Annals of Mathematics, **106** (1977), 101-119

The homeomorphism group of a compact Hilbert cube manifold is an ANR

By Steve Ferry*

Abstract

In this paper, we prove that the homeomorphism group, $H(M)$, of a compact Q-manifold is an ANR. Results of Geoghegan and Torunczyk then show that $H(M)$ is an l_2-manifold.

As by-products of the proof, we obtain a CE approximation theorem for l_2-manifolds, a Vietoris theorem for simple homotopy theory (generalizing the result that a CE map between complexes is simple), and a proof that the nerve of a suitably nice open cover of a complex is simple homotopy equivalent to the complex.

1. Introduction

Let $Q = \prod_{i=1}^{\infty}[-1, 1]$ be the Hilbert cube and let l_2 be separable Hilbert space. A separable metric space M is called a *Q-manifold* or an *l_2-manifold* if it is a manifold modelled on Q or l_2, respectively. If M is a locally compact separable metric space, we will let $H(M)$ denote the group of self-homeomorphisms of M with the compact-open topology. If M is compact, we will consider $H(M)$ to be a metric space under the norm $d(f, g) = \sup_{m\in M} d(f(m), g(m))$.

In [A-B], Anderson and Bing posed the question: If *M is a compact manifold modelled on R^n or on Q, is $H(M)$ an l_2-manifold*? The answer is "*yes*" for 2-manifolds ([M], [L-M], [G], [T]) and the question is unsolved for finite dimensional manifolds of dimension $n>2$. In this paper we show that the homeomorphism group of a compact Q-manifold is an l_2-manifold.

In the years since Anderson and Bing raised the question, several authors have made important inroads on the problem. In particular, our proof relies on the following versions of theorems of Geoghegan and Torunczyk.

THEOREM (Geoghegan [G]). *If $A\subsetneq M$ is a closed subset of a (finite dimensional or Q-) manifold M, then $H_A(M)\approx H_A(M)\times l_2$, where $H_A(M)$ is the group of homeomorphisms of M which fix A.*

THEOREM (Torunczyk [T]). *The product of l_2 with any complete separ-*

* Partially supported by NSF Grant.

able metric ANR *is an l_2-manifold.*

These results reduce the problem of showing that $H(M)$ is an l_2-manifold to the problem of showing that $H(M)$ is an ANR. We will prove that $H_A(M)$ is an ANR, where M is a compact Q-manifold and $A \subset M$ is a Z-set (see § 2 for the definition of Z-set).

We need some definitions. If α is an open cover of a space Y, then two maps $f, g: X \to Y$ are *α-close* if for each $x \in X$ there is $U_x \in \alpha$ such that $f(x)$, $g(x) \in U_x$. A homotopy $h: X \times I \to Y$ is said to be an α-homotopy if for each $x \in X$ there is a $U_x \in \alpha$ such that $h(\{x\} \times I) \subset U_x$. If h is an α-homotopy from f to g, we write $h: f \overset{\alpha}{\simeq} g$ or simply $f \overset{\alpha}{\simeq} g$. In such a case we will say that f is *α-homotopic to g*. A homotopy $h: X \times I \to Y$ is *stationary* on $X_0 \subset X$ if $h(x, t) = h(x, 0)$ for each $t \in I, x \in X_0$. If $U \in \alpha$, $\mathrm{St}(U, \alpha) = \bigcup\{V \in \alpha \mid V \cap U \neq \emptyset\}$. By St α we will mean the cover $\{\mathrm{St}(U, \alpha) \mid U \in \alpha\}$.

A map $f: X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. The symbol $f \overset{\alpha}{\simeq}_p g$ will mean that f is proper homotopic to g. If α is an open cover of Y, a proper map $f: X \to Y$ is said to be an *α-equivalence* if there is a proper map $g: Y \to X$ such that $g \circ f \overset{f^{-1}(\alpha)}{\simeq} \mathrm{id}$ and $f \circ g \overset{\alpha}{\simeq}_p \mathrm{id}$, where $f^{-1}(\alpha)$ is the cover $\{f^{-1}(U) \mid U \in \alpha\}$. Now g is called an α-inverse for f. If X and Y are l_2-manifolds, we drop the requirement that f, g and the homotopies be proper. A map $f: X \to Y$ is called a *fine homotopy equivalence* if f is an α-equivalence for each open cover α of Y.

Haver [H], Kozlowski [K], Lacher [L], and Price [P] have characterized the fine homotopy equivalences between locally compact ANRs as being those maps whose point-inverses are cell-like. These maps are called CE. Armentrout [A_1], [A_2], Siebenmann [S], and Chapman [Ch_1] have proven CE approximation theorems for 3-manifolds, manifolds of dimension $n \geqq 5$, and for Hilbert cube manifolds, respectively. A CE approximation theorem says that if $f: N \to M$ is a CE map between manifolds of the same (finite or infinite) dimension and β is an open cover of M then f is β-close to a homeomorphism.

Several of our results are CE approximation-type theorems. Theorem 3.3 says that if M and N are l_2-manifolds and $f: N \to M$ is an α-equivalence then f is St α-close to a homeomorphism. This implies that fine homotopy equivalences between l_2-manifolds can be approximated arbitrarily closely by homeomorphisms. We would like to thank F. Ancel for bringing this question to our attention.

Theorem 3.1, which contains the main ideas of this paper, is a generalization of Chapman's CE approximation theorem. Theorem 3.1 says roughly that an α-equivalence $f: N \to M$ is close to a homeomorphism. Moreover, the smallness of α needed to insure this depends only on M. This immediately implies (Cor. 3.2) that if K is a polyhedron, then there is an open cover α of K such that if L is a polyhedron and $f: L \to K$ is an α-equivalence, then f is a simple homotopy equivalence. This generalizes Chapman's theorem [Ch_2] that a CE map between polyhedra is a simple homotopy equivalence.

To prove that $H(M)$ is an ANR, we develop the basics of Q-manifold theory parameterized over an arbitrary separable metric space. We combine this with the ideas of Theorem 3.1 to exhibit $H(M)$ as a retract of a suitable space of homotopy equivalences.

We would like to thank T. A. Chapman for his encouragement and advice during the course of our work. H. Torunczyk has also obtained many of the results of this paper.

2. Preliminaries

In this section we will establish our notation and state some basic results from Q-manifold theory which are needed in the sequel. All spaces in this (and all other) sections will be separable and metric. Proofs of the results stated here, except for Theorem 2.5, may be found in [Ch_4, Chapters I–IV].

A closed subset A of a Q-manifold M is said to be a *Z-set* if for each open cover α of M there is a map $f: M \to M - A$ with $f \overset{\alpha}{\simeq} \mathrm{id}$. This is equivalent to requiring an α-homotopy $f: M \times I \to M$ with $f_0 = \mathrm{id}$ and $f_t(M) \subset M - A$ for all $t > 0$. An imbedding $f: X \to M$ is called a Z-imbedding if $f(X)$ is a Z-set in M.

The basic result on Z-sets in Q-manifolds is

THEOREM 2.1 (Z-set unknotting [A-Ch]). *If $f, g: X \to M$ are proper homotopic Z-imbeddings then there is a homeomorphism $H: M \to M$ such that $H \circ f = g$. If the homotopy is limited by an open cover α of M then H can be chosen α-close to the identity.*

We will also need stability and collaring theorems.

