hypothesis of Theorem* 2.1, *if* $m=2k\geqq 6$ *then* f *is homotopic to a map* $g\colon M\to M'$ *such that there isa codim* 1 *submanifold* $N\subset M$ *with* $g_N = g\mid N\colon N\to N'$ *satisfying the following conditions*:

(A) $K_i(A_N,N)=0$ *for* $i\leqq k$ *and* $K_i(B_N,N)=0$ *for* $i<k$.

(B) $K_k(B_N,N)$ *is finitely generated projective* R*-modules.*

Proof. $K_i(A_N,N)=K_i(B_N,N)=0$ for $i<k$ is in Lemma 3.5. Let us proceed to carry out the argument of Lemma 3.5 to eliminate $K_k(A_N,N)$. The only place where it may stop us is to represent $b_1,\cdots,b_s$ by disjoint embeddings, because there is one dimension off for applying Whitney's embedding range. However, we may represent them by immersions with isolated intersection points in the interior of $\boldsymbol{D}^i$. Since $\pi_1 N\to\pi_1\overline{A_N - tA_N}$ is an isomorphism, we may pipe out the intersection points [36, § 4], and represent thsm by embeddings. By the argument of Lemma 3.5, we have (A). Since $K_j(A_N,N)=0$ for $j\leqq k$, $g_N\colon(A_N,N)\to(A_{N'},N')$ is $(k+1)$-connected. Hence $K_j(A_N,N;\mathscr{B})=0$ for $j\leqq k$ and any R-module $\mathscr{B}$. Using Poincaré duality, $K^j(A_N,\mathscr{B})=0$ for $j\geqq k$. By the argument of Lemma 3.1, $K^{j+1}(B_N,N;\mathscr{B})=0$ for any R-module $\mathscr{B}$ and $j\geqq k$. By [34], $K_k(B_N,N)$ is a finitely generated projective R-module. This completes the proof of the lemma.

LEMMA 4.2. *Under the hypothesis of Theorem 2.1, if* $m=2k+1\geqq 7$, *then* f *is homotopic to a map* $g\colon M\to M'$ *such that there is a codim 1 submanifold* $N\subset M$ *with* $g_N=g\mid N\colon N\to N'$ *satisfying the following conditions:*

(A) $K_i(A_N,N)=K_i(B_N,N)=0$ *for* $i\leqq k$,

(B) $K_{k+1}(A_N,N)$ *and* $K_{k+1}(B_N,N)$ *are finitely generated projective* R-*modules.*

Proof. (A) follows from Lemma 3.5. By Lemma 3.2, $K_{k+1}(B_N,N)$ is a finitely generated R-module. Since

$$K^j(A_N,N;\mathfrak{B})\oplus K^j(B_N,N;\mathfrak{B})=K^{j-1}(N;\mathfrak{B})$$

for any R-module $\mathfrak{B}$ by a similar argument of Lemma 3.1 and $K^{j-1}(N;\mathfrak{B})=0$ for $j-1>k$ by Poincaré duality, $K^j(A_N,N;\mathfrak{B})=K^j(B_N,N;\mathfrak{B})=0$ for $j>k+1$. By [33], $K_{k+1}(A_N,N)$ and $K_{k+1}(B_N,N)$ are projective R-modules.

For (N,g) satisfying the conclusion of Lemma 4.1 or Lemma 4.2, we call it an *almost splitting.* Recall that t_* is a α-linear nilpotent endomorphism of $K_k(B_N,N)$ for $m=2k$ or $K_{k+1}(B_N,N)$ for $m=2k+1$ for N an almost splitting.

Let us define

$$\begin{aligned}(26)\qquad 0(f)&=(-1)^{k-1}[K_k(B_N,N),t_*]\ \text{for}\ m=2k\geqq 6,\\ &=(-1)^{k}[K_{k+1}(B_N,N),t_*]\ \text{for}\ m=2k+1\geqq 7\end{aligned}$$

$\in C(R,\alpha)$.

LEMMA 4.3. $0(f)$ *is independent of the choice of the almost splitting. In fact,* $0(f)$ *is equal to* $p\tau(f)$ *where* $\tau(f)$ *is the torsion of* f *and* p *is the projection mapof* (4). *Moreover, if* f *is splittable, then* $0(f)=0$.

Proof. Let U, V and W be the mapping sylinder of g_Y, g_B and g_N respectively. It follows from the definition and Poincaré duality that

(A) $H_i(U,Y_M;R)=0$ for all i,

(B) $H_i(V,B_N\cup W;R)=0$ for $i\neq k+1$, $m=2k$ or $i\neq k+2$, $m=2k+1$.

(C) $(H_{k+1}(V,B_N\cup W;R),t_*)\cong(K(B_N),t_*)$ for $m=2k$,
$(H_{k+2}(V,B_N\cup W;R),t_*)\cong(K_{k+1}(B_N,N),t_*)$ for $m=2k+1$

as objects in $\mathscr{B}(R,\alpha)$ of [12]. It follows from the argument of [14][11, Chapter III] that

$$\begin{aligned}(27)\qquad p\tau(f)&=(-1)^{k+1}[H_{k+1}(V,B_N\cup W;R),t_*]\ \textit{for}\ m=2k,\\ &=(-1)^{k+2}[H_{k+2}(V,B_N\cup W;R),t_*]\ \text{for}\ m=2k+1,\end{aligned}$$

where $\tau(f)$ denotes the torsion of f and p is the projectioi of (4). Therefore, the lemma follows readily.

5. Realizing the object in $0(f)$ by an almost splitting and the proof of Theorem 2.1. In this section, we shall show that every object[8] in $0(f)$ is realizable by an almost splitting. In particular, we have a proof of Theorem 2.1.

LEMMA 5.1. *Suppose that $m \geqq 2k \geqq 6$ and (N_1, g_1) is an almost splitting. If*

$$0 \to (P_1, v_1) \to (P, v) \to (F_1, f_1) \to 0 \tag{28}$$

is a short exact sequence in $\mathscr{C}(R, \alpha)$ such that $(K_k(B_N, N), t_*) \cong (P, v)$ and (F_1, f_1) is a triangular object,[9] then there is another almost splitting (N, g) such that $(K_k(B_N, N), t_*) \cong (P, v)$.

Proof. By an easy induction argument, it suffices to prove the assertion for $(F_1, f_1) \cong (R, 0)$. Let $a \in P$ which projects onto a generator of R. Set $x = v(a) \in (P_1, v_1) \cong (K_k(B_N, N), t_*)$. Let n be a large integer such that there is $y \in K_k(B_{N_1} - t^{-n}B_{N_1}, N_1)$ (where $K_k(B_{N_1} - t^{-n}B_{N_1}, N_1)$ denotes the kernel of $H_k(B_{N_1} - t^{-n}B_{N_1}, N_1) \to H_k(B_{N'} - t^{-n}B_{N'}, N'))$ with the image of y in $K_k(B_{N_1}, N_1)$ equal to x. Consider $z = t_*^{-1}y \in K_k(t^{-1}B_{N_1} - t^{-n-1}B_{N_1}, t^{-1}N_1)$. It is clear that the image of z under the homomorphism

$$K_k(t^{-1}B_{N_1} - t^{-n-1}B_{N_1}, t^{-1}N_1) \to K_k(t^{-1}B_{N_1}, t^{-1}N_1) \tag{29}$$

is just $t_*^{-1}x$. Let us consider the exact sequence

$$\begin{aligned} &\to K_k(t^{-1}B_{N_1}, t^{-1}N_1) \cong K_k(B_{N_1}, \overline{B_{N_1} - t^{-1}B_{N_1}}) \xrightarrow{\ \partial\ } \\ &K_{k-1}(\overline{B_{N_1} - t^{-1}B_{N_1}}, N_1) \to K_{k-1}(B_{N_1}, N_1) = 0 \to \cdots. \end{aligned} \tag{30}$$

Let $u = \partial(t_*^{-1}x)$. Using the argument of Lemma 3.5, we may assume that u is in the image of the following composite map

$$\begin{aligned} \pi_{k+1}(V; B_{N_1} - t^{-n}B_{N_1}, W) &\xrightarrow{\ \partial\ } \pi_k(B_{N_1} - t^{-n}B_{N_1}, W) \\ &\xrightarrow{\ H\ } H_k(B_{N_1} - t^{-n}B_{N_1}, W) \end{aligned} \tag{31}$$

where V, W are the mapping cylinder of the mappings

[8] See [12] for the definition of an object in $\mathscr{C}(R, \alpha)$.

