A SIMPLE-HOMOTOPY APPROACH TO THE FINITENESS OBSTRUCTION

Steve Ferry*

§0. INTRODUCTION

The purpose of this paper is to develop Wall's finiteness obstruction ([Wa_1], [Wa_2]) from an extremely geometrical point of view. There is an analogy, which will be made precise, to the situation regarding the theory of Whitehead torsion where there are two treatments in somewhat differing styles. The first treatment is that of Whitehead [W] which quickly reduces statements in PL topology to statements above chain complexes and proceeds on this basis. The second treatment is that of M. Cohen [Co_1] in which the Whitehead group is defined geometrically. All of the *formal* properties of the theory (functoriality, sum theorem, product theorem, etc.) are easily derivable from this geometric definition. Of course, the same reduction to chain complexes is necessary in order to calculate Whitehead groups and show that the theory is non-vacuous.

Cohen's approach has proven to be very influential. Siebenmann's infinite simple homotopy theory [S] and Hatcher's higher simple homotopy theory [H] are developed along these geometric lines. It is also possible to develop a controlled simple homotopy theory (in the sense of Chapman-Connell-Hollingswoth-Quinn) along these lines. These last two cases are noteworthy because the corresponding reduction to algebra is either difficult to carry out or reduces to heretofore unknown algebraic territory.

The approach of Wall's original papers on the finiteness obstruction [Wa_1,Wa_2] is philisophically very similar to that of Whitehead. We will redevelop this theory along Cohen's more geometrical lines.

One payoff is that there is a well-understood procedure for passing from the study of Wall's finiteness obstruction to the study of stable PL or Q-manifold missing-boundary problems. See [Ch-S] or the section entitled "Splitting Theorem" in almost any paper by Chapman. Thus, a geometric version of the Chapman-Connell-Hollingsworth-Quinn controlled simple homotopy theory should be usable to develop a Q-manifold version of Quinn's controlled end theory [Q].

The techniques used in this paper are derived from Mather's influential [Ma] and from "Siebenmann's variation on West's proof that compact ANR's have finite type," an unpublished manuscript of R.D. Edwards. These techniques have been exploited relentlessly during the past few years by T.A. Chapman and the author ([Ch_1],[Ch_2],[F]). The present paper is, to the author's knowledge, the first time that this approach has been used to set up a nontrivial obstruction theory. As stated above, the main value of this approach is that it is extremely formal and therefore is adaptable to a variety of situations.

---

* Partially supported by NSF grants and the A.P. Sloan Foundation.

§1. SIMPLE HOMOTOPY PRELIMINARIES

We will assume that the reader is *completely* familiar with §4-6 of [Co_1]. For our purposes, we will need to develop this geometric theory in slightly greater generality.

DEFINITION 1.1. Let X be a topological space. A *finite relative cell complex* (over X) is a pair (L,X) where $L = L_n \supset L_{n-1} \supset \dots \supset L_o = X$ and L_{i+1} is obtained from L_i by attaching a cell of some dimension. Note that cells are attached in no particular order and with possibly noncellular attaching maps.

DEFINITION 1.2. If (K_1,X) and (K_2,X) are finite relative cell complexes, we say that K_1 collapses to K_2 by an *elementary collapse* if $K_1 = K_2 \cup e^{n-1} \cup e^n$ and the attaching map for e^n is a homeomorphism over (when restricted to the inverse image of) the interior of e^{n-1}. A finite sequence of operations each of which is an elementary expansion or elementary collapse is called a *formal deformation*.

DEFINITION 1.3. $Wh(X) = \{(K,X) \mid K \hookrightarrow X$ is a homotopy equivalence and (K,X) is a finite relative cell complex$\}/\sim$ where $(K_1,X) \sim (K_2,X)$ iff K_1 formally deforms to K_2 rel X. We will denote the equivalence class of (K_1,X) by $[K_1,X]$ or $\tau(K_1,X)$.

DEFINITION 1.4. If $f : X \to Y$ we define $f_* : Wh(X) \to Wh(Y)$ by $f_*([K,X]) = [K \cup_f Y,Y]$.

One easily checks that Wh(X) is an abelian group. The only alteration of the proof on p. 21 of [Co] is to use reduced mapping cylinders M_D in which the copy of $X \times I$ has been collapsed to a single copy of X.

