HOMOTOPING ε-MAPS TO HOMEOMORPHISMS.

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1. Introduction, Notation, and Preliminaries. If Y is a metric space, a continuous function $g: Y \to X$ is called an *ε-map* if $\operatorname{diam}(g^{-1}(x)) < \varepsilon$ for each $x \in X$. Kirby and Siebenmann [10, p. 220; 16] have conjectured that if M is a compact n-manifold without boundary and $\varepsilon > 0$ is given, then there is a $\delta > 0$ such that if N is another compact n-manifold without boundary and $g: M \to N$ is a surjective δ-map, then g is homotopic through ε-maps to a homeomorphism. The purpose of this paper is to verify this conjecture (along with an appropriate modification dealing with noncompact and/or bounded manifolds) for manifolds of dimension greater than four.

If α is an open cover of a space Y, maps $f, g: X \to Y$ are said to be *α-homotopic* if there is a homotopy $h_t: X \to Y$, $0 \leq t \leq 1$, such that $h_0 = f$, $h_1 = g$, and such that for each $x \in X$ there is a $U_x \in \alpha$ which contains $h_t(x)$ for $0 \leq t \leq 1$. A map $f: X \to Y$ is an *α-domination* if there is a map $g: Y \to X$ such that $f \circ g$ is α-homotopic to the identity. In such a situation, g is called a *right α-inverse* for f. $f: X \to Y$ is called an *α-equivalence* if f is an α-domination and for some right α-inverse g, $g \circ f$ is $f^{-1}(\alpha)$-homotopic to the identity, where $f^{-1}(\alpha)$ denotes the cover $\{f^{-1}(U) | U \in \alpha\}$ of X. We call g an *α-inverse* for f. A map $g: Y \to X$ is called an *α-map* if for each $x \in X$ there is a $U_x \in \alpha$ such that $g^{-1}(x) \subset U_x$. α-dominations, α-equivalences, and α-maps of pairs are defined by requiring that all maps and homotopies be maps and homotopies of pairs. A map $f: X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. $f: X \to Y$ is a *proper α-equivalence* if f, g, and the homotopies are proper. If α is an open cover of Y, then for each $B \subset Y$, $\operatorname{St}(B, \alpha) = \bigcup \{U \in \alpha | U \cap B \neq \emptyset\}$ and $\operatorname{St}\alpha = \{\operatorname{St}(U, \alpha) | U \in \alpha\}$. $\operatorname{St}^2\alpha = \operatorname{St}(\operatorname{St}\alpha)$. The terms ε-map, ε-domination, ε-equivalence, etc., are defined by letting α be the open cover of Y by open balls of radius ε in the definitions above. Here is the statement of our main theorem.

Theorem 1. *If M is an n-manifold and α is an open cover of M, then there is an open cover β of M such that if N is an n-manifold and $g: (M, \partial M) \to (N, \partial N)$ is a proper β-map, then g is homotopic through α-maps to a homeomor-*

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phism. This is true provided that either

i. $n \geqslant 6$ *or*

ii. $n=5$ *and* $g|\partial M$ *is a homeomorphism onto* ∂M.

Theorem 1 is a direct consequence of the following three theorems.

THEOREM 2. *If* X *and* Y *are locally compact ANRs and* Y *is a closed subset of* X, *then for each open cover* α *of* X *there is an open cover* β *of* X *such that if* $g: X \to Z$ *is a surjective proper* β*-map, then there is a proper* α*-domination of pairs* $f:(Z, g(Y)) \to (X, Y)$ *with right* α*-inverse* g.

THEOREM 3. *If* M *is a connected topological manifold and* α *is an open cover of* M, *then there is an open cover* β *of* M *such that if* N *is a connected topological manifold of the same dimension and* $f:(N, \partial N) \to (M, \partial M)$ *is a proper* β*-domination of pairs, then* f *is a proper* α*-equivalence of pairs.*

THEOREM 4 (α-approximation theorem). *If* M^n *is a topological manifold and* α *is an open cover of* M, *then there is an open cover* β *of* M *such that if* N *is a topological manifold of the same dimension and* $f:(N, \partial N) \to (M, \partial M)$ *is a proper* β*-equivalence of pairs, then* f *is* α*-close to a homeomorphism. This is true provided that either* (i) $n \geqslant 6$ *or* (ii) $n=5$ *and* $f|\partial N$ *is a homeomorphism onto* ∂M.

Theorem 4 is the main technical tool used in this paper. It is the main theorem of [5]. It says that "small" homotopy equivalences can be approximated by homeomorphisms. Note that Theorems 3 and 4 combine to show that small *homotopy dominations* can be approximated by homeomorphisms. This strengthened version of Theorem 4 is stated as Corollary 3.7.

In Section 5 we use ε-maps to give a new proof of a characterization of manifolds-with-boundary due to Cernavskii and Seebeck. This theorem is stated in Section 5.

R. D. Edwards has pointed out an interesting consequence of Theorem 1.

THEOREM 6. *If* $f: M^n \to X$ *is a CE map of an* n*-manifold without boundary,* $n \geqslant 5$, *onto an ANR* X, *then* f *is a near-homeomorphism if and only if for each pair of open covers* α *of* M *and* β *of* X *there are maps* $p: M \to X$ *and* $q: X \to M$ *such that* p *is* β*-close to* f *and* $q \circ p$ *is* α*-close to* id.

The remainder of this paper is organized as follows. Section 2 contains the proof of Theorem 2, Section 3 contains the proof of Theorem 3, Section 4 contains the proof of Theorem 1, Section contains our proof of Theorem 5, and Section 6 contains the proof of Theorem 6.

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2. The Proof of Theorem 2. Let l_2 be separable Hilbert space with its usual norm.

PROPOSITION 2.1. *If β is an open cover of l_2 and $g: l_2 \to Y$ is a proper β-map onto a metric space Y, then there is a continuous function $f: Y \to l_2$ such that for each $x \in l_2$, $f \circ g(x)$ lies in the convex hull of some element of β which contains x.*

Proof. For each $U \in \beta$, let $V_U = \{ y \in Y \mid g^{-1}(y) \subset U \}$. Since g is proper, one easily verifies that $\mathcal{V} = \{ V_U \mid U \in \beta \}$ is an open cover of Y. Let $\mathcal{W}$ be an open cover of Y such that $\mathrm{St}\,\mathcal{W}$ refines $\mathcal{V}$, and let $\{\varphi_W\}_{W \in \mathcal{W}}$ be a partition of unity subordinate to $\mathcal{W}$. For each $W \in \mathcal{W}$ choose $x_W \in g^{-1}(W)$. The map $f: Y \to l_2$ is defined by $f(y) = \Sigma \varphi_W(y) x_W$. If $x \in l_2$, $\bigcup \{ W \mid \varphi_W(g(x)) \neq 0 \} \subset \mathrm{St}(g(x), \mathcal{W}) \subset V_U$ for some $U \in \beta$. Since $g^{-1}(V_U) \subset U$, $\{x\} \bigcup \{ x_W \mid \varphi_W(g(x)) \neq 0 \} \subset U$ and $\Sigma \varphi_W(g(x)) x_W$ is contained in the convex hull of U. Q.E.D.