THEOREM 2.2 (Stability [A-S]). (i) *Let M be a Q-manifold and let α be an open cover of M. Then there is a homeomorphism $\gamma: M \times Q \to M$ which is α-close to projection.*

(ii) *The same statement holds for l_2-manifolds when Q is replaced by l_2.* ■

THEOREM 2.3 (Collaring [Ch_3]). *Let N and M be Q-manifolds and let $f: N \to M$ be a Z-imbedding. Then there is an open imbedding $F: N \times [0, 1) \to M$*

such that $F(n, 0) = f(n)$. ∎

If $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ is a finite or infinite direct system of maps, we define $\text{Map}(X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots)$ to be the space obtained from the disjoint union $\amalg_{i=1} X_i \times [0, 1]$ by identifying $(x, 1) \in X_i \times [0, 1]$ with $(f_i(x), 0) \in X_{i+1} \times [0, 1]$. Note (see figure 1) that $\text{Map}(X \xrightarrow{f} Y)$ is not the usual mapping cylinder of f.

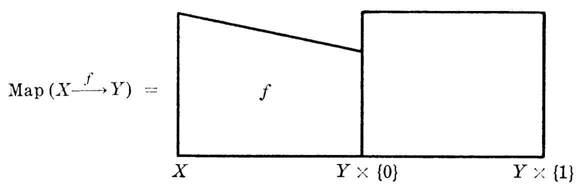


FIGURE 1.

The point of this is that $Y \times \{1\}$ is a Z-set in $\text{Map}(X \xrightarrow{f} Y)$, while the base of the usual mapping cylinder need not be a Z-set in the mapping cylinder.

An easy corollary of the stability and collaring theorems (compare Cor. 4.10) is

THEOREM 2.4 (Weak mapping cylinder theorem). *If* $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$ *is a sequence of Q-manifolds and Z-imbeddings and* α *is an open cover of* X_n, *then the collapse* c: $\text{Map}(X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n) \to X_n \times \{1\}$ *is* α*-close to a homeomorphism.* ∎

The following general deformation result of Fathi and Visetti is crucial.

THEOREM 2.5 [F-V]. *Let* $C \subset U \subset M$ *where M is a Q-manifold, C is compact, and U is open. Let* η: $U \to M$ *be the inclusion and let* $I(U, M)$ *be the space of open imbeddings of U into M with the compact-open topology. Then there is a neighborhood P of* η *in* $I(U, M)$ *and a map* Φ: $P \to H(M)$ *such that* $\Phi(g)|C = g|C$, $\Phi(\eta) = \text{id}$ *and* $\Phi(g)|(M - U) = \text{id}$.

If C is closed but not compact, then for each open cover α *of M there is an open cover* β *of M such that if* g: $U \to M$ *is* β*-close to the inclusion then there is a homeomorphism* $\bar{g}$: $M \to M$ *such that* $\bar{g}|U \overset{\alpha}{\simeq} g$, $\bar{g}|C = g|C$, *and* $\bar{g}|(M - U) = \text{id}$. ∎

3. A generalization of the CE approximation theorem

In this section we will prove our generalization of the CE approximation theorem and derive some of its corollaries. A parameterized version of this theorem is used in Section 5 to show that $H(M)$ is an ANR. The theorem of this section cannot be derived directly from the parameterized version of

Section 5 because (1) the manifolds in Section 5 are required to be compact and (2) technicalities in the parameterized version obscure the dependence of β on α and M.

In what follows, a map such as $p_A: A \times B \to A$ will denote projection onto A.

THEOREM 3.1 (α-approximation theorem). *Let M be a Q-manifold. For each open cover α of M there is an open cover β of M such that if N is a Q-manifold and $f: N \to M$ is a β-equivalence then f is α-close to a homeomorphism.*

Proof. Let g be a β-inverse for f. Without loss of generality, f and g are Z-imbeddings.

Step I. The construction of an approximation to $f \times \mathrm{id}: N \times [0, \infty) \to M \times [0, \infty)$.

Pf. (I). Let $X = \mathrm{Map}(N \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{f} M \to \cdots)$. We first construct homeomorphisms $H: N \times [0, \infty) \to X$ and $K: X \to M \times [0, \infty)$ such that $p_M \circ (K \circ H)$ is St β-close to $f \circ p_N$. See Figure 2.

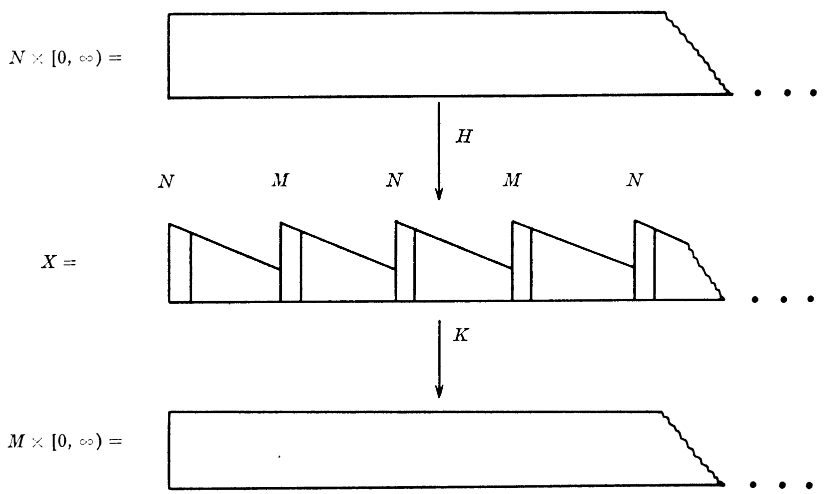


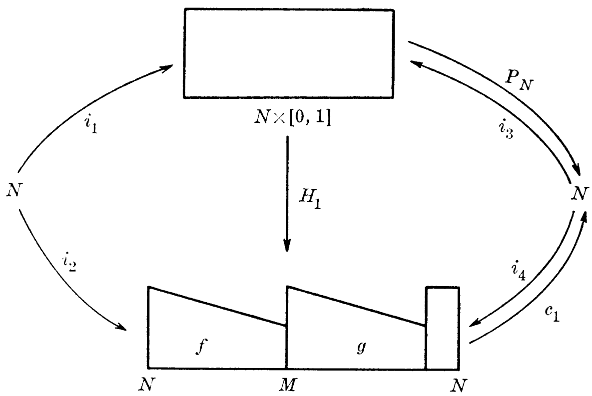
FIGURE 2.

We proceed to construct H and K.

A. *There is a homeomorphism H_1 from $N \times [0, 1]$ to* $\mathrm{Map}(N \xrightarrow{f} M \xrightarrow{g} N)$ *such that*

(i) *p_N is $f^{-1}(\beta)$-close to $c_1 \circ H_1$, where c_1 is the collapse of* $\mathrm{Map}(N \xrightarrow{f} M \xrightarrow{g} N)$ *to N.*

(ii) *$H_1 \circ i_1 = i_2$ and $H_1 \circ i_3 = i_4$.*



Proof. Let $\gamma_1: N \times [0, 1] \to N$ and $\gamma_2: \mathrm{Map}(N \to M \to N) \to N$ be homeomorphisms closely approximating p_N and c_1, respectively. Then $\gamma_1 \circ i_1$ is $f^{-1}(\beta)$-homotopic to $\gamma_2 \circ i_2$ and $\gamma_1 \circ i_3$ is $f^{-1}(\beta)$-homotopic to $\gamma_2 \circ i_4$ (actually, we can make the last homotopy as small as we want). Thus, there is a homeomorphism $\gamma_3: N \to N$ such that γ_3 is $f^{-1}(\beta)$-close to the identity and such that $\gamma_3 \circ \gamma_1 \circ i_1 = \gamma_2 \circ i_2$ and $\gamma_3 \circ \gamma_1 \circ i_3 = \gamma_2 \circ i_4$. We set $H_1 = \gamma_2^{-1} \circ \gamma_3 \circ \gamma_1$. Then

