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Verdier Duality for Systems of Coefficients over Simplicial Sets

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Abstract. Let $X.$ be a simplicial set. A cohomological system of coefficients $F.$ on $X.$ with values in an abelian category $\mathcal{A}$ is given by an object $F_\sigma \in \mathcal{A}$ for every $\sigma \in X.$ and by a compatible set of morphisms $F_{\alpha^*\sigma} \to F_\sigma$ for every map of ordered sets $\alpha : \underline{m} \to \underline{n}$, $m, n = 0, \ldots$ and $\sigma \in X_n$. We denote by $SH(X.)$ the category of cohomological systems $F.$ of coefficients on $X.$ such that the maps $F_{\alpha^*\sigma} \to F_\sigma$ are isomorphisms if α is surjective. This category is equivalent to a certain subcategory of the category of sheaves $SH(X)$ on the geometric realization $X = |X.|$ of the simplicial set $X.$.

The present paper gives a construction of a duality theory for $SH(X.)$ which is analogue to the topological Verdier duality for $SH(X)$. It consists of a construction of a right adjoint to the cohomology with compact support functor and a resulting duality theory given by a dualizing complex. These constructions are much more explicit than in the topological case. We compare the simplicial and topological constructions via the embedding $SH(X.) \to SH(X)$. In particular, we get an explicit description of the dualizing sheaf complex for topological spaces $X = |X.|$.

1. Introduction

Let $X. = \{X_n\}_{n=0,\ldots}$ be a simplical set. If $\alpha : \underline{m} \to \underline{n}$ is a map of odered sets we denote by $\alpha^* : X_n \to X_m$ the induced map. A cohomological system of coefficients $F.$ on $X.$ with values in an abelian category $\mathcal{A}$ is given by the following data:

1. An object $F_\sigma \in \mathcal{A}$ for every $\sigma \in X.$.
2. A morphism $F_{\alpha^*\sigma} \to F_\sigma$ for every $\alpha : \underline{m} \to \underline{n}$, $m, n = 0, \ldots$ and $\sigma \in X_n$. These morphisms are assumed to be functorial in α.

We are mainly interested in the following examples of cohomological systems of coefficients:

(i) Let $X = |X.|$ be the geometric realization of the simplicial set $X.$. Let $SH(X)$ be the category of sheaves on X with values in the category of modules over some

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noetherian ring and let $\mathcal{F} \in SH(X)$. For every $\sigma \in X_n$ we denote by $[\sigma]$ the canonical map from the topological n-dimensional standard simplex Δ_n to X. Assume that for all $\sigma \in X.$ the sheaf $[\sigma]^*\mathcal{F}$ is constant on the subset of inner points $inn(\Delta_n)$ of Δ_n for all $\sigma \in X.$. Denote the stalk at such a point by F_σ. Then $F.$ carries a natural structure of a cohomological system of coefficients on $X.$.

(ii) Let X be a noetherian scheme. Denote by $SH(X)$ the category of quasicoherent sheaves on X and let $\mathcal{F} \in SH(X)$. Let $X. = S.(X)$ be the simplicial set of flags of irreducible closed subschemes. Given $\sigma \in X.$, the local adeles $F_\sigma = \mathbb{A}(\{\sigma\}, \mathcal{F})$ as defined by BEILINSON [1] carry the structure of a cohomological system of coefficients.

(iii) Let X be a reductive algebraic group over a finite field $\mathbb{F}_q$. We denote by $SH(X)$ the category of finite dimensional representations of the finite group $X(\mathbb{F}_q)$ and let $M \in SH(X)$. Let $X. = \Delta.(X)$ be the combinatorial building of X. Associated to a simplex $\sigma \in X.$ we have a parabolic subgroup $P \subset X$. We denote by $R_u(P)(\mathbb{F}_q)$ the group of rational points of the unipotent radical of P. Then a cohomological system of coefficients $F.$ is given by $F_\sigma = M^{R_u(P)} \subseteq M$ with inclusion maps for different σ.

These examples have two properties in common:

Firstly, the considered cohomological systems of coefficients $F.$ have the additional property that the maps $F_{\alpha^*\sigma} \to F_\sigma$ are isomorphisms if α is surjective. Let us call such a system of coefficients a sheaf and denote the category of sheaves by $SH(X.)$.

Secondly, under some additional assumptions, there are duality theories on the derived categories $D^b_c(X)$ of the abelian categories $SH(X)$ with a certain finiteness condition on cohomology (see [16] for (i), [11] for (ii), the duality for (iii) is induced by the duality on the irreducible representations given in [5]). They consist of an antiselfduality

$$D^b_c(X) \longrightarrow D^b_c(X) \tag{1.1}$$

which is given in the first two cases by the inner homomorphisms with values in a dualizing sheaf complex $\mathcal{D}_X \in D^b(X)$.

A. A. BEILINSON [2] first conjectured the existence of a duality theory for cohomological systems of coefficients on simplicial sets which is related to the duality on topological spaces and schemes via the constructions in (i) and (ii).

The present paper provides a construction of a such a duality theory on the derived category $D^b_c(X.)$ of $SH(X.)$ and a discussion of its relation to the topological duality. The relationship between this duality and the duality of representations on algebraic groups via (iii) is considered in [7], where the results and methods of this paper are needed. But we hope that the more foundational material presented here will be useful.

The relation to (ii) is not well understood, we refer to [8] for further discussions.

Here are more details:

In Section 3 we study the structure of the category $SH(X.)$ and construct the geometric realization functor which is the inverse of the functor in Example (i). This allows the identification of $SH(X.)$ with the full subcategory of $SH(|X.|)$ consisting of sheaves that satisfy the condition of Example (i) (see Corollary 3.18).

