TOPOLOGICAL CHARACTERIZATION OF FLAT AND ALMOST FLAT RIEMANNIAN MANIFOLDS M^n $(n \neq 3, 4)$

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0. Introduction. A connected manifold M^n is aspherical provided $\pi_i M^n = 0$ for all $i > 1$. Eilenberg obstruction theory shows that an aspherical manifold is determined up to homotopy equivalence by its fundamental group. This fact is one reason for the following conjecture.

CONJECTURE. *A closed (i.e., compact without boundary) aspherical manifold is determined up to homeomorphism by its fundamental group.*

A. Borel mentioned to R. Szczarba in 1966 that this conjecture was plausible in light of Bieberbach's classical theorems on flat manifolds and Mostow's work on solvmanifolds; Szczarba related this conjecture to the present authors. In 1968, the conjecture was verified for aspherical manifolds with abelian fundamental groups of rank greater than 4 and shortly afterwards, for aspherical manifolds with poly-**Z** fundamental groups of rank (Hirsh number) greater than 4. (A poly-**Z** group is a group possessing a normal series with ∞-cyclic factor groups). These results are due to Wall [27], [28] and Hsiang-Shaneson [20].

In this paper, we verify the conjecture for aspherical manifolds (of dimensions greater than 4) whose fundamental groups contain nilpotent subgroups of finite index. (See Theorem 5.1). In particular, the conjecture is true for (high dimensional) aspherical manifolds whose fundamental groups are virtually abelian; i.e., contain an abelian subgroup with finite index. Since this is the class of fundamental groups of flat Riemannian manifolds, we see that any manifold the homotopy type of a closed flat Riemannian manifold M^n of dimension $n \neq 3, 4$ is homeomorphic to M^n. (See Theorem 6.1). This leads, in Theorem 6.2, to a topological character-

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ization of (high dimensional) flat Riemannian manifolds as aspherical manifolds with virtually abelian fundamental groups.

Recently, M. Gromov introduced, in [19], the definition of an almost flat Riemannian manifold. Roughly speaking, a manifold is almost flat if it supports a sequence of Riemannian metrics whose sectional curvatures converge to zero but whose diameters stay bounded away from ∞. Gromov showed that such manifolds are aspherical with virtually nilpotent fundamental groups. His result together with the main result of this paper show that these two conditions topologically characterize almost flat Riemannian manifolds (of dimensions different from 3 and 4). (See Theorem 6.4).

Theorem 6.1, which was announced in [15], extends our earlier results proven in [13] and [14]. Although the present paper is more or less independent of [14], we suggest that the reader looks at [14] before studying this paper in detail. The outlines of the two papers are similar; but the difficulties (U Nil problems, etc.) which prevented (when [14] was written) the proof of Theorem 6.1 (and the more general Theorem 5.1) are here overcome in principally two ways. First, we use a more detailed algebraic classification of crystallographic groups (Theorem 1.1) which includes those with non-trivial torsion; second, we systematically use a metric vanishing criterion (Theorem 3.1) which is based on recent works of F. Quinn [25] extending results of T. Chapman and S. Ferry. See the announcement [15] for a more detailed description of the differences between [14] and the present paper. In proving Theorem 5.1, it is also necessary to make use of results of another recent paper of ours [16] where we showed that the Whitehead group of a torsion-free virtually poly-**Z** group vanishes.

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1. Group Theoretic Preliminaries. In this section, we will study crystallographic groups; i.e., discrete cocompact subgroups of the group of rigid motions, denoted by $E(n)$, of n-dimensional Euclidean space. (The group of affine motions of Euclidean space is denoted by $A(n)$.) A torsion-free crystallographic group is called a Bieberbach group. The main object of this section is to prove a more refined version of Theorem

1.1 of [14]. First, we fix our notation and recall some basic facts about crystallographic groups. (Good references are [5], [10] and [29]).

Crystallographic groups are algebraically characterized as those finitely generated groups Γ which contain a maximum (under inclusion) subgroup A among its abelian subgroups of finite index and A is torsion-free. Note A is a characteristic (hence normal) subgroup of Γ; we call it the translation subgroup of Γ. The factor group $G = \Gamma/A$ acts effectively on A; G is called the holonomy group of Γ and A considered as a G-module is called the holonomy representation of Γ. We define the rank of Γ to be the rank of A. For any positive integer s, define $\Gamma_s = \Gamma/sA$ and $A_s = A/sA$ where sA is the subgroup of A consisting of all elements divisible by s; Γ_s is an extension of A_s by G and in fact a semidirect product if $(s, |G|) = 1$.

Throughout this paper, we denote the infinite cyclic group by T and the finite cyclic group of order n by T_n. We reserve the notation $\mathbf{Z}$ and $\mathbf{Z}_n$ for the ring of integers and the ring of integers modulo n, respectively. (However in Section 4, we do use $\mathbf{Z}^2 = \mathbf{Z} \oplus \mathbf{Z}$ to denote the additive subgroup of integral lattice points in $\mathbf{R}^2$.) The expression $G = N \rtimes S$ means a group G is the semi-direct product of a subgroup S and a normal subgroup N.

Recall a hyperelementary group is an extension of a cyclic group of order n by a p-group where $(n, p) = 1$. We now state the main result of this section.

THEOREM 1.1. *Let Γ be a crystallographic group of rank n and holonomy group G, then either*

(i) $\Gamma = \pi \rtimes T$ *where π is a crystallographic subgroup of rank $n - 1$;*

(ii) *there is an epimorphism from Γ to a non-trivial crystallographic group $\hat{\Gamma}$ with holonomy group $\hat{G}$ and an infinite sequence of positive integers $s \equiv 1 \bmod |\hat{G}|$ such that any hyperelementary subgroup of $\hat{\Gamma}_s$ which projects onto $\hat{G}$ (via the canonical map) projects isomorphically onto $\hat{G}$; or*

(iii) *G is an elementary abelian 2-group and either*

(a) $\Gamma = A \rtimes T_2$ *and $T_2 = G$ acts on A via multiplication by -1, or*

(b) Γ *maps epimorphically onto a crystallographic group $\hat{\Gamma}$ with holomony group $T_2 \oplus T_2$ and translation subgroup $T \oplus T$ such that the image of the holonomy representation in $GL_2(\mathbf{Z})$ is either*

$$\left\{\begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix} \middle| i, j = \pm 1\right\} \quad or$$

$$\left\{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \middle| i = \pm 1\right\}.$$

Proof. First, observe that if Γ admits a non-trivial homomorphism $\phi:\Gamma \to T$, then Γ satisfies (i). This is because $\Gamma = \ker \phi \rtimes T$ and $\ker \phi$ is a crystallographic group since it's a normal subgroup of one (cf. [10, Theorem 17]).

We now assume that Γ satisfies neither (i) nor (ii) and proceed to show this forces Γ to satisfy (iii). By Theorem 3.1 of [16] (which is an almost verbatim extension of Theorem 1.1 of [14] from Bieberbach groups to crystallographic groups) and the above observation, $\Gamma = B *_H D$ where $[B:H] = [D:H] = 2$. Hence Γ admits an epimorphism to $T_2 * T_2$. Therefore by [9, Theorem 5], A contains an infinite cyclic subgroup S such that S is normal in Γ and A/S is torsion-free. Consider Γ/S; it may not be crystallographic but it is an extension of the torsion-free abelian A/S by G. By modifying the Auslander-Kuranishi argument [1], one can construct a crystallographic quotient Γ_1 of Γ/S such that rank Γ_1 = rank A/S = rank $\Gamma - 1$. We briefly sketch the argument. Using group cohomology, an isomorphism f_1 of Γ/S to a subgroup of finite index in $A/S \rtimes G$ can be constructed. Since G is finite, one can construct (by an averaging process) a homomorphism f_2 of $A/S \rtimes G$ into $E(n-1)$, where $n - 1$ = rank A/S, such that $f_2|(A/S)$ is monic. Then the composite $f_2 f_1$ maps Γ/S epimorphically onto the desired crystallographic group Γ_1.

We now inductively continue this process constructing a sequence of crystallographic groups Γ_i ($i = 0, 1, 2, \ldots, n$) and epimorphisms $\phi_i:\Gamma_{i-1} \to \Gamma_i$ such that $\Gamma_o = \Gamma$, $|\Gamma_n| = 1$ and rank $\Gamma_i = n - i$. Let $\psi_i:\Gamma \to \Gamma_i$ be the composite $\phi_i\phi_{i-1} \cdots \phi_1$ and $K_i = \ker \psi_i$; then

$$K_1 \subset K_2 \subset \cdots \subset K_n = \Gamma, \tag{1.1}$$

each K_i is normal in Γ and rank $K_i = i$. Put $A_i = K_i \cap A$; then

$$A_1 \subset A_2 \subset \cdots \subset A_n = A, \tag{1.2}$$

each A_i is normal in Γ and rank $A_i = i$. Define groups $\hat{A}_i$ ($i = 1, 2, \ldots, n$) by

(1.3) $$\hat{A}_i = \{x \in A \mid sx \in A_i \text{ for some } s > 0\};$$

then

(1.4) $$\hat{A}_1 \subset \hat{A}_2 \subset \cdots \subset \hat{A}_n = A,$$

each $\hat{A}_i$ is normal in Γ, $\hat{A}_i/\hat{A}_{i+1} \simeq T\ (i = 2, 3, \ldots, n)$ and $\hat{A}_1 \simeq T$. Pick a basis $e_1, e_2, \ldots, e_n$ for A such that for each $i = 1, 2, \ldots, n$ $\{e_1, e_2, \ldots, e_i\}$ spans $\hat{A}_i$. For each $g \in G$, the automorphism of A determined by the holonomy representation is represented in terms of this basis by an upper triangular $n \times n$ matrix M_g with integral entries. Each diagonal entry of M_g is either 1 or -1, hence $(M_g)^2 = M_{g^2}$ is upper triangular and all of its diagonal entries are 1. Since $(M_g)^2$ has finite order, $(M_g)^2$ must be the identity matrix. Therefore each element $g \in G$ has order 2 and consequently G is an elementary abelian 2-group.

If $G = T_2$, then the holonomy representation A is a direct sum of T_2-modules of the following 3 types: T^+, T^- and $\mathbf{Z}(T_2)$. As abelian groups T^+ and T^- are T; T_2 acts trivially on T^+ and the non-trivial element in T_2 acts on T^- via multiplication by -1; $\mathbf{Z}(T_2)$ is the integral group ring of T_2 and T_2 acts on it in the ordinary way. Since $(T^+)^{T_2} \neq 0$ and $(\mathbf{Z}(T_2))^{T_2} \neq 0$, each summand of A must be T^-; otherwise, $A^{T_2} \neq 0$ and $\Gamma = \pi \rtimes T$ by [9, Lemma 7] contradicting the assumption that Γ does not satisfy (i). Hence if $G = T_2$, then Γ satisfies case (a) of (iii) since $H^2(T_2, T^-) = 0$.

We now assume $|G| > 2$ and proceed to show case (b) of (iii) is satisfied. Let G_i denote the holonomy group of Γ_i and B_i its translation subgroup. Since rank $\Gamma_{n-1} = 1$ and (i) is not satisfied, $G_{n-1} = T_2$. Let s be the smallest integer such that $G_s = T_2$; then by the last paragraph, $\Gamma_s = B_s \rtimes T_2$ and the non-trivial element of T_2 acts on B_s via multiplication by -1. Since $\phi_s : \Gamma_{s-1} \to \Gamma_s$ maps B_{s-1} into B_s, it induces an epimorphism $\hat{\phi}_s : G_{s-1} \to G_s$. Let $\hat{G}_{s-1} = \ker \hat{\phi}_s$; by our hypotheses, $|\hat{G}_{s-1}| > 1$. Choose a basis $e_1, e_2, \ldots e_j$ for B_{s-1} $(j = n - (s - 1))$ such that e_1 generates $\ker(\phi_s | B_{s-1})$. For each $g \in G_{s-1}$, let M_g be the $j \times j$ matrix representing the action of g on B_{s-1} under the holonomy representation in terms of the basis $\{e_i, e_2, \ldots, e_j\}$. If $g \in \hat{G}_{s-1}$, then

(1.5) $$M_g = \left(\begin{array}{c|c} \pm 1 & * \\ \hline 0 & I \end{array}\right);$$

i.e., M_g is upper triangular with diagonal entries $(M_g)_{ii} = 1$ if $i > 1$. Consequently, $|\hat{G}_{s-1}| = 2$ and hence $G_{s-1} \cong T_2 \oplus T_2$, We can enumerate the elements in G_{s-1} as e, h, k, ℓ where e is the identity element and $h \in \hat{G}_{s-1}$ such that the matrices M_g ($g = e, h, k, \ell$) have the following form

$$M_e = I, \qquad M_k = -I, \qquad M_\ell = -M_g, \tag{1.6}$$

$$M_g = \left(\begin{array}{c|c} -1 & a \\ \hline 0 & I \end{array}\right).$$

Consider B_{s-1} as a $T_2 = \hat{G}_{s-1}$-module; it decomposes as a direct sum of modules T^+, T^- and $\mathbf{Z}(T_2)$. Clearly, exactly one summand is different from T^+ because of (1.6). Hence we can decompose B_{s-1} as $\hat{G}_{s-1}$-modules into

$$B_{s-1} = \hat{B}_{s-1} \oplus D_{s-1} \tag{1.7}$$

where D_{s-1} is isomorphic to either $\mathbf{Z}(T_2)$ or $T^- \oplus T^+$. Because of (1.6), $\hat{B}_{s-1}$ and D_{s-1} are G_{s-1}-submodules of B_{s-1}. Let $\hat{\Gamma} = \Gamma_{s-1}/\hat{B}_{s-1}$, then $\hat{\Gamma}$ is a crystallographic group which is an epimorphic image of Γ via the composite of $\phi_{s-1}:\Gamma \to \Gamma_{s-1}$ and the canonical map $\Gamma_{s-1} \to \hat{\Gamma}$. Clearly, the holonomy group of $\hat{\Gamma}$ is $T_2 \oplus T_2$ and the image of the holonomy representation in $GL_2(\mathbf{Z})$ is one of the two subgroups listed in case (b) of (iii) depending on whether D_{s-1} is isomorphic to $T^- \oplus T^+$ or $\mathbf{Z}(T_2)$. This completes the proof of Theorem 1.1.

We will also need the following result.

LEMMA 1.2. *Let $\phi:\Gamma \to \hat{\Gamma}$ be an epimorphism between crystallographic groups $\Gamma \subseteq E(n)$ and $\hat{\Gamma} \subseteq E(m)$. Then there exists a ϕ-equivariant affine surjection $F:\mathbf{R}^n \to \mathbf{R}^m$.*

Proof. Let A, $\hat{A}$ and G, $\hat{G}$ be the translation subgroups and holonomy groups of Γ, $\hat{\Gamma}$, respectively. Pick an origin for Euclidean space $\mathbf{R}^n$; identifying $\mathbf{R}^n$ with the group of translations of $\mathbf{R}^n$, $E(n) = \mathbf{R}^n \rtimes 0(n)$ where $0(n)$ denotes the orthogonal group. Identify G with its image in $0(n)$, then

$$\Gamma \subseteq \mathbf{R}^n \rtimes G \subseteq E(n). \tag{1.8}$$

Let $K = \ker(\phi | A)$, V be the $\mathbf{R}$-subspace of $\mathbf{R}^n$ spanned by K and $V^\perp$ the orthogonal complement of V. Both V and $V^\perp$ are sub-G-modules; identify $V^\perp$ to $\mathbf{R}^m$ by an isometry, then G acts orthogonally on $\mathbf{R}^m$ but perhaps not faithfully. Let $\bar{G}$ be the image of G in $0(m)$ and $F_1 : \mathbf{R}^n \to \mathbf{R}^m$ be the orthogonal projection where $\ker F_1 = V$. Then F_1 extends canonically to a group homomorphism

$$\mathbf{R}^n \rtimes G \to \mathbf{R}^m \rtimes \bar{G} \subseteq E(m) \tag{1.9}$$

also denoted by F_1. Let $\bar{\Gamma} = F_1(\Gamma)$ and $\psi = F_1 | \Gamma$; it can be shown that ker ϕ = ker ψ and hence ϕ factors as a composite $\eta\psi$ where $\eta : \bar{\Gamma} \to \hat{\Gamma}$ is an isomorphism. Also $\bar{\Gamma}$ is a discrete cocompact subgroup of $E(m)$; i.e., $\bar{\Gamma}$ is a crystallographic group. By Bieberbach's Theorem [cf. 10, Theorem 19], there is a ψ-equivariant, invertible, affine map $F_2 : \mathbf{R}^m \to \mathbf{R}^m$. Then the composite $F_2 F_1$ is the map F posited in Lemma 1.2.

Let Γ be a finitely generated, torsion-free, virtually nilpotent group. We recall from [16] the notion of a *fibering apparatus* $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ *for* Γ. *This consists of a crystallographic group* $\hat{\Gamma} \subseteq E(m)$ *where* $m > 0$, *a group epimorphism* $\phi : \Gamma \to \hat{\Gamma}$, *a properly discontinuous* (*hence free*) *action of* Γ *on* $\mathbf{R}^n$ *with compact orbit space, and a* ϕ*-equivariant fiber bundle map* $f : \mathbf{R}^n \to \mathbf{R}^m$ *with fiber diffeomorphic to* $\mathbf{R}^{n-m}$. In particular, if $\phi : \Gamma \to \hat{\Gamma}$ is an epimorphism between crystallographic groups where Γ is torsion-free; then $(\hat{\Gamma}, \phi, F)$ is a fibering apparatus for Γ where F is the map posited in Lemma 1.2.

If Γ is a group, $cd\ \Gamma$ denotes its cohomological dimension; e.g., if Γ is a Bieberbach group, then $cd\ \Gamma$ = rank Γ.

LEMMA 1.3. *Let* Γ *be a finitely generated, torsion-free, virtually nilpotent group with* $cd\ \Gamma > 1$, *then there exists a fibering apparatus* $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ *for* Γ *such that rank* $\hat{\Gamma} \geq 2$.

Proof. In [16, Lemma 1.2] a fibering apparatus $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ was constructed for Γ. In this construction, rank $\hat{\Gamma} = \dim L/[L, L]$ where L is a simply connected (connected) nilpotent Lie group which contains a discrete cocompact subgroup N isomorphic to a subgroup of finite index in Γ and $[L, L]$ is the commutator subgroup of L. Hence $\dim L \geq 2$ which implies that $\dim L/[L, L] \geq 2$. (Hint: all 2-dimensional connected nilpotent Lie groups are abelian.)

COROLLARY 1.4. *Let* Γ *be a finitely generated, torsion-free, virtually nilpotent group and* $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ *be a fibering apparatus for* Γ

where $\mathbf{R}^n/\Gamma$ *denotes the orbit space of the action of* Γ *on* $\mathbf{R}^n$ *occurring in* $\mathfrak{A}$. *If* $\hat{\Gamma} = \pi \rtimes T$, *then* $\mathbf{R}^n/\Gamma$ *fibers over a circle with a closed aspherical manifold as fiber and the fundamental group of the fiber is* $\phi^{-1}(\pi)$. *Note* $\phi^{-1}(\pi)$ *is also a finitely generated, torsion-free, virtually nilpotent group and* $cd(\phi^{-1}(\pi)) = (cd\,\Gamma) - 1$.

Proof. Let $\psi:\hat{\Gamma} = \pi \rtimes T \to T \subseteq E(1)$ be the canonical map. By Lemma 1.2, there is a ψ-equivariant affine surjection $F:\mathbf{R}^m \to \mathbf{R}$; hence, $(T, \psi\phi, Ff)$ is a fibering apparatus for Γ. Therefore, the composite Ff induces the desired fiber bundle projection between the orbit spaces $\mathbf{R}^n/\Gamma$ and $\mathbf{R}/T$. (Note $\mathbf{R}/T$ is the circle.)

2. Induction Theorems. In this section, we apply Frobenius induction theory to study the group of homotopy-topological structures on an aspherical manifold M^n. (A connected manifold M^n is *aspherical* provided $\pi_i M^n = 0$ for all $i > 1$.)

Throughout this section, let M^n be a closed aspherical manifold such that $Wh(\pi_1 M^n \oplus T^i) = 0$ for all $i \geq 0$ where T^i denotes the free abelian group of rank i. Let I^k denote the closed k-disc and E^{n+k} be the total space of an I^k-bundle with base space M^n. Recall [28] [22] a (homotopy-topological) structure on E^{n+k} is represented by a pair (N^{n+k}, f) where N^{n+k} is a compact manifold and $f:N^{n+k} \to E^{n+k}$ is a homotopy equivalence which maps ∂N onto ∂E homeomorphically. Another structure $(\bar{N}, \bar{f})$ is equivalent to (N, f) if there exists a homeomorphism $g:N \to \bar{N}$ with $\bar{f}g$ homotopic to f rel ∂. Let $\mathcal{S}(E^{n+k})$ denote the set of equivalence classes of structures on E^{n+k}; it is an abelian group when $n + k > 4$. This group is essentially periodic of period 4; namely the following result is due to Siebenmann [22] using result of Quinn [24].

LEMMA 2.1. *If* $n + k > 4$ *and* $k > 0$, *then* $\mathcal{S}(E^{n+k}) \simeq \mathcal{S}(E^{n+k} \times I^4)$. *Also, if* $n > 4$ *and* $\mathcal{S}(M^n \times I^4) = 0$; *then* $\mathcal{S}(M^n) = 0$.