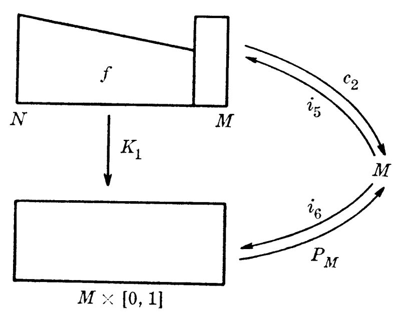
$$c_1 \circ H_1 = c_1 \circ \gamma_2^{-1} \circ \gamma_3 \circ \gamma_1 \simeq_p \gamma_3 \circ \gamma_1 \overset{f^{-1}(\beta)}{\simeq_p} \gamma_1 \simeq_p p_N .$$

That $H_1 \circ i_1 = i_2$ and $H_1 \circ i_3 = i_4$ is clear.

B. *There is a homeomorphism K_1 from* Map $(N \xrightarrow{f} M)$ *to* $M \times [0, 1]$ *such that*

(iii) $i_6 = K_1 \circ i_5$ *and*

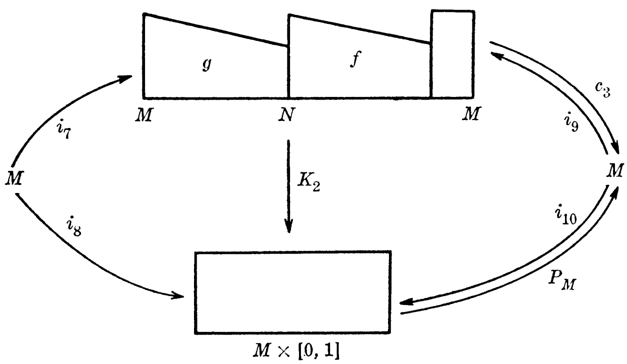
(iv) $p_M \circ K_1 \simeq_p c_2$, *where c_2 is the collapse. K_1 may be chosen so that the homotopy is as small as we please.*



C. *There is a homeomorphism K_2:* Map $(M \xrightarrow{g} N \xrightarrow{f} M) \to M \times [0, 1]$ *such that*

(v) $K_2 \circ i_7 = i_8$ *and* $K_2 \circ i_9 = i_{10}$.

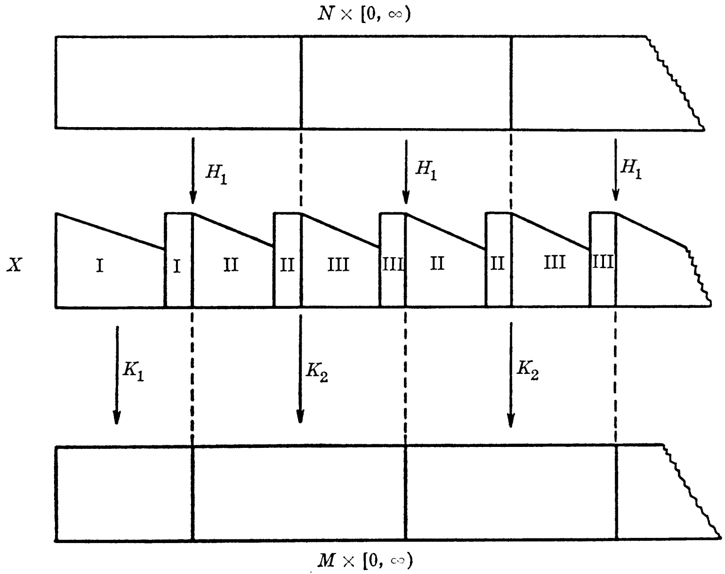
(vi) $p_M \circ K_2 \overset{\beta}{\simeq}_p c_3$, *where c_3 is the collapse* Map $(M \xrightarrow{g} N \xrightarrow{f} M) \to M$.



Proof. The proofs of B and C are entirely analogous to the proof of A.

D. *The construction of H and K.*

Pf.



H is constructed by gluing together infinitely many copies of H_1. K is constructed from one copy of K_1 and infinitely many copies of K_2.

E. *The verification that $F = K \circ H$ has the desired properties.*

We consider three cases according to whether $H(x)$ lies in Regions I, II, or III of X as shown.

Case 1. $H(x) \in$ Region I:

$$f \circ p_N(x) \overset{\beta}{\simeq}_p f \circ c_1 \circ H_1(x) = f \circ g \circ c_2 \circ H_1(x)$$
$$\overset{\beta}{\simeq}_p c_2 \circ H_1(x) \simeq_p p_M \circ K_1 \circ H_1(x) = p_M \circ F(x) .$$

Case 2. $H(x) \in$ Region II.

$$f \circ p_N(x) \overset{\beta}{\simeq}_p f \circ c_1 \circ H_1(x) = c_3 \circ H_1(x)$$
$$\overset{\beta}{\simeq}_p p_M \circ K_2 \circ H_1 = p_M \circ F(x) .$$

Case 3. $H(x) \in$ Region III.

$$f \circ p_N(x) \overset{\beta}{\simeq}_p f \circ c_1 H_1(x) = f \circ g \circ c_3 \circ H_1(x)$$
$$\overset{\beta}{\simeq}_p c_3 \circ H_1(x) \overset{\beta}{\simeq}_p p_M \circ K_2 \circ H_1(x) = p_M \circ F(x) .$$

This completes the proof of Step I.

Step II. The construction of an approximation to $f \times \mathrm{id}: N \times [0, 1] \to M \times [0, 1]$.

Pf. (II). Let $\rho: [0, \infty) \to [0, 1)$ be a homeomorphism and define $h_1: N \times [0, 1) \to M \times [0, 1)$ by $h_1 = (\rho \times \mathrm{id}_M)(F)(\rho \times \mathrm{id}_N)^{-1}$, where F is as in Step I. By choosing ρ appropriately and requiring β to be fine we can obtain any desired degree of approximation to $f \times \mathrm{id}$. Let $h_2: N \times (0, 1] \to M \times (0, 1]$ be defined by $h_2(n, t) = h_1(n, 1 - t)$.

Consider $h_1 h_2^{-1} | (M \times (1/4, 3/4)): M \times (1/4, 3/4) \to M \times [0, 1]$. If h_1 and h_2 are γ-close to $f \times \mathrm{id}$, we have

$$h_1 h_2^{-1} \overset{\gamma}{\sim} (f \times \mathrm{id}) h_2^{-1} \overset{\gamma}{\sim} h_2 h_2^{-1} = \mathrm{id} ,$$

so $h_1 h_2^{-1} | M \times (1/4, 3/4)$ is $St(\gamma)$-close to id. By Theorem 2.5, if γ is sufficiently fine, there is an extension $h_3: M \times [0, 1] \to M \times [0, 1]$ of $h_1 h_2^{-1} | M \times [1/3, 2/3]$ which is close to the identity. Thus, for sufficiently fine β and appropriate ρ, the map $h: N \times [0, 1] \to M \times [0, 1]$ defined by

$$h(n, t) = \begin{cases} h_1(n, t), & t \leqq 1/2 \\ h_3 \circ h_2(n, t), & t \geqq 1/2 \end{cases}$$

is well-defined and satisfies $p_M \circ h \overset{\alpha}{\sim} f \circ p_N$. This completes Step II.

The remainder of the proof of Theorem 3.1 is easy. If we choose γ_N and γ_M to be homeomorphisms close to $p_N: N \times [0, 1] \to N$ and $p_M: M \times [0, 1] \to M$, $\gamma_M \circ h \circ \gamma_N^{-1}$ is a homeomorphism α-close to f. ■

COROLLARY 3.2. *Let L be a locally compact* ANR. *Then there is an open cover α of L such that if K is another locally compact* ANR *and $f: K \to L$ is an α-equivalence then f is a simple homotopy equivalence.*

Proof. By a result of R.D. Edwards (see [Ch_4, §44]), $L \times Q$ is a Q-manifold. By Theorem 3.1, there is a cover α_1 of $L \times Q$ such that any α_1-equivalence from a Q-manifold to $L \times Q$ is homotopic to a homeomorphism. Refine α_1 to a cover by sets of the form $U \times V$ where $U \subset L$ and $V \subset Q$ are open. For each $l \in L$ let $U_1^l \times V_1^l, \cdots, U_{k_l}^l \times V_{k_l}^l$ be a finite cover of $\{l\} \times Q$ and write $U^l = \bigcap_{i=1}^{k_l} U_i^l$. Let $\alpha = \{U_l \mid l \in L\}$.

If $f: K \to L$ is an α-equivalence with α-inverse g, then $f \times \mathrm{id}: K \times Q \to L \times Q$ is an α_1-equivalence with α_1-inverse $g \times \mathrm{id}$. Thus, $f \times \mathrm{id}$ is proper homotopic to a homeomorphism and, by [Ch_2], f is a simple homotopy equivalence. ■

COROLLARY 3.3. *Let X be a compact* ANR. *Then there is an $\varepsilon > 0$ such that if $\mathfrak{U}$ is a finite open cover of X by open sets such that*

(i) diam $U < \varepsilon$ *for each $U \in \mathfrak{U}$ and*

(ii) *intersections of subcollections of $\mathfrak{U}$ are either empty or contractible then X and $N(\mathfrak{U})$ have the same simple homotopy type. Here, $N(\mathfrak{U})$ is the nerve of $\mathfrak{U}$.*

Proof. If ε is sufficiently small, X is homotopy equivalent to $N(\mathfrak{U})$ in such a way that Corollary 3.2 applies. The proof is a modification of Example 1 of Chapter 5 of [Su], which is a proof that the Lubkin nerve of a complex is homotopy equivalent to the complex. One can prove statements analogous to Corollary 3.3 about sufficiently fine Lubkin nerves and brick decompositions ([B, 13.2]). ■

THEOREM 3.4. *Let M and N be l_2-manifolds. If $f: M \to N$ is an α-equivalence then f is* $\mathrm{St}(\alpha)$*-close to a homeomorphism.*

Proof. The proof of Step I of Theorem 3.1 works equally well for l_2-manifolds. Let $h: N \times [0, \infty) \to M \times [0, \infty)$ be a homeomorphism approximating $f \times \mathrm{id}$. Since $l_2 \cong l_2 \times [0, \infty)$, Theorem 2.2 implies that we can choose homeomorphisms $\gamma_N: N \times [0, \infty) \to N$ and $\gamma_M: M \times [0, \infty) \to M$ as close as we like to projection. Thus, $\gamma_M h \gamma_N^{-1}$ is a homeomorphism approximating f. ■

One can modify the argument of Theorem 3.1 above to prove a local theorem.

Definition 3.5. Let M be a Q-manifold, let $U \subset M$ be an open set, and let α be an open cover of M. A map $f: N \to M$ is said to be an *α-equivalence over U* if there is a map $g: U \to N$ such that $g \circ f \mid f^{-1}(U)$ and $f \circ g$ are $f^{-1}(\alpha)$- and α-homotopic to the appropriate inclusions.

THEOREM 3.6. *If C is a closed subset of M, U is an open subset of M containing C, and α is an open cover of M, then there is an open cover β of M such that if $f: N \to M$ is a β-equivalence over U then f is α-close to a map $\bar{f}: N \to M$ such that $\bar{f} \mid \bar{f}^{-1}(C)$ is a homeomorphism and $f = \bar{f}$ over $M - U$.* ■

Remark 3.7. The reader who is familiar with completely regular maps will note that Theorem 1(1) of [Ch-F] follows from Theorem 3.1 by a selection argument. Similarly, Theorem 1(i) of [F] follows from Theorem 3.1 and R.D. Edwards' result that the product of a locally compact ANR with Q is a Q-manifold. ■

4. Parameterized preliminaries

In this section we will prove parameterized versions of some basic theorems of Q-manifold theory. Our starting point is a slightly weakened version of a canonical Z-set unknotting theorem due to Chapman.

PROPOSITION 4.1 ([Ch_5, Thm. 5.1]). *Let A be a compact metric space and let M be a Q-manifold. Then there is a continuous map ϕ: $C^*(A\times I, M\times s)\to H(M\times Q)$ such that $\phi(F)\circ F_0 = F_1$. Here, $C^*(A\times I, M\times s)$ denotes $\{F: A\times I\to M\times s \mid F|A\times\{0\}$ and $F|A\times\{1\}$ are $1-1\}$ (with the C-O topology) and $F_t = F|A\times\{t\}$ ($s\prod_{i=1}^{\infty}(-1, 1)\subset Q$).*

The difficulty in using this theorem is that it is not expressed in topologically invariant form. By defining sliced Z-sets and imitating Chapter 2 of [Ch_4], we prove a parameterized analogue of Theorem 2.1.

Definition 4.2. A map f: $X\times B\to Y\times B$ is said to be *fiber-preserving* (f.p.) if $p_B\circ f = p_B$. A closed set $A\subset M\times B$ is said to a *sliced Z-set* if for each open cover β of $M\times B$ there is an f.p. map f: $M\times B\to(M\times B)-A$ which is β-close to id. An f.p. map f: $A\times B\to M\times B$ is said to be a *sliced Z-imbedding* if $f(A\times B)$ is a sliced Z-set.

Remark 4.3. The following modification of Example 5.2 of [Ch_5] shows that a set $A\subset M\times B$ such that $A\cap M\times\{b\}$ is a Z-set in $M\times\{b\}$ is not necessarily a sliced Z-set. Let $Q\times Q$ be fibered over Q by projection onto the first factor. The diagonal $\Delta\subset Q\times Q$ is not a sliced Z-set. An f.p. map h: $Q\times Q\to Q\times Q-\Delta$ would have the form $h(q_1, q_2)=(q_1, h_1(q_1, q_2))$ with $h_1(q_1, q_2)\neq q_1$. Then $\bar{h}(q_1, q_2)=(h_1(q_1, q_2), 0)$ would be a map from $Q\times Q$ to itself with no fixed point, a contradiction.

Wong [W_1] has shown that if B is a finite-dimensional polyhedron, then a closed set $A\subset M\times B$ is a sliced Z-set if and only if $A\cap M\times\{b\}$ is a Z-set in $M\times\{b\}$ for each $b\in B$. Wong has developed an entirely satisfactory Q-manifold theory parameterized over finite complexes. ■

For the remainder of this section, M will be a compact Q-manifold and B will be a separable metric space. Our main objective is to prove the theorem below.

THEOREM 4.4 (Z-set unknotting). *Let A be compact and let $f, g: A\times B\to M\times B$ be* f.p. *homotopic sliced Z-imbeddings. Then*

(i) *there is an* f.p. *homeomorphism $H: M\times B\to M\times B$ such that $H\circ f=g$;*

(ii) *if A_0 is a closed subset of A, $f|(A_0\times B) = g|(A_0\times B)$, and the homotopy from f to g is stationary on $A_0\times B$, then $H|(A_0\times B) = \mathrm{id}$;*

(iii) *if the homotopy is limited by an open cover β of $M\times B$, then H can be chosen β-close to the identity;*

(iv) *if B_0 is a closed subset of B and the homotopy is stationary on $A\times B_0$ then we can choose $H|(M\times B_0) = \mathrm{id}$.*

We will prove some intermediate propositions before arriving at a proof of Theorem 4.4. We remark that Theorem 4.4 remains valid for noncompact M and A if all maps and homotopies are proper.

PROPOSITION 4.5. *Let A be compact and let $f, g: A\times B\to M\times s\times B$ be* f.p. *imbeddings such that f and g are* f.p. *homotopic in $M\times Q\times B$. Then there is an* f.p. *homeomorphism $H: M\times Q\times B\to M\times Q\times B$ such that $H\circ f=g$. Furthermore, if the homotopy from f to g is limited by β, then H may be chosen β-close to the identity.*

Proof. The original homotopy can be approximated arbitrary closely by a homotopy in $M\times s\times B$. The unestimated version then follows directly from Proposition 4.1. The estimated version follows from the unestimated version by the Anderson-McCharen trick. See [A-M], [Ch_4, § 19] or the proof of our Theorem 4.4 (which follows Proposition 4.8) for details. ∎

We will need an f.p. version of the stability theorem.

THEOREM 4.6 (Fibered stability). *Let M be a Q-manifold and let β be an open cover of $M\times B$. Then there is an* f.p. *homeomorphism $h: M\times Q\times B\to M\times B$ which is β-close to projection.*

Proof. Let $h_1: M\times Q'\to M$ be a homeomorphism. It suffices to find an f.p. homeomorphism $\alpha: M\times Q'\times Q\times B\to M\times Q'\times B$ which is $(h_1\times \mathrm{id})^{-1}(\beta)$-close to projection.

In [W_2], R. Wong constructs a map $\bar{h}: Q\times Q\times[1,\infty)\to Q$ such that

(i) $\bar{h}_t: Q\times Q\to Q$ is a homeomorphism, $1\leqq t<\infty$,

(ii) if $n\leqq t$ then $\bar{h}_t$ is 2^{-n}-close to projection on the first factor. (See also [Ch_4, § 13].)

One can now construct a function $\rho: M\times B\to[1,\infty)$ such that the homeomorphism defined by $\alpha(mq_1, q_2, b) = (m, \bar{h}(q_1, q_2, \rho(m, b)), b)$ has the desired properties. ∎

PROPOSITION 4.7. *Let $A\subset M\times Q\times B$ be a sliced Z-set and let β be an* *open cover of* $M \times Q \times B$. *If* X *is compact and* $f\colon X \times B \to M \times Q \times B$ *is an* f.p. *map, then there is an* f.p. *imbedding* $\bar{f}\colon X \times B \to M \times s \times B - A$ *such that* $\bar{f}$ *is* β*-close to* f.

Proof. Let β_1 be a star refinement of β. Since A is a sliced Z-set, there is an f.p. map $f'\colon X \times B \to M \times Q \times B$ which is β_1-close to f such that $f'(X \times B) \cap A = \varnothing$. Let β_2 be an open cover of $M \times Q \times B$ such that $\operatorname{St}\beta_2$ refines β_1 and such that if f^* is $\operatorname{St}\beta_2$-close to f', then $f^*(X \times B) \cap A = \varnothing$.

Choose f.p. f'' β_2-close to f' such that $f''(X \times B) \subset M \times s \times B$. Our task reduces to finding $f'''\colon X \times B \to M \times s \times B$ such that f''' is 1-1 and f''' is β_2-close to f''.

Let s' be a copy of s and let $\gamma\colon s' \times s \times B \to s \times B$ be an f.p. homeomorphism such that $\mathrm{id} \times \gamma\colon M \times s' \times s \times B \to M \times s \times B$ is β_2-close to $p_{M \times s \times B}$. Let $g\colon X \times B \to s'$ be an imbedding. Then $f'''(x, b) = (\mathrm{id} \times \gamma)(p_M \circ f''(x, b), g(x, b), p_s \circ f''(x, b), b)$ has the desired properties. ∎

PROPOSITION 4.8. *Let* $A \subset M \times Q \times B$ *be a sliced* Z*-set and let* β *be an open cover of* $M \times Q \times B$. *Then there is a homeomorphism* $h\colon M \times Q \times B \to M \times Q \times B$ *which is* β*-close to the identity and such that* $h(A) \subset M \times s \times Q$.

Proof. We sketch a proof along the lines of [Ch_4, § 10]. The details may be found there. Let $W = \{1\} \times \prod_{i=1}^{\infty} [-1, 1]$ be a face of Q. Given an open cover β_1 of $M \times Q \times B$, we construct a homeomorphism h_1 which is β_1-close to the identity and such that $h_1(A) \cap W = \varnothing$.

Let $k_t\colon Q \to Q$ be an isotopy such that $k_0 = \mathrm{id}$ and $k_t(W) \subset s$, $0 < t \leq 1$. Such an isotopy may be constructed as follows. Let $k'\colon W \times [0, 1] \to Q$ be an imbedding such that k' is the inclusion and $k'(W \times (0,1]) \subset s$. k' is a Z-imbedding, so there is an isotopy (see [A-M] or the discussion following the proof of this proposition) $k_t\colon Q \to Q$ with $k_0 = \mathrm{id}$ and $k_t(w) = k'(w, t)$ for each $(w, t) \in W \times [0, 1]$. This is the desired isotopy.

Define $g\colon M \times Q \times I \times B \to M \times Q \times B$ by $g_t(m, q, b) = (m, k_t(q), b)$. Then $g^{-1}(\beta_1)$ is an open cover of $M \times Q \times I \times B$. Let $\rho\colon M \times Q \times B \to (0, 1)$ be a continuous function such that for each m, q, and b the set $\{m\} \times \{q\} \times [0, \rho(m, q, b)] \times \{b\}$ is contained in a single element of $g^{-1}(\beta_1)$. Then $\bar{g}(m, q, b) = (m, k_{\rho(m,q,b)}(q), b)$ is an f.p. homeomorphism limited by β_1 such that $\bar{g}(M \times W \times B) \subset M \times s \times B$. Since A is a Z-set, there is an f.p. imbedding $j\colon M \times W \times B \to (M \times s \times B) - A$ which is as close as we like to $\bar{g} | M \times W \times B$. By Proposition 4.5, there is an f.p. homeomorphism $J\colon M \times Q \times B \to M \times Q \times B$ such that $J\bar{g}(M \times W \times B) \cap A = \varnothing$ and $J\bar{g}$ is β_1-close to the identity. $h_1 = (J\bar{g})^{-1}$ is, therefore, our desired homeomorphism.