If $f : (K,X) \to (L,X)$ is a homotopy equivalence of finite relative cell complexes which is the identity on X, we define $\tau(f) = f_*([M_X(f),K]) \in Wh(L)$ where $M_X(f)$ is the reduced mapping cylinder.

If $K \supset L$ are finite simplicial complexes with L a strong deformation retract of K, then $[K,L] = 0$ in Wh(L) if and only if there exist a finite simplicial complex $\overline{K}$ and PL maps $c : \overline{K} \to K$, $d : \overline{K} \to L$ with contractible point-inverses such that the diagram

$$\begin{tikzcd} & \overline{K} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ L \arrow[rr, hook] & & K \end{tikzcd}$$

homotopy commutes. "Only if" follows from the fact that K and L have PL homeomorphic regular neighborhoods in high-dimensional Euclidean spaces. "If" is a result of M. Cohen [Co_2]. The reader is urged to attempt to prove this result for himself using the Sum Theorem (Prop. 1.6) for Whitehead torsion. The "if" implication is not used in this paper.

PROPOSITION 1.6. ([Co_1,p.76]). Suppose that $K = K_1 \cup K_2$, $K_0 = K_1 \cap K_2$, $L = L_1 \cup L_2$, $L_0 = L_1 \cap L_2$ and that $f : K \to L$ is a map which restricts to homotopy equivalences $f_\alpha : K_\alpha \to L_\alpha$, $\alpha = 0,1,2$. Let $j_\alpha : L_\alpha \to L$ and $i_\alpha : K_\alpha \to K$ be the inclusions. Then f is a homotopy equivalence and $\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) - j_{0*}\tau(f_0)$. If f is an inclusion, $\tau(L,K) = i_{1*}\tau(L_1,K_1) + i_{2*}\tau(L_2,K_2) - i_{0*}\tau(L_0,K_0)$.

PROOF. We prove the second statement. The first follows easily. Let $r_t : L_0 \to L_0$ be a homotopy rel K_0 from id to a retraction $r_1 : L_0 \to K_0$. By the simple homotopy extension theorem, L_1 deforms rel K_0 to $L_1 \cup_{r_1} L_0$. Similarly, L_2 and L deform rel K_0 to $L_2 \cup_{r_1} L_0$ and $L \cup_{r_1} L_0$. But one easily sees that

$$(L \cup_{r_1} L_0) \underset{K_0}{\cup} L_0 = (L_1 \cup_{r_1} L_0) \underset{K_0}{\cup} (L_2 \cup_{r_1} L_0) .$$

This completes the proof of Proposition 1.6. ∎

REMARK: Proposition 1.6 could be proven for finite relative complexes, but this result is not needed in the sequel, so we omit it. We will need the following formula for the torsion of a composition.

PROPOSITION 1.7. If $f : K \to L$ and $g : L \to M$ are homotopy equivalences between finite simplicial complexes, then $\tau(g \circ f) = \tau(g) + g_*\tau(f)$.

PROOF. This follows immediately from Fact 2 on p. 22 of [Co_1]. Again, a version for finite relative complexes is possible but unnecessary.

We will need several easy lemmas about mapping cylinders in the sequel.

LEMMA MC1. If $f : X \to Y$ is a map, then the mapping cylinder $M(f)$ is homotopy equivalent to Y. ∎

LEMMA MC2. If $f : X \to Y$ and $g : Y \to Z$ are maps then the mapping cylinder $M(g \circ f)$ is homotopy equivalent to the union along Y of $M(f)$ and $M(g)$. This homotopy equivalence is the identity on $X \cup Z$. ∎

LEMMA MC3. If $f,g : X \to Y$ are homotopic maps, then $M(f)$ is homotopy equivalent to $M(g)$ via a homotopy equivalence which is the identity on $X \cup Y$. ∎

See [Co_1,§5] or [F] for proofs.

§2. WALL'S FINITENESS OBSTRUCTION

Let X be a topological space and let $d : K^n \to X$ be a domination with right inverse u. Here, K^n is a finite n-dimensional CW-complex.

PROPOSITION 2.1. (Mather [M]). X is homotopy equivalent to an (n+1)-dimensional CW-complex.

PROOF. Let $\alpha = u \circ d : K \to K$ and let $T(\alpha)$ be the space obtained from the mapping cylinder $M(\alpha)$ by identifying the top and the bottom of $M(\alpha)$ using the identity map. Following Mather, we show that X is homotopy equivalent to a cyclic cover of $T(\alpha)$.

The argument is most easily understood via a picture (Figure 1). $T(\alpha)$ is seen to be homotopy equivalent to the intermediate space Y. This uses MC2 and the fact that $\alpha = u \circ d$. Y is then seen to be homotopy equivalent to $X \times S'$ using MC2 and the fact that $d \circ u \simeq id$.

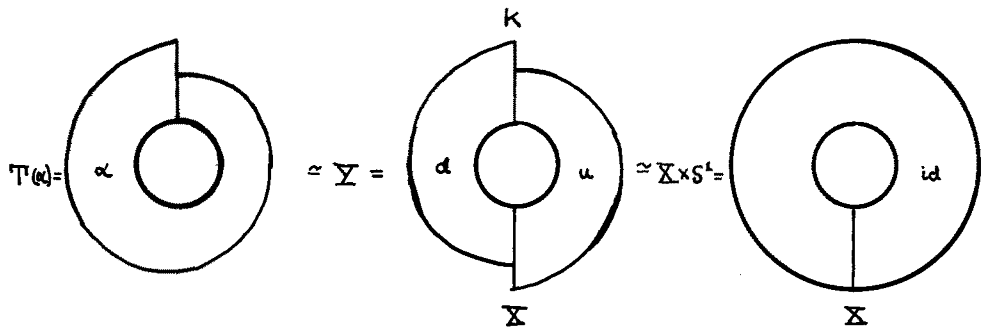


Figure 1

X is homotopy equivalent to the infinite cyclic cover of $X \times S'$ obtained by unwrapping the S^1 factor. X is therefore homotopy equivalent to the covering space $I(\alpha)$ of $T(\alpha)$ illustrated in Figure 2.

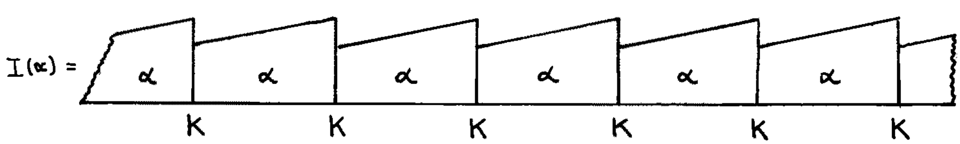


Figure 2

This completes the proof of Proposition 2.1. ∎

DEFINITION 2.2. Let $T : X \times S^1 \to X \times S^1$ be the homeomorphism $(\chi,\theta) \to (\chi,-\theta)$. Let $\phi : T(\alpha) \to X \times S^1$ be the homotopy equivalence defined in Proposition 2.1. Then we define $\sigma(X) = \phi_* \tau(\phi^{-1} T\phi) \in Wh(X \times S^1)$, where ϕ^{-1} is a homotopy inverse to ϕ.

Let X be a finitely dominated topological space. Then

THEOREM 2.3. $\sigma(X)$ is well-defined and $\sigma(X) = 0$ if and only if X is homotopy equivalent to some finite CW-complex.

PROOF. Let $d_1 : K_1 \to X$ and $d_2 : K_2 \to X$ be finite dominations with right inverses u_1 and u_2, respectively.

STEP 1. We wish to show that the elements of $Wh(X \times S^1)$ obtained by the process of Proposition 2.1 are the same.

<u>Case I.</u> Suppose that $K_1 \supset K_2$, $d_2 = d_1|K_2$, and $u_1 = u_2$.

Let $\alpha_1 = u_1 \circ d_1$ and let $\alpha_2 = u_2 \circ d_2$. In this case there is a collapse $g : T(\alpha_1) \to T(\alpha_2)$ and the diagram

$$\begin{tikzcd} T(\alpha_1) \arrow[rr, "g"] \arrow[dr, "\phi_1"'] & & T(\alpha_2) \arrow[dl, "\phi_2"] \\ & X \times S' & \end{tikzcd}$$

commutes up to homotopy. Then

$$\begin{aligned} \phi_{1*}\tau(\phi_1^{-1}T\phi_1) &= \phi_{2*}g_*\tau(g^{-1}\phi_2^{-1}T\phi_2 g) \\ &= \phi_{2*}g_*[0 + g_*^{-1}\tau[\phi_2^{-1}T\phi_2] + 0] \\ &= \phi_{2*}\tau[\phi_2^{-1}T\phi_2] \ . \end{aligned}$$