Next, we describe transfer maps for $\mathcal{S}(E^{n+k})$. Let S be a subgroup of finite index in $\pi_1 M^n = \pi_1 E^{n+k}$ and $p_S:E_S \to E$ denote the covering space corresponding to S. (Note E_S is the total space of an I^k-bundle whose base space is M_S—the total space of the covering space of M corresponding to S.) If (N^{n+k}, f) represents an element $b \in \mathcal{S}(E^{n+k})$, then the top row $(\bar{N}, \bar{f})$ of the following pullback diagram represents its transfer $i^*(b) \in \mathcal{S}({E_S}^{n+k})$ where $i:S \to \pi_1 M^n$ is the inclusion map

(2.1)
$$\begin{array}{ccc} \bar{N} & \xrightarrow{\bar{f}} & E_S \\ \downarrow & & \downarrow{\scriptstyle p_S} \\ N & \xrightarrow{f} & E \end{array}$$

Also, recall the Sullivan-Wall surgery exact sequence [28] [22]

(2.2)
$$\cdots \to L_{n+k+1}(S, \omega_S) \overset{d_S}{\to} \mathcal{S}(E_S^{n+k}) \overset{\tau_S}{\to} [E_S^{n+k} \operatorname{rel} \partial, G/\mathrm{Top}] \overset{\theta_S}{\to} L_{n+k}(S, \omega_S)$$

where $\omega_S: \pi_1 E_S \to T_2$ is the first Stiefel-Whitney class of E_S. (When $S = \pi_1 M^n$, we abbreviate the notation for ω_S, d_S, τ_S and θ_S to ω, d, τ and θ; respectively.) The following two theorems, which extend results used in [13] and [14], are immediate consequences of Andrew Nicas's Ph.D. Thesis [23]; they depend heavily on Dress's work [7]. Let G be a finite factor group of $\pi_1 M^n$ and $p: \pi_1 M^n \to G$ be the canonical map.

THEOREM 2.2. *Let $b \in \mathcal{S}(E^{n+k})$ where $k \geq 1$. If b vanishes under transfer to $\mathcal{S}(E_S^{n+k})$ for every hyperelementary subgroup H of G where $S = p^{-1}H$, then $\tau(b) = 0$.*

THEOREM 2.3. *Assume $k > 0$ and*

$$\tau_S: \mathcal{S}(E_S^{n+k}) \to [E_S^{n+k} \operatorname{rel} \partial, G/\mathrm{Top}]$$

is the zero homomorphism for each hyperelementary subgroup H of G where $S = p^{-1}H$. If $b \in \mathcal{S}(E_S^{n+k})$ vanishes under transfer to $\mathcal{S}(E_S^{n+k})$ for every hyperelementary subgroup H of G where $S = p^{-1}H$, then $b = 0$.

3. A Metric Criterion for Vanishing. We keep the notation from Section 2. Let $y \in \mathcal{S}(E^{n+k} \times I)$ where $n + k > 4$, then it can be represented by a homotopy $h_t: E \to E$ $(0 \leq t \leq 1)$ such that $h_o = id$, $h_t | \partial E = id$ for $0 \leq t \leq 1$ and h_1 is a homeomorphism; i.e., the homotopy equivalence $\hat{h}: E \times I \to E \times I$ defined by

(3.1) $$\hat{h}(x, t) = (h_t(x), t) \quad \text{where} \quad x \in E, t \in [0, 1]$$

represents y. Let Γ be a torsion-free, finitely generated, virtually nilpotent group and $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ. let $M^n = \mathbf{R}^n/\Gamma$ and $p: E^{n+k} \to M^n$ be an I^k-bundle; consider the following pullback diagram

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\bar{p}} & \mathbf{R}^n \\ \downarrow & & \downarrow \\ E & \xrightarrow{p} & M^n \end{array} \tag{3.2}$$

where the vertical arrows are the universal covering spaces of M^n, E^{n+k} (respectively) and $\bar{p}: \bar{E} \to \mathbf{R}^n$ is an I^k-bundle. We proceed to define a non-negative real number associated to y and $\mathfrak{A}$ called the *$\mathfrak{A}$-diameter* of y. Lift the homotopy h_t to the homotopy $\bar{h}_t$ of $\bar{E}$ such that $\bar{h}_o = id$ and define a family $\{\alpha_x | x \in \bar{E}\}$ of paths in $\mathbf{R}^m$ by

$$\alpha_x(t) = f\bar{p}\bar{h}_t(x), \qquad t \in [0, 1]. \tag{3.3}$$

The diameter of α_x, denoted $\|\alpha_x\|$, is defined by

$$\|\alpha_x\| = \max\{|\alpha_x(t) - \alpha_x(0)| \,|\, t \in [0, 1]\} \tag{3.4}$$

and the *$(\mathfrak{A}, h_t)$-diameter* of y is $\max\{\|\alpha_x\| \,|\, x \in \bar{E}\}$. The *$\mathfrak{A}$-diameter* of y is the greatest lower bound of the set of $(\mathfrak{A}, h_t)$-diameters of y as h_t varies over all homotopies representing y.

The fibering apparatus $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ is called *admissible* if for each finite subgroup H of $\hat{\Gamma}$ and each I^k-bundle $\bar{E}^{m+k}$ $(m + k > 4)$ over N^m where N^m is any closed aspherical manifold with $\pi_1(N^m) = \phi^{-1}(H)$, then $\mathcal{S}(\bar{E}^{m+k}) = 0$. We now state the main result of this section.

THEOREM 3.1. *Let $\hat{\Gamma} \subseteq E(m)$ be a crystallographic group, then there is a number $\epsilon > 0$ such that the following is true. Let $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ be any admissible fibering apparatus for a finitely generated, torsion-free, virtually nilpotent group Γ and E^{n+k} $(n + k > 4)$ be any I^k-bundle over $M^n = \mathbf{R}^n/\Gamma$, then the only element in $\mathcal{S}(E \times I)$ whose $\mathfrak{A}$-diameter is less than ϵ is zero.*

Proof. Substituting Theorem A from the Appendix for Corollary 4.2 of [16], the proof follows more or less by a similar line of reasoning as the proof of [16, Theorem 2.1]. However we repeat the whole argument here for the reader's convenience.

First, recall some elementary facts about a smooth action of a finite group G on a closed manifold M. (Some general references are [2] [6].) By averaging the metric if necessary, we can always assume that M is Riemannian and G is a subgroup of isometries of M. For $x, y \in M$, we say that x, y are of the same orbit type if their isotropy subgroups G_x, G_y are conjugate. If (H) is the conjugacy class of the subgroup H of G, then $M_{(H)}$ denotes the submanifold (generally not closed) of M consisting of the points whose isotropy subgroup is in (H); i.e.,

(3.5) $$M_{(H)} = \{x \mid x \in M, G_x \in (H)\};$$

$M_{(H)}$ is an invariant subset of M under G. Define $Cl(M_{(H)})$ by

(3.6) $$Cl(M_{(H)}) = \cup_{(H') \supseteq (H)} M_{(H')}$$

where $(H') \supseteq (H)$ means H is a subgroup of some member of (H'). Clearly, $Cl(M_{(H)})$ is a closed invariant subset of M containing $M_{(H)}$. (Warning, $Cl(M_{(H)})$ is not always the topological closure of $M_{(H)}$.)

We can choose equivariant neighborhoods $N(M_{(H)})$ of $M_{(H)}$ satisfying the following conditions:

(3.7) (i) The set $\bar{M}_{(H)}$ defined by

$$\bar{M}_{(H)} = M_{(H)} - \cup_{(H') \supsetneqq (H)} N(M_{(H')}),$$

is a closed invariant submanifold of M and $\bar{M}_{(H)}$ is an equivariant deformation retract of $M_{(H)}$.

(ii) Each $N(M_{(H)})$ has an equivariant disc bundle structure

$$\xi_{(H)} : \mathbf{D} \to N(M_{(H)}) \overset{r_{(H)}}{\to} M_{(H)}$$

with the associated sphere bundle

$$\partial_0 \xi_{(H)} : \partial \mathbf{D} \to \partial_0 N(M_{(H)}) \to M_{(H)}$$

such that if we define $N(\bar{M}_{(H)})$ to be the closure of

$$N(M_{(H)}) - \cup_{(H') \supsetneqq (H)} N(M_{(H')}),$$

then we have an induced equivariant disc bundle structure

$$\bar{\xi}_{(H)}: \mathbf{D} \to N(\bar{M}_{(H)}) \xrightarrow{r_{(H)}} \bar{M}_{(H)}$$

with associated sphere bundle

$$\partial_0 \bar{\xi}_{(H)}: \partial \mathbf{D} \to \partial_0 N(\bar{M}_{(H)}) \to \bar{M}_{(H)}.$$

Moreover, there exists a constant $C > 0$ such that for $x, y \in N(\bar{M}_{(H)})$

$$d(r_{(H)}(x), r_{(H)}(y)) \leq Cd(x, y)$$

where $d(\ ,\ )$ denotes the distance function of M.

(iii) The set

$$\cup_{(H') \supsetneqq (H)} N(M_{(H')})$$

is an equivariant "regular" neighborhood of $Cl(M_{(H)})$. We can produce a constant $C' > 0$ and an equivariant retraction

$$\hat{r}_{(H)}: \cup_{(H') \supseteq (H)} N(M_{(H')}) \to Cl(M_{(H)})$$

such that

$$\hat{r}_{(H)} | N(M_{(H)}) = r_{(H)} \quad \text{and}$$

$$d(\hat{r}_{(H)}(x), \hat{r}_{(H)}(y)) \leq C'd(x, y) \quad \text{for}$$

$$x, y \in \cup_{(H') \supseteq (H)} N(M_{(H')}).$$

(iv) If $(H) \subseteq (H')$ and $x \in N(M_{(H)}) \cap N(M_{(H')})$, then

$$r_{(H')}(r_{(H)}(x)) = r_{(H')}(x).$$

Note that the properties of the above depend on the choices of $N(M_{(H)})$, $r_{(H)}$, etc. We shall call $\{N(M_{(H)})\}$ a neighborhood system of $\{M_{(H)}\}$. Suppose that $\{N(M_{(H)})\}$ is given. For a conjugacy class (H_0), we can choose a small $\bar{\epsilon} > 0$ to enlarge $N(\bar{M}_{(H_0)})$ by an amount $\bar{\epsilon}$ in the following sense. We have a new neighborhood system $\{N'(M_{(H)})\}$ such that

$N'(M_{(H)}) = N(M_{(H)})$ for $(H) \neq (H_0)$ and $N'(M_{(H_0)}) \supsetneq N(M_{(H_0)})$ and satisfies the following conditions.

(3.8) (i) Define $\bar{M}'_{(H)}$ to be the closure of

$$M_{(H)} - \bigcup_{(H') \supsetneq \not\supseteq (H)} N'(M_{(H')}).$$

For $(H) \subsetneq (H_0)$, $\bar{M}'_{(H)}$ is contained in $\bar{M}_{(H)}$ and $d(\bar{M}'_{(H)}, \bar{M}_{(H_0)}) > \bar{\epsilon}$ where $d(A, B)$ denotes the distance between the subsets $A, B \subset M$. Otherwise, $\bar{M}'_{(H)} = \bar{M}_{(H)}$.