We can now (as in [Ch_4, § 10]) choose a sequence $\{h_i\}$ of homeomorphisms such that the infinite left composition $\cdots \circ h_n \circ h_{n-1} \circ \cdots \circ h_1$ converges to a homeomorphism h such that h is β-close to the identity and $h(A) \subset M \times s \times B$. The idea is to use h_i to push A off of $M \times W_i \times B$ where W_i is the i^{th} face of Q. ■

We now proceed to prove Theorem 4.4. Write M as $M_1 \times Q$. Then we have homotopic sliced Z-imbeddings $f, g\colon A \times B \to M_1 \times Q \times B$. By Proposition 4.5, there is an f.p. homeomorphism $H\colon M_1 \times Q \times B \to M_1 \times Q \times B$ such that $H(f(A \times B) \cup g(A \times B)) \subset M_1 \times s \times B$. Theorem 4.4 (i) now follows from Proposition 4.5. The estimated version (iii) follows by the Anderson-McCharen trick. We modify this trick slightly to prove (ii).

Let $A' = A \times I/\sim$ where $(a, t) \sim (a', t')$ if $a = a' \in A_0$. The idea is to consider the homotopy to be a map of $A' \times B$ into $M \times Q \times B$. Approximating this map (rel f and g) by a sliced Z-imbedding $G\colon A' \times B \to M \times Q \times B$ we can use part (i) of the theorem to find a homeomorphism $H\colon M \times Q \times B \to M \times Q \times I \times B$ such that $H \circ G(A' \times B)$ is nicely positioned with respect to the I-factor (see Fig. 3).

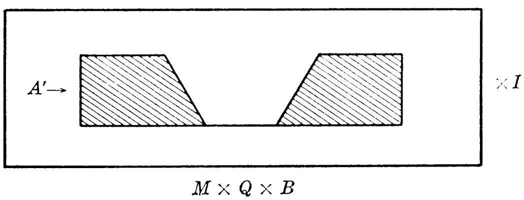


FIGURE 3.

It is now easy to construct the required homeomorphism by hand. For details see [A-M] or [Ch_4].

(iv) If $f, g\colon A \times B \to M \times B$ are sliced Z-imbeddings which are f.p. homotopic via a homotopy F which is the identity over B_0, we define a cover β of $M \times (B - B_0)$ as follows. For each $b \in B$ define functions $\delta_1(b)$ and $\delta_2(b)$ by $\delta_1(b) = 1/2\, d(b, B_0)$ and $\delta_2(b) = 2 \sup_{a \in A} \operatorname{diam} F(a \times I \times b)$. δ_1 and δ_2 are continuous functions. Note that $\delta_1^{-1}(0) = \delta_2^{-1}(0) = B_0$. Let

$$U(m, b) = \{(m', b') \mid d(m, m') < \delta_2(b),\ d(b, b') < \delta_1(b)\}$$

and let $\beta = \{U(m, b) \mid (m, b) \in M \times B\}$. Then $F \mid M \times (B - B_0)$ is a β-homotopy and there is an f.p. homeomorphism $\bar{H}\colon M \times (B - B_0) \to M \times (B - B_0)$ which is limited by β. We claim that $\bar{H}$ can be continuously extended by the identity to an f.p. map $H\colon M \times B \to M \times B$ such that $H \circ f = g$.

If $\lim (m_i, b_i) = (m, b)$, $b \in b_0$, let $(m_i', b_i') \in M \times (B - B_0)$ be such that $(m_i, b_i) \in U(m_i', b_i')$. Since $d(b_i, b_i') < 1/2\, d(b_i', b)$, $d(b_i', b) < 2d(b_i, b)$ and $\lim b_i' = b$. Thus,

$$\lim_{i\to\infty} \delta_1(b_i') = \lim_{i\to\infty} \delta_2(b_i') = 0, \text{ and } \lim_{i\to\infty} \bar{H}(m_i, b_i) = (m, b).$$

This completes the proof of Theorem 4.4. ∎

PROPOSITION 4.9 (Fibered mapping approximation). *Let A be compact with $A_0 \subset A$ a closed subset. If $f: A \times B \to M \times B$ is an* f.p. *map such that $f|(A_0 \times B)$ is a sliced Z-imbedding and β is an open cover of $M \times B$ then there is an* f.p. *sliced Z-imbedding $\bar{f}$, β-close to f, such that $\bar{f}|(A_0 \times B) = f|(A_0 \times B)$.*

Proof. Let $\gamma: M \times Q \times B \to M \times B$ be an f.p. homeomorphism close to $p_{M\times B}$. By Theorem 4.4 (i), we may assume that $\gamma^{-1} \circ f | A_0 \times B \subset M \times \{0\} \times B$. Let $\bar{g}: A/A_0 \to s \subset Q$ be an imbedding with $\bar{g}([A_0]) = 0$ and let $g: A \to s$ be the map induced by $\bar{g}$. The map $f': A \times B \to M \times s \times B$ defined by $f'(a, b) = (p_M \circ \gamma^{-1} \circ f(a, b), g(a), b)$ is an imbedding (and therefore a sliced Z-imbedding), so $\bar{f} = \gamma \circ f'$ is the desired sliced Z-imbedding approximating f. ∎

COROLLARY 4.10 (Collaring theorem). *Let M and N be Q-manifolds and let $f: N \times B \to M \times B$ be a sliced Z-imbedding. Then there is an* f.p. *open imbedding $F: N \times [0, 1) \times B \to M \times B$ such that $F(n, 0, b) = f(n, b)$.*

Proof. The idea is to show that $f(N \times B)$ is locally f.p. collared and to apply Brown's collaring theorem. The details are similar to those in [Ch-F, Prop. 2.5]. ∎

COROLLARY 4.11 (Weak mapping cylinder theorem). *If M and N are Q-manifolds and $f: N \times B \to M \times B$ is a sliced Z-imbedding and β is an open cover of $M \times B$ then the collapse of* Map $(N \times B \xrightarrow{f} M \times B)$ *to $M \times B$ is β-close to an* f.p. *homeomorphism.*

Proof. Perform the collapse in two stages. c_1: Map $(N \times B \xrightarrow{f} M \times B) \to M \times I \times B$ and $c_2: M \times I \times B \to M \times B$. The collaring theorem allows us to approximate c_1 by a homeomorphism. If $k: M \times Q \times B \to M \times B$ is a homeomorphism β'-close to projection, and $h: Q \times I \to Q$ is a homeomorphism then $M \times B \times I \xrightarrow{k^{-1} \times \mathrm{id}} M \times Q \times B \times I \xrightarrow{\mathrm{id}_{M\times B} \times h} M \times Q \times B \xrightarrow{k} M \times B$ is a homeomorphism St β'-close to projection. Thus, c_2 may be approximated as closely as we like by a homeomorphism and the desired approximation to c exists. ∎

COROLLARY 4.12. *Let M be a Q-manifold and let $A \subset M$ be a Z-set. Then there is an open set $U \subset H(M)$ such that $H_A(M) \subset U \subset H(M)$ and $H_A(M)$ is a retract of U.*

Proof. Choose a cover α of M so fine that maps into M which are α-close are canonically homotopic. Let $U \subset H(M)$ be the set $\{h | h|A$ is α-close to inclusion$\}$. Define $F: M \times U \to M \times U$ by $F(m, h) = (h(m), h)$ and note that there is an f.p. homotopy from $F|A \times U$ to $\mathrm{id}|A \times U$. This homotopy is stationary on $A \times H_A(M)$.

$A \times U$ is a sliced Z-set in $M \times U$. Since F is an f.p. homeomorphism, $F(A \times U)$ is also a sliced Z-set. Thus, by Theorem 4.4 there is an f.p. homeomorphism $H: M \times U \to M \times U$ with $H \circ F | A \times U = \mathrm{id} | A \times U$ and $H = \mathrm{id}$ on $M \times H_A(M)$. The map $h \mapsto H \circ F | M \times \{h\}$ is the desired retraction of U onto $H_A(M)$. ■

Remark. It is not difficult to adapt this construction to prove the weak handle lemma of Fathi and Visetti [F-V]. This avoids the torus trick used in their paper. ■

5. $H(M)$ is an ANR

In this section, M and N will denote compact Q-manifolds and $\mathcal{C}(X, Y)$ will denote the space of continuous functions from X to Y with the compact-open topology.