COROLLARY 2.2. *If X is a closed subset of l_2, then for each positive function $\varepsilon: X \to (0, \infty)$ there is a positive function $\delta: X \to (0, \infty)$ such that if $g: X \to Y$ is a proper map of X onto a metric space Y with $\operatorname{diam} g^{-1} g(x) < \delta(x)$ for each $x \in X$, then there is a function $f: Y \to l_2$ such that $\|x - f \circ g(x)\| < \varepsilon(x)$ for each $x \in X$.*

Proof. Extend ε to a continuous function $\bar{\varepsilon}: l_2 \to (0, \infty)$. By Proposition 2.1, there is a continuous function $\bar{\delta}: l_2 \to (0, 1)$ such that if $\bar{g}: l_2 \to Z$ is a proper map onto a metric space such that $\operatorname{diam} \bar{g}^{-1} \bar{g}(x) < \bar{\delta}(x)$ for each $x \in l_2$, then there is a map $\bar{f}: Z \to l_2$ such that $\|\bar{f} \circ \bar{g}(x) - x\| < \bar{\varepsilon}(x)$ for all $x \in l_2$. Let $\delta = \bar{\delta} | X$.

If $g: X \to Y$ satisfies the hypotheses of the corollary, let $Z = l_2 \bigcup_g Y$ and let $\bar{g}: l_2 \to Z$ be the quotient map. Since Z is metrizable, there is a map $\bar{f}: Z \to l_2$ as described above. The restriction of $\bar{f}$ to Y is the desired map f. Q.E.D.

COROLLARY 2.3. *If X is a complete separable metric ANR, then for each open cover α of X there is an open cover β of X such that if $g: X \to Y$ is a proper β-map onto a metric space Y, then there is a map $f: Y \to X$ such that $f \circ g$ is α-homotopic to the identity.*

Proof. X can be embedded as a closed subset of l_2. Since X is an ANR, there is a neighborhood U of X in l_2 and a retraction $r: U \to X$. One can now choose β to be so fine that the map $f': Y \to l_2$ guaranteed by Corollary 2.2 has the property that the line segment from x to $f' \circ g(x)$ is contained in U. If β is chosen to be sufficiently fine, we can set $f = r \circ f'$, and the homotopy $h_t(x) = r(tx + (1-t) f' \circ g(x))$ will be the desired α-homotopy. Q.E.D.

We can complete the proof of the absolute version of Theorem 2 by showing that β can be chosen so that f and the homotopy are proper.

For this, it suffices to assume that α is a locally finite cover by relatively compact open sets. If $K \subset X$ is compact, $g^{-1} \circ f^{-1}(K) \subset \mathrm{St}(K, \alpha)$ and $f^{-1}(K) \subset g(\overline{\mathrm{St}(K, \alpha)})$. This shows that $f^{-1}(K)$ is compact. A similar argument shows that the homotopy is proper.

The relative version now follows easily. One first constructs f' so that $f' \circ g$ is close to the identity. If $f'|g(Y)$ maps into a small neighborhood of Y, then, since Y is an ANR, we can homotop f' to a map $f: Z \to X$ which carries $g(Y)$ into Y such that $f \circ g$ is close to the identity. This uses only the estimated homotopy extension theorem (see, for instance, [5]). We can now homotop $f \circ g$ to id by first homotoping $f \circ g|Y$ to id (and extending this homotopy to all of X) and then following this homotopy with a homotopy of the resulting map to the identity rel Y. The details are left to the reader. Q.E.D.

Remarks. The compact case of Theorem 2 is due to Eilenberg [6]. Further generalizations of Theorem 2 are known to Kozlowski.

3. The Proof of Theorem 3. In this section we will show that a small domination between manifolds is a small homotopy equivalence. It is easy to show that a homotopy domination between compact simply connected manifolds without boundary is a homotopy equivalence. The domination and its inverse are degree one maps. A standard argument (see the proof of Proposition 3.3) using Poincaré duality shows that degree one maps induce surjections on homology. Since the inverse of the domination is injective on homology as well, it is a homology equivalence and, by the Whitehead theorem, a homotopy equivalence. An extension of this argument works in the nonsimply connected case.

Our strategy is to work through a local version of this argument to show that if $f: M \to N$ is a small domination, then the inverse image of a contractable open set contracts to a point in the inverse image of a slightly larger open set. Our proof that f is a small homotopy equivalence is then analogous to Lacher's proof [13] that a CE map between ANRs is a homotopy equivalence.

Proposition 3.1 is a local analog of the Hurewicz theorem.

PROPOSITION 3.1. *For each $k \geqslant 0$ there is an integer $n_k > 0$ such that if $A_1 \subset A_2 \subset \ldots \subset A_{n_k}$ is a sequence of connected ANRs with $i_{\#}: \pi_1(A_j) \to \pi_1(A_{j+1})$ and $i_*: H_l(A_j) \to H_l(A_{j+1})$ equal to zero for all j and for all l between 0 and k, then each map of a k-complex into A_1 is homotopic to a constant map in A_{n_k}.*

Proof. The proof is by induction on k.

i. $k = 0$. Since A_1 is a connected ANR, each map of a 0-complex into A_1 is homotopic to a constant map. Thus $n_0 = 1$.

ii. $k=1$. A map of a 1-complex into A_1 is homotopic in A_1 to a map which takes all vertices to a common point in A_1. Since $i_{\#}:\pi_1(A_1)\to\pi_1(A_2)$ is zero, we can take $n_1=2$.

iii. Suppose that the theorem is true for $k\leqslant l$ and that $f:K\to A_1$ is a map of an $(l+1)$-dimensional complex into A_l. According to the inductive hypothesis, f is homotopic in A_{n_l} to a map $f':K\to A_{n_l}$ which takes the l-skeleton of K to a point. Each $(l+1)$-simplex of K determines an element of $\pi_{l+1}(A_{n_l})$. We will be done if we can show that there exists n_{l+1} so that $i_{\#}:\pi_{l+1}(A_{n_l})\to\pi_{l+1}(A_{n_{l+1}})$ is zero.

We will sketch a proof that $n_{l+1}=2n_l+1$ suffices. The details are similar to the details in the proof of the Hurewicz theorem [9]. If $[\alpha]\in\pi_{l+1}(A_{n_l})$, let $\varphi([\alpha])\in H_{l+1}(A_{n_l})$ be the image of α under the Hurewicz homomorphism. By our assumptions on H_*, $i_*\varphi([\alpha])$ is zero, where $i_*:H_{l+1}(A_{n_l})\to H_{l+1}(A_{n_l+1})$. Thus, if $\alpha:(\Delta^{l+1},\dot\Delta^{l+1})\to(A_{n_l},*)$ represents $[\alpha]$, then α bounds a singular chain in $(A_{n_l+1},*)$. By the inductive hypothesis, α bounds a singular chain $c=\sum n_i f_i$ in $(A_{2n_l+1},*)$ such that each map $f_i:\Delta^{l+2}\to A_{2n_l+1}$ takes the (l)-skeleton of Δ^{l+2} to the basepoint.

The restriction of f_i to any $(l+1)$-face of Δ^{l+2} determines an element of $\pi_{l+1}(A_{2n_l+1})$. By the homotopy addition theorem [9], the sum of these homotopy elements over each Δ^{l+2} is zero. On the other hand, the sum of these elements over the entire chain is $[\alpha]$. Thus, $[\alpha]=0$. Q.E.D.

COROLLARY 3.2. *If the ANR A_1 in Proposition 3.1 is k-dimensional, then A_1 contracts to a point in A_{n_k}.*

Proof. Since A_1 is a k-dimensional ANR, there exist a k-complex K and maps $d:K\to A_1$ and $u:A_1\to K$ such that $d\circ u\simeq$ id. Thus, the inclusion map $i:A_1\to A_{n_k}$ is homotopic to the map $i\circ d\circ u$. By Proposition 3.1, $i\circ d$ is homotopic to a constant map, so i is homotopic to a constant map into A_{n_k}. Q.E.D.

We say that a map $f:X\to Y$ is an *ε-domination over* a subset A of Y if there is a map $g:A\to X$ such that $f\circ g$ is ε-homotopic to the inclusion. Let $rB^n\subset R^n$ be the subset $\{x\in R^n \mid \|x\|<r\}$. Let $R^n_+=\{x=(x_1,\dots,x_n)\in R^n \mid x_n\geqslant 0\}$, and let $rB^n_+=rB^n\cap R^n_+$. Let $\partial R^n_+=\{x\in R^n \mid x_n=0\}$.

PROPOSITION 3.3.

i. *There is an $\varepsilon>0$ such that if a proper map $f:V^n\to R^n$ is an ε-domination over B^n with right inverse g such that $g(B^n)\supset f^{-1}(\tfrac{1}{2}B^n)$, then $f^{-1}(\tfrac{1}{2}B^n)$ contracts to a point in $f^{-1}(B^n)$.*

ii. *There is an $\varepsilon>0$ such that if a proper map $f:(V^n,\partial V^n)\to(R^n_+,\partial R^n_+)$ is an ε-domination of pairs over $(B^n_+,B^n_+\cap\partial R^n_+)$ with right inverse g such that*

$g(B^n_+)\supset f^{-1}(\frac{1}{2}B^n_+)$, then $f^{-1}(\frac{1}{2}B^n_+)$ contracts to a point in $f^{-1}(B^n_+)$ in such a way that $f^{-1}(\frac{1}{2}B^n_+)\cap\partial V$ remains in ∂V.

Proof. (i) Let $rC=f^{-1}(rB^n)$, and let $rD=g^{-1}(rC)$, where g is a right ε-inverse for f. Since $f\circ g$ is ε-close to the identity, $(r+\varepsilon)D\supset rB^n\supset(r-\varepsilon)D$. Thus, $(r-\varepsilon)D$ contracts to a point in $(r+\varepsilon)D$.

For each $r<1-2\varepsilon$, $g:rD\to rC$ is a proper map. If $K\subset rC$ is compact, then $g^{-1}(K)$ is a closed subset of $\overline{(1-\varepsilon)B^n}$ and is therefore compact. Consider the composition

$$rD\xrightarrow{g}rC\xrightarrow{f}rB^n.$$

Since $f\circ g$ is ε-homotopic to the identity, $f\circ g$ is (properly) ε-homotopic to a map which is the identity on $(r-2\epsilon)B^n$. Let rD^* be the component of rD containing $(r-2\varepsilon)B^n$, and let g map rD^* to the component rC^* of rC.

The restrictions $g:rD^*\to rC^*$ and $f:rC^*\to rB$ are proper maps, so the composition $f\circ g:rD^*\to rB$ is a proper map of degree one (rD^* and rB have orientations in locally finite homology with Z-coefficients). Thus, rC^* has an orientation class, and the maps $g:rD^*\to rC^*$ and $f:rC^*\to rB$ have degree one.

Note that $g_\#:\pi_1(rD^*)\to\pi_1(rC^*)$ is onto. Otherwise, g would factor through a covering space of rC^* corresponding to $g_\#(\pi_1(rD^*))$ and could not be a proper map of degree one. Thus, we have a commutative diagram

$$\begin{array}{ccc}\pi_1((r-2\varepsilon)D^*) & \longrightarrow & \pi_1(rD^*)\\ {\scriptstyle g_\#}\downarrow{\scriptstyle \text{onto}} & & {\scriptstyle g_\#}\downarrow{\scriptstyle \text{onto}}\\ \pi_1((r-2\varepsilon)C^*) & \xrightarrow{i_\#} & \pi_1(rC^*)\end{array}$$

which shows that $i_\#=0$.

Let H_c^* denote cohomology with compact supports. By Poincaré duality [8], there are isomorphisms $H_c^k(sC^*)\cong H_{n-k}(sC^*)$ and $H_c^k(sD^*)\cong H_{n-k}(sD^*)$ obtained by capping with orientation classes μ_{sC^*} and μ_{sD^*}. By naturality of cap products, we have

$$\mu_{sC^*}\cap\beta=\left(g_*\mu_{sD^*}\cap\beta\right)=g_*(\mu_{sD^*}\cap g^*\beta).$$

Since every element of $H_*(sC^*)$ has the form $\mu_{sC^*}\cap\beta$ for some β, $g_*:H_*(rD^*)\to H_*(rC^*)$ is onto. As in the π_1 case, it is now easy to see that $i_*:H_*((r-2\varepsilon)C^*)\to H_*(rC^*)$ is zero.