[9] For the definition of a triangular object, see [12].

$$B_{N_1} - t^{-n}B_{N_1} \to B_{N'} - t^{-n}B_{N'}, \quad N_1 \to N'$$

respectively, and H denotes the Hurewicz homomorphism. Hence, we have an embedding by Whitney's embedding theorem

$$u: (\boldsymbol{D}^{k-1}, \partial \boldsymbol{D}^{k-1}) \to (B_{N_1} - t^{-1}B_{N_1}, N_1) \tag{32}$$

representing the class u. Thickening this embedded disc, we have a cobordism $(U; N_1, N)$. We may exchange this cobordism to A_{N_1} (i.e., exchange the handle $u(\boldsymbol{D}^{k-1})$ to A_{N_1}), and we have a new almost splitting (M, g) such that $g(U) \subset N'$. Let us consider the following exact sequence

$$\begin{array}{lll} (33) & \to H_k(W, N_1) = 0 \to K_k(B_{N_1}, N_1) \to K_k(B_{N_1}, U) & \\ & & \cong \uparrow \text{excision} \\ & & K_k(B_N, N) \\ & \to H_{k-1}(U, N_1) \to K_k(B_{N_1}, N_1) = 0 \to . & \end{array}$$

We see from (33) that $K_k(B_N, N) \cong K_k(B_N, N) \oplus R$, and it follows from the construction of U that $t_*^{-1}x$ is a generator of the second summand. In fact, (33) may be identified with

$$0 \to (P_1, v_1) \to (P, v) \to (R, 0) \to 0 \tag{34}$$

with the image of $t_*^{-1}x$ generating R. Here we use the fact that $t_*(t_*^{-1}x) = x \in K_k(B_{N_1}, N_1)$. Since $t_*(t_*^{-1}x) = x$, (34) is isomorphic to the exact sequence of the objects in $\mathcal{C}(R, \alpha)$ as stated in the lemma. The proof of the lemma is thereby completed.

LEMMA 5.2. *Suppose that $m = 2k + 1 \geqq 7$ and (N_1, g_1) is an almost splitting. If*

$$0 \to (P_1, v_1) \to (P, v) \to (F_1, f_1) \to 0 \tag{35}$$

is a short exact sequence in $\mathcal{C}(R, \alpha)$ such that $(K_{k+1}(B_{N_1}, N), t_*) \cong (P_1, v_1)$ and (F_1, f_1) is a triangular object, then there is another almost splitting (N, g) such that $(K_{k+1}(B_N, N), t_*) \cong (P, v)$.

The proof of this lemma is exactly the same as Lemma 5.1 except some number change. We leave it to the reader.

LEMMA 5.3. *Suppose that $m = 2k \geqq 6$ and (N_1, g_1) is an almost splitting. If*

$$0 \to (F_1, f_1) \to (P_1, v_1) \to (\mathrm{P}, \mathrm{v}) \to \tag{36}$$

is a short exact sequence in $\mathscr{C}(R,\alpha)$ *such that* $(K_k(B_{N_1},N_1),t_*)\cong(P_1,v_1)$ *and* (F_1,f_1) *is a triangular object, then there is another almost splitting* (N,g) *such that* $(K_k(B_N,N),t_*)\cong(P,v)$.

Proof. Again, it suffices to prove the assertion for $(F_1,f_1)\cong(R,0)$. Let a be a generator of $F_1\cong R$. Suppose that $t_*{}^l=0$ and $t_*{}^{l-1}\neq 0$. It is easy to see that $a\in t_*{}^{l-1}(K_k(B_{N_1},N_1))$. By the argument of Lemma 3.5, a is in the image of the following composite maps.

$$\pi_{k+1}(V;B_{N_1}-t^{-1}B_{N_1},W)\xrightarrow{\ \partial\ }\pi_k(B_{N_1}-t^{-1}B_{N_1},N_1)\xrightarrow{\ H\ }H_k(B_{N_1},N_1) \tag{37}$$

where V, W are the mapping cylinders of the mappings $B_{N_1}-t^{-1}B_{N_1}\to B_{N'}-t^{-1}B_{N'}$, $N_1\to N'$ respectively, and H denotes the Horewicz homomorphism. Let $c\in\pi_{k+1}(V;B_{N_1}-t^{-1}B_1,\ W)$ such that the image of $b=\partial c$ in $H_k(B_{N_1},N_1)$ represents a. Using the piping out argument [36, § 4], we have an embedding

$$b:(\boldsymbol{D}^k,\partial\boldsymbol{D}^k)\to(B_{N_1}-t^{-1}B_{N_1},N_1) \tag{38}$$

representing b. Thickening this embedding, we have a cobordism $(U;N_1,N)$. We may exchange W to A_{N_1} to define a new splitting (N,g) as we did in Lemma 5.1. Consider the following exact sequence

$$\begin{array}{c} 0\to H_k(U,N_1)\to K_k(B_{N_1},N_1)\to K_k(B_{N_1},U)\to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{excision}\Big\uparrow\cong \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad K_k(B_N,N). \end{array} \tag{39}$$

It is easy to see that (39) respects the action of t_* and it is isomorphic to

$$0\to(F_1,f_1)\to(P_1,v_1)\to(P,v)\to 0$$

as exact sequences of $\mathscr{C}(R,\alpha)$. In particular, $(K_k(B_{N_1},N_1),t_*)\cong(P,v)$, and this completes the proof of the lemma.

LEMMA 5.4. *Suppose that* $m=2k+1\geqq 7$ *and* (N_1,g_1) *is an almost splitting. If*

$$0\to(F_1,f_1)\to(P_1,v_1)\to(P,v)\to 0 \tag{40}$$

is a short exact sequence in $\mathscr{C}(R,\alpha)$ *such that* $(K_{k+1}(B_{N_1},N_1),t_*)\cong(P_1,v_1)$, *then there is another almost splitting* (N,g) *such that* $(K_{k+1}(B_N,N),t_*)\cong(P,v)$.

Proof. Again, we may assume that $(F_1, f_1) \cong (R, 0)$ without loss of generality. By the same argument of Lemma 5.3, there is a generator a of $F_1$and an element c of $\pi_{k+2}(V; B_{N_1} - t^{-1}B_{N_1}, W)$ (where V, W are the mapping cylinder of the mappings $B_{N_1} -. B_{N_1} \to B_{N'} - t^{-1}B_{N'}$, $N_1 \to N'$ respectively) such that a is the image of c under the following composite of maps:

$$(41) \qquad \pi_{k+2}(V; B_{N_1} - t^{-1}B_{N_1}, W) \xrightarrow{\partial} \pi_{k+1}(B_{N_1} - t^{-1}B_{N_1}, N_1)$$
$$\xrightarrow{H} H_{k+1}(B_{N_1} - t^1{}_1 B_{N_1}, N_1).$$

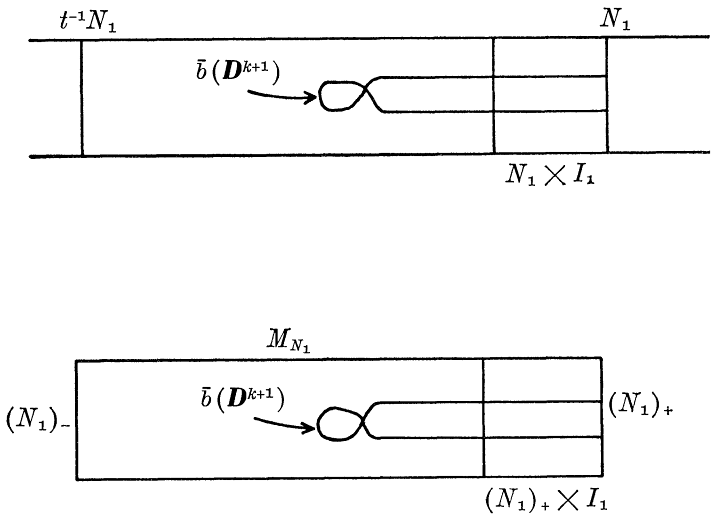
We have an immersion

$$(42) \qquad b: (\boldsymbol{D}^{k+1}, \partial \boldsymbol{D}^{k+1}) \to (B_{N_1} - t^{-1}B_{N_1}, N_1)$$

representing $b = \partial c \in {}_{k+1}(B_{N_1} - t^{-1}B_{N_1}, N_1)$. We are now completely off the range for Whitney embedding theorem, and we do not have enough connectivity to pipe out the sigularity. What we are going to do is to follow a trick of [36, § 4]. Let us assume that

$$(43) \qquad \bar{b}: \partial \boldsymbol{D}^{k+1} \times I_1 \to N_1 \times I_1$$

FIGURE 6



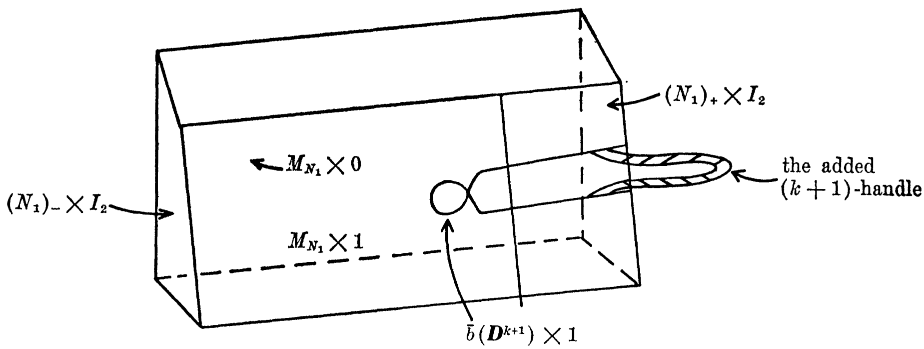
is a level-preserving embedding where $I_1 = [-1, 0]$ and $\partial \boldsymbol{D}^{k+1} \times I_1$, $N_1 \times I_1$ are the collars of $\partial \boldsymbol{D}^{k+1}$, N_1 in $\boldsymbol{D}^{k+1}$ and $B_{N_1} - t^{-1}B_{N_1}$ respectively. This is because $\pi_1 N_1 \to \pi_1 B_{N_1} - t^{-1}B_{N_1}$ is a isomorphism and we can apply Whitney's trick to the singularities on the boundary $\partial \boldsymbol{D}^{k+1}$ of the immersion (42). Let M_{N_1} be the manifold gotten from N_1 by cutting M along N_1 as we did before. $B_{N_1} - t^{-1}B_{N_1}$ is diffeomorphic to $M_{N_1} - (N_1)_-$ under the obvious projection. and we have the immersion b of $\boldsymbol{D}^{k+1}$ into $M_{N_1} - (N_1)_-$ by the identification. Let $(N_1)_+ \times I_1 = (N_1)_+ \times [-1, 0]$ be the corresponding collar of $(N_1)_+$. Then $\bar{b}\colon \partial \boldsymbol{D}^{k+1} \times I_1 \to (N_1)_+ \times I_1$ is a level-preserving embedding.

Let $I_2 = [0,1]$ and let us attach a $(k+1)$-handle to the normal disc bundle of

(44) $$\bar{b} \times \mathrm{id}\colon \partial \boldsymbol{D}^{k+1} \times [-\tfrac{3}{4}, -\tfrac{1}{4}] \to (N_1)_+ \times [-\tfrac{3}{4}, -\tfrac{1}{4}] \times 1$$

We have a cobordism W on M_{N_1}.

FIGURE 7



Since the added $(k+1)$-handle does not touch $(N_1)_+ \times I_2$ and $(N^1)_- \times I_2$, we may identify $(N_1)_+ \times I_2$ with $(N^1)_- \times I_2$ in the obvious way to form a cobordism U on $M \times I_2$ with a $(k+1)$-handle added.

In fact, the cobordism $(U; M, M_1)$ contains $N_1 \times I_2$ such that when cut along $N \times I_2$, we get W back. We can define a retract

(45) $$rf_1\colon U \to M \times I_2$$

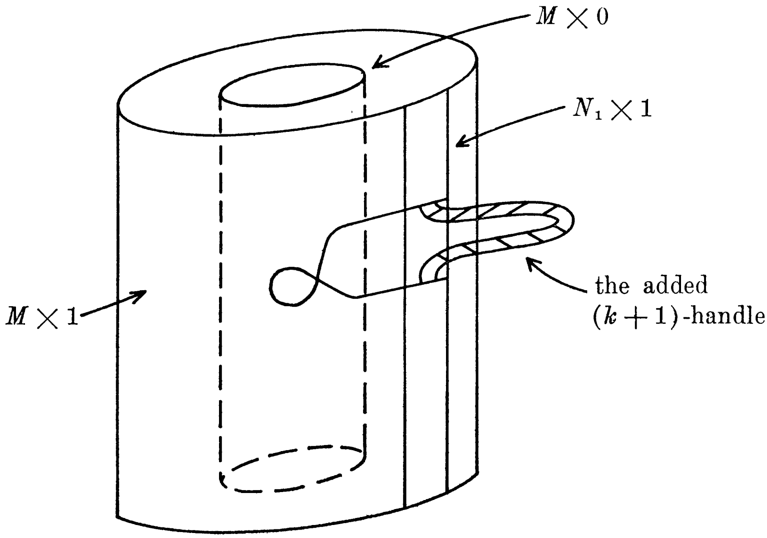
as follows. We first retract the added handle to its central disc $\boldsymbol{D}^{k+1}$ and then we map the central disc into $M \times 1$ by $\bar{b} \times \mathrm{id}$. This composite map is r_1. In fact, r_1 induces a degree 1 map of triads.

(46) $$r\colon (U; M, M_1) \to (M \times I_2; M \times 0, M \times 1).$$

Let us consider the following exact sequence of kernels of r and r_1:

(47) $$0 \to K_{k+1}(M_1) \to K_{k+1}(U) \to K_{k+1}(U, M_1) \to K_k(M_1) \to 0.$$

FIGURE 8



It follows from the construction that

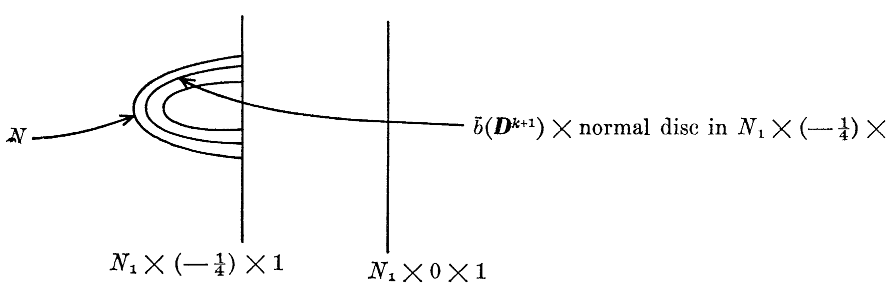
(48) $$0 \to K_{k+1}(M_1) \xrightarrow{\cong} K_{k+1}(U) \to 0,$$

and hence we also have

(49) $$0 \to K_{k+1}(U, M_1) \to K_k(M_1) \to 0.$$

In fact, they are all isomorphic to R. Let us look at $K_k(M_1)$ a little closer and pick up a generator for it. Recall that U is gotten from $M \times I_2$ by attaching a $(k+1)$-handle to $N_1 \times [-\frac{3}{4}, -\frac{1}{4}] \times 1$ where $N \times [-1, 0] \times 1$ is the collar of $N_1 \times 1$ gotten from $(N_1)_+ \times [-1, 0] \times 1$. Therefore, we have a cobordism V inside of M_1 such that V is gotten from $N_1 \times [-\frac{1}{4}, 0] \times 1$

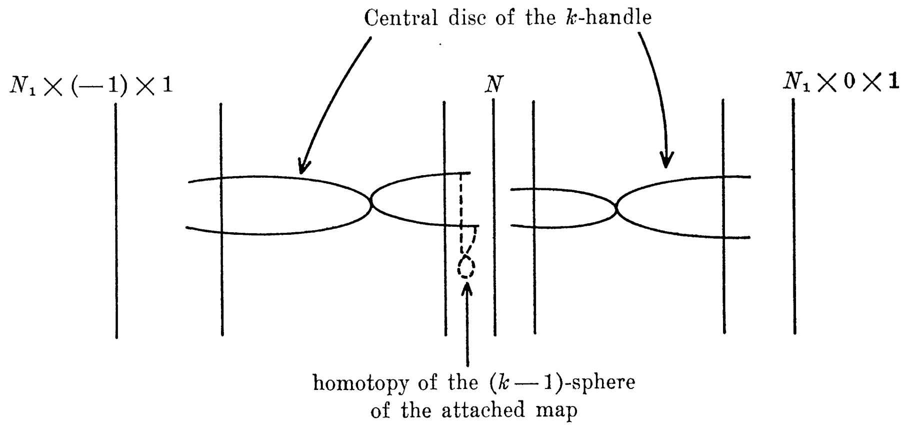
FIGURE 9



by adding a $(k+1)$-handle which was induced from the attached $(k+1)$-handle of U.

In fact, the $(k+1)$-handle of V is contained in the $(k+1)$-handle of U and the latter is a 1-dim thickening of the former. Write the elementary cobordism V as $(V; N_1, N)$ with $N_1 = N_1 \times (-1) \times 1$, and consider V as gotten from N by adding a k-handle. It is not difficult to see that the attaching map of this k-handle is actually trivial in N. Therefore, the central disc of this k-handle together with the homotopy of the $(k-1)$-sphere of the attaching map in N gives us an element $c \in \pi_k(V)$.

FIGURE 10



We see that

$$c: \boldsymbol{S}^k \to V \to M_1 \tag{50}$$

represents a generator of $K_k(M_1)$. In fact, we may assume that $\operatorname{Im}(c) \subset \operatorname{Int}(V)$ without loss of generality. Let ls analyze the element $c \in \pi_k(M_1)$ with a little more care. c induces a map

$$\bar{c}: \boldsymbol{S}^k \to V \to W. \tag{51}$$

$\bar{c}$ extends to a map

$$\bar{d}: \boldsymbol{D}^{k+1} \to W \tag{52}$$

such that the Hurewicz homomorphism sends the elements in $\pi_{k+1}(W, M_1)$ represented by the pair of mappings $(\bar{d}, \bar{c})$ to a generator of

$$K_{k+1}(W; M_{N_1}, (M_1)_{N_1})$$

which denotes the kernel of the retraction

$$r_2\colon (W; M_{N_1}, M_1)_{N_1}) \to (M_{N_1} \times I_2; M_{N_1} \times 0, M_{N_1} \times 1)$$

where r_2 is induced from the retraction r of (46) by cutting along $N_1 \times I_2$. Using the trick of piping out of singularity of [36, § 4], we may assume that $\bar{d}$ is an embedding and $\bar{c}(\boldsymbol{S}^k) \subset V$. Applying handle subtraction to the embedded disc $\bar{d}(\boldsymbol{D}^{k+1})$ [36, § 4] and gluing the resultant along $N_1 \times I_2$ again, we have a cobordism $(U_1; M, M_2)$ and a map

(53) $$r_3\colon (U_1; M, M_2) \to (M \times I_2; M \times 0, M \times 1)$$

induced from r_2 and the handle subtraction. It is easy to check that r_3 is a homotopy equivalence and $r_3(N_1 \times I_2) \subset N_1 \times I_2$. Therefore U_1 is an h-cobordism on M. Since we have never touched $N \times I_2$ during the construction of U_1, we see that the torsion $\tau(U_1, M) \in \mathrm{Wh}\, G$. Let us now consider the following composite of maps:

(54) $$M_2 \xrightarrow{i} U_1 \xrightarrow{r_3} M \times I_2 \xrightarrow{p_1} M \xrightarrow{g_1} M'$$

where i is the inclusion and p_1 is the projection map. Now, the generator a of F bounds an embedded $(k+1)$ dim disc in $B_{N_1} - t^{-1}B_{N_1}$ satisfying all the requirements. Following the argument of Lemma 5.3, the generator a of F' may be eliminated by the almost splitting (N, g) *of* M_2. Now, observe that

(55) $$\tau(U_1, M_2) = f\bar{\tau}(U_1, M) \in \mathrm{Wh}\, G.$$

U may be gotten from $M \times L$ by attaching 2-handles and 3-handles without touching $N \times I \subset M \times I$. Therefore, there is an almost splitting (N, g) in M such that $(K_{k+1}(B_N, N), t_*) \cong (P, v)$. This completes the proof of the lemma.

The following theorem is a summary of all the above lemmas from § 3 to § 5.

THEOREM 5.5. *Under the hypothesis of Theorem* 2.1, *if* (N_1, g_1) *is an almost splitting such that*

$$\begin{aligned} 0(f) &= (-1)^{k-1}[K_k(B_{N_1}, N_1), t_*] \text{ for } m = 2k \geqq 6 \\ &= (-1)^k[K_{k+1}(B_{N_1}, N_1), t_*] \text{ for } m = 2k+1 \geqq 7 \end{aligned}$$

defined by (26) *and Lemma* 4.3, *then for any object* $(P, v) \in \mathscr{C}(R, \alpha)$ *in the class of* $0(f)$, *there is another splitting* (N, g) *such that*

$$(K_k(B_N, N), t_*) \cong (P, v) \text{ for } m = 2k \geqq 6,$$

$$(K_{k+1}(B_N, N), t_*) \cong P, v) \text{ for } m = 2k+1 \geqq 7.$$

Proof of Theorem 2.1. The necessity is trivial. So let us assume $0(f)=0$. Then, $(0,t_*)$ is an object in $[K_k(B_{N_1},N_1),t_*]$ for $m=2k\geqq 6$ and $[K_k(B_{N_1},N_1).t_*]$ for $m=2k+1\geqq 7$. By Theorem 5.5, there is an almost splitting (N,g) such that $K_i(B_N,N)=0$ for $i\leqq k$ and $m=2k\geqq 6$, and $K_i(B_N,N)=0$ for $i\leqq k+1$ and $m=2k+1\geqq$ G. By Poincaré duality and Lemma 3.1, we see that $K_i(N)=0$ for $i\geqq 0$ in both cases. Hence, g_N is a homotopy equivalence. This completes the proof of the splitting theorem.

6. Obstruction to finding a homotopy strip. Starting from this section, we shall give some applications of the splitting theorem, Theorem 2.1. For other applications, see [31], [16], [36]. We first give a geometric interpretation of the components of the algebraic decomposition of $C(R,\alpha)$ into $\tilde{C}(R,\alpha)\oplus\tilde{K}_0G$ [12]. We shall see that the component of $0(f)$ in $C(R,\alpha)$ for f is essentially the obstruction to finding a homotopic open strip, and then the component of $0(f)$ in $\tilde{K}_0G$ is the Novikov-Siebenmann obstruction to splitting $M\times R$ [29], [26].

Let (M',N') and $f\colon M\to M'$ be given as in § 2. Let us ask what is the obstruction to finding a map $g\colon M\to M'$ and an open submanifold U of M such that

(56) (A) $g(U)\subset N'$ and $g_U\colon U\to N'$ is a homotopy equivalenve.
(B) g is homotopic to f.

When such (U,g) exists, we say (U,g) is a *splitting of f by a homotopy open strip.* Let

$$q_1\colon C(R,\alpha)\to\tilde{C}(R,\alpha),\quad q_2\colon C(R,\alpha)\to\tilde{K}_0G \tag{57}$$

be the projections of the direct decomposition $C(R,\alpha)=C(R,\alpha)\oplus\tilde{K}_0G$ [12].

THEOREM 6.1. *Suppose that $f\colon M\to M'$ is given as Theorem 2.1. f is splittable by a homotopy open strip if and only if $q_10(f)=0$.*

Proof. Let us first prove the necessity. Let (U,g) be such a splitting. It is easy to see that U is an open manifold with two tame ends $\mathcal{E}_+$, $\mathcal{E}_-$. Following the argument of [29, Chap. V], we find a codimension 1 submanifold $N\subset U$ satisfying the following conditions:

(58) (A) N divides U into C_N, D_N such that $C_N\cap D_N=N$,
(B) $H_i(D_N,N)=0$ for $i\neq k$ and $m=2k\geqq 6$, or
for $i\neq k+1$ and $m=2k+1\geqq 7$.

It follows from Lemma 3.2 and Lemma 4.2 that $H_k(D_N,N)$ for $m=2k\geqq 6$ or $H_{k+1}(D_N,N)$ for $m=2k+1\geqq 7$ is finitely generated projective module.

Let us consider the pair (N, g) induced from (U, g). It is easy to check that (N, g) is an almost splitting and we can choose $C_N \subset A_N$, $D_N \subset B_N$. Lifting U into Y_M, we have the following maps of exact sequences:

$$
(59)\qquad
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \| & & \\
\cdots \to K_i(N) & \to & K_i(C_N) \oplus K_i(D_N) & \to & K_i(U) & \to & \\
\downarrow = & & \downarrow & & \downarrow & & \\
\cdots \to K_i(N) & \to & K_i(A_N) \oplus K_i(B_N) & \to & K_i(Y_M) & \to & \\
 & & & & \| & & \\
 & & & & 0 & &
\end{array}
$$

$$
(60)\qquad
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \| & & & & \\
\cdots \to K_i(C_N) & \to & K_i(U) & \to & H_i(U, C_N) & \cong & H_i(D_N, N) \to \\
\downarrow & & \downarrow & & & & \downarrow \\
\cdots \to K_i(A_N) & \to & K_i(Y_M) & \to & K_i(Y_M, A_N) & \cong & K_i(B_N, N) \to \\
 & & \| & & & & \\
 & & 0 & & & &
\end{array}
$$

$$
(61)\qquad
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \| & & & & \\
\cdots \to K_i(D_N) & \to & K_i(U) & \to & H_i(U, D_N) & \cong & H_i(C_N, N) \to \\
\downarrow & & \downarrow & & & & \downarrow \\
\cdots \to K_i(B_N) & \to & K_i(Y_M) & \to & K_i(Y_M, B_N) & \cong & K_i(A_N, N) \to \\
 & & \| & & & & \\
 & & 0 & & & &
\end{array}
$$

where $K_i(C_N)$, $K_i(D_N)$ denote the kernels of the mappings of $C_N \to N'$, $D_N \to N'$ respectively. We see that

$$
(62)\qquad
\begin{aligned}
&K_i(C_N) \cong K_i(A_N), && K_i(D_N) \cong K_i(B_N) \\
&K_i(C_N, N) \cong K_i(A_N, N), && K_i(D_N, N) \cong K_i(B_N, N)
\end{aligned}
$$

where the isomorphisms are all induced by inclusions. Since $U \subset t^{-1}(A_N)$, (62) implies that the endomorphism t_* in $(K_k(B_N, N), t_*)$ for $m = 2k$ or in $(K_{k+1}(B_N, N) t_*)$ for $m = 2k + 1$ is zero. Therefore, $q_1 0(f) = 0$ by the definition.

Before we prove the sufficiency we need the following lemma which is of interest in its own right.

LEMMA 6.2. *Let W^m be an elementary cobordism on an $(m-1)$-dim manifold N^{m-1} with s j-handles $(3 \leqq j \leqq m-3)$ attached on $N^{m-1} \times 1$ of $N^{m-1} \times I$. If we can decompose $H_j(W,N) = P \oplus Q$ with P, Q projective R-modules $(R = \mathbf{Z}(G),\ G = \pi_1 N = \pi_1 W)$ such that the boundary homomorphism*

$$(63) \qquad \partial: H_j(W,N) \to H_j(N)$$

of the exact sequence of the pair (W,N) is isomorphic to the projection onto the second summand Q and $\mathrm{Im}(\partial)$ *is a direct summand of $H_{j-1}(N)$, then there is an open subset of W which is a manifold with boundary N and has a tame end $\mathcal{E}$ such that $\sigma(\mathcal{E}) = [Q]$ where $\sigma(\mathcal{E})$ denotes the obstruction of Novikov-Siebenmann ([25], [29]), to adding a boundary to $\mathcal{E}$. (Note that we suppress the coefficient group of the homology groups again. We always use $F = \mathbf{Z}(G)$, $G = \pi_1 N = \pi_1 W$ as the coefficient group.)*