Definition 5.1. Let $HE(N, M) \subset \mathcal{C}(N, M) \times \mathcal{C}(M, N) \times \mathcal{C}(N)^I \times \mathcal{C}(M)^I$ be the set $\{(f, g, h, k) | h(0) = g \circ f,\ h(1) = \mathrm{id},\ k(0) = f \circ g,\ k(1) = \mathrm{id}\}$. This is the space of homotopy equivalences, homotopy inverses, and homotopies to the identity. Note that the closed subspace of $HE(N, M)$ consisting of homeomorphisms, their inverses, and constant homotopies is homeomorphic to $H(N, M)$, the space of homeomorphisms from N to M. We will identify $H(N, M)$ with this subspace. We study $H(N, M)$ rather than $H(M)$ to avoid confusing the domain and range during the argument.

PROPOSITION 5.2. *$HE(N, M)$ is an* ANR.

Proof. We show that $HE(N, M)$ is a retract of an open subset of $\mathcal{C}(N, M) \times \mathcal{C}(M, N) \times \mathcal{C}(N)^I \times \mathcal{C}(M)^I$. Let α_M be an open cover of M such that maps into M which are α_M-close are canonically homotopic. Let α_N be a similar cover of N. Consider the neighborhood U of $HE(N, M)$ in $\mathcal{C}(N, M) \times \mathcal{C}(M, N) \times \mathcal{C}(N)^I \times \mathcal{C}(M)^I$ defined by

$$U = \{(f, g, h, k) | h(0) \overset{\alpha_N}{\simeq} g \circ f,\ h(1) \overset{\alpha_N}{\simeq} \mathrm{id},\ k(0) \overset{\alpha_M}{\simeq} f \circ g,\ k(1) \overset{\alpha_M}{\simeq} \mathrm{id}\}.$$

The retraction is defined by $(f, g, h, k) \mapsto (f, g, h', k')$ where h' and k' are homotopies from $g \circ f$ and $f \circ g$. Here is the definition of h'. k' is defined similarly.

$$h'_t = \begin{cases} \text{canonical homotopy from } g \circ f \text{ to } h(0) & 0 \leq t \leq \dfrac{d(g \circ f, h(0))}{3(1 + d(g \circ f, h(0)))} \\ \text{homotopy } (h) \text{ from } h(0) \text{ to } h(1) & \\ \qquad \dfrac{d(g \circ f, h(0))}{3(1 + d(g \circ f, h(0)))} \leq t \leq \dfrac{d(h(1), \mathrm{id})}{3(1 + d(h(1), \mathrm{id}))} & \\ \text{canonical homotopy from } h(1) \text{ to id} & \dfrac{d(h(1), \mathrm{id})}{3(1 + d(h(1), \mathrm{id}))} \leq t \leq 1 \end{cases}$$

Each homotopy is reparameterized linearly to fit into its assigned interval. ∎

THEOREM 5.3. *There is a continuous retraction of a neighborhood of* $H(N, M)$ *in* $HE(N, M)$ *onto* $H(N, M)$.

This theorem is the goal of the remainder of this section.

Definition 5.4. An f.p. proper map $f: X\times B \to Y\times B$ is called an f.p. β-equivalence, β a cover of $Y\times B$, if there exist an f.p. proper map $g: Y\times B\to X\times B$ and f.p. proper homotopies h and k such that $h: g\circ f \overset{f^{-1}(\beta)}{\sim} \mathrm{id}$ and $k: f\circ g \overset{\beta}{\sim} \mathrm{id}$.

PROPOSITION 5.5. *Let N and M be Q-manifolds and let $f: N\times B\to M\times B$ be an f. p. β-equivalence. Then there is an f. p. homeomorphism $F: N\times[0,\infty)\times B\to M\times[0,\infty)\times B$ such that $p_{M\times B}\circ F$ is* St *β-close to $p_{M\times B}\circ(f\times \mathrm{id})$ and such that $d(p_{[0,\infty)}\circ F, p_{[0,\infty)}) < 3$.*

Proof. The proof of Step I of Theorem 3.1 can be translated word for word into this context. ∎

PROPOSITION 5.6. *There is a continuous map* $\Phi: HE(N, M)\to H(N\times[0,1), M\times[0, 1))$ *such that* $\Phi(h, h^{-1}, \mathrm{id}, \mathrm{id}) = h\times \mathrm{id}$.

Proof. The idea is to map $HE(N, M) - H(N, M)$ to $H(N\times[0, 1), M\times[0, 1))$, exercising sufficient control that we can extend to all of $HE(N,M)$ in a way which fulfills our requirements.

Define $\delta_1, \delta_2: HE(N, M)\to[0, \infty)$ by

$$\delta_1(f, g, h, k) = 2\max\{\mathrm{diam}\, h, \mathrm{diam}\, k, \mathrm{diam}\, f\circ h\}, \text{ where } h, k, \text{ and } f\circ h \text{ are considered as paths in } \mathcal{C}(N) \text{ or } \mathcal{C}(M).$$

$$\delta_2(f, g, h, k) = 1/2\, d((f, g, h, k), H(N, M)) .$$

Note that $\delta_1^{-1}(0) = \delta_2^{-1}(0) = H(N, M)$. Let $\beta = \{U(m; f, g, h, k)\}$ where

$$U(m; f, g, h, k) = \{(m'; f', g', h', k') \mid d(m, m') < \delta_1(f, g, h, k) \text{ and } d((f, g, h, k), (f', g', h', k')) < \delta_2(f, g, h, k)\} .$$

β is an open cover of $M\times HE_0(N, M)$, where $HE_0(N, M) = HE(N, M) - H(N, M)$. Define an f.p. β-equivalence F (with β-inverse G and homotopies H and K) from $N\times HE_0(N, M)$ to $M\times HE_0(N, M)$ by

$$F(n, f, g, h, k) = (f(n), f, g, h, k) ,$$
$$G(m, f, g, h, k) = (g(m), f, g, h, k) ,$$
$$H_t(n, f, g, h, k) = (h_t(n), f, g, h, k) ,$$
$$K_t(m, f, g, h, k) = (k_t(m)f, g, h, k) .$$

By Proposition 5.5 there is an f.p. homeomorphism $\bar{F}: N\times[0, \infty)\times HE_0(N, M)\to M\times[0, \infty)\times HE_0(N, M)$ such that $p_{M\times HE_0(N,M)}\circ \bar{F}$ is β-close to $F\circ p_{N\times HE_0(N,M)}$

and $d(p_{[0,\infty)}, p_{[0,\infty)}\circ\bar{F})<3$. We define $\psi_N: N\times[0,\infty)\times HE_0(N, M)\to N\times[0,1)\times HE_0(N, M)$ by the formula

$$\psi_N(n, t(f, g, h, k)) = \left(n, \frac{\delta_2(f, g, h, k)t}{1+\delta_2(f, g, h, k)t}, (f, g, h, k)\right).$$

