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SHAPE EQUIVALENCE DOES NOT IMPLY CE EQUIVALENCE[1]

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ABSTRACT. We give an example of shape equivalent compacta X and Y such that there is no compactum Z with cell-like maps $Z \twoheadrightarrow X$ and $Z \twoheadrightarrow Y$.

A space X is said to be *cell-like* if for some imbedding of X in an ANR, X has the property that for each neighborhood U of X, X contracts to a point in U. This is an intrinsic property of X and is independent of the choice of ANR and embedding. A continuous map $f\colon Z \to Y$ between compacta is said to be cell-like (CE) if f is surjective and $f^{-1}(y)$ is cell-like for each $y \in Y$. If X and Y are compacta, we say that X and Y are *CE equivalent* if there exist compacta $X = X_0, X_1, \ldots, X_{2k} = Y$ and CE maps $f_{2i}\colon X_{2i+1} \to X_{2i}$ and $f_{2i+1}\colon X_{2i+1} \to X_{2i+2}$ for $i = 0, 1, \ldots, k-1$.

In [$\mathbf{F_1}$] it is shown that two compacta which are homotopy equivalent must be CE equivalent. In fact, more is shown. The maps constructed have sections and contractible point-inverses. It is natural to seek a Čech analog of this theorem for general compacta. Thus, we are led to study the question: "If X and Y are shape equivalent compacta, must X and Y be CE equivalent?"

In this note we will exhibit a simple example which shows that this is not the case. Let X be a plane compactum which is the union of a circle C and a ray R

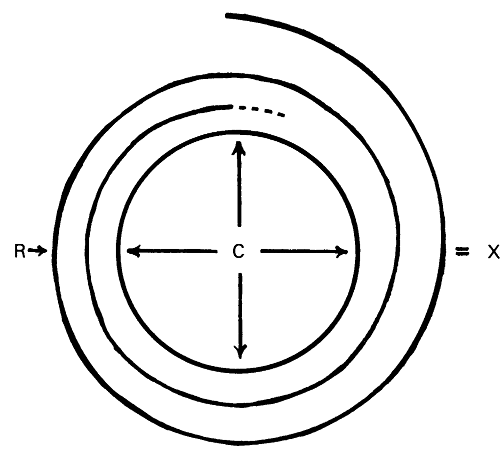

FIGURE 1

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which spirals into C. See Figure 1. X is shape equivalent to S^1. We will show that X is not CE equivalent to S^1.

DEFINITION 1. We will say that a compactum Z *is an acyclic image* if there exist a compactum W with $\check{H}^*(W) = \check{H}^*(\mathrm{pt})$ and a continuous surjection $f\colon W \to Z$.

LEMMA 1. *Let P and Q be CE equivalent compacta. Then P is an acyclic image if and only if Q is an acyclic image.*

PROOF. It suffices to consider the case in which there is a CE map $r\colon P \to Q$. It is clear that Q is an acyclic image if P is an acyclic image. Suppose, then, that Q is an acyclic image. Let $f\colon W \to Q$ be a surjection as in Definition 1 and let E be the pullback in the diagram below.

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & P \\ \downarrow \tilde r \text{ CE} & & \downarrow r \text{ CE} \\ W & \xrightarrow{f} & Q \end{array}$$

E is compact, $\tilde f$ is surjective, and $\tilde r$ is CE. A cell-like set has the Čech cohomology of a point, so the Vietoris-Begle theorem [S] implies that $\tilde r$ induces an isomorphism of Čech cohomology. Thus, E has the Čech cohomology of a point and P is an acyclic image. □

LEMMA 2.[2] *The space X of Figure 1 is not an acyclic image.*

PROOF. Suppose not. Let $f\colon W \to X$ be a surjection as in Definition 1. Let $r\colon X \to C$ be a radial retraction and let $e\colon E^1 \to C$ be the universal cover. Since $\check{H}^1(W) \cong [W, C] = 0$, the composition $r \circ f\colon W \to C$ lifts to E^1 and there is a map $\tilde f\colon W \to E^1$ such that $e \circ \tilde f = r \circ f$.

Let $W' = f^{-1}(R)$. Choose a map $\tilde r\colon R \to E^1$ so that $e \circ \tilde r = r|R$ and so that $\tilde r \circ f = \tilde f$ for some point $w_0 \in W'$. Let $W'' = \{w \in W' | \tilde r \circ f(w) = \tilde f(w)\}$. The usual argument shows that W'' is open in W' and therefore in W. W'' cannot be closed in W since W is connected and W'' is neither empty nor all of W.

There is therefore a sequence $\{w_i\} \in W''$ converging to a point $w^* \in W - W'$. Thus, $\lim f(w_i) \in C$ and $\{\tilde r \circ f(w_i)\}$ is unbounded in E^1. On the other hand, $\{\tilde r \circ f(w_i)\} = \{\tilde f(w_i)\} \subset \tilde f(W)$, which is compact. This is the desired contradiction. □

This completes the proof of our main result, since there is a continuous map of $[0, 1]$ onto S^1.

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[2] Lemma 2 is essentially Theorem 1 of M. K. Fort [F0].

It would be interesting to find shape equivalent UV^1 compacta which are not CE equivalent. Parts of [$\mathbf{F_2}$] are relevant to this problem.

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