(ii) We have inclusions of equivariant disc bundles

$$\begin{array}{ccccc}
\xi_{(H_0)}: \mathbf{D} & \longrightarrow & N(M_{(H_0)}) & \longrightarrow & M_{(H_0)} \\
\downarrow & & \downarrow & & \downarrow = \\
\xi'_{(H_0)}: \mathbf{D}' & \longrightarrow & N'(M_{(H_0)}) & \longrightarrow & M_{(H_0)}
\end{array}$$

$$\begin{array}{ccccc}
\bar{\xi}_{(H_0)}: \mathbf{D} & \longrightarrow & N(\bar{M}_{(H_0)}) & \longrightarrow & \bar{M}'_{(H_0)} \\
\downarrow & & \downarrow & & \downarrow = \\
\bar{\xi}'_{(H_0)}: \mathbf{D}' & \longrightarrow & N'(\bar{M}_{(H_0)}) & \longrightarrow & \bar{M}_{(H_0)}
\end{array}$$

such that the distance between the sphere bundles $\partial_0 \bar{\xi}_{(H_0)}$ and $\partial_0 \bar{\xi}'_{(H_0)}$ is greater than $\bar{\epsilon}$.

We can also shrink $N(M_{(H_0)})$ by a small amount $\bar{\epsilon} > 0$ satisfying conditions similar to (3.8).

The manifold M is equivariantly stratified by $\{M_{(H)}\}$ and the orbit space M/G has the induced stratification $\{M_{(H)}/G\}$. Abusing language, we shall call $\{\bar{M}_{(H)}\}$, $\{\bar{M}_{(H)}/G\}$ the closed strata of M, M/G respectively. (Beware that they depend on the choice of $N(M_{(H)})$!) At the orbit space level, we have the cone bundle

$$cL \to N(\bar{M}_{(H)})/G \to \bar{M}_{(H)}/G \tag{3.9}$$

over $\bar{M}_{(H)}/G$ where L is the link of $M_{(H)}/G$ in M/G, $cL = L \times [0, 1]/L \times 0$. If $\mathbf{D}$ is the fiber of the bundle (3.7, ii), then $cL = \mathbf{D}/H$ with c corresponding to the image of the origin of $\mathbf{D}$ in $\bar{M}_{(H)}/G$. In fact, $N(\bar{M}_{(H)}) \cap M_{(H')}$ (where $(H') \subsetneq (H)$) is an equivariant neighborhood of $\bar{M}_{(H)}$ in $\bar{M}_{(H)}$

$\cup \bar{M}_{(H')}$ and $(N(\bar{M}_{(H)}) \cap M_{(H')})/G$ has a subcone bundle structure over $\bar{M}_{(H)}/G$ and these subcone bundles over $\bar{M}_{(H)}/G$ together form the cone bundle (3.9).

Let $q: M \to M/G$ be the natural map; M/G has an induced metric from M. Namely, $d_{M/G}$ is defined by

$$d_{M/G}(a, b) = d(q^{-1}(a), q^{-1}(b)) \tag{3.10}$$

where $a, b \in M/G$.

Now, let us consider the action of $\hat{\Gamma}$ on $\mathbf{R}^m$. The decomposition $1 \to A \to \hat{\Gamma} \to G \to 1$ factors the action into two steps. The subgroup A acts on $\mathbf{R}^n$ freely and the orbit space is the flat torus T^m and the finite group G acts on T^m as a group of isometries such that $\mathbf{R}^m/\hat{\Gamma} = T^m/G$. Apply the facts about a finite group action to the present situation. Let

$$g: \mathbf{R}^n/\Gamma \to \mathbf{R}^m/\hat{\Gamma} = T^m/G \tag{3.11}$$

be the projection induced from the given admissible fibering apparatus $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$. For $\epsilon > 0$ and B a subset of a metric space X, define B^ϵ by

$$B^\epsilon = \{x \mid x \in X, d(x, B) < \epsilon\}. \tag{3.12}$$

Enumerate the conjugacy classes of subgroups of G as follows. Let $H_1, H_2, \ldots, H_s$ be a complete non-redundant list of conjugacy class representatives such that

(3.13) (a) $H_1 = 1$,
(b) $H_s = G$ and
(c) if $(H_i) \subseteq (H_j)$, then $i \leq j$.

Let $\{N(T^m{}_{(H_i)}) \mid 1 \leq i \leq s\}$ be a neighborhood system for (T^m, G) — the holonomy action of G on T^m. Enlarge this system several times to obtain a family of sets $\{N_{ij} \mid 1 \leq j \leq i \leq s\}$ such that for each i

(3.14) (a) $N(T^m{}_{(H_i)}) = N_{i1} \subseteq N_{i2} \subseteq \cdots \subseteq N_{ii}$ and
(b) $\mathfrak{N}_i = \{N_{1,1}, N_{2,2}, \ldots, N_{ii}, N_{i+1,i}, \ldots, N_{si}\}$

is a neighborhood system for (T^m, G) where $\mathfrak{N}_{i+1}$ is obtained from $\mathfrak{N}_i$ by enlarging in succession $N_{i+1,i}$ to $N_{i+1,i+1}$, $N_{i+2,i}$ to $N_{i+2,i+1}$, $\ldots$, N_{si} to $N_{s,i+1}$ via the method described in (3.8) and each enlargement should have size larger than some small number $\bar{\epsilon} > 0$. Let $\{\bar{T}_{ij} | j \leq i \leq s\}$ be the closed strata of (T^m, G) relative to $\mathfrak{N}_i$ as described in (3.7) and let $\bar{N}_{ij} = N(\bar{T}_{ij})$ relative to $\mathfrak{N}_j$. (Note that $\bar{T}_{ii} \subseteq \bar{T}_{i,i-1} \subseteq \cdots \subseteq \bar{T}_{i1} = \bar{T}^m{}_{(H_i)}$). Define $X_i \subseteq T^m/G$ and $C_i \subseteq X_i$ by

$$X_i = (\cup_{j \geq i} T^m{}_{(H_j)})/G \quad \text{and} \tag{3.15}$$

$$C_i = \bar{T}_{ii}/G.$$

Define subsets R_i, R'_i of T^m/G by

$$R_i = \cup_{j>i} (N_{ji}/G) \quad (\text{if } i > 0), \qquad R_0 = T^m/G \quad \text{and} \tag{3.16}$$

$$R'_i = (N_{ii}/G) \cup R_{i-1} \quad (\text{where } i > 0).$$

There exists a small positive number $\hat{\epsilon}$ (with $3\hat{\epsilon} < \bar{\epsilon}$) such that if we define R''_i by

$$R''_i = (N_{ii}/G) \cup (R_{i-1}{}^{\hat{\epsilon}}), \tag{3.17}$$

then there is a map $r_i : R''_i \to X_i$ (constructable as in (3.7), (3.8)) such that r_i is a fiber bundle projection over $C_i^{\hat{\epsilon}} \subseteq X_i$; in fact, it is essentially the cone bundle projection induced from the equivariant disc bundle $\xi_{(H_i)}$ where $\xi_{(H_i)}$ is defined relative to the neighborhood system

$$\mathfrak{N}'_i = \{N_{1,1}, N_{2,2}, \ldots, N_{i,i}, N_{i+1,i-1}, \ldots, N_{s,i-1}\} \tag{3.18}$$

for (T^m, G). For technical purposes, we assume there is an intermediate enlargement $\hat{\mathfrak{N}}_i$ between $\mathfrak{N}_{i-1}$ and $\mathfrak{N}'_i$ where

$$\hat{\mathfrak{N}}_i = \{N_{1,1}, N_{2,2}, \ldots, N_{i-1,i-1}, \hat{N}_{i,i}, N_{i+1,i-1}, \ldots, N_{s,i-1}\} \tag{3.19}$$

with $N_{i,i-1} \subseteq \hat{N}_{i,i} \subseteq N_{i,i}$ and $\hat{\mathfrak{N}}_i$ is (at least) an $\bar{\epsilon}$-enlargement of $\mathfrak{N}_{i-1}$ while $\mathfrak{N}'_i$ is (at least) an $\bar{\epsilon}$-enlargement of $\hat{\mathfrak{N}}_i$. Let $\hat{R}_i \subseteq R'_i$ be defined by

$$\hat{R}_i = (\hat{N}_{ii}/G) \cup R_{i-1}. \tag{3.20}$$

Denote now $\mathbf{R}^n/\Gamma$ by M^n and let $M_{i-1} \subseteq \hat{M}_i \subseteq M'_i \subseteq M^n$ be defined by

$$M_{i-1} = g^{-1}(R_{i-1}), \qquad \hat{M}_i = g^{-1}(\hat{R}_i) \quad \text{and} \quad M'_i = g^{-1}(R_i) \tag{3.21}$$

and define $M_{(i-1)} \supseteq \hat{M}_{(i)} \supseteq M'_{(i)}$ to be the closure of the complements (in M^n) of M_i, $\hat{M}_i$ and M'_i, respectively. Let $\pi: E \to M^n$ be an I^k-bundle ($n + k > 4$) over M^n. Consider the composite map

$$p: E \times I \xrightarrow{p_0} E \xrightarrow{\pi} M^n \xrightarrow{g} \mathbf{R}^m/\hat{\Gamma} = T^m/G \tag{3.22}$$

where $E \times I \xrightarrow{p_0} E$ is the obvious projection. Define $E_{i-1} \times I$, $\hat{E}_i \times I$, $E'_i \times I$, $E_{(i-1)} \times I$, $\hat{E}_{(i)} \times I$ and $E'_{(i)} \times I$ to be the inverse images $(\pi p_0)^{-1}(\quad)$ of M_{i-1}, $\hat{M}_i$, M'_i, $M_{(i-1)}$, $\hat{M}_{(i)}$ and $M'_{(i)}$, respectively.

The composite of p and r_i defines a map

$$e_i: E'_i \times I \to X_i \tag{3.23}$$

which is a fibration over $C_i^{\hat{\epsilon}}$ such that the fiber is a disc bundle over $\mathbf{R}^{n-m}/\phi^{-1}(G_x)$ where $G_x \in (H_i)$. By the hypotheses of Theorem 3.1 the fibration e_i satisfies the hypotheses of the Appendix. (Note the main result of [16] is also used here; i.e., Theorem 5.2 stated below in Section 5.) So, we may choose a sequence of pairs of positive numbers δ_i, ϵ_i ($1 \leq i \leq s$) satisfying

(3.24) (a) $\delta_i \ll \epsilon_i \ll \delta_{i+1}$,
(b) $\epsilon_s \ll \hat{\epsilon} < (\bar{\epsilon}/3)$ and
(c) $\delta_i \ll \bar{\delta}_i$

where $\bar{\delta}_i$ is the number δ posited in Theorem A when we consider the projection e_i and set $\epsilon = \epsilon_i$, $X = C_i^{\hat{\epsilon}} \subseteq X_i$ and $C = C_i$. The symbol $a \ll b$ in (3.24) means the ratio b/a is very large—exactly how large depends on the geometry of our particular choice of $\{N_{ij} \mid 1 \leq j \leq i \leq s\}$; e.g., on the constants C, C', $\bar{\epsilon}$, etc., occurring in (3.7), (3.8).