Proof. Suppose that W is gotten from $N \times I$ by attaching $a_1, \cdots, a_s$ j-handles on $N \times 1$. Let us introduce s pairs of complementary $(j-1)$-handles and j-handles $(b_1, c_1), \cdots, (b_s, c_s)$ into W. We can view that $b_1, \cdots, b_s$ are handles attached to $N \times \frac{1}{2}$ by trivial embeddings

$$(64) \qquad b_i: \partial \mathbf{D}^{j-1} \times \mathbf{D}^{m-j} \to N \times \tfrac{1}{2} \quad i = 1, \cdots, s$$

contained in $\mathbf{D}^{m-1} \subset N \times \frac{1}{2}$. By trivial embeddings, we mean that $\{b_i(\partial \mathbf{D}^{j-1} \times x_0),\ i = 1, \cdots, s\}$ bound disjoint discs $\{\mathbf{D}_i^{j-1},\ i = 1, \cdots, s\}$ in $\mathbf{D}^{m-1} - \bigcup_{i=1}^{s} b_i(\mathbf{D}^{j-1} \times \mathrm{Int}\ \mathbf{D}^{m-j})$, where $x_0 \in \partial \mathbf{D}^{m-i}$ is a base point. Therefore

$$(65) \qquad b_i(\mathbf{D}^{j-1} \times x_0) \cup \mathbf{D}_i^{j-1} \quad i = 1, \cdots, s$$

are embedded spheres in $N \times [0, \frac{1}{2}] \cup \{\bigcup_{i=1}^{s} b_i(\mathbf{D}^{j-1} \times \mathbf{D}^{m-j})\}$. For simplicity, let us denote the cobordism $N \times [0, \frac{1}{2}] \cup \{\bigcup_{i=1}^{s} b_i(\mathbf{D}^{m-j})\}$ on N by $(W_{\frac{1}{2}}; N_0, N_{\frac{1}{2}})$. Let us pick up a base point $y_0 \in N_{\frac{1}{2}}$ and join paths from y_0 to the embedded spheres of (65). It is easy to see that these based embedded spheres represent a set of free generators $\{e_j\}$ of a direct summand R^s $(R = \mathbf{Z}(G),\ G = \pi_1 N\,!)$ of $H_{j-1}(N_{\frac{1}{2}})$. In fact, if $j - 1 \neq (m-1)/2$, $H_{j-1}(N_{\frac{1}{2}}) \cong H_{j-1}(N) \oplus R^s$ and $\{e_i \mid i = 1, \cdots, s\}$ is a set of generators of R^s; while

$$j - 1 = (m-2)/2,\ H_{j-1}(N_{\frac{1}{2}}) \cong H_j(N) \oplus R^s \oplus R^s$$

such that $\{e_i \mid i = 1, \cdots, s\}$ generates one of the summands R^s. Set $V_{\frac{1}{2}} = W - W_{\frac{1}{2}}$. $V_{\frac{1}{2}}$ is a cobordism on $N_{\frac{1}{2}}$ with $2s$ j-handles attached to $N_{\frac{1}{2}}$,

i. e., $V_{\frac{1}{2}}$ is diffeomorphic to a cobordism gotten from $N_{\frac{1}{4}} \times I$ by attaching $2s$ j-handles to $N_{\frac{1}{4}} \times 1$. In fact, they are just $a_1, \cdots, a_s$, $c_1, \cdots, c_s$. So, $H_j(V_{\frac{1}{2}}, N_{\frac{1}{2}}) = F_1 \oplus F_2$ where F_1 is a free R-module generated by $a_1, \cdots, a_s$ and F_2 is the free R-module generated by $c_1, \cdots, c_s$ (when we join paths in $V_{\frac{1}{2}}$ from $y_0 \in N_{\frac{1}{2}}$ to $a_1, \cdots, a_s$, $c_1, \cdots, c_s$. Let us examine the boundary homomorphism

$$\partial\colon H_j(V_{\frac{1}{2}}, N_{\frac{1}{2}}) = F_1 \oplus F_2 \to H_{j-1}(N_{\frac{1}{2}}) \tag{66}$$

of the exact sequence of the pair $(V_{\frac{1}{2}}, N_{\frac{1}{2}})$. We find that $\partial \mid F_2$ is an isomorphism onto the R-module $H_{j-1}(N_{\frac{1}{2}})$ generated by $b_1, \cdots, b_s$ and $\partial \mid F_2$ is maped into the direct summand of $H_{j-1}(N_{\frac{1}{2}})$ which is identified with $H_{j-1}(N)$. In fact, under this identification, $\mathrm{Im}(\partial \mid F_1)$ is identified with the direct summand isomorphic to Q of $H_{j-1}(N)$ given in the hypothesis. Let us decompose

(67) $F_1 = P_1 \oplus Q_1$ with $\partial \mid P_1 = 0$ and $\partial \mid Q_1$ an isomorphism onto $Q \subset H_{j-1}(N)$,

$$F_2 = P_2 \oplus Q_2 \tag{68}$$

where P_2, Q_2 are isomorphic to P, Q respectively. We may also decompoce the direct summand of $H_{j-1}(N_{\frac{1}{2}})$ generated oy $b_1, \cdots, b_s$ by $P' \oplus Q'$ with $\partial\colon P_2 \to P'$, $\partial\colon Q_2 \to Q'$ isomorphisms. Let us rearrange the direct decompositions of (67), (68) such that

$$F_1 \oplus F_2 = (P_1 \oplus Q_2) \oplus (P_2 \oplus Q_1). \tag{69}$$

Since $(P_2 \oplus Q_1)$ is a free R-module of rank s, we can realize a basis of it by s j-handles on $N_{\frac{1}{2}}$. Let $V_{\frac{3}{4}}$ be the elementary cobordism on $N_{\frac{1}{2}}$ by attaching these s j-handles and let $N_{\frac{3}{4}}$ be the other end of the cobordism of $V_{\frac{3}{4}}$. Set $U_1 = V_{\frac{1}{2}} \cup V_{\frac{3}{4}}$ and $W_1' = \overline{W - U}$. It is easy to see that W_1' contains an elementary cobordism W_1 on $N_{\frac{3}{4}}$ such that the inclusion $W_1 \subset W_1'$ is a homotopy equivalence and W_1 verifies all the hypotheses W. Applying the same procedure to W_1, we have U_2 and W_2, etc. Immediately, we construct U_i, W_i $(i \geqq 1)$. Set $U = \bigcup_{\infty} U_i$. It is easy to check from the construction that $H_j(U, N) \cong Q$ is the only non-vanishing homology group of the pair (U, N) This completes the proof of the lemma.

Let us now prove the sufficiency of Theorem 6.1. We divide the proof into two cases according to $m = 2k$ or $m = 2k + 1$.

Case I. $m = 2k \geqq 6$. It follows from Theorem 5.5 that there is an almost splitting (N_0, g_0) of f such that

(A) $K_i(B_{N_0}, N_0) = 0$ for $i \neq k$,

(B) $K_k(B_{N_0}, N_0)$ is a finitely generated projective R-module,

(C) t_* is the zero endomorphism of $K_k(B_{N_0}, N_0)$.

Let $a_1^0, \cdots, a_s^0$ be a set of generators of $K_k(B_{N_0}, N_0)$. By the argument of Lemma 5.3, there are maps

$$(71) \qquad a_i^0 : (\boldsymbol{D}^k, \partial \boldsymbol{D}^k) \to (B_{N_0} - t^{-1}B_{N_0}, N_0) \quad i = 1, \cdots, s$$

representing $a_i^0, \cdots 5, a_s^0$. We can make them as disjoint embeddings. Thickening up these discs, we have an elementary cobordism $(W_0; N_0, N_1)$. We may modify g_0 such that $g_0(W_0) \subset N'$. W_0 satisfies the hypothesis of Lemma 6.2. By the argument of Lemma 6 2, we may decompose $W_0 = W_- \cup W_+$ such that W_+, W_- are elmentary cobordisms on N satisfying the following conditions:

(72)
(A) $W_+ \cap W_- = N$,
(B) $H_i(W_-, N) = 0$ for $i \neq k$ and
$H_i(W_+, N) = 0$ for $i \neq k+1$,
(C) N is an almost splitting.

By Lemma 6.2, we have U_+ in W_+ add U_- in W_- such that

$$K_{k+1}(U_+, N) \cong K_{k+1}(A_N, N) \text{ and } K_k(U_-, N) \cong K_k(B_N, N).$$

Set $U = U_+ \cup U_-$. It is easy to see that (U, g) is a splitting by open strip of f.

Case II. $m = 2k+1 \geqq 7$. Let us follow the argument of Lemma 5.4 to find a cobordism W similar to that of Case I. Let (N_0, g_0) be an almost splitting of f satisfying the conditions:

(B) $K_{k+1}(B_{N_0}, N_0)$ is a finitely generated projective R-module,

(73) (A) $K_i(B_{N_0}, N_0) = 0$ for $i \neq k+1$,

(C) t_* is the zero endomorphism of $K_{k+1}(B_{N_0}, N_0)$.

Let $a_1^0, \cdots, a_s^0$ be a set of generators of $K_{k+1}(B_{N_0}, N_0)$. By the argument of Lemma 5.4, we may assume that they are represented by disjoint embeddings

$$(74) \qquad a_i^0 : (\boldsymbol{D}^{k+1}, \partial \boldsymbol{D}^{k+1}) \to (B_{N_0} - t^{-1}B_{N_0}, N_0).$$

Thickening up thesediscs, wehave an elementary cobordism $(W; N_0, N_1)$. Except some number changing, the same argument for Case I works for Case II. This completes the proof of Theorem 6.1.

THEOREM 6.3. *If (U,g) is a splitting of f by an open homotopy strip, then U is of the proper homotopy type of $N' \times R$, and hence U has two tame ends $\mathcal{E}_+$, $\mathcal{E}_-$ with $\sigma(\mathcal{E}_-) = q_2 0(f)$.*

Proof. It follows from definition that $g \mid U\colon U \to N'$ is a homotopy equivalence. With a little care, we may create a proper homotopy equivalence $g_U\colon U \to N' \times R$ such that $U \xrightarrow{g_U} N' \times R \xrightarrow{\text{proj}} N'$ is homotopic to g. Hence, U has two tame ends $\mathcal{E}_+$, $\mathcal{E}_-$. In order to compute $\sigma(\mathcal{E}_-)$, we follow the sufficiency part of Theorem 6.1. Let $N \subset U$ be the codimension 1 submanifold satisfying the conditions of (56). The exact sequence (61) shows that $H_i(D_N, N) \cong K_i(B_N, N)$. It follows from the definition that $\sigma(\mathcal{E}_-) = 0(f)$.

THEOREM 6.4.[10] *Let $f\colon M \to M'$ be a homeomorphism of closed manifolds with $\pi_1 M = \pi_1 M' = G \times_\alpha T$ where G is a finitely presented group. (We identify $\pi_1 M$ with $\pi_1 M'$ via f_*.) Then $\tau(f)$ has no components in $C(R,\alpha)$ and $C(\mathrm{R},\alpha^{-1})$ of the decomposition* (2).

Proof Since M_1, M_2 are closed manifolds, it follows from [22] that $\tau(f) = (-1)^{m+1}\bar{\tau}(f)$ where $m = \dim M$. By [12], the conjugation '—' interchanges $\tilde{C}(R,\alpha)$ and $\tilde{C}(R,\alpha^{-1})$. It suffices to show that $\tau(f)$ has zero component in $C(R,\alpha)$. Moreover, we may assume that $m = \dim M = \dim M' \geqq 6$ without loss of generality by applying the product formula of Whitehead torsion to $f \times \mathrm{id}\colon M \times P \to M' \times P$ where P is a high dimensional simply-connected manifold with Euler number 1. Since G is finitely presented, we may interpolate a closed submanifold N' of codim 1 in M' such that $\pi_1 N' \to \pi_1 M'$ is a monomorphism onto the subgroup G of $G \times_\alpha T$. Let U' be the tubular neighborhood of N' and $r\colon U' \to N'$ is a deformation retraction. r extends to a homtopy to the identity $f'\colon M' \to M'$ by 'homotopy extension theorem.' Set $U = f^{-1}(U')$ and $g = f'f\colon M \to M'$. Clearly (U,g) is a splitting of ϕ by a homotopy strip. The theorem follows from Theorem 6.1.

COROLLARY 6.5 [13]. *Let M_m $(m \geqq 5)$ be a closed manifold with $\pi_1 M — T_{p^2} \times T^n$ where T_{p^2} is the cyclic group of p^2, and $n \geqq 2$. There are infinitely many h-cobordisms on M which are not homeomorphic to $M \times I$.*

Proof. Let $G = T_{p^2} \times T^{n-1}$ and $\alpha = \mathrm{id}$. We can write $\pi_1 M = G \times T$. Acording to [5], $\tilde{C}(R,\alpha)$ is an infinite group where $R = Z(G)$. Pick $x \in \mathrm{Wh}\,\pi_1 M$ which has non-zero component in one of the $\tilde{C}(R,\alpha)$ of the decomposition (2). Building an h-cobordism W_x on M with $\tau(W_x, M) = x$.

[10] After the work of [19], [20], the result is obsolete, of course! We include it here as an illustration.

Let M' be the other end of W_x. If there is a homeomorphism $F\colon W_x \to M \times I$, there is a homseomorphism $f\colon M' \to M$ such that the induced isomorphism f_* is the same as that induced by the composite map

$$M' \xrightarrow{\text{incl}} W_x \xrightarrow{\text{def. retract}} M. \tag{75}$$

By [12], $\tau(f)$ has $\pm z$ in the second and the third component of (2) where z denotes the component of x in the second component. Hence, $q_1 0(f) \neq 0$ and this contradicts Theorem 6.4.

7. Novikov's problem on homotopy invariance of L-genus.[11] In this section, we shall answer a problem raised by Novikov in [25], [28], [15] about L-genus. Let M^m be an orientable closed manifold. Suppose that $z \in H_{4k}(M^m; \boldsymbol{Z})$ be an element whose Poincaré dual is

$$Dz = y_1, \cdots, y_n \quad (m = n + 4k) \tag{76}$$

where $y_1, \cdots, y_n \in H^1(M^n; \boldsymbol{Z})$ form a linearly independent set. Novikov asked in [25], [26]:

$$\text{`Is } (L_k(M^m), z) \text{ a homotopy invariant?'} \tag{77}$$

($L_k(M^m) \in H^{4k}(M^m; \boldsymbol{Q})$ is the k-th L-genus of (M^m.)

THEOREM 7 ([25], [28], [15]). *$L_k(M^m), z)$ is a homotopy invariant where $L_k(M^m)$ and z are given as above.*

Proof. By multiplying a fixed simply-connected closed manifold P with non-zero signature, we may assume that $k \geqq 2$ without loss of generality. Let M^m be given as above and let $f\colon M^m \to M_1{}^m$ be a homotopy equivalence. We have the following commutative diagram

$$\begin{array}{ccc}
\pi_1 M & \xrightarrow[\cong]{f_*} & \pi_1 M_1 \\
\downarrow H & & \downarrow H \\
H_1(M; \boldsymbol{Z}) & \xrightarrow[\cong]{f_*} & H_1(M_1; \boldsymbol{Z}) \\
\cup & & \cup \\
A & \xrightarrow[\cong]{} & A
\end{array} \tag{78}$$

where A, A_1 are the torsion subgroups of $H_1(M; \boldsymbol{Z})$ and $H_1(M_1; \boldsymbol{Z})$ respectively. Since $m \geqq 8$, the kernel of $\pi_1 M_1 \to H_1(M_1; \boldsymbol{Z})/A_1$ are carried by a

[11] Lusztig has proved the result by a different method.

set of finitely many embedded circles. Making f transversely regular with respect to these embedded circles and performing surgery on the inverse images, we may assume that $\pi_1 M$ is also represented by embedded circles, f is a diffeomorphism when we restrict it to the tubular neighborhood of the circles, and the inverse image of the circles in M_1 are those corresponding ones in M. Performing simultaneous surgeries to these circles, we have a new homotopy equivalence

$$(79) \qquad g: L^m \to L_1{}^m$$

where L, L_1 are the manifolds gotten from M, M_1 by these surgeries. It is easy to see that g is a homotopy equivalence and $\pi_1 L^m = \pi_1 L_1{}^m$ is a free abelian group. By a 'diagram chasing,' it is easy to see that it is sufficient to prove the theorem for g. Following [28], it suffices to assume that $f_* z$ is represented by the intersection of n connected codim 1 submanifold $N_1{}^{(1)}, \cdots, N_1{}^{(n)}$ of L such that the intersection of any number of them, $N_1{}^{(j_1)} \cdots N_1{}^{(j_l)}$ $(j_i \neq j_{i'}$
of L such that the intersection of any number of them, $N_1{}^{(j_1)} \cdots N_1{}^{(j_l)}$ $(j_i \neq j_{i'}$
for $i \neq i')$ is connected. Then

$$(80) \qquad (L_k(L_1), g_* z) = \text{the signature of the manifold } \bigcap_{i=1}^{n} N_1{}^{(i)}.$$

If we can find a corresponding set of codim 1 submanifold $N_1{}^{(1)}, \cdots, N_1{}^{(n)}$ of L satisfying the conditions of $N_1{}^{(1)}, \cdots, N_1{}^{(n)}$ and a map $g': L \to L'$, homotopic to g such that $g \mid \bigcap_1^n N^{(i)}: \bigcap_1^n N^{(i)} \to \bigcap_1^n N_1{}^{(i)}$ is a homotopy equivalence, then the theorem follows from Hirzebruch's theorem. This can be achieved by a successive application of Theorem 2.1 and making use of the fact that $\mathbf{Z}[T^l]$ is a regular ring and hence $C(\mathbf{Z}[T^l], \mathrm{id}) = 0$. This completes the proof of the theorem.

Remark. Novikov proved it for $n = 1$ in [25] and Rohlin proved for $n = 2$ in [28]. We can generalize the above result to nilpotent π_1 by the result of [12].

Appendix. A weakly splitting theorem for 5-manifolds.[12] In this Appendix, we shall prove a weaker version of Theorem 2.1 for 5-manifolds. Let us now set up the problem and state the result. Let $f: M^5 \to M'^5$ be a homotopy equivalence of closed 5-manifolds with the fundamental group $\pi_1 = G \times_\alpha T$, a semi-direct product of a finitely presented normal subgroup

[12] S. Cappell and J. Shaneson have generalized this theorem by a different approach (cf. Theorem 5.1 of [37]).

G and the infinitely cyclic group T. (As always, we identify $\pi_1 M^5$ with $\pi_1 M'^5$ via f_*.) Let $N'^5 \subset M'^5$ be a codimension 1 submanifold such that the inclusion $G \subset G \times_\alpha T$. Cutting M' along N', we have a manifold $M'_{N'}$. Suppose that we have an embedding $S^2 \subset \operatorname{Int} M'_{N'}$ representing an element of $\pi_2(M'_{N'})$. Joining a tube from the normal sphere bundle of S^2 in $M'-N'$ to N', we have a new codimension 1 submanifold of M'. Repeating this procedure a finite number of times,we have a codimension 1 submanifold L'^4 of M'^5 and the inclusion $L' \subset M'$ again induces the π_1 inclusion $G \subset G \times_\alpha T$. We say that f is '*weakly splittable*' along N'^4 if we can find such a submanifold $L'^4 \subset M'^5$ and a codimension 1 submanifold L^4 of M^5 together with a map of pairs $g\colon (M^5, L^4) \to M'^5, L'^4)$ such that $g \mid M^5$ is homtopic to the original f.

THEOREM A. *If* $0(f)=0$, *then* f *is 'weakly splittable' along* N'^4.

The reason that one should expect Theorem A to be true is because of the feling that 4-manifolds behave better when we stabilize them by performing connected sums with many copies of $S^2 \times S^2$. In the present paper, we have no control of the number of copies of $S^2 \times S^2$. However, it seems to me that it is possible to estimate the numbsr of copies if we are careful enough in some special cases.

Proof of Theorem A. Following § 3, we may assume that $f^{-1}(N') = N$ is a codim 1 submanifold of M^5 such that f_N induces a π_1-isomorphism. Recall that we have

(81) $$0(f) = [K_3(B_N, N), t_*] \in \tilde{C}(R, \alpha)$$

and it follows from the hypothesis that there is a $_1$ltration

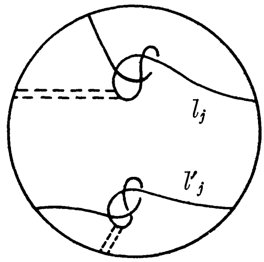
(92) $$0 = F_0 \subset \cdots \subset F_i \subset \cdots \subset F_n = K_3(B_N, N)$$

of submodules of $K_3(B_N, N)$ such that $t_* F_{i+1} \subset F_i$ and $F_{i+1}/F \cong Z(G)$ for $i=1,\cdots,n-1$. Let $[a]$ be a generator of F_1. Since $t_* F_1 = 0$, we have an immersion

(83) $$a\colon (\boldsymbol{D}^3, \boldsymbol{S}^2) \to (M^5, N^4)$$

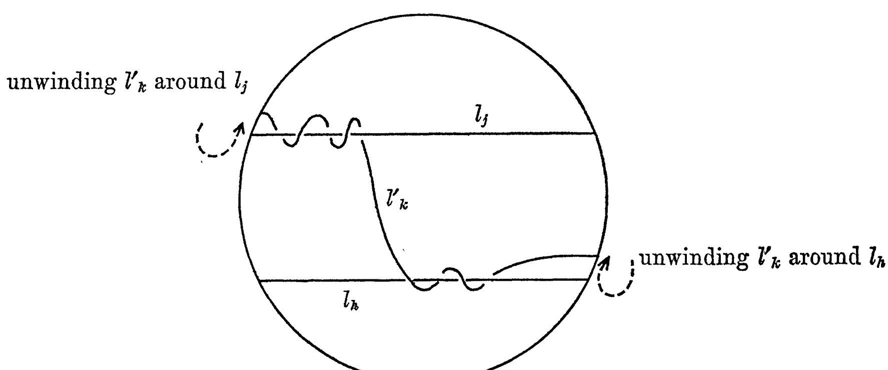
such that $a(\operatorname{Int}\boldsymbol{D}^3) \subset M^5 - N^4$ and it represents the homology class $[a]$ when we lift it up to (B_N, N). Without loss of generality, we may assume that the preimages of the self-intersection points of a consists of pairs of arcs $l_1, l_1'; \cdots, l_j, l_j'; \cdots, l_m, l_m'$ with $a(l_j) = a(l_j')$ $(j=1,\cdots,m)$. Of course an arc l_j (or l_j') may be knotted with itself. For example, l_j is one of such arcs.)See Fig. 11)

FIGURE 11



We can join two arcs from the knotted arc l_j to S^2 to break it into two arcs $\bar{l}_j$, $\bar{\bar{l}}_j$ such that $\bar{l}_j$, $\bar{\bar{l}}_j$ have simpler knots than l_j and the number of knots is not increased. We join two arcs from the corresponding points of l_j' to S^2 such that l_j' is also broken into two arcs $\bar{l}_j'$, $\bar{l}_j''$. We can choose the arcs joining from l_j' to S^2 so carefully that no new knot is introduced and the arcs $l_1, l_1'; l_2, l_2'; \cdots; \bar{l}_j, \bar{l}_j'; \bar{\bar{l}}_j, \bar{\bar{l}}_j''; \cdots; l_m, l_m'$ are disjoint from each other. Now we can perform a regular homotopy $a_t (0 \leqq t \leqq 1)$ on the immersion a such that $a_0 = a$ and a_1 has self-intersections $l_1, l_1'; \cdots; \bar{l}_j, \bar{l}_j'; \bar{\bar{l}}_j, \bar{\bar{l}}_j'; \cdots; l_m, l_m'$ with $a_1(l_k) = a_1(l_k')$ for $k \neq j$, $a_1(\bar{l}_j) = a_1(\bar{l}_j')$, $a_1(\bar{\bar{l}}_j) = a_1(\bar{\bar{l}}_j)$ and $a_t(\operatorname{Int} D^3) \subset M^5 - N^4$ for $0 \leqq t \leqq 1$. Therefore, we may assume from the very beginning that each l_j (or l_j') for $j = 1, \cdots, m$ is never knotted with itself without loss of generality. It may see that l_j winds around some other arcs, but wecan easily unwind it by a regular homotopy. (For example, see Fig. 12).

FIGURE 12



Therefore we may assume that $l_1, l_1'; \cdots; l_m, l_m'$ are parallel arcs and l_j, l_j' $(j = 1, \cdots, m)$ bound disjoint $\boldsymbol{D}_j^2$, $\boldsymbol{D}_j'^2 (j = \cdots, m)$ in $\boldsymbol{D}^3$ such that $\partial \boldsymbol{D}_j^2$, $\partial \boldsymbol{D}_j'^2$ $(j = 1, \cdots, m)$ are contained in $\boldsymbol{S}^2$ without loss of generality.

$a(\boldsymbol{D}_j^2 \cup \boldsymbol{D}_j'^2)$ $(j=1,\cdots,m)$ represents an element of $\pi_2(M_N, N)$ where M_N is gotten from M by cutting along N. Since $\partial\colon \pi_2(M_N, N) \to \pi_1(N)$ is trivial; $a(\boldsymbol{D}_j^2 \cup \boldsymbol{D}_j'^2)$ represents an element $[b_j] \in \pi_2(M - N)$ $(j=1,\cdots,m)$. Let $f_{M_N}\colon M \to M'_{N'}$ be the induced map and let $b_j'\colon S^2 \to M' - N'$ $(j=1,\cdots,m)$ be an embedding representing the element $(f_{M_N})_*[b_j] \in \pi_2(M'_{N'})$. Joining a tube from the normal sphere bundles of $b_j'(S^2)$ to N'^4, we have a new codimension 1 submanifold L_1' of M'^5 and the inclusion induces the π_1 inclusion $G \subset G \times_\alpha T$. When we thicken the 2-handles $a(\boldsymbol{D}_j^2 \cup \boldsymbol{D}_j'^2)$ $(j=1,\cdots,m)$ on N^4, we have a new codimension 1 submanifold L_1^4 of M^5. After we examine the evaluation of

(84) $$w_2(M^5) = f^* w_2(M^5) \in H^2(M^5; Z_2)$$

on the 2-cycles represented by $[b_j]$ $(j=1,\cdots,m)$, it is easy to construct a map homotopic to f,

(85) $$g_1\colon M^5 \to M'^5$$

with $g_1^{-1}(L_1'^4) = L_1^4$. We can lift L_1^4, $L_1'^4$ onto Y_M and $Y_{M'}$ respectively and let us denote the corresponding triads gotten from L_1, L_1' by $(Y_M; A_1, B_1)$, $(Y_{M'}; A_1', B_1')$ respectively. It is not difficult to see that we have an isomorphism

(86) $$K_3(B_N, N) \to K_3(B_i, L_1).$$

If we denote the image of $[a]$ in $K_3(B_1, L_1; ZG)$ by $[a_1]$, then $[a_1]$ is represented geometrically as follows. When we thicken the 2-handles $a(\boldsymbol{D}_j^2 \cup \boldsymbol{D}_j'^2)$ $(j=1,\cdots,m)$ on N in order to obtain L_1, we may do it so carefully that there are disjoint regular neighborhoods $\boldsymbol{D}_j^3$, $\boldsymbol{D}_j'^3$ $(j=1,\cdots,m)$ of $\boldsymbol{D}_j^2$, $\boldsymbol{D}_j'^2$ respectively in $\boldsymbol{D}^3$ that if we put $\boldsymbol{D}_1^3 = \boldsymbol{D}^3 - \bigcup_{j=1}^{m} (\boldsymbol{D}_j^3 \cup \boldsymbol{D}_j'^3)$, then the immersion a induces an embedding

(87) $$a_1\colon (\boldsymbol{D}_1^3, \partial \boldsymbol{D}_1^3) \to (M_{L_1}, L_1)$$

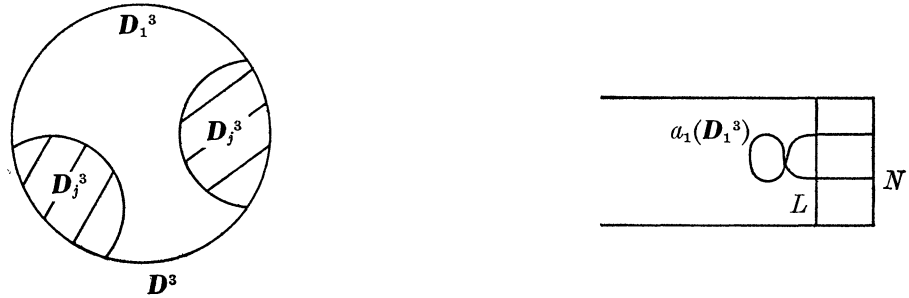
representing $[a_1] \in K_3(A_1, L_1; ZG)$ where M_{L_1} denotes the manifold with boundaries from M by cutting along L_1.