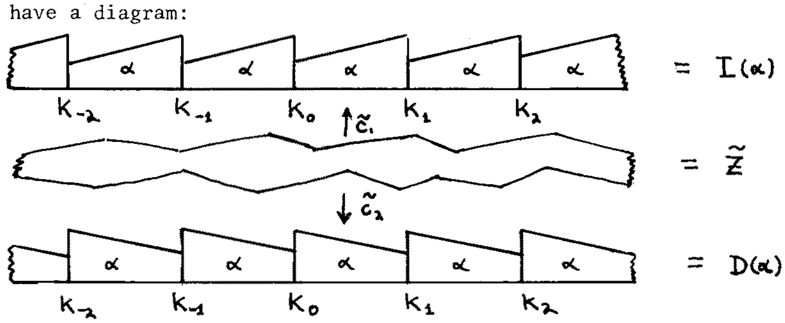
This calculation uses the fact that $\tau(g) = 0$ and the composition formula Proposition 1.7. This completes the proof of Case I.

<u>Case II.</u> The general case.

Given $d_1 : K_1 \to X$ and $d_2 : K_2 \to X$ form $K_3 = M(u_2 \circ d_1)$. Define $d_3 : K_3 \to X$ by $d_3 = d_2 \circ c$, where $c : M(u_2 \circ d_1) \to K_2$ is the mapping cylinder collapse. Define $u_3 : X \to K_3$ to be the composition of u_2 with the inclusion map. Now, $d_3|K_1 \sim d_2 \circ u_2 \circ d_1 \sim d_1$ and the composition of u_1 with the inclusion map is homotopic to $u_2 \circ d_1 \circ u_1 = u_2 = u_3$. Thus we are reduced to Case I and $\sigma(X)$ is well-defined.

STEP 2. $\sigma(X) = 0 \Rightarrow X$ is homotopy equivalent to some finite complex.

If $\sigma(X) = 0$, then $\tau(\phi^{-1}T\phi) = 0$ and there exist a finite polyhedron Z and CE-PL maps $c_1 : Z \to T(\alpha)$ and $c_2 : Z \to T(\alpha)$ so that $c_2 \sim \phi^{-1}T\phi c_1$. Passing to infinite cyclic covers, we have a diagram:



where the map $\tilde{c}_2 \circ (\tilde{c}_1)^{-1} : I(\alpha) \to I(\alpha)$ reverses the ends. We refer to this reversed copy of $I(\alpha)$ as $D(\alpha)$. If we choose N large enough, the region of $\tilde{Z}$ trapped between $(\tilde{c}_1)^{-1}(K_N)$ $(K_1, K_2, \ldots$ are all copies of K) and $(\tilde{c}_2)^{-1}(K_{-N})$ is a strong deformation retract of $\tilde{Z}$ and therefore has the homotopy type of $\tilde{Z}$, that is to say, the homotopy type of X.

STEP III. If X is homotopy equivalent to a finite complex, then $\sigma(X) = 0$.

In this case, let $d : K \to X$ be a homotopy equivalence and let $u : X \to K$ be d^{-1}. Then $\alpha \sim id : K \to K$ and $T(\alpha) \approx K \times S^1$. In this case the torsion of $\phi^{-1}T\phi$ is clearly zero. ∎

We will need the following proposition later.

PROPOSITION 2.3. If $K \underset{u}{\overset{d}{\rightleftarrows}} X$ is a finite domination and $\sigma(X) = 0$, then there exist a finite complex $\overline{K} \supset K$ and an extension $\overline{d} : \overline{K} \to X$ of d such that $u \circ \overline{d} \sim id$.

PROOF. This follows easily from the proof that $\sigma(X)$ is well-defined. ∎

(Sum Theorem)

PROPOSITION 2.4. If $X = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ and each X_i is a finitely dominated CW-complex (ANR), then $\sigma(X) = i_{1*}\sigma(X_1) + i_{2*}\sigma(X_2) - i_{0*}\sigma(X_0)$.

PROOF. An ANR X is finitely dominated if and only if there is a homotopy $h_t : X \to X$ with $h_0 = id$ and $\overline{h_1(X)}$ compact. Using the homotopy extension theorem and the finite domination of X_i, $i = 0,1,2$, one easily constructs such an h_t which respects X_i, $i = 0,1,2$. The result then follows immediately from the Sum Theorem for Whitehead torsion. ∎

The following naturality result also follows easily from this approach.

PROPOSITION 2.5. If X and Y are finitely dominated spaces and $f : X \to Y$ is a homotopy equivalence, then $f_*\sigma(X) = \sigma(Y)$.

PROOF. If $K \underset{u}{\overset{d}{\rightleftarrows}} X$ is a domination, then so is $K \underset{u \circ f^{-1}}{\overset{f \circ d}{\rightleftarrows}} Y$. Then $\alpha_X = u \circ d$ and $\alpha_Y = u \circ f^{-1} \circ f \circ d \simeq u \circ d$. $\phi_Y : \alpha_Y \to Y \times S^1$ is simply $\phi_X : \alpha_X \simeq \alpha_Y \to X \times S^1$ composed with $f \times id$. Thus, $\psi_X = \phi_X^{-1} T_X \phi_X$ and $\psi_X = \phi_Y^{-1} T_Y \phi_Y$ are equal. The result follows. ∎

COROLLARY 2.6. If X is finitely dominated and $f : X \to X$ is a homotopy equivalence, then $f_*\sigma(X) = \sigma(X)$.

§3. REALIZATION

It is not true that for each $\tau \in Wh(K \times S^1)$ there is an X with $\sigma(X) = \tau$. We will identify the appropriate "obstruction subgroup" of $Wh(K \times S^1)$.

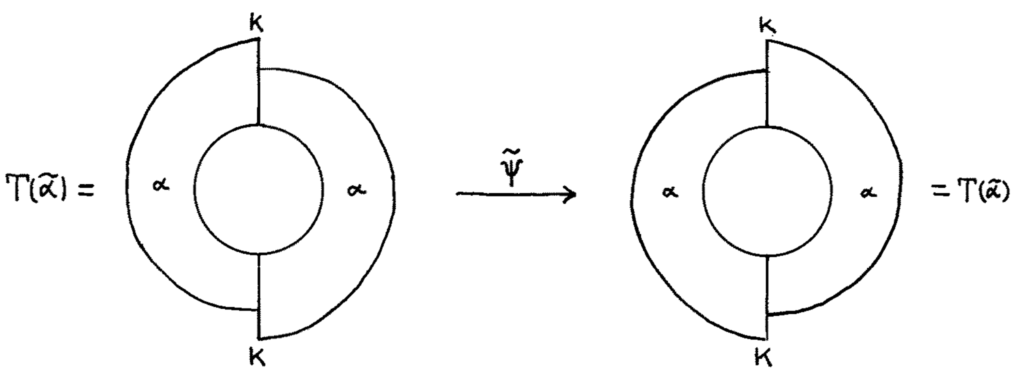
DEFINITION 3.1. Let $p_1 : S^1 \to S^1$ be a finite covering map. This induces a finite covering $p = p_1 \times \mathrm{id} : X \times S^1 \to X \times S^1$. If $[L, X \times S^1] \in \mathrm{Wh}(X \times S^1)$, let $\tilde{L}$ be the covering of L induced by p using the pullback diagram below:

$$\begin{tikzcd} \tilde{L} \arrow[r] \arrow[d] & L \arrow[d, "\text{retraction}"] \\ X \times S^1 \arrow[r, "p \times \mathrm{id}"'] & X \times S^1 \end{tikzcd}$$

The strong deformation retraction $L \searrow X \times S^1$ lifts to a strong deformation retraction $\tilde{L} \searrow X \times S^1$ and we obtain an element $[\tilde{L}, X \times S^1] \in \mathrm{Wh}(X \times S^1)$. We call this element $p^*[L, X \times S^1]$. The map $p^* : \mathrm{Wh}(X \times S^1) \to \mathrm{Wh}(X \times S^1)$ is a homomorphism called the *transfer*. Let $\mathrm{Wh}_0(X \times S^1)$ be the subgroup of $\mathrm{Wh}(X \times S^1)$ consisting of elements invariant under such transfer maps.

THEOREM 3.2. (a) If X is finitely dominated then $\sigma(X) \in \mathrm{Wh}_0(X \times S^1)$.
(b) If $\tau \in \mathrm{Wh}_0(K \times S^1)$ and $\tau \neq 0$, then there is a finitely dominated X with $\sigma(X) \neq 0$.