ψ_M is defined similarly. We then define $\Phi: HE(N, M)\to H(N\times[0,1), M\times[0,1))$ by

$$\Phi(f, g, h, k)(n, t) = \begin{cases} p_{M\times[0,1)}\psi_M\bar{F}\,\psi_N^{-1}(n, t, (f, g, h, k)) & (f, g, h, k)\in HE_0(N, M) \\ f\times \mathrm{id} & (f, g, h, k)\in H(N, M)\,. \end{cases}$$

The proof that Φ is continuous is similar to the proof of Theorem 4.4 (iv). ■

Consider the space of open imbeddings $I(M\times(1/4, 3/4), M\times[0,1])$. By Theorem 2.5, there are a neighborhood P of the inclusion η and a map $\psi: P\to H(M\times[0,1])$ such that $\psi(\eta)=\mathrm{id}$ and $\psi(g)|(M\times[1/3,2/3])=g|(M\times[1/3,2/3])$. Let

$$U\subset H(N\times[0,1), M\times[0,1))\times H(N\times(0,1], M\times(0,1]))$$

be

$$\{(h_1, h_2)\,|\,h_1h_2^{-1}|(M\times(1/4, 3/4))\in P\}\,.$$

PROPOSITION 5.7. *There is a continuous function* $\sigma: U\to H(N\times[0,1], M\times[0,1])$ *such that* $\sigma(h\times\mathrm{id}, h\times\mathrm{id}) = h\times\mathrm{id}$.

Proof. Define

$$\sigma(h_1, h_2)(n, t) = \begin{cases} h_1(n, t) & (n, t)\in h_1^{-1}(M\times[0,1/2]) \\ \psi(h_1h_1^{-1})\circ h_2 & (n, t)\notin h_1^{-1}(M\times[0,1/2])\,. \end{cases}$$

This is well-defined since $\psi(h_1h_2^{-1}) = h_1h_2^{-1}$ on $M\times[1/3, 2/3]$. The second statement follows from the fact that $\psi(\eta)=\mathrm{id}$. ■

PROPOSITION 5.8. *There is a continuous map* $\tau: H(N\times[0,1], M\times[0,1])\to H(N, M)$ *such that* $\tau(h\times\mathrm{id}) = h$.

Proof. Let $B = H(N\times[0,1], M\times[0,1])$ and let $B_0\subset B$ be the set of homeomorphisms of the form $h\times\mathrm{id}$. Let $\alpha_N: N\times[0,1]\times B\to N\times B$ be an f.p. map such that $\alpha_N|N\times[0,1]\times(B-B_0)$ is a homeomorphism and such that $\alpha_N|N\times[0,1]\times B_0$ is the projection map. Such a map is easily constructed using Theorem 4.9. We write $\alpha_N(n, t, h) = (\bar{\alpha}_N(h)(n, t), h)$. $\bar{\alpha}_M$ is defined similarly. We write $\tau(h)(n) = [\bar{\alpha}_M(h)]h[\bar{\alpha}_N(h)]^{-1}(n)$. Note that this is well-defined (even over B_0) and, therefore, continuous. ■

It is now an easy matter to complete the proof of Theorem 5.3. Let

Φ_1: $HE(N, M) \to H(N\times[0, 1), M\times[0, 1))$ and Φ_2: $HE(N, M) \to H(N\times(0, 1], M\times(0, 1])$ be maps as in Proposition 5.6 and let σ, τ and U be as in Propositions 5.7 and 5.8. The map $\tau\circ\sigma\circ(\Phi_1\times\Phi_2)$: $(\Phi_1\times\Phi_2)^{-1}(U)\to H(N, M)$ is a retraction. ■

UNIVERSITY OF KENTUCKY, LEXINGTON

REFERENCES

[A-B] R. D. ANDERSON and R. H. BING, A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines, Bull. A.M.S. **74** (1968), 771–792.

[A-Ch] R. D. Anderson and T. A. Chapman, Extending homeomorphisms to Hilbert cube manifolds, Pacific J. Math. **38** (1971), 281–293.

[A-M] R. D. Anderson and John McCharen, On extending homeomorphisms to Fréchet manifolds, Proc. A.M.S. **25** (1970), 283–289.

[A-S] R. D. ANDERSON and R. SCHORI, Factors of infinite-dimensional manifolds, Trans. A. M. S. **142** (1969), 315–330.

[A_1] S. ARMENTROUT, Concerning cellular decompositions of 3-manifolds with boundary, Trans. A.M.S. **137** (1969), 231–236.

[A_2] ———, Cellular decompositions of 3-manifolds that yield 3-manifolds, Memoir **107**, A.M.S., 1971.

[B] K. BORSUK, *Theory of Retracts*, Polish Scientific Publishers, Warsaw, 1967.

[Ch_1] T. A. CHAPMAN, Cell-like mappings of Hilbert cube manifolds: Solution of a handle problem, General Top. and Appl. **5** (1975), 123–145.

[Ch_2] ———, Topological invariance of Whitehead torsion, Am. J. Math. **96** (1974), 488–497.

[Ch_3] ———, On the structure of Hilbert cube manifolds, Composition Math. **24** (1972), 392–353.

[Ch_4] ———, Lectures on Hilbert cube manifolds, CBMS regional conference series in Mathematics; no. 28, A.M.S., Providence, R.I., 1976.

[Ch_5] ———, Canonical extensions of homeomorphisms, General Top. and Appl. **2** (1972), 227–247.

[Ch-F] T. A. CHAPMAN and S. FERRY, Hurewicz fiber maps with ANR fibers, Topology **16** (1977), 131–144.

[Ch-W] T. A. CHAPMAN and R. WONG, On homeomorphisms of infinite-dimensional bundles III, Trans. A.M.S. **191** (1974), 269–276.

[F-V] A. FATHI and Y. M. VISETTI, Deformation of open embeddings of Q-manifolds, Trans. A.M.S. **224** (1976), 427–436.

[F] S. FERRY, Strongly regular mappings with ANR fibers are Hurewicz fibrations, preprint.

[G] R. GEOGHEGAN, On spaces of homeomorphisms, embeddings and functions–I, Topology **11** (1972), 159–177.

[H] W. HAVER, Mappings between ANRs that are fine homotopy equivalences, Pacific J. of Math. **58** (1975), 457–461.

[K] G. KOZLOWSKI, Factorization of certain maps up to homotopy, Proc. A.M.S. **21** (1969), 88–92.

[L] R. C. LACHER, Cell-like mappings I, Pacific J. Math. **30** (1969), 717–731.

[L-M] R. LUKE and W. K. MASON, The space of homeomorphisms on a compact two-manifold is an absolute neighborhood retract, Trans. A.M.S. **161** (1971), 185–205.

[M] W. K. MASON, The space of all homeomorphisms of a two-cell which fix the cell's boundary is an absolute retract, Trans. A.M.S. **161** (1971), 185–205.

[P] T. M. PRICE, Homotopy properties of decomposition spaces, Trans. A.M.S. **122** (1966), 427–435.

[S] L. SIEBENMANN, Approximating cellular maps by homeomorphisms, Topology **11** (1972), 271–294.

[Su] D. SULLIVAN, Geometric topology, part I: Localization, periodicity, and Galois symmetry, M.I.T. Notes, 1970.

[T] H. TORUNCZYK, Absolute retracts as factors of normed linear spaces, Fund. Math. (1974), 51–67.

[W_1] R. WONG, On homeomorphisms of infinite-dimensional bundles I, Trans. A.M.S. **191** (1974), 245–259.

[W_2] ———, On homeomorphisms of certain infinite-dimensional spaces, Trans. A.M.S. **128** (1967), 48–54.

(Received September 22, 1976)