We consider the usual homological algebra, i.e., functors of global sections, global sections with compact support and their relative variants, direct image, direct image with compact support and inverse image, tensor product and inner homomorphism for sheaves and cohomological systems of coefficients in Sections 4 – 7. For every functor we check its relation to the corresponding topological functor.

Section 8 contains the construction of the dualizing sheaf complex $\mathcal{D}_{X.}$. More generally, we show the existence of a right adjoint of the functor of global sections with compact support, see Theorem 8.8. The main Theorem 8.9 of this paper states the analogue of (1.1) under a minor assumption on the simplicial set $X.$. In Proposition 8.20 we show that the geometric realization of $\mathcal{D}_{X.}$ is a dualizing sheaf complex for the topological space $|X.|$. This gives an explicit description of the dualizing sheaf complex for topological spaces X, which are given as geometric realizations of simplicial sets.

2. Notations

Let $\underline{n}$ be the ordered set $\{0 < 1 < \cdots < n\}$ for every $n = -1, 0, 1, \ldots$. By Δ (resp. $\tilde{\Delta}$) we denote the category consisting of the ordered sets $\underline{n}$ for $n = 0, 1, \ldots$ (resp. $n = -1, 0, \ldots$) and morphisms of ordered sets. As usually $\partial_i : \underline{n} \to \underline{n+1}$, for $i = 0, \ldots, n+1$, denotes the injective map, s. t. $i \notin \mathrm{Im}(\partial_i)$, and $s_j : \underline{n+1} \to \underline{n}$, for $j = 0, \ldots, n$, the surjective map with $s_j(j) = s_j(j+1)$. In general we will use the letters $\alpha, \beta, \ldots$ for arbitrary maps of ordered sets δ, ∂ for injective maps and s, t for surjective maps.

For a simplicial (resp. cosimplicial) object in some category $\mathcal{C}$ we will write α^* (resp. α_*) for the morphism in $\mathcal{C}$ corresponding to α.

By $\Delta.[\underline{n}]$ we denote the simplicial n-dimensional standard simplex and by Δ_n the geometric realization, the topological n-dimensional standard simplex. As functors in $\underline{n}$ they form cosimplicial objects in the category of simplicial sets (topological spaces). By $inn(\Delta_n)$ we denote the open subset of inner points of Δ_n and by $\partial\Delta_n$ the boundary.

For a simplicial set $X.$ and a simplex $\sigma \in X_n$ for some n we define the map of simplicial sets

$$[\sigma] : \Delta.[\underline{n}] \longrightarrow X. \qquad \underline{i} \xrightarrow{\alpha} \underline{n} \longmapsto \alpha^*\sigma .$$

By the same letter we also denote its geometric realization $[\sigma] : \Delta_n \to |X.|$. In this case we will write $\Delta.[\sigma]$, resp. Δ_σ, for $\Delta.[\underline{n}]$, resp. Δ_n.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a functor between small categories and let $\mathcal{A}$ be some category with arbitrary projective limits. Then it is well known (see for example [13] or [14], Ch. X, § 3, Th. 1 and Cor. 2) that the morphism of the categories of functors

$$f^*_{cat} : \begin{array}{ccc} \mathrm{Func}(\mathcal{Y}, \mathcal{A}) & \longrightarrow & \mathrm{Func}(\mathcal{X}, \mathcal{A}) \\ \mathcal{G} : \mathcal{Y} \to \mathcal{A} & \longmapsto & f^*(\mathcal{G}) := \mathcal{G} \circ f \end{array} \tag{2.1}$$

has a right adjoint functor (the so called right Kan extension)

$$f_{cat,*} : \begin{array}{ccc} \mathrm{Func}(\mathcal{X}, \mathcal{A}) & \longrightarrow & \mathrm{Func}(\mathcal{Y}, \mathcal{A}) \\ \mathcal{F} : \mathcal{X} \to \mathcal{A} & \longmapsto & f_{cat,*}(\mathcal{F}) \end{array} \tag{2.2}$$

which can be given by

$$f_{cat,*}(\mathcal{F})(Y) := \varprojlim_{Y \backslash f} \mathcal{F} \circ pr .$$

pr being the projection from the left homotopy fiber $Y\backslash f$ to $\mathcal{X}$. Remember that the objects of the left homotopy fiber $Y\backslash f$ are given by pairs $(X, \alpha : Y \to f(X))$ and morphisms

$$(X, \alpha : Y \to f(X)) \longrightarrow (X', \alpha' : Y \to f(X'))$$

are given by morphisms $\varphi : X \to X'$ with $\alpha' = f(\varphi)\alpha$. The induced map for $Y \to Y'$ is the natural one.

3. Sheaves on simplicial sets

Let $X.$ be a simplicial set. We define the corresponding category $\mathcal{X}$ as follows:

objects : arbitrary $\sigma \in X.$

morphisms : $\sigma \overset{\alpha}{\to} \sigma'$ is a morphism $\alpha : \underline{n} \to \underline{n}'$ in Δ for which $\sigma \in X_n$, $\sigma' \in X_{n'}$ and $\alpha^*\sigma' = \sigma$.

We call a morphism $\alpha : \sigma \to \sigma'$ of $\mathcal{X}$ injective, resp. surjective, if the underlying map of ordered sets $\underline{n} \to \underline{n}'$ is injective, resp. surjective.

If $f. : X. \to Y.$ is a morphism of simplicial sets we have the induced functor between the associated categories $\mathcal{X} \to \mathcal{Y}$ given by $\sigma \mapsto f(\sigma)$ with $\sigma \in X. = Ob(\mathcal{X})$.

In the introduction we had defined the notion of a cohomological system of coefficients. In terms of the category $\mathcal{X}$ a cohomological system of coefficients $F.$ is a covariant functor $F. : