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## SHAPE EQUIVALENCE DOES NOT IMPLY CE EQUIVALENCE<sup>1</sup>

STEVE FERRY

**ABSTRACT.** We give an example of shape equivalent compacta  $X$  and  $Y$  such that there is no compactum  $Z$  with cell-like maps  $Z \rightarrow X$  and  $Z \rightarrow Y$ .

A space  $X$  is said to be *cell-like* if for some imbedding of  $X$  in an ANR,  $X$  has the property that for each neighborhood  $U$  of  $X$ ,  $X$  contracts to a point in  $U$ . This is an intrinsic property of  $X$  and is independent of the choice of ANR and embedding. A continuous map  $f: Z \rightarrow Y$  between compacta is said to be cell-like (CE) if  $f$  is surjective and  $f^{-1}(y)$  is cell-like for each  $y \in Y$ . If  $X$  and  $Y$  are compacta, we say that  $X$  and  $Y$  are *CE equivalent* if there exist compacta  $X = X_0, X_1, \dots, X_{2k} = Y$  and CE maps  $f_{2i}: X_{2i+1} \rightarrow X_{2i}$  and  $f_{2i+1}: X_{2i+1} \rightarrow X_{2i+2}$  for  $i = 0, 1, \dots, k-1$ .

In  $[F_1]$  it is shown that two compacta which are homotopy equivalent must be CE equivalent. In fact, more is shown. The maps constructed have sections and contractible point-inverses. It is natural to seek a Čech analog of this theorem for general compacta. Thus, we are led to study the question: "If  $X$  and  $Y$  are shape equivalent compacta, must  $X$  and  $Y$  be CE equivalent?"

In this note we will exhibit a simple example which shows that this is not the case. Let  $X$  be a plane compactum which is the union of a circle  $C$  and a ray  $R$

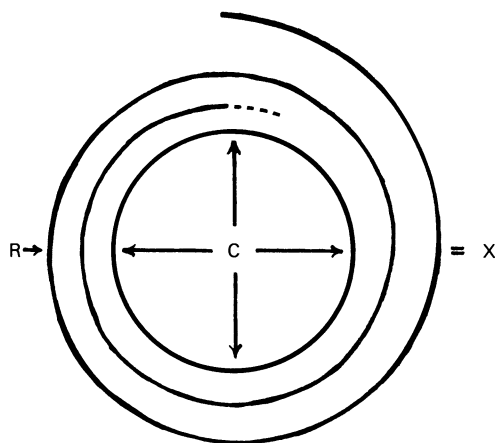


FIGURE 1

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which spirals into  $C$ . See Figure 1.  $X$  is shape equivalent to  $S^1$ . We will show that  $X$  is not CE equivalent to  $S^1$ .

DEFINITION 1. We will say that a compactum  $Z$  is an *acyclic image* if there exist a compactum  $W$  with  $\check{H}^*(W) = \check{H}^*(\text{pt})$  and a continuous surjection  $f: W \rightarrow Z$ .

LEMMA 1. Let  $P$  and  $Q$  be CE equivalent compacta. Then  $P$  is an acyclic image if and only if  $Q$  is an acyclic image.

PROOF. It suffices to consider the case in which there is a CE map  $r: P \rightarrow Q$ . It is clear that  $Q$  is an acyclic image if  $P$  is an acyclic image. Suppose, then, that  $Q$  is an acyclic image. Let  $f: W \rightarrow Q$  be a surjection as in Definition 1 and let  $E$  be the pullback in the diagram below.

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & P \\ \downarrow \tilde{r} \text{ CE} & & \downarrow r \text{ CE} \\ W & \xrightarrow{f} & Q \end{array}$$

$E$  is compact,  $\tilde{f}$  is surjective, and  $\tilde{r}$  is CE. A cell-like set has the Čech cohomology of a point, so the Vietoris-Begle theorem [S] implies that  $\tilde{r}$  induces an isomorphism of Čech cohomology. Thus,  $E$  has the Čech cohomology of a point and  $P$  is an acyclic image.  $\square$

LEMMA 2.<sup>2</sup> The space  $X$  of Figure 1 is not an acyclic image.

PROOF. Suppose not. Let  $f: W \rightarrow X$  be a surjection as in Definition 1. Let  $r: X \rightarrow C$  be a radial retraction and let  $e: E^1 \rightarrow C$  be the universal cover. Since  $\check{H}^1(W) \cong [W, C] = 0$ , the composition  $r \circ f: W \rightarrow C$  lifts to  $E^1$  and there is a map  $\tilde{f}: W \rightarrow E^1$  such that  $e \circ \tilde{f} = r \circ f$ .

Let  $W' = f^{-1}(R)$ . Choose a map  $\tilde{r}: R \rightarrow E^1$  so that  $e \circ \tilde{r} = r|_R$  and so that  $\tilde{r} \circ f = \tilde{f}$  for some point  $w_0 \in W'$ . Let  $W'' = \{w \in W' \mid \tilde{r} \circ f(w) = \tilde{f}(w)\}$ . The usual argument shows that  $W''$  is open in  $W'$  and therefore in  $W$ .  $W''$  cannot be closed in  $W$  since  $W$  is connected and  $W''$  is neither empty nor all of  $W$ .

There is therefore a sequence  $\{w_i\} \in W''$  converging to a point  $w^* \in W - W'$ . Thus,  $\lim f(w_i) \in C$  and  $\{\tilde{r} \circ f(w_i)\}$  is unbounded in  $E^1$ . On the other hand,  $\{\tilde{r} \circ f(w_i)\} = \{\tilde{f}(w_i)\} \subset \tilde{f}(W)$ , which is compact. This is the desired contradiction.  $\square$

This completes the proof of our main result, since there is a continuous map of  $[0, 1]$  onto  $S^1$ .

<sup>2</sup> Lemma 2 is essentially Theorem 1 of M. K. Fort [F0].

It would be interesting to find shape equivalent  $UV^1$  compacta which are not CE equivalent. Parts of [F<sub>2</sub>] are relevant to this problem.

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