PROOF. (a) Consider the double cover of ψ pictured below:



We have drawn the second picture of $T(\tilde{\alpha})$ backwards to emphasize the fact that $\tilde{\psi}$ reverses orientation over S^1. Using the fact that $\alpha \circ \alpha \sim \alpha$, we can use MC2 to define a simple homotopy equivalence $S : T(\tilde{\alpha}) \to T(\alpha)$ such that the diagram below commutes:

$$\begin{tikzcd} T(\tilde{\alpha}) \arrow[r, "\tilde{\psi}"] \arrow[d, "s"] & T(\tilde{\alpha}) \arrow[d, "s"] \arrow[dr, "\tilde{\phi}"] & \\ T(\alpha) \arrow[r, "\psi"'] & T(\alpha) \arrow[r, "\phi"'] & X \times S^1 \end{tikzcd}$$

$$\begin{aligned} p^*(\sigma(X)) &= \tilde{\phi}_* \tau(\tilde{\psi}) = \tilde{\phi}_* \tau(s^{-1}\psi s) = \phi_* s_* \tau(s^{-1}\psi s) \\ &= \phi_* s_* [\tau(s^{-1}) + s_*^{-1}\tau(\psi) + s_*^{-1}\psi_* \tau(s)] \\ &= \phi_* s_* s_*^{-1} \tau(\psi) = \phi_* \tau(\psi) \end{aligned}$$

This completes the proof of part (a).

PROOF. (b) Let $\tau \in Wh_0(K \times S')$, $\tau \neq 0$. Let $(L, K \times S')$ be a representative for τ and let $r : L \to K \times S'$ be a PL retraction. Lifting to the cyclic cover, we have a PL retraction $\tilde{r} : \tilde{L} \to K \times \mathbf{R}'$. Let $W = \tilde{r}^{-1}(K \times [0,\infty))$. W is a polyhedron. We need:

LEMMA 3.3. W is finitely dominated.

PROOF. Let $\tilde{r}_t$, $0 \leq t \leq 1$, be a homotopy from $\tilde{r}$ to id rel $K \times R^1$. This homotopy is proper. Thus, there is a compact set $C \subset W$ such that $\tilde{r}_t(W - C) \subset W$ for all t, $0 \leq t \leq 1$. Let $\phi : W \to [0,1]$ be a function with $\phi(c) = 0$ such that $\phi^{-1}[0,1)$ is compact. Then $h_t(\chi) = \tilde{r}_{t \cdot \phi(\chi)}(\chi)$ deforms W into the union of $K \times [0,\infty)$ with a finite complex. It is easy to modify h_t to $\overline{h}_t$ which deforms W into a finite subcomplex L of W relative to a neighborhood of $\tilde{r}^{-1}[K \times 0]$. Setting $d = i : L \hookrightarrow W$ and $u = \overline{h}_1 : W \to L$, we see that W is finitely dominated.

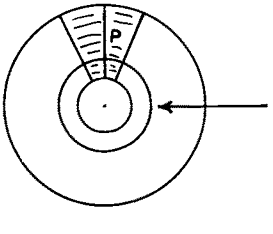
The next step in the proof is the following lemma:

LEMMA 3.4. If $\sigma(W) = 0$ there exist an n and a finite bicollared polyhedron $P \subset L \times I^n$ which separates the ends such that $P \hookrightarrow L$ is a homotopy equivalence.

PROOF. Let $W_0 \subset W$ be the subcomplex $\tilde{r}^{-1}(K \times 0)$. We may assume that r was chosen to make W_0 bicollared in $\tilde{L}$. If $\sigma(W) = 0$, then there exist $\overline{W} \supset W$ and $\overline{\overline{h}}$ extending $\overline{h}_1$ so that $\overline{\overline{h}} : \overline{W} \to W$ is a homotopy equivalence. If n is large, we may assume that $\overline{\overline{h}}$ is a PL imbedding $\overline{\overline{h}} : \overline{W} \to W \times I^n \times \{1\} \subset W \times I^n \times I$. Let N be a regular neighborhood of $\overline{\overline{h}}(\overline{W})$ in $W \times I^n \times \{1\}$ and let $P_0 = (W_0 \times I^n \times I \cup N) - \mathring{N}$, where $\mathring{N}$ is the interior of N in $W \times I^n$. $P_0 \hookrightarrow W - \mathring{N}$ is a homotopy equivalence since the inclusions $W - \mathring{N} \hookrightarrow W$ and $P_0 \hookrightarrow W$ are homotopy equivalences.