It now follows from Corollary 3.2 that rC^* contracts to a point in $f^{-1}(B^n)$ if $r<1$ and ε is small. To complete the proof of (i), we need only show that $rC^*\supset f^{-1}(\frac{1}{2}B^n)$ for $r>\frac{1}{2}$ and ε small.

Let $x \in f^{-1}(\frac{1}{2}B^n)$. By hypothesis, $x = g(y)$ for some $y \in B^n$. Since $f \circ g(y) = f(x) \in \frac{1}{2}B^n$ and $f \circ g$ is ϵ-close to id, $y \in (\frac{1}{2}+\epsilon)B^n$ and $f^{-1}(\frac{1}{2}B^n) \subset g\big((\frac{1}{2}+\varepsilon)B^n\big)$. Moreover, $f \circ g\big((\frac{1}{2}+\varepsilon)B^n\big) \subset (\frac{1}{2}+2\varepsilon)B^n$, so $g\big((\frac{1}{2}+\varepsilon)B^n\big) \subset (\frac{1}{2}+2\varepsilon)C$. Since $g\big((\frac{1}{2}+\varepsilon)B^n\big)$ is connected, $f^{-1}(\frac{1}{2}B^n) \subset (\frac{1}{2}+2\varepsilon)C^*$. This completes the proof of part i.

(ii) Let $f:(V,\partial V) \to (R^n_+, \partial R^n_+)$ be a proper map which is an ϵ-domination of pairs over $(B^n_+, \partial B^n_+)$. Let $g:(B^n_+, \partial B^n_+) \to (V, \partial V)$ be a right inverse for f. We first note that the collars on ∂B^n and ∂V allow us to construct maps $f_1:(V,\partial B) \to (\mathrm{Ri}^n_+, \partial R^n_+)$ and $g_1:(B^n_+, \partial B^n_+) \to (V, \partial V)$ arbitrarily close to f and g so that $f_1^{-1}(\partial R^n_+) = \partial V$ and $g_1^{-1}(\partial V) = \partial B^n_+$. Moreover, we can require that there be an ϵ-homotopy of pairs from $f_1 \circ g_1$ to id which takes only boundary points into the boundary.

We now proceed as in part i, defining $rC = f_1^{-1}(rB^n_+)$ and $rD = g_1^{-1}(rC)$. We note as before that $(r+\varepsilon)D \supset rB \supset (r-\varepsilon)D$. By the argument of part i, we can choose $\varepsilon > 0$ small enough so that $f_1^{-1}(\frac{1}{3}\partial B^n_+)$ contracts to a point in $f_1^{-1}(\frac{2}{3}\partial B^n_+)$ and so that $f_1^{-1}(\frac{2}{3}B^n_+ - \frac{2}{3}\partial B^n_+)$ contracts to a point in $f_1^{-1}(B^n_+ - \partial B^n_+)$. Since $f_1^{-1}(\frac{2}{3}B^n_+)$ can be displaced into $f_1^{-1}(\frac{2}{3}B^n_+ - \frac{2}{3}\partial B^n_+)$ by an arbitrarily small homotopy, $f_1^{-1}(\frac{2}{3}B^n_+)$ contracts to a point in $f_1^{-1}(B^n_+)$. Using the homotopy extension theorem, this contraction can be performed in such a way that one point in $f^{-1}(\partial B^n_+)$ remains fixed.

We can now contract $f_1^{-1}(\frac{1}{3}B^n_+)$ to a point relative to the boundary in $f_1^{-1}(B^n_+)$ by first contracting $f_1^{-1}(\frac{1}{3}\partial B^n_+)$ to a point * in $f_1^{-1}(\frac{2}{3}\partial B^n_+)$ and then contracting $f_1^{-1}(\frac{2}{3}B^n_+)$ to * (rel *) in $f_1^{-1}(B^n_+)$. Q.E.D.

Remark. Parts of this argument are modeled on arguments from [7] and [2].

Definition 3.4. A map $f:(X,X_1) \to (Y,Y_1)$ has the *α-homotopy lifting property of pairs* with respect to (Z,Z_1) if for each pair of maps $s_1, s_2:(Z,Z_1) \to (X,X_1)$ such that $f \circ s_1$ and $f \circ s_2$ are α-homotopic as maps from (Z,Z_1) to (Y,Y_1), there is a $\mathrm{St}^2 f^{-1}\alpha$-homotopy (of pairs) from s_1 to s_2.

The next proposition characterizes α-equivalences in terms of the α-homotopy lifting property.

PROPOSITION 3.5.

i. *If $f:(X,X_1) \to (Y,Y_1)$ is an α-equivalence of pairs, then f has the α-homotopy lifting property with respect to any pair (Z,Z_1).*

ii. *If (X,X_1) is an n-dimensional ANR pair and f is a proper α-domination of pairs which has the α-lifting property with respect to n-complexes, then f is a $\mathrm{St}^2\alpha$-equivalence of pairs.*

Proof. (i) Let $s_1, s_2 : (Z, Z_1) \to (X, X_1)$ be maps with $f \circ s_1 \overset{\alpha}{\simeq} f \circ s_2$. Then

$$s_1 \overset{f^{-1}\alpha}{\simeq} g \circ f \circ s_1 \overset{\mathrm{St} f^{-1}\alpha}{\simeq} g \circ f \circ s_2 \overset{f^{-1}\alpha}{\simeq} s_2,$$

where all homotopies are homotopies of pairs and the middle homotopy is obtained by applying g to the given homotopy from $f \circ s_1$ to $f \circ s_2$.

(ii) Since (X, X_1) is an n-dimensional ANR pair, for each open cover γ of X one can find a pair (K, K_1) of n-dimensional complexes and a γ-domination of pairs $d : (K, K_1) \to (X, X_1)$.

The maps $s_1 = d$ and $s_2 = g \circ f \circ d : (K, K_1) \to (X, X_1)$ have the property that $f \circ s_1 \overset{\alpha}{\simeq} f \circ s_2$. Thus,

$$d \overset{\mathrm{St}^2 f^{-1}\alpha}{\simeq} g \circ f \circ d.$$

This implies that

$$d \circ u \overset{\mathrm{St}^2 f^{-1}\alpha}{\simeq} g \circ f \circ d \circ u,$$

where u is a right γ-inverse for d. Thus, if γ is sufficiently fine,

$$\mathrm{id} \overset{\mathrm{St} f^{-1}\alpha}{\simeq} g \circ f. \qquad \text{Q.E.D.}$$

PROPOSITION 3.6. *Let $f : X \to Y$ be a proper map, and let α be a locally finite open cover of Y such that $\bar{U}$ is compact for each $U \in \alpha$. If $g : Y \to X$ is a map such that*

$$f \circ g \overset{\alpha}{\simeq} \mathrm{id} \quad \text{and} \quad g \circ f \overset{f^{-1}\alpha}{\simeq} \mathrm{id},$$

then f is a proper α-equivalence with α-inverse g.