Then, we can hicken up this 3-handles on L_1 and exchange it to A_1. Now we have a new codimension 1 submanifold and a new map $M^5 \to M'^5$ which we shall again denote by L_1 and g_1 such that

(88) $$K_3(B_1, L_1) \cong K_3(B_N, N)/F_1.$$

But unfortunately, we may mess up the π_1 isomorphism. However, we still

FIGURE 13



have the epimorphism $(g_1)_*\colon \pi_1 L_1 \to \pi_1 L_1'$. Let $\{x_1, \cdots, x_s\}$ be a finite set of elements such that their normal closure in $\pi_1 L_1$ is the kernel of $(g_1)_*$. Following § 3, wemay assume that x_i $(i = 1, \cdots, s)$ are represented by disjont embeddings

$$x_i\colon S^1 \to L_1^4 \quad (i = 1, \cdots, s) \tag{89}$$

such that $x_1(S^1)$ bounds a 2-disc D^2 in A_1 or B_1, say B_1, and we can choose the embedding of D^1 in $B_1 - t^{-1}B_1$. We can choose the embedding so carefully that if we thicken it up and exchange it to A_1 and let us denote the new codimension 1 submanifold by L_2 the induced triad by $(Y_M; A_2, B_2)$ with $t(A_2) \subset A_2$, etc., then we shall kill x_1 and $x_2(S^1)$ will bound a 2-disc in $A_2 - tA_2$, etc. After we carry on this procedure s times, we have a new codimension 1 submanifold of M^5 and it will be again denoted by L_2^5. Of course, if we modify the map g_1 in the same way as we did in [35], we may just complete a vicious circle or worse. Now, let us make the following observation. Since the composite map $S_1 \xrightarrow{x_1} L_1 \to L_1'$ is trivial, the composite map

$$(D^2, S^1) \xrightarrow{y_1} (B_1 - t^{-1}B_1, L_1) \to \text{A}B' - t^{-1}B', L_1') \tag{90}$$

with $y_1 \mid S^1 = x_1$ actually represents an element in $\pi_2(M'_{L_1'})$. In fact, we can do anything so carefully that it represents the trivial element. Choose a trivially embedded $S^2 \times S^2$ in $\operatorname{Int} M'_{L_1'}$ and join a tube from $S \times S^2$ to L_1' from the B_1' side. Denote the new codimensional submanifold of M'^5 by $L_2'^4$. Clearly, the inclusion $L_2' \subset M'$ again induces the π_1 inclusion.

REFERENCES.

[1] D. Barden, "Structure of manifolds," Ph. D. thesis, Cambridge University, 1963.

[2] H. Bass, K-theory and stable algebra," *Publ. de L'Inst. des Hautes Études Sci.*, No. 22 (1964), pp. 5-60.

[3] ———, *Algebraic K-Theory*, Benjamin, New York, 1968.

[4] H. Bass, A. Heller and R. Swan, "The Whitehead group of a polynomial extension." *Publ. de L'Inst. des Hautes Études Sci.*, No. 22 (1964), pp. 61-79.

[5] H. Bass and M. P. Murthy, "Grothendieck groups and Picard groups of abelian group rings," *Annals of Mathsmatics*, vol. 86 (1968), pp. 16-73.

[6] W. Browder, "Homotopy type of differentiable manifolds," *Colloq. on Alg. Top. Notes*, Aarhus (1962), pp. 42-46.

[7] ———, "Manifolds with $\pi_1 = Z$," *Bull. AMS*, vol. 72 (1966), pp. 238-244.

[8] ———, "Structure on $M \times R$," *Proc. Camb. Phil. Soc.*, vol. 61 (1965), pp. 337-345.

[9] W. Browder and J. Levine, "Fibering manifolds over circle," *Comm. Math. Helv.*, vol. 40 (1966), pp. 153-160.

[10] S. Cappell, "A splitting theorem for manifolds and surgery groups," *Bull. AMS*, vol. 77 (1971), pp. 281-286.

[11] F. T. Farrell, "The obstruction to fibring a manifold over a circle," thesis, Yale University, 1967; also *Bull. AMS*, vol. 73 (1967), pp. 737-740.

[12] F. T. Farrell and W. C. Hsiang, "A formula for $K_1R_2[\tau]$," *Proc. Symp. in Pure Math.*, vol. 17 (Category Algebra), Amer. Math. Soc., 1970, pp. 192-218.

[13] ———, "*H*-cobordant manifolds are not necessarily homeomorphic," *Bull. AMS*, vol. 73 (1967), pp. 741-744.

[14] ———, "A geometric intepretation of the Künneth formula for algebraic K-theory," *Bull. AMS*, vol. 74 (1968), pp. 548-553.

[15] W. C. Hsiang, "A splitting theorem and the Künneth formula in algebraic K-theory," *Algebraic K-theory and its Geometric Applications*, Lecture notes 108, Springer-Verlag, 17969, pp. 72-77.

[16] W. C. Hsiang and J. L. Shaneson, "Fake tori," *Proc. of* 1969 *Georgia Conf. in Topology*; also *Proc. Nat. Acad. Sci.*, vol. 62 (1969), pp. 687-691.

[17] W. C. Hsiang and C. T. C. Wall, "On homotopy tori II," *Bull. London Math. Soc.*, vol. 1 (1969), pp. 341-342.

[18] L. Jones, "Patch space," (to appear).

[19] R. C. Kirby and L. C. Siebenmann, "On the triangulation of manifolds and the Hauptvermutung," *Bull. AMS*, vol. 75 (1969), pp. 742-749.

[20] R. C. Kirby, "Lectures on triangulation of manifolds," *Notes* U. C. L. A. ,1969.

[21] R. Lashof and M. Rothenberg, "On the Hauptvermutung, triangulation of manifolds, and h-cobordism," *Bull. AMS*, vol. (1969).

[22] J. Milnor, "Whitehead torsion," *Bull. AMS*, vol. 72 (1966), pp. 358-426.

[23] S. P. Novikov, "Homotopy equivalent smooth manifolds I," *Izv. Akad. Nauk SSSR Ser. Mat.*, vol. 28 (1964), pp. 365-474; English transl. Amer. Math. Soc. transl. (2), vol. 48 (1965), pp. 271-396.

[24] ———, "Rational Pontrjagin classes, homeomorphism and homotopy type of closed manifolds," *Izv. Akad. Nauk SSSR Ser. Mat.*, vol. 29 (1965), pp. 1373-1388; English transl. Amer. Math. Soc. transl. (2), vol. 66 (1968), pp. 214-230.

[25] ———, "Manifolds with free Abelian fundamental group and their applications (Pontroagin classes,smoothnesses, multidimensional knots," *Izv. Akad.*

SSSR Ser. Mat., vol. 30 (1966), pp. 207-246; English transl. Amer. Math. Soc. transl. (2), vol. 71 (1968).

[26] ———, "Pontrjagin classes, the fundamental group and some problems of stable algebra," International Congress of Mathematicians in Moscow, 1966. English transl. Amer. MathH. Soc. transl. (2), vol. 70 (1968).

[27] G. de Rham, S. Maumary and M. Kervaire, "Torsion et type simple d'homotopotie," *Lecture Notes in Mathematics*, vol. 47 (1967), Springer-Verlag.

[28] V. A. Rohlin, "A Pontrjagin-Hirzebruch class of codimensionality 2," *Izv. Akad. Nauk SSSR Ser. Mat.*, vol. 30 (1966), pp. 705-718; English transl. Amer. Math. Soc. transl. (2), vol. 71 (1968).

[29] L. C. Siebenmann, doctoral dissertation, Princeton University, 1965.

[30] ———, "On the homotopy type of compact topological manifolds, *Bull. AMS.*

[31] J. L. Shaneson, "Wall's surgery group for $Z \times G$," *Annals of Mathematics*, vol. 90 (1969), pp. 269-334.

[32] J. Stallings, "On fibring certain 3-manifolds," *Topologly for 3-manifolds and related topics*, Prentice-Hall, 1962, pp. 95-99.

[33] C. T. C. Wall, "Finiteness condition for *CW*-complexes I," *Annals of Mathematics*, vol. 81 (1965), pp. 56-69; II, *Proc. Roy. Soc.*, A. 295 (1966), pp. 129-139.

[34] ———, "Poincaré complexes I," *Annals of Mathematics.*

[35] ———, "Surgery of non-simply connected manifolds," *Annals of Mathematics*, vol. 84 (1966), pp. 217-276.

[36] ———, *Surgery of compact manifolds*, Academic Press, 1970.