Let $V = (\tilde{L} \times I^n \times I) - (W - \mathring{N})$ and repeat this process on the other side of P_0, obtaining a PL inclusion $\overline{\overline{k}} : \overline{V} \hookrightarrow V \times I^m \times \{1\} \subset V \times I^m \times I$ so that $\overline{\overline{k}}(\overline{V}) \supset P_0 \times I^m \times I$. Now, let M be a regular neighborhood of $\overline{\overline{k}}(\overline{V})$ in V and let $P = (P_0 \times I^m \times I \cup M) - \mathring{M}$. As before, P is a strong deformation retract of the other side, $(W \times I^{n+m+1} \times I - V \times I^m \times I) \cup M$, since $P \cup M$ has already been shown to be a strong deformation retract of this space and P is a strong deformation retract of $P \cup M$ since M is a regular neighborhood of a set in $V \times I^m \times \{1\}$. ∎

We continue with the proof of Theorem 3.2. Since τ is invariant under transfer, we may assume that the projection $\tilde{L} \times I^{n+m} \to L \times I^{n+m}$ imbeds a bicollar neighborhood $P \times I$ of P.



$K \times S^1$

The complement of $\widehat{\overset{\circ}{P\times I}}$ in $L \times I^{n+m}$ is homeomorphic to the space between two translated copies of P in $\tilde{L} \times I^{n+m}$. It is therefore homotopy equivalent to K and we can construct a strong deformation retraction $r_t : L \times I^{n+m} \searrow K \times S'$ which takes $P \times I$ into itself and $(L \times I^{n+m} - P \times I)$ into itself. The sum theorem for Whitehead torsion [Co_1,p.76] shows that $\tau \in \operatorname{im}(Wh(K) \to Wh(K \times S^1))$. But it is geometrically clear that if $p : S^1 \to S^1$ is a double cover, then $p^*\tau = 2\tau$. The invariance of τ under transfer then shows that $\tau = 0$. ∎

COROLLARY (of the proof). If $\tau \in Wh(K \times S^1)$ and $p : S^1 \to S^1$ is the double cover, then $p^*\tau = \tau$ implies that $\sigma(W) = 0$, where W is defined as in the theorem above. Thus, if there is a nonzero element $\tau \in Wh(K \times S^1)$ which is invariant under passage to the double cover, then there is a nonzero element $\tau \in Wh(K \times S^1)$ which is invariant under passage to all finite covers. ∎

REMARK: This corollary is also clear from the Bass-Heller-Swann decomposition of $Wh(K \times S^1)$.

REFERENCES

[Ch-S] T.A. Chapman and L.C. Siebenmann, Finding a boundary for a Hilbert cube manifold, Acta Mathematica 137 (1976), 171-208.

[Ch_1] T.A. Chapman, Homotopy conditions which detect simple homotopy equivalences, Pacific J. Math. 80 (1979), 13-45.

[Ch_2] ____________, Whitehead torsion and the Wall finiteness obstruction, to appear.

[Co_1] M. Cohen, *A Course in Simple-Homotopy Theory*, Springer-Verlag, Berlin and New York.

[Co_2] ________, Simplicial structures and transverse cellularity, Annals of Math. 85 (1967), 218-245.

[H] A. Hatcher, Higher simple homotopy theory, Annals of Math. 102 (1975), 101-139.

[F] S. Ferry, Homotopy, simple homotopy, and compacta, Topology 19 (1980), 101-110.

[M] M. Mather, Counting homotopy types of topological manifolds, Topology 4 (1965), 93-94.

[Q] F. Quinn, Ends of maps, I, Annals of Math. 110 (1979), 275-331.

[S] L.C. Siebenmann, Infinite simple homotopy types, Indag. Math. 32 (1970), 479-495.

[Wa_1] C.T.C. Wall, Finiteness conditions for CW-complexes, Annals of Math. 106 (1977), 1-18.

[Wa_2] ___________, Finiteness conditions for CW-complexes II, Proc. Roy. Soc. Ser. A., 295 (1966), 129-139.

[W] J.H.C. Whitehead, Simple homotopy types, Amer. J. Math. 72 (1952), 1-57.

University of Kentucky
Department of Mathematics
Lexington, Kentucky 40506