Proof. We need only show that g and the two homotopies $h_t : f \circ g \simeq \mathrm{id}$ and $k_t : g \circ f \simeq \mathrm{id}$ are proper. h_t is proper, since if $K \subset Y$ is compact, then $h^{-1}(K)$ is a closed subset of the compact space $\overline{\mathrm{St}(K, \alpha) \times I}$.

To show that k is proper, note that if K is a compactum in X, then $f^{-1}f(K) \supset f^{-1}f(k) \supset k$ is also compact. Then

$$k^{-1}(K) \subset k^{-1}f^{-1}(f(K)) \subset (f \circ k)^{-1}(f(K)).$$

Since $f \circ k_t$ is α-close to f for each t, $(f \circ k)^{-1}(f(K)) \subset f^{-1}\left(\overline{\mathrm{St}(f(K), \alpha)}\right) \times I$, which is also compact. Thus, k is proper. A similar argument shows that g is proper. Q.E.D.

We now complete the proof of Theorem 3 by showing that for each manifold M^n, $k \geqslant 0$, and open cover α of M^n there is an open cover β of M^n such that if $f:(N, \partial N) \to (M, \partial M)$ is a proper β-domination of pairs, then f has the α-homotopy lifting property for pairs of k-complexes. For simplicity, we consider only the unbounded case. The general case is entirely similar. The proof is by induction on k.

(i) If $k=0$, let $s_1, s_2 : K \to N$ be maps of a 0-complex into N so that $f \circ s_1 \overset{\alpha}{\simeq} f \circ s_2$. Let $h_t : K \times I \to M$, $0 \leqslant t \leqslant 1$, be an α-homotopy with $h_0 = f \circ s_1$ and $h_1 = f \circ s_2$. If β is sufficiently fine, $g \circ h_t$ is an $f^{-1}\alpha$-homotopy from $g \circ f \circ s_1$ to $g \circ f \circ s_2$. Thus, it suffices to find $f^{-1}\alpha$-homotopies from s_1 to $g \circ f \circ s_1$ and from s_2 to $g \circ f \circ s_2$.

Choose a locally finite cover of M by open sets V_i such that each V_i is contained in a W_i with $(W_i, V_i) \cong (B^n, \frac{1}{2}B^n)$ and such that the cover $\{W_i\}$ of M refines α. If β is chosen to be sufficiently fine, $f^{-1}(V_i)$ will contract in $f^{-1}(W_i)$ for each i and for each element x of K, $s_1(x)$ and $g \circ f \circ s_1(x)$ will be contained in $f^{-1}(V_i)$ for some i. One can now use the contractions of $f^{-1}(V_i)$ in $f^{-1}(W_i)$ to define an $f^{-1}\alpha$-homotopy from s_1 to $g \circ f \circ s_1$. The homotopy from s_2 to $g \circ f \circ s_2$ is constructed in a similar fashion.

(ii) Assume that for each α there exists β so that if $f : N \to M$ is a proper β-domination, then f has the α-lifting property for complexes of dimension $\leqslant l$. Let $s_1, s_2 : K \to N$ be maps of an $(l+1)$-complex into N such that $f \circ s_1 \overset{\alpha}{\simeq} f \circ s_2$. As before, it suffices to show that β can be chosen sufficiently fine so that

$$s_1 \overset{f^{-1}\alpha}{\simeq} g \circ f \circ s_1 \quad \text{and} \quad s_2 \overset{f^{-1}\alpha}{\simeq} g \circ f \circ s_2.$$

We consider the s_1 case.

Let $\{V_i\}$ and $\{W_i\}$ be locally finite open covers of M such that $V_i \subset W_i$, $(W_i, V_i) \cong (B^n, \frac{1}{2}B^n)$, and $\{W_i\}$ refines α. Choose an open cover γ so that $\mathrm{St}^3\gamma$ refines $\mathcal{V} = \{V_i\}$. Now choose β so fine that:

1. $f^{-1}(V_i)$ contracts in $f^{-1}(W_i)$.
2. If $q_1, q_2 : P \to N$ are maps of an l-complex into N with $f \circ q_1 \overset{\gamma}{\simeq} f \circ q_2$, then $q_1 \overset{\mathrm{St}^2 f^{-1}\gamma}{\simeq} q_2$.
3. β refines γ.

Subdivide K so that the track of each simplex of K under the β-homotopy from $f \circ g \circ f \circ s_1$ to $f \circ s_1$ lies in an element of γ. By condition 2, there is a $\mathrm{St}^2 f^{-1}\gamma$-homotopy from $s_1|K^l$ to $g \circ f \circ s_1|K^l$. For each $(l+1)$-simplex Δ of K, then, $s_1|\Delta$, $g \circ f \circ s_1|\Delta$, and the restriction of the homotopy to $\partial\Delta \times I$ define a map of S^{l+1} into N. The image of every such map lies in an element of

$f^{-1}\mathrm{St}^3\gamma$, which refines $f^{-1}(\mathcal{V})$. Thus, we can use the contractions guaranteed by (1) to extend the homotopy over K^{l+1}. This completes the inductive step and the proof of Theorem 3. Q.E.D.

COROLLARY 3.7. *If M^n is a topological manifold and α is an open cover of M, then there is an open cover β of M such that a proper β-domination $f:(N,\partial N)\to(M,\partial M)$ ($n\geq 6$ or $n=5$ and $f|\partial N$ a homeo) is α-homotopic to a homeomorphism.*

Proof. This strengthened version of Theorem 4 is an easy corollary to Theorem 3. Q.E.D.

We will now state a local version of Theorem 3 which will be useful in the next two sections. First, we need a definition.