Choose the real number ϵ of Theorem 3.1 such that $\epsilon \ll \delta_1$. Let $\hat{h}: E \times I \to E \times I$ represent an element in $\mathcal{S}(E \times I)$ whose $\mathcal{A}$-diameter is less than ϵ. We will inductively construct homeomorphisms (homotopic to $\hat{h}$) over

$$E_{(1)} \times I, E_{(2)} \times I, \ldots, E_{(s)} \times I = E \times I$$

(rel $\partial(E \times I)$). (Note we will have proven Theorem 3.1 when this construction is finished.) To be precise, we will construct embeddings $k_i: E_{(i)} \times I$

$\to E \times I$ such that $k_i(x) \in \partial(E \times I)$ if and only if $x \in \partial(E \times I)$ and $\hat{h}(k_i(x)) = x$ whenever $x \in \partial(E \times I)$. Furthermore, denoting image (k_i) by $\overline{\mathcal{E}}_{(i)}$, we will construct homotopies $\hat{h}_i(\ ,\ ): \mathcal{E}_{(i)} \times I \to E \times I$ (rel $\partial(E \times I)$) such that, for each i, k_i and $\hat{h}_i$ satisfy

(3.25) (a) $\hat{h}_i(\ ,0) = \hat{h} \,|\, \mathcal{E}_{(i)}$;
(b) $\hat{h}_i(\ ,1) = k_i^{-1}$;
(c) if $k_i^{-1}(x) \notin \hat{E}_i \times I$, then $\hat{h}_i(x, u) = \hat{h}_{i-1}(x, u)$ for all $u \in [0, 1]$;
(d) if $k_i^{-1}(x) \in \hat{E}_i \times I$, then $\hat{h}_i(x, u) \in E'_i \times I$ (for all $u \in [0, 1]$) and the path $u \to e_i(\hat{h}_i(x, u))$ has diameter less than $2\epsilon_i$ in X_i.

Assuming $k_1, k_2, \ldots, k_{i-1}$ and $\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{i-1}$ satisfying (3.25) have already been constructed, we now show how to construct k_i and $\hat{h}_i$. In Theorem A of the appendix, replace ϵ, X, C, E and p by ϵ_i, $C_i^{\hat{\epsilon}}$, C_i, $e_i^{-1}(C_i^{\hat{\epsilon}})$ and $e_i^{-1}|e_i^{-1}(C_i^{\hat{\epsilon}})$, respectively. Let E' in the statement of Theorem A be the closure of

$$E \times I - k_{i-1}(\hat{E}_{(i)} \times [0, 1]); \tag{3.26}$$

also, let $\partial_0 E' = k_{i-1}(\partial_0 E = e_i^{-1}(C_i^{\hat{\epsilon}}))$. Tapering $\hat{h}\,|\,E'$ into the homeomorphism $k_{i-1}{}^{-1}$ near $\partial_0 E'$, we obtain a δ_i-equivalence (over $C^{2\epsilon_i}$) $f: E' \to E$ such that $f\,|\,\partial_0 E' = k_{i-1}{}^{-1}\,|\,\partial_0 E'$. Since $\delta_i \ll \bar{\delta}_i$ and $\bar{\delta}_i$ is the number δ posited in Theorem A for this situation, f is (properly) ϵ_i-homotopic to a map f' which is a homeomorphism over C^{ϵ_i} and agrees with $k_{i-1}{}^{-1}$ on $\partial_0 E'$. Hence, we can glue the embeddings $(f')^{-1}|p^{-1}(C)$ and $k_{i-1}|(\hat{E}_{(i)} \cap E_{(i)} \times I)$ together to define k_i; the homotopy $\hat{h}_i$ is similarly constructed using the above data so that the pair k_i, $\hat{h}_i$ satisfies (3.25). This completes the induction step and the proof of Theorem 3.1.

4. Metric Properties of Crystallographic Groups. In this section, we discuss the metric properties of a crystallographic group Γ which correspond to the classification of Γ given in Theorem 1.1. These properties are elaborations of the Epstein-Shub result [8].

Let $\Gamma \subseteq E(m)$ be a crystallographic group with holonomy group G and maximal abelian subgroup of finite index A. We say a monomorphism $\psi: \Gamma \to \Gamma$ is *s-expansive if ψ induces multiplication by s on A* (*where s is a positive integer*) *and the identity map on G.*

THEOREM 4.1. *For any positive integer* $s \equiv 1 \bmod |G|$, *there exists an s-expansive endomorphism* ψ *of* Γ. *Furthermore, for any s-expansive endomorphism* ψ *of* Γ, *there exists a* ψ*-equivariant diffeomorphism* $g: \mathbf{R}^m \to \mathbf{R}^m$ *such that* $|dg(X)| = s|X|$ *for each vector X tangent to* $\mathbf{R}^m$ *where* $|\ |$ *is the Euclidean metric on* $\mathbf{R}^m$.

This result is a restatement of [16, Theorem 2.2] which itself is an easy generalization of the main result of [8].

Now we examine in much more detail the 2-dimensional crystallographic groups relevant to Theorem 1.1. First, we divide these into 3 types. The group $(T \oplus T) \rtimes T_2 \subset E(2)$ where $G = T_2$ acts on $A = T \oplus T$ via multiplication by -1 is the only group of type 1; explicitly, $(T \oplus T) \rtimes T_2$ is that subgroup $\Gamma \subset E(2) = \mathbf{R}^2 \rtimes 0(2)$ such that $A = \Gamma \cap \mathbf{R}^2 = \mathbf{Z}^2$ and which is generated by A together with the motion $x \mapsto -x$ (where $x \in \mathbf{R}^2$).

To describe the type 2 groups, identify G with

$$(4.1) \qquad \left\{ \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix} \middle| \, i, j = \pm 1 \right\}$$

and A with $\mathbf{Z} \oplus \mathbf{Z} \subseteq \mathbf{R}^2 \subseteq E(2)$; then pick a single crystallographic group $\Gamma_\theta \subset E(2)$ representing each cohomology class $\theta \in H^2(G; A(2))$ of group extension where $A(2)$ denotes A with the G-module structure determined by (4.1). The collection $\{\Gamma_\theta | \theta \in H^2(G; A(2))\}$ are the groups of type 2. Similarly, we define the groups of type 3 to be the collection $\{\Gamma_\omega | \omega \in H^2(G; A(3))\}$ where G is identified with

$$(4.2) \qquad \left\{ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \middle| \, i = \pm 1 \right\},$$

A is identified with $\mathbf{Z}^2$, $A(3)$ is A with its G-module structure determined by (4.2), and we pick a single group $\Gamma_\omega \subset E(2)$ representing each $\omega \in H^2(G; A(3))$.

Let D_∞ denote the ∞-dihedral group; explicitly, $D_\infty \subset E(1)$ is the subgroup generated by $x \mapsto x + 1$ and $x \mapsto -x$ (where $x \in \mathbf{R}$).

If Γ is a type 1, 2 or 3 group and p is an odd prime, then a hyperelementary subgroup E of Γ_p is called *special* if $|E| = p|G|$. Note if E and E' are both special and $E \cap A_p = E' \cap A_p$, then E and E' are conjugate subgroups of Γ_p.

THEOREM 4.2. *Let Γ be a crystallographic group of either type* 1, 2 *or* 3; *p be an odd prime and E be a special hyperelementary subgroup of Γ_p; then there exists a second crystallographic group Π of the same type as Γ together with a group monomorphism $\psi:\Pi \to \Gamma$ and a group surjection $\eta\ \Pi \to D_\infty$ satisfying $\psi(\Pi) = q^{-1}(E')$ where $E' \subseteq \Gamma_p$ is conjugate to E and $q:\Gamma \to \Gamma_p$ is the canonical projection. Furthermore, there exists a ψ-equivariant bijection $g:\mathbf{R}^2 \to \mathbf{R}^2$ and an η-equivariant affine surjection $h:\mathbf{R}^2 \to \mathbf{R}$ such that*

$$|d(hg^{-1})(X)| \leq \left(\frac{2}{\sqrt{p}}\right)|X|$$

for each vector X tangent to $\mathbf{R}^2$.

The proof of this theorem for the type 1 group depends on the following result.

LEMMA 4.3. *Let p be a prime and r be an integer such that $0 < r < p$, then there exist integers a and b such that*

(i) $(a, b) = 1$,
(ii) $|b| < \sqrt{p}$ *and*
(iii) $|ap - br| < \sqrt{p}$

Proof. By Euclid's algorithm, for each integer i where $1 \leq i \leq \sqrt{p} + 1$, there exist integers x_i and y_i with $0 \leq y_i < p$ such that

$$i = x_i p - y_i r. \tag{4.3}$$

There must be two numbers, say y_j and y_k, in the set $\{y_i | 1 \leq i \leq \sqrt{p} + 1\}$ such that

$$0 < (y_j - y_k) < \sqrt{p}. \tag{4.4}$$

Let $b' = y_j - y_k$ and $a' = x_j - x_k$; then subtracting equation (4.3.k) from (4.3.j), we obtain

$$j - k = a'p - b'r. \tag{4.5}$$

Hence, $|b'| < \sqrt{p}$ and $|a'p - b'r| < \sqrt{p}$. Let d be the greatest common divisor of a' and b', then $a = a'/d$ and $b = b'/d$ satisfy Lemma 4.3.

Proof of Theorem 4.2. We consider first the case where $\Gamma = (T \oplus T) \rtimes T_2$ is the type 1 crystallographic group. (Represent elements in $T \oplus T$ or $T_p \oplus T_p$ by column vectors with entries in $\mathbf{Z}$ or $\mathbf{Z}_p$, respectively). In this case, $\Pi = \Gamma$ and both ψ and g are induced in the canonical way by matrix multiplication using a 2×2 matrix M with integral entries. Also, η and h are induced by matrix multiplication using a 1×2 matrix N with integral entries. In order to describe M and N, let v generate $A_p \cap E$ where

$$(4.6) \qquad v = \begin{pmatrix} 0 \\ [1] \end{pmatrix}, \quad \begin{pmatrix} [1] \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} [1] \\ [r] \end{pmatrix}.$$

(Here, $[r]$ denotes the congruence class in $\mathbf{Z}_p$ of the integer r where $0 < r < p$). If the first possibility in (4.6) holds, let

$$(4.7) \qquad M = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N = (0, 1);$$

for the second possibility, let

$$(4.8) \qquad M = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad N = (1, 0).$$

If the third possibility holds, let

$$(4.9) \qquad M = \begin{pmatrix} 1 & 0 \\ r & p \end{pmatrix} \quad \text{and} \quad N = (a, b)$$

where a and b are the integers posited in Lemma 4.3 relative to p and r.