Definition 3.8. Let Y be a space, and let C be a closed subset of Y. If α is an open cover of Y, then $f:X\to Y$ is said to be a *proper α-equivalence over C* if $f|f^{-1}C$ is proper and there exist a proper map $g:C\to X$ and proper homotopies

$$f\circ g\overset{\alpha}{\simeq}\mathrm{id}|C \quad\text{and}\quad g\circ f\overset{f^{-1}\alpha}{\simeq}\mathrm{id}|f^{-1}C.$$

THEOREM 3.9. *Let M be an n-manifold. If C is a closed subset of M and C_1, C_2 are closed neighborhoods of C with $\mathring{C}_2\supset C_1$, then for each open cover α of C_2 there is an open cover β of C_2 such that if N is another n-manifold and $f:(N,\partial N)\to(M,\partial M)$ is a proper β-domination of pairs over $(C_2, C_2\cap\partial M)$ with inverse g such that $g(C_2)\supset f^{-1}(C_1)$, then f is a proper α-equivalence of pairs over C.*

Proof. The proof of Theorem 3 generalizes to show that f has an appropriate lifting property over C_0 with $\mathring{C}_1\supset C_0\supset \mathring{C}_0\supset C$. One then uses the nerve of a fine open cover of $f^{-1}(C)$ to generalize Proposition 3.5 (ii) and complete the proof. Q.E.D.

4. The Proof of Theorem 1. To prove Theorem 1 is now an easy matter. Let M be a topological manifold and let α be an open cover of M. Then:

1. By Theorem 4, there is an open cover β such that any proper β-equivalence of pairs $f:(N,\partial N)\to(M,\partial M)$ is α-homotopic to a homeomorphism provided that either $n\geq 6$ or $n\geq 5$ and $f|\partial N$ is a homeomorphism.
2. By Theorem 3, there is an open cover β_1 of M such that any proper β_1-domination of pairs $f:(N,\partial N)\to(M,\partial M)$ is a proper β-equivalence.

3. By Theorem 2, there is an open cover β_2 of M such that any proper β_2 map of pairs $g:(M,\partial M)\to(N,\partial N)$ is the right inverse of a proper β_1-domination $f:(g(M),g(\partial M))\to(M,\partial M)$. Note, however, that since there is a proper homotopy of pairs from $f\circ g$ to id, we have $(f\circ g)_*\mu=\mu$, where μ is the orientation class of $(M,\partial M)$ in locally finite homology with Z_2 coefficients. Thus, $(g(M),g(\partial M))$ supports the orientation class of $(N,\partial N)$, g is onto, and f is defined on all of N. Moreover, if $g|\partial M$ is a homeomorphism, we may take $f|\partial N=g^{-1}$. This is an easy application of the homotopy extension theorem.

It now follows that a proper β_2-map $g:(M,\partial M)\to(N,\partial N)$ is a β-inverse for a proper β-equivalence $f:(N,\partial N)\to(M,\partial M)$ which is α-homotopic to a homeomorphism, h. We therefore have an $f^{-1}\mathrm{St}\,\alpha$-homotopy from g to h^{-1} given by $h^{-1}\simeq g\circ f\circ h^{-1}\simeq g\circ h\circ h^{-1}=g$. This is the desired homotopy from g to a homeomorphism. It is clear that each stage of the homotopy is a $\mathrm{St}\,\alpha$-map.

Q.E.D.

We wish to prove a local version of Theorem 1. This will be useful in proving the taming theorem of the next section. We will first need an invariance of domain result for ε-maps. Let rD^n be the closure of rB^n in R^n.

PROPOSITION 4.1. *For each $r<1$ there is an $\varepsilon>0$ such that if V^n is a manifold without boundary and $g:\bar{B}^n\to V^n$ is an ε-map, then $g(rB^n)$ is contained in the interior of $g(B^n)$.*

Proof. By Theorem 2, for small ε there is a map

$$f:(g(\bar{B}^n),g(\partial\bar{B}^n))\to(\bar{B}^n,\partial\bar{B}^n)$$

such that $f\circ g$ is close to the identity. Moreover, for ε small, $g(rB^n)$ is contained in a single component U of $V-g(\partial\bar{B}^n)$.

Consider the induced maps $\bar{g}:\bar{B}^n/\partial\bar{B}^n\to g(\bar{B}^n)/(g(\bar{B}^n)-U)$ and $\bar{f}:g(\bar{B}^n)/(g(\bar{B}^n)-U)\to\bar{B}^n/(\bar{B}^n-rB^n)$. $\bar{f}\circ\bar{g}$ is a degree one map between manifolds. We may consider $g(\bar{B}^n)/(g(\bar{B}^n)-U)$ to be a subset of the relative manifold $V/(V-U)$. Since $g(\bar{B}^n)/(g(\bar{B}^n)-U)$ supports an n-dimensional homology class, $g(\bar{B}^n)/(g(\bar{B}^n)-U)=V/(V-U)$ and $g(\bar{B}^n)\supset U\supset g(rB^n)$.

Q.E.D.

Remark. Similarly we can show that for each $r<1$ there is an $\epsilon>0$ such that if $g:(\bar{B}^n_+,\partial\bar{B}^n_+)\to(V^n,\partial V^n)$ is an ϵ-map, then $g(rB^n_+)$ is contained in the interior of $g(\bar{B}^n_+)$.

THEOREM 4.2. *Let M^n be a manifold and let C_1,C_2 be closed subsets of M with $C_1\subset\mathring{C}_2$. Then for each open cover α of C_2 there is an open cover β of*

C_2 such that if $g:(C_2, C_2 \cap \partial M) \to (V^n, \partial V)$ is a proper β-map, and either $n \geqslant 6$ or $n=5$ and $g|C \cap \partial M$ is a homeomorphism then $g|C_1$ is $g(\mathrm{St}\,\alpha)$-homotopic to a homeomorphism. By this we mean that there is a homotopy $h: C_1 \times I \to V$ with $h_0 = g|C_1$, h_1 a homeomorphism, and such that for each $x \in C_1$, $h(\{x\} \times I) \subset g(\mathrm{St}(\{x\}, \alpha))$.

Proof. Choose closed sets C_3 and C_4 so that $\mathring{C}_2 \supset C_3 \supset \mathring{C}_3 \supset C_4 \supset \mathring{C}_4 \supset C_1$. By the proof of Theorem 2, given an open cover γ_1 of M, we can choose an open cover β of M which is so fine that g a proper β-map implies that there is a map $f: g(C_2) \to M$ such that $f \circ g$ is γ_1-homotopic to the inclusion. By Proposition 4.1, we can also choose β so that $g(C_3)$ is contained in the interior of $g(C_2)$.

Let U be the interior of $g(C_2)$. We will show that $f|U: U \to M$ is a proper γ_1-domination over C_3. This will follow by Proposition 3.6 once we have verified that f is proper over C_3. This is easy. If $K \subset C_3$ is compact, $g^{-1} \circ f^{-1}(K)$ is a closed subset of $\overline{\mathrm{St}(K, \gamma_1)}$. This is compact whenever elements of γ_1 have compact closures. Since the image of g contains U, $f^{-1}(K) = g \circ g^{-1} \circ f^{-1}(K)$ is compact.