If Γ is a type 2 crystallographic group; i.e., $\Gamma = \Gamma_\theta$ where $\theta \in H^2(G; A(2))$; then $E \cap A_p$ is generated by v where

$$(4.10) \qquad v = \begin{pmatrix} [1] \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ [1] \end{pmatrix}.$$

The argument for each of the two possibilities in (4.10) is symmetric; hence, we consider only the first possibility. Let M be the 2×2 matrix given by

(4.11) $$M = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

then matrix multiplication by M induces a G-module monomorphism $\bar{\psi}: A(2) \to A(2)$. Since $(|\text{cok } \bar{\psi}|, |G|) = 1$, $\bar{\psi}^*: H^2(G; A(2)) \to H^2(G; A(2))$ is an isomorphism. Let $\theta' \in H^2(G; A(2))$ be such that $\bar{\psi}^*(\theta') = \theta$, then $\Pi = \Gamma_{\theta'}$ and there is an extension $\psi: \Gamma_{\theta'} \to \Gamma_\theta$ of $\bar{\psi}$ which induces id on G. By Bieberbach's Theorem (cf. [8]), there is a ψ-equivariant affine bijection $g: \mathbf{R}^2 \to \mathbf{R}^2$. For each $\alpha \in \Gamma_{\theta'}$, define $\alpha' \in E(1)$ by requiring $\alpha'(y)$ to be the second co-ordinate of $\alpha(x, y)$ where $(x, y) \in \mathbf{R}^2$. Let $D' = \{\alpha' \mid \alpha \in \Gamma_{\theta'}\}$, then D' is a non-abelian crystallographic group in $E(1)$ whose translational subgroup is either $\mathbf{Z}$ or $1/2\ \mathbf{Z}$. Hence, there is an affine bijection $k: \mathbf{R} \to \mathbf{R}$ which induces an isomorphism $\bar{\eta}: D' \to D_\infty$ and such that $|dk(Y)| \leq 2|Y|$ for each vector Y tangent to $\mathbf{R}$. Let $\eta: \Gamma_{\theta'} \to D_\infty$ be defined by $\eta(\alpha) = \bar{\eta}(\alpha')$ for $\alpha \in \Gamma_{\theta'}$, and $h: \mathbf{R}^2 \to \mathbf{R}$ be given by $h(x, y) = k(y)$. One easily checks that h is η-equivariant and $|d(hg^{-1})(X)| \leq (2/\sqrt{p})|X|$ if X is tangent to $\mathbf{R}^2$. This verifies Theorem 4.2 for type 2 groups. The verification for type 3 groups is similar and we leave it as an exercise to the reader.

We will apply the above results together with the vanishing criterion of Section 3 to analyze the transfer maps discussed in Section 2. Let Γ be a torsion-free finitely generated virtually nilpotent group and $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ. Let $M^n = \mathbf{R}^n/\Gamma$ denote the orbit space of the action of Γ on $\mathbf{R}^n$ determined by $\mathfrak{A}$. (Because of Theorem 5.2 of Section 5, the definitions in Section 2 apply to M^n). We say that $\mathfrak{A}$ is a *special* fibering apparatus provided either

(4.12) (i) $\hat{\Gamma}$ is a type 1, 2 or 3 crystallographic group or
(ii) there is an infinite sequence of positive integers $s \equiv 1 \bmod |\hat{G}|$ (where $\hat{G}$ is the holonomy group of $\hat{\Gamma}$) such that any hyperelementary subgroup of $\hat{\Gamma}_s$ which projects onto $\hat{G}$ (under the canonical map) is isomorphic to $\hat{G}$.

THEOREM 4.4. *Let $\mathfrak{A} = (\hat{\Gamma}, \phi, f)$ be both a special and an admissible fibering apparatus for Γ and let E^{n+k} be an I^k-bundle (where $n + k > 4$) over $M^n = \mathbf{R}^n/\Gamma$ and let $y \in \mathcal{S}(E^{n+k} \times I)$. Then there exists an infinite set $\mathcal{P}(y)$ of positive integers each of which is relatively prime to $|\hat{G}|$ and satisfying the following property. For each $s \in \mathcal{P}(y)$ and each hyperelementary subgroup H of $\hat{\Gamma}_s$ such that $|\hat{G}|$ divides $|H|$, y vanishes under transfer to $\mathcal{S}(E_S{}^{n+k} \times I)$ where $S = \phi^{-1}(q^{-1}(H))$ and $q: \hat{\Gamma} \to \hat{\Gamma}_s$ denotes the canonical map.*

Proof. We first consider the case where $\hat{\Gamma}$ satisfies condition (ii) in (4.12). Let $\epsilon > 0$ be the number posited in Theorem 3.1 for $\hat{\Gamma}$, then $\mathcal{P}(y)$ consists of all the integers s given in (4.12.ii) subject to the following constraint

$$(4.13) \qquad (\mathcal{A} - \text{diameter of } y) < s\epsilon.$$

We proceed to verify that $\omega^*(y) = 0$; here, $\omega^*: \mathcal{S}(E \times I) \to \mathcal{S}(E_S \times I)$ is the transfer map where $\omega: S \to \Gamma$ denotes the inclusion map. Let $\psi: \hat{\Gamma} \to \hat{\Gamma}$ and $g: \mathbf{R}^m \to \mathbf{R}^m$ be the s-expansive endomorphism and ψ-equivariant diffeomorphism, respectively, posited in Theorem 4.1. Note that $\psi(\hat{\Gamma}) = q^{-1}(H')$ where H' is a subgroup of $\hat{\Gamma}_s$ conjugate to H. Since the vanishing under transfer of y with respect to a given subgroup depends only on the conjugacy class of the subgroup, we may assume that $H' = H$; i.e., $\psi(\hat{\Gamma}) = q^{-1}(H)$. Then, $\mathcal{A}_S = (\hat{\Gamma}, \psi^{-1} \circ (\phi | S), g^{-1}f)$ is an admissible fibering apparatus for S where the action of S on $\mathbf{R}^n$ is the restriction (to S) of the action of Γ determined by $\mathcal{A}$. It is easily checked that the $\mathcal{A}_S$ $-$ diameter of $\omega^*(y)$ is $(1/s)$ ($\mathcal{A}$ $-$ diameter of y); hence by (4.13) and Theorem 3.1, $\omega^*(y) = 0$.

Next, we consider the remaining case; i.e., when Γ is a type 1, 2 or 3 crystallographic group. In this case, let $\epsilon > 0$ denote the number posited by Theorem 3.1 for the ∞-dihedral group D_∞, and let $\mathcal{P}(y)$ be the set of all odd primes p such that

$$(4.14) \qquad (\mathcal{A} - \text{diameter of } y) < \frac{\sqrt{p}}{2}\epsilon.$$

One easily sees that there is a special hyperelementary subgroup H' containing H. (Special hyperelementary subgroups were defined in the paragraph preceeding Theorem 4.2). Hence, to show that y vanishes under transfer; we may assume that H is a special hyperelementary subgroup of $\hat{\Gamma}_p$. By Theorem 4.2, there is a crystallographic group Π of the same type as $\hat{\Gamma}$, group homomorphisms $\psi: \Pi \to \hat{\Gamma}$ and $\eta: \Pi \to D_\infty$, and affine maps $g: \mathbf{R}^2 \to \mathbf{R}^2$ and $h: \mathbf{R}^2 \to \mathbf{R}$ which are ψ and η-equivariant, respectively. Furthermore, ψ is a monomorphism, η is an epimorphism and $\psi(\Pi) = q^{-1}(H)$. In this case, let $\mathcal{A}_S = (D_\infty, \eta \circ \psi \circ (\phi | S), hg^{-1}f)$; then $\mathcal{A}_S$ is an admissible fibering apparatus for S. One easily checks that the $\mathcal{A}_S$-diameter of $\omega^*(y)$ is smaller than

$$\frac{2}{\sqrt{p}}(\mathcal{A} - \text{diameter of } y);$$

hence by (4.14) and Theorem 3.1, $\omega^*(y) = 0$. This completes the proof of Theorem 4.4.

5. The Main Result. We now state the main result of this paper.

THEOREM 5.1. *Let M^n be a closed aspherical manifold whose fundamental group is virtually nilpotent and let E^{n+k} be the total space of an I^k-bundle whose base space is M^n $(n + k > 4)$, then $\mathcal{S}(E^{n+k}) = 0$; in particular, $\mathcal{S}(M^n) = 0$ when $n > 4$.*

Before we can prove this theorem, we need some preliminary results. First, we recall the main result of [16] because of which the results of Section 2 are applicable in proving Theorem 5.1.

THEOREM 5.2. *Let Π be a torsion-free virtually poly-$\mathbf{Z}$ group, then $Wh\ \Pi = 0$; in particular, if Γ is a finitely generated torsion-free virtually nilpotent group, then $Wh(\Gamma \times T^n) = 0$ for all $n \geq 0$.*

Recall a poly-$\mathbf{Z}$ group is a group with a normal series where each of the successive factor groups in ∞-cyclic.

LEMMA 5.3. *Let M^n be a closed aspherical manifold which fibers over the circle with fiber N^{n-1}, if $\pi_1 M^n$ is a virtually poly-$\mathbf{Z}$ group and E^{n+k} $(n + k > 5)$ is an I^k-bundle over M^n, then there is an exact sequence*

$$\mathcal{S}(\bar{E} \times I) \to \mathcal{S}(E) \to \mathcal{S}(\bar{E})$$

where $\bar{E}^{(n-1)+k}$ is the restriction of the bundle E to N^{n-1}.

LEMMA 5.4. *Let M^n be a closed aspherical manifold with $\pi_1(M^n)$ a virtually poly-$\mathbf{Z}$ group and let E^{n+k} be an I^k-bundle over M^n $(n + k > 4)$, then there is an exact sequence*

$$\mathcal{S}(E \times I) \to \mathcal{S}(E \times S^1) \to \mathcal{S}(E) \to 0.$$

Proofs. These two Lemmas follow directly from [11] or [12] using Theorem 5.2 (cf. [3]).

COROLLARY 5.5. *Let M^n and $\bar{M}^n$ be two closed aspherical manifolds such that $\pi_1(M)$ is virtually nilpotent and $\pi_1(M) \cong \pi_1(\bar{M})$. Assume that $\mathcal{S}(E^{n+k}) = 0$ for every I^k-bundle over M^n where $n + k > 4$. Then $\mathcal{S}(\bar{E}^{n+k}) = 0$ for every I^k-bundle over $\bar{M}^n$ where $n + k > 4$.*

Proof. If $n < 3$, then clearly M^n and $\bar{M}^n$ are homeomorphic and there is nothing to prove; hence assume $n \geq 3$. Applying Lemma 5.3

twice, we see that $\mathcal{S}(\hat{E}^{n+k}) = 0$ for any I^k-bundle $(n + k > 6)$ over $M^n \times S^1 \times S^1$. In particular, $\mathcal{S}(M^n \times S^1 \times S^1 \times I^4) = 0$; hence $\mathcal{S}(M^n \times S^1 \times S^1) = 0$ by Lemma 2.1. But, there is a homotopy equivalence $h : \bar{M}^n \to M^n$; hence $h \times id : \bar{M}^n \times S^1 \times S^1 \to M^n \times S^1 \times S^1$ represents an element of $\mathcal{S}(M^n \times S^1 \times S^1)$; therefore $\bar{M}^n \times S^1 \times S^1$ and $M^n \times S^1 \times S^1$ are homeomorphic. Consequently, $\mathcal{S}(\hat{E}^{n+k}) = 0$ for any I^k-bundle $(n + k > 6)$ over $\bar{M}^n \times S^1 \times S^1$. Let $\bar{E}^{n+k}$ $(n + k > 4)$ be an I^k-bundle over $\bar{M}^n$, then $\bar{E}^{n+k} \times S^1 \times S^1$ is an I^k-bundle over $\bar{M}^n \times S^1 \times S^1$; hence $\mathcal{S}(\bar{E} \times S^1 \times S^1) = 0$. Now a double application of Lemma 5.4 shows that $\mathcal{S}(\bar{E}^{n+k}) = 0$.

If $\hat{\Gamma} \subseteq E(m)$ is a crystallographic group, we call the integer m the *dimension* of $\hat{\Gamma}$ and denote it by dim $\hat{\Gamma}$. (Note that if $\hat{\Gamma}$ is torsion-free, then dim $\hat{\Gamma} = cd(\hat{\Gamma})$. In general, dim $\hat{\Gamma} = vcd(\hat{\Gamma})$; i.e., it is the same as the virtual cohomological dimension of $\hat{\Gamma}$.)

Let Γ be a finitely generated, torsion-free, virtually nilpotent group and let $(\mathbf{R}^n, \Gamma)$ be a free properly discontinuous action with compact orbit space $\mathbf{R}^n/\Gamma$. We define the *holonomy number* of this action, denoted by $h(\mathbf{R}^n, \Gamma)$, to be the minimum order of the holonomy group of a crystallographic group $\hat{\Gamma}$ that can occur in a fibering apparatus $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ for Γ where $vcd\ \hat{\Gamma} \geq 2$ and $(\mathbf{R}^n, \Gamma)$ is the action of Γ on $\mathbf{R}^n$ occurring in $\mathcal{A}$. If there is no such apparatus $\mathcal{A}$ in which $(\mathbf{R}^n, \Gamma)$ occurs, we set $h(\mathbf{R}^n, \Gamma) = \infty$. And we define the *holonomy number of* Γ, denoted by $h(\Gamma)$, to be the minimum of $h(\mathbf{R}^n, \Gamma)$ as $(\mathbf{R}^n, \Gamma)$ varies over all free properly discontinuous actions with $\mathbf{R}^n/\Gamma$ compact. Note that if $cd(\Gamma) > 1$, then $h(\Gamma) < \infty$; this is a consequence of Lemma 1.3. (Warning, the definition of $h(\Gamma)$ made above is slightly different from the one used in [16].)

Proof of Theorem 5.1. Let $\Gamma = \pi_1(M^n)$; we will proceed by induction first on $n = \dim M^n$ and next on $h(\Gamma)$; i.e., we assume $\mathcal{S}(\bar{E}^{m+k}) = 0$ for any I^k-bundle whose base space is a closed aspherical manifold N^m where $\pi_1(N^m)$ is virtually nilpotent and either $m < n$, or $m = n$ and $h(\pi_1(N^m)) < h(\Gamma)$. If $n = 0$, then M^n is a point and $E^{n+k} = I^k$; in this case, Theorem 5.1 is a consequence of Smale's h-cobordism theorem, the Alexander trick and Lemma 2.1. When $n = 1$, M^n is a circle and Theorem 5.1 is a consequence of the case $n = 0$ together with Lemmas 5.3 and 2.1. Hence, we may assume that $n \geq 2$.

If $h(\Gamma) = 1$, then there is a fibering apparatus $\mathcal{A} = (\hat{\Gamma}, f, \phi)$ for Γ such that $\hat{\Gamma}$ is free abelian of rank at least 2. Let $(\mathbf{R}^n, \Gamma)$ denote the Γ action in $\mathcal{A}$; then by Corollary 1.4, $\mathbf{R}^n/\Gamma$ fibers over a circle with a closed aspherical manifold N^{n-1} for fiber where $\pi_1(N^{n-1})$ is virtually nilpotent.

Hence, we see $\mathcal{S}(E^{n+k}) = 0$ by Lemmas 5.3 and 2.1 together with Corollary 5.5 and our induction hypothesis. Therefore, we can assume both $cd(\Gamma) \geq 2$ and $h(\Gamma) \geq 2$.

The induction argument now splits into two steps. First, we show that the normal map

$$\tau : \mathcal{S}(\bar{E}^{n+k} \times I^4) \to [\bar{E}^{n+k} \times I^4 \operatorname{rel} \partial, G/\mathrm{Top}] \tag{5.1}$$

vanishes where $\bar{E}^{n+k}$ is any I^k-bundle $(n + k > 4)$ with base space $\mathbf{R}^n/\Gamma$ and $h(\mathbf{R}^n, \Gamma) = h(\Gamma)$. To do this step, let $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ with $(\mathbf{R}^n, \Gamma)$ its Γ-action and such that $vcd(\hat{\Gamma}) \geq 2$ and $|\hat{G}| = h(\Gamma) < \infty$ where $\hat{G}$ is the holonomy group of $\hat{\Gamma}$.

Now $\hat{\Gamma}$ satisfies one of the three possibilities listed in Theorem 1.1. If it satisfies possibility (i), then again $\mathbf{R}^n/\Gamma$ fibers over a circle and $\mathcal{S}(\bar{E}^{n+k} \times I^4) = 0$ by the same argument as before.

In the case of possibility (iii), $\hat{\Gamma}$ maps onto a crystallographic group $\bar{\Gamma}$ of either type 1, 2 or 3; let $\psi : \hat{\Gamma} \to \bar{\Gamma}$ denote this epimorphism. By Lemma 1.2, there is a ψ-equivariant affine surjection $F : \mathbf{R}^{m'} \to \mathbf{R}^2$ where $\hat{\Gamma} \subseteq E(m)$. Composing F with f and ψ with ϕ, we obtain a new fibering apparatus $(\bar{\Gamma}, \psi\phi, Ff)$ for Γ which is a *special* fibering apparatus. (See 4.13).

In the case of possibility (ii), we proceed analogously (to the above method) to obtain a *special* fibering apparatus $(\bar{\Gamma}, \eta, \bar{F})$ with $(\mathbf{R}^n, \Gamma)$ as its Γ-action and where $\bar{\Gamma}$ is a crystallographic group satisfying (4.12.ii) in which $\hat{\Gamma}$ is replaced by $\bar{\Gamma}$ and $\hat{G}$ by the holonomy group of $\bar{\Gamma}$. We can also assume that $vcd(\bar{\Gamma}) \geq 2$ since there are (up to isomorphism) only 2 crystallographic groups T and $T_2 * T_2$ of dimension 1 and $T_2 * T_2$ does not satisfy (4.12.ii) while if $\bar{\Gamma} = T$, then $\hat{\Gamma}$ would satisfy possibility (i) which we have already discussed.

Because of the above remarks, we may as well assume when possibilities (ii) or (iii) occur that $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ is a *special* fibering apparatus for Γ with $(\mathbf{R}^n, \Phi)$ as its Γ-action and with $vcd(\hat{\Gamma}) \geq 2$. By our induction hypothesis, $\mathcal{A}$ is also an admissible fibering apparatus. Let $\hat{E} = \bar{E}^{n+k} \times I^3$ and $y \in \mathcal{S}(\hat{E} \times I)$. By Theorem 4.4, there is a positive integer s which is relatively prime to $|\hat{G}|$ and such that y transfers to 0 in $\mathcal{S}(\hat{E}_S \times I)$ for each hyperelementary subgroup H of $\hat{\Gamma}_s$ whose order $|H|$ is divisible by $|\hat{G}|$, where $S = \phi^{-1}(q^{-1}(H))$ and $q : \hat{\Gamma} \to \hat{\Gamma}_s$ is the canonical map. We will use Theorem 2.2 (in its statement, replace G by $\hat{\Gamma}_s$), to conclude that $\tau(y) = 0$. To show that Theorem 2.2 is applicable, it remains to show that y vanishes under transfer to $\mathcal{S}(\hat{E}_S \times I)$ where $S = \phi^{-1}(q^{-1}(H))$ and H is a hyperelementary subgroup of $\hat{\Gamma}_s$ such that $|\hat{G}|$ does not divide $|H|$. Consider $\mathcal{A}_H = (q^{-1}(H), \phi|S, f)$; it is a fibering apparatus for S and $vcd(q^{-1}(H)) = vcd(\hat{\Gamma}) \geq 2$. Note that the holonomy group of $q^{-1}(H)$ is a proper subgroup of $\hat{G}$; hence $h(S) < h(\Gamma)$. Therefore, $\mathcal{S}(\hat{E}_S \times I) = 0$ by our induction hypothesis; consequently, Theorem 2.2 is applicable; this completes the verification of step 1.

Our second step is to show that $\mathcal{S}(\bar{E}^{n+k} \times I^4) = 0$ where $\bar{E}^{n+k}$ is as above; i.e., $\bar{E}^{n+k}$ is an I^k-bundle $(n + k > 4)$ over $\mathbf{R}^n/\Gamma$ where $h(\mathbf{R}^n, \Gamma) = h(\Gamma)$. We also keep the same fibering apparatus $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ as before; in particular, if $\hat{\Gamma}$ satisfies either possibility (ii) or (iii) of Theorem 1.1 (but not possibility (i)), then $vcd(\hat{\Gamma}) \geq 2$ and $\mathcal{A}$ is a special fibering apparatus which by our inductive assumption is also admissible.

If possibility (i) is satisfied by $\hat{\Gamma}$, then $\mathbf{R}^n/\Gamma$ fibers over a circle and (as was argued above) $\mathcal{S}(\bar{E}^{n+k} \times I^4) = 0$. If either possibility (ii) or (iii) (but not (i)) is satisfied by $\hat{\Gamma}$, let $y \in \mathcal{S}(\hat{E} \times I)$ (where $\hat{E} = \bar{E} \times I^3$) and s be one of the positive integers given by Theorem 4.4 and such that $(s, |\hat{G}|) = 1$ and y vanishes when transferred to $\mathcal{S}(\hat{E}_S \times I)$ for each $S = \phi^{-1}(q^{-1}(H))$ where H is any hyperelementary subgroup of $\hat{\Gamma}_s$ with $|H|$ divisible by $|\hat{G}|$. As we showed above using our induction hypothesis, y also vanishes when we transfer it to $\mathcal{S}(\hat{E}_S \times I)$ where $S = \phi^{-1}(q^{-1}(H))$ and H is any other hyperelementary subgroup of $\hat{\Gamma}_s$ since in this situation the group $\mathcal{S}(\hat{E}_S \times I)$ itself is zero. Hence, if Theorem 2.3 is applicable; then $y = 0$ and we will have verified step 2.

To show that Theorem 2.3 is applicable, we must see that the normal map

$$\tau_S : \mathcal{S}(\hat{E}_S \times I) \to [\hat{E}_S \times I \operatorname{rel} \partial; G/\mathrm{Top}] \tag{5.2}$$

is the zero homomorphism where $S = \phi^{-1}(q^{-1}(H))$ and H is any hyperelementary subgroup of $\hat{\Gamma}_s$. If $|\hat{G}|$ does not divide $|H|$, we saw above that $\mathcal{S}(\hat{E}_S \times I) = 0$ and hence $\tau_S = 0$. If $|\hat{G}|$ divides $|H|$, then $\mathcal{A}_H = (q^{-1}(H), \phi|S, f)$ is a fibering apparatus for S such that $vcd(q^{-1}(H)) = vcd(\hat{\Gamma}) \geq 2$ and the holonomy group of $q^{-1}(H)$ is isomorphic to $\hat{G}$. Hence, either $h(S) < h(\Gamma)$ or $h(\mathbf{R}^n, S) = h(S) = h(\Gamma)$. In the first case, $\mathcal{S}(\hat{E}_S \times I)$ vanishes (and hence $\tau_S \equiv 0$) by our induction hypothesis. In the second case, step 1 is applicable to the pair $(\mathbf{R}^n, S)$ which also shows that $\tau_S \equiv 0$. (Note that although S need not be isomorphic to Γ, it is a torsion-free, finitely generated, virtually nilpotent group such that $cd(S) = cd(\Gamma)$ and $h(\mathbf{R}^n, S) = h(S) = h(\Gamma)$; and the arguments above actually showed that step 1 is valid for any pair $(\mathbf{R}^n, S)$ satisfying these hypotheses.) This completes the verification of step 2. Theorem 5.1 now follows immediately from step 2 together with Lemma 2.1.