By Theorem 3.9, given γ_2 we can choose γ_1 so that if f is a proper γ_1-domination over C_3, then f is a proper γ_2-equivalence with inverse g over C_4. The α-approximation theorem of Theorem 4 is proved by a handle induction and therefore has a local version (see Theorem 5.3 of [5] for a statement). Thus, we can choose γ_2 so fine that if f is a proper γ_2-equivalence over C_4, then $f|f^{-1}C_1$ is α-homotopic to a homeomorphism. Given α, we need only choose γ_2, γ_1, and β in that order to complete the proof that f is α-homotopic to a homeomorphism.

To see that g is $g(\mathrm{St}\,\alpha)$-homotopic to f^{-1}, note that we have

$$f^{-1} \overset{f^{-1}\alpha}{\simeq} g \circ f \circ f^{-1} = g.$$

Thus, we need only show that an $f^{-1}\alpha$-homotopy is a $g(\mathrm{St}\,\alpha)$-homotopy. Suppose that $x_1, x_2 \in f^{-1}(U)$, $U \in \alpha$. Since g is onto, there exist y_1, y_2 with $x_1 = g(y_1)$, $x_2 = g(y_2)$, and we see that y_1 is $\mathrm{St}\,\alpha$-close to y_2, since y_1 is α-close to $f \circ g(y_1) = f(x_1)$, which is α-close to $f(x_2)$, which in turn is α-close to y_2. This completes the proof. Q.E.D.

5. A Theorem of Černavskii and Seebeck.

Definition. Let X be a metric space, and let M be a subset of X. M is said to be *k-LCC in X* if for each $x \in M$ and $\varepsilon > 0$ there is a $\delta > 0$ such that each map $f: S^i \to B(\delta, x) \cap (X - M)$, $i \leqslant k$, is homotopic to a constant map in $B(\varepsilon, x) \cap (X - M)$.

THEOREM 5. *Suppose that U is a noncompact n-manifold without boundary, $n \geqslant 5$, and that M is an $(n-1)$-manifold without boundary. Suppose that $X = U \cup M$ is a locally compact metric space such that $U \cap M = \varnothing$, U is dense in X, and M is $(n-1)$-LCC in X. Then X is an n-manifold with boundary M.*

Remark. This theorem was announced by Černavskii in [3]. He says that the proof is a difficult engulfing argument. Details have never appeared. Seebeck has recently circulated a proof [14, 15] which is indeed a difficult engulfing argument. We will show that this result is a straightforward corollary to Theorem 4.2.

Let X, U, and M be as in the statement of Theorem 5.

LEMMA 5.1. *For each open cover α of X,*

i. *there is a map $f: M \to U$ such that f is α-close to the inclusion, and*
ii. *there is an open cover β of X such that two maps $f_1, f_2: M \to U$ which are β-close to the inclusion are α-homotopic in U.*

Proof. The proof of this lemma is standard. If M is triangulable, the required maps and homotopies are easily constructed by induction on skeleta. Otherwise one works through the nerve of an appropriate open cover of M. Q.E.D.

LEMMA 5.2. *For each open cover α of $M \times [0,1)$ there is a proper α-map $g: M \times [0,1) \to U$ such that g extends continuously by the identity to a map $\bar{g}: M \times [0,1] \to X$.*

Proof. For simplicity, assume that M is compact. Choose a sequence of points $0 = t_1 < t_2 < \dots$ with $\lim t_i = 1$ and a sequence $\{\alpha_i\}$ of open covers of M so that $\{U_i \times [t_i, t_{i+2}] \mid U_i \in \alpha_i,\ i = 1, 2, \dots\}$ refines α.

Let γ_1 be an open cover of X such that $\mathrm{St}\,\gamma_1 | M$ refines α_1, and let β_1 be an open cover of X such that two maps $f_1, f_2: M \to U$ which are β_1-close to inclusion are γ_1-homotopic. Let $g | M \times \{0\}$ be a map from M to U which is β_1-close to the inclusion. Let γ_2 be an open cover of X such that $(\mathrm{St}^3 \gamma_2) | M$ refines α_2, γ_2 refines γ_1, and such tthat no element of $\mathrm{St}\,\gamma_2$ meets both M and $g(M \times \{0\})$. Let β_2 be an open cover of X such that maps of M into U which are β_2-close to the inclusion are γ_2-homotopic. Let $g | M \times \{t_2\}$ be a map from M to U which is β_2-close to the inclusion, and extend g to $M \times [0, t_2]$ using the γ_1-homotopy promised by our choice of β_1.

In general, if g is defined on $M \times [0, t_i]$, choose γ_{i+1} so that γ_{i+1} refines γ_i, $(\mathrm{St}^3 \gamma_{i+1}) | M$ refines α_{i+1}, and no element of $\mathrm{St}\,\gamma_{i+1}$ meets both M and $g(M \times [0, t_1])$. Choose β_{i+1} so that maps β_{i+1}-close to inclusion are γ_{i+1}-homotopic, and let $g|M\times\{t_{i+1}\}$ be a map β_{i+1}-close to inclusion. Use the γ_i-homotopy guaranteed by our choice of β_i to extend g over $M\times[t_i, t_{i+1}]$.

Our choice of $\{\gamma_i\}$ implies that every point inverse $g^{-1}g(x,t)$ is contained in $M\times[t_i,t_{i+2}]$ for some i. Moreover, if $(x,t)\in M\times[t_1,t_{i+2}]$, then $g(x,t)$ is γ_i-close to $g(x,t_i)$, which is β_i-close to x. Thus, $g(x,t)$ is $\mathrm{St}\,\gamma_i$-close to x. Thus, if $g^{-1}g(x,t)\subset M\times[t_i,t_{i+2}]$, then $g(x,t)=g(x',t')$ implies that x is $(\mathrm{St}^2\gamma_i)|M$-close to x' and therefore that $g^{-1}g(x,t)\subset U\times[t_i,t_{i+2}]$, where $U\in(\mathrm{St}^3\gamma_i)|M$. Since $(\mathrm{St}^3\gamma_i)|M$ refines α_i, g is an α-map.

It is clear that the covers β_i may be chosen so that g extends by the identity to $M\times[0,1]$. This guarantees that g is proper. Q.E.D.