6. Applications. In this section, we deduce several consequences of Theorem 5.1. Our first result is a specialization of this theorem to the case of virtually abelian fundamental groups.

THEOREM 6.1. *Let N^n be a closed connected flat Riemannian manifold where $n \neq 3, 4$ and let M^n be an aspherical manifold such that $\pi_1(M^n)$ is isomorphic to $\pi_1(N^n)$, then N^n and M^n are homeomorphic.*

This improves on Theorem A of [14]; in the eariler result, we had to assume the holonomy group of N^n had odd order. (To deduce Theorem 6.1 from Theorem 5.1, recall that two aspherical manifolds with isomorphic fundamental groups are homotopically equivalent.)

Next, we topologically characterize (high dimensional) flat Riemannian manifolds. (This result improves on Theorem A′ of [14].)

THEOREM 6.2. *Let M^n $(n \neq 3, 4)$ be a closed connected manifold. It supports a flat Riemannian structure if and only if $\pi_i(M^n) = 0$ for all $i > 1$ and $\pi_1(M^n)$ contains an abelian subgroup with finite index.*

The implication in one direction is a consequence of Bieberbach's theorem that a crystallographic group is virtually abelian together with Killing's result that the univeral cover of a closed connected flat Riemannian manifold is $\mathbf{R}^n$. In the other direction, the implication follows immediately from Theorem 6.1 together with a result due to Zassenhaus [30] and Auslander and Kuranishi [1] that any finitely generated, torsion-free, virtually abelian group is crystallographic.

Recall an infranilmanifold (cf. [21], [26], [28]) is a double coset space $\Gamma\backslash G/K$ with $G = L \rtimes K$ where L is a simply connected nilpotent Lie group, K is a compact Lie group, and Γ is a discrete cocompact subgroup of G. Theorem 5.1 also has the following immediate consequence.

THEOREM 6.3. *Let N^n $(n \neq 3, 4)$ be a closed connected infranilmanifold and M^n be an aspherical manifold with $\pi_1(M^n)$ isomorphic to $\pi_1(N^n)$, then N^n and M^n are homeomorphic.*

If N^n is a nilmanifold, this result was proven 10 years ago by Wall [28, Theorem 15 B.1]; and if N^n is the n-torus, the result was proven earlier yet by Wall [27], and Hsiang and Shaneson [20].

If $b(\ ,\ )$ is a Riemannian metric on a compact manifold M^n, let $d(M^n, b)$ denote the *diameter* of M^n with respect to $b(\ ,\ )$ and let $c(M^n, b)$

denote the *maximum of the sectional curvatures* of M^n relative to $b(\, , \,)$. Following the terminology introduced by Gromov in]19], an *almost flat* structure on a closed connected smooth manifold M^n is a sequence of Riemannian metrics $b_i(\, , \,)$, where $i = 1, 2, 3, \ldots$, such that

(6.1) (a) $\lim_{i \to \infty} c(M^n, b_i) = 0$ and

(b) $\{d(M, b_i) \,|\, i = 1, 2, \ldots,\}$ has a finite upper bound.

Our final application is a topological characterization of (high dimensional) almost flat manifolds.

THEOREM 6.4. *A closed connected manifold M^n ($n \neq 3, 4$) supports an almost flat smooth structure if and only if $\pi_i(M^n) = 0$ for all $i > 1$ and $\pi_1(M^n)$ contains a nilpotent subgroup with finite index.*

The implication in one direction is a consequence of the Main Theorem in Gromov's paper [19]. The implication in the other direction follows from Theorem 6.3 together with the fact that there exists an infranilmanifold N^n with $\pi_1(N^n)$ isomorphic to $\pi_1(M^n)$ (cf. [16, proof of Lemma 1.2]) and the following result whose proof was outlined to us by Dan Burns.

LEMMA 6.5. *Any infranilmanifold supports an almost flat structure.*

Proof. We follow Gromov's argument [19, p. 235] extending it from nilmanifolds to infranilmanifolds. Let M^n be an infranilmanifold, then recall $M^n = \Gamma \backslash G/K$ with $G = L \rtimes K$ where L is a simply connected (connected) nilpotent Lie group, K is a compact Lie group (in fact, we can assume K is finite) and Γ is a discrete cocompact subgroup of G. Let $\mathfrak{l}$ denote the Lie algebra of L, then the action of K on L is uniquely determined by an action of K on $\mathfrak{l}$ via Lie algebra automorphisms. Let

$$\mathfrak{l} = C^1\mathfrak{l} \supset C^2\mathfrak{l} \supset \cdots C^m\mathfrak{l} = 0 \tag{6.2}$$

be the descending central series for $\mathfrak{l}$, then the action of K on $\mathfrak{l}$ respects this filtration; i.e., $gx \in C^i\mathfrak{l}$ provided $x \in C^i\mathfrak{l}$ and $g \in K$. Since K is compact (finite), we can decompose the representation relative to this filtration; i.e., there are K-invariant $\mathbf{R}$-subspaces $A_1, A_2, \ldots, A_m$ of $\mathfrak{l}$ such that $C^i\mathfrak{l} = A_i \oplus C^{i+1}\mathfrak{l}$ for $i = 1, 2, \ldots, m$. (Let $C^{m+1}\mathfrak{l} = 0$). For each j, choose a K-invariant inner product $b^j(\, , \,)$ on A_j and define an infinite sequence $\hat{b}_i$ of K-invariant inner products on $\mathfrak{l}$ determined by

$$\hat{b}_i(x_s, y_t) = \begin{cases} 0, & \text{if } s \neq t \\ \mu_{i,j} b^j(x_s, y_t), & \text{if } s = t = j, \end{cases} \tag{6.3}$$

where $x_s \in A_s$, $y_t \in A_t$ and $\mu_{i,j}$ is an array of positive real numbers indexed by $i = 1, 2, 3, \ldots$ and $j = 1, 2, \ldots, m$ such that

(6.4) (a) $\mu_{i,1} = \dfrac{1}{i}$ and

(b) $(\mu_{i,j})^{n+4} < \mu_{i-1,j}$ for all i and j.

Then, each $\hat{b}_i$ determines a left invariant Riemannian metric $\bar{b}_i$ on L; furthermore, K acts on L via isometries of this metric. Identifying L with G/K, we see that $\Gamma \subseteq (L \rtimes K) = G$ acts on L via isometries of $\bar{b}_i$; let b_i be the induced Riemannian metric on $M^n = \Gamma \backslash G/K$. Then, by [19; 4.4], we see that condition (6.1.a) is satisfied. To see that condition (6.1.b) is also satisfied, consider the nilmanifold $N^n = (\Gamma \cap L)\backslash L$ which is a finite sheeted cover of M^n and let $\{b'_i \mid i = 1, 2, \ldots,\}$ be the set of induced Riemannian metrics on N^n. Clearly,

$$d(N^n, b'_i) \geq d(M^n, b_i); \tag{6.5}$$

but, as observed in [19; 4.5], $\{d(N^n, b'_i) \mid i = 1, 2, \ldots,\}$ has a finite upper bound. This completes the proof of Lemma 6.5.

7. Appendix. In this section, we shall state a variance of Chapman-Ferry-Quinn theory [18], [4], [24] for the structure set. The proof is almost the same as that in [4] except for some modifications; it is carried out in detail in [17]; in fact, a more general result than stated here is proven there.

Let M^n be a closed aspherical manifold satisfying the following conditions where T^ℓ denotes the ℓ-dimensional torus:

(7.1) (i) $Wh(\pi_1(M^n \times T^\ell)) = 0$ for $\ell \geq 0$;

(ii) if $F^{n+\ell+a} \to M^n \times T^\ell$ is an I^a-bundle over $M^n \times T^\ell$ ($\ell \geq 0$), then $\mathcal{S}(F^{n+\ell+a}) = 0$ for $n + \ell + a \geq 6$.

Consider a locally trivial bundle

(7.2) $$F \to E \xrightarrow{p} X$$

where $F = F^{n+\ell+a}$ is as in (7.1) and X is a topological manifold (possibly open, possibly with boundary) with metric d. For a compact submanifold (or subset) $C \subseteq X$, we have the ϵ-neighborhood C^ϵ of C defined to be

(7.3) $$C^\epsilon = \{x \mid d(x, C) < \epsilon\},$$

Let $f:E' \to E$ be a homotopy equivalence. We say that f is a δ-homotopy equivalence (δ-equivalence) over a subset Y of X if there exists a (proper) homotopy inverse $g:E \to E'$ and (proper) homotopies $F:E' \times I \to E'$ connecting gf to $Id_{E'}$ and $G:E \times I \to E$ connecting fg to Id_E satisfying the following conditions:

(7.4) (i) each path $pfF(x \times I)$ (where $x \in (fp)^{-1}(Y)$) has diameter less than δ (in X);

(ii) each path $pG(x \times I)$ has diameter less than δ provided $p(x) \in Y$.

(Note f need only be a homotopy equivalence over Y; in particular, F and G need only be defined over Y). In the case $a > 0$ in (7.1) and (7.2), we modify the notion of δ-equivalence as follows. Let $\partial_0 E$ be the sub-bundle of (7.2) whose fiber is ∂F and let $\partial_0 E'$ be a codimension-0 submanifold of ∂E. Then f restricted to $\partial_0 E'$ is a homeomorphism onto $\partial_0 E$ (such that $f^{-1}(\partial_0 E) = \partial_0 E'$) with inverse $g \mid \partial_0 E$ and the homotopies $F \mid (\partial_0 E' \times I)$, $G \mid (\partial_0 E \times I)$ are constant.

Similarly, we can define ∂-homotopies of maps.

THEOREM A. *Let $p:E \to X$ be given as above and let C be a compact subset of X and $\epsilon > 0$ be a small number. Then, there exists a number $\delta > 0$ depending only on C, X and ϵ (in particular, δ does not depend on E) such that every δ equivalence (over $C^{2\epsilon}$) $f:E' \to E$ (where* $\dim E = \dim E' > 5$) *is ϵ-homotopic to a map $f':E' \to E$ such that*

$$f' \mid (pf')^{-1}(C^\epsilon):(pf')^{-1}(C^\epsilon) \to p^{-1}(C^\epsilon)$$

is a homeomorphism. Furthermore, f' agrees with f on $\partial_0 E'$ and the homotopy between them is constant there.

ADDENDUM. *If f is already a homeomorphism over a codimension-0 submanifold of ∂X, then we can arrange (in Theorem A) for f' to agree with f over this part of the boundary and for the homotopy to be constant there.*

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