The proof of Theorem 5 is now an easy exercise. Let α be an open cover of $M\times[0,1)$ such that $(x,t)\in U\in\alpha$ implies $\operatorname{diam} U<(1-t)/2$. Let β be an open cover of $M\times[0,1)$ such that if $g:M\times[0,1)\to U$ is a proper β-map, then $g|M\times[\frac{1}{2},1)$ is $g(\mathrm{St}^2\alpha)$-homotopic to a homeomorphism

$$h:M\times\left[\tfrac{1}{2},1\right)\xrightarrow{\text{into}} U.$$

Since g extends by the identity to $M\times[0,1]$, for each $\varepsilon>0$ there is a $T<1$ such that $d(g(x,t),g(x',t'))<d(x,x')+\varepsilon$ for $t,t'>T$. This means that if $(x,t)\in U\in \mathrm{St}^2\alpha$ and t is large, then $\operatorname{diam} g(U)$ is small. h therefore extends by the identity to a homeomorphism

$$\bar{h}:M\times\left[\tfrac{1}{2},1\right]\xrightarrow[\text{into}]{} X.$$

$\bar{h}(M\times(\frac{1}{2},1])$ is both open and closed in $X-h(M\times\frac{1}{2})$, so the image of $M\times[\frac{1}{2},1]$ under $\bar{h}$ is a collar neighborhood of M, and X is a manifold. Q.E.D.

J. W. Cannon has pointed out that this argument can be used to prove a taming theorem which does not appear to be accessible by the techniques of Seebeck's proof of the Černavskii-Seebeck theorem. Let M^{n-1} $(n\geqslant 5)$ be a space such that $M\times R^1$ is an n-dimensional manifold, and let $X\supset M$ be a space such that $X-M$ consists of two components U_1 and U_2, each of which is an n-manifold. If M is $(n-1)$-LCC in $\bar{U}_1$ and in $\bar{U}_2$, then X is a manifold.

Every point in $X-M$ already has a Euclidean neighborhood. By repeating our proof of the Cernavskii-Seebeck theorem in $\bar{U}_1$ and again in $\bar{U}_2$, we obtain a bicollar of M in X. This provides the required Euclidean neighborhoods of points in M. If X is a homology manifold, then the $(n-1)$-LCC condition is automatic (by Alexander duality and Proposition 3.1) if M is known to be 1-LCC in $\bar{U}_1$ and $\bar{U}_2$. Q.E.D.

6. The Proof of Theorem 6. Theorem 6 is due to R. D. Edwards. For simplicity, we consider only the case where M is unbounded.

(i) *Only if*: Let $f:M\to X$ be a near-homeomorphism. Then for each open cover β of X there is a homeomorphism $h:M\xrightarrow{\text{onto}}X$ such that h is β-close to f. Letting $p=h$ and $q=h^{-1}$ satisfies the conditions of the theorem.

(ii) *If*: We wish to apply Bing's shrinking criterion [4, p. 45], which states that f is a near-homeomorphism if and only if for each pair of open covers $\mathcal{U}$ of M and $\mathcal{V}$ of X there is a homeomorphism $h:M\to M$ such that fh is $\mathcal{V}$-close to f and such that each $hf^{-1}(y)$ lies in an element of $\mathcal{U}$.

Choose an open cover β of X so that $\mathrm{St}^2|\beta$ refines $\mathcal{V}$. Choose an open cover α_2 of M so that $\mathrm{St}\,\alpha_2$ refines $\mathcal{U}$ and so that α_2 refines $f^{-1}\beta$. Choose an open cover α_1 of M so that every α_1-domination from M to itself is α_2-homotopic to a homeomorphism. Choose α_1 so that α_1 refines $f^{-1}\beta$. Now choose p and q so that p is β-homotopic to f and $q\circ p$ is α_1-homotopic to the identity.

Consider the map $q\circ f:M\to M$. According to Lacher [13], for each open cover γ of X there is a map $\mu:X\to M$ such that

$$f\circ\mu\overset{\gamma}{\simeq}\mathrm{id}\quad\text{and}\quad\mu\circ f\overset{f^{-1}\gamma}{\simeq}\mathrm{id}.$$

If μ is such a map, we see that $(q\circ f)\circ(\mu\circ p)\simeq q\circ p\simeq\mathrm{id}$. If γ is chosen sufficiently fine, this is an α_1-homotopy and $(q\circ f)$ is an α_1-domination. Thus $q\circ f$ is α_2-homotopic to a homeomorphism $h:M\to M$.

If $f(m_1)=f(m_2)$, then $h(m_1)$ is α_2-close to $q\circ f(m_1)=q\circ f(m_2)$, and $q\circ f(m_2)$ is α_2-close to $h(m_2)$. This implies that $hf^{-1}(x)$ is contained in an element of $\mathrm{St}\,\alpha_2$ (which refines $\mathcal{U}$) for each $x\in X$. We need only show that $f\circ h$ is $\mathcal{V}$-close to f.

Now, $f\circ h\overset{\beta}{\simeq}f\circ q\circ f$. We complete the proof by showing that $f\circ q\overset{\mathrm{St}\beta}{\simeq}\mathrm{id}$. Note that $f\circ q\circ(p(x))\overset{\beta}{\simeq}f(x)\overset{\beta}{\simeq}p(x)$. Thus, it will suffice to show that p is onto. This is true for homological reasons.

Recall from [13] that for each $U\subset X$, $f:f^{-1}(U)\to U$ is a proper homotopy equivalence. If $V\subset M^n$ is a connected open set such that $\bar{V}$ is compact, then $H^n(M,M-V;Z_2)=Z_2$ and $H^n(M-m,\ M-V;\ Z_2)=0$ for each $m\in V$. Let $p:M\to X$ be a map which is β-homotopic to f such that $p(M)\subset X-\{x_0\}$. If β is a locally finite cover by relatively compact open sets, then we can use the homotopy extension theorem to construct a map $\bar{p}:M\to X$ such that $\bar{p}|f^{-1}(X-U)=f|f^{-1}(X-U)$, where U is a relatively compact connected open subset of X, such that $\bar{p}(M)\subset X-x_0$, and such that $\bar{p}$ is homotopic to f rel $f^{-1}(X-U)$.

It follows from Corollary 4 of [11] that f induces an isomorphism from $H^n(X,X-U;Z_2)$ to $H^n\big(M,M-f^{-1}(U);Z_2\big)$. Thus, both groups are isomorphic to Z_2. On the other hand, $p^*:H^n(X,X-U)\to H^n\big(M,M-f^{-1}(U)\big)$ factors through $H^n(X - x_0, X - U; Z_2)$. f induces an isomorphism of this latter group with $H^n(M - f^{-1}(x_0), M - f^{-1}(U); Z_2)$, which is zero. Since f is homotopic to $\bar{p}$ rel $M - f^{-1}(U)$, this is a contradiction, which completes the proof of Theorem 6. Q.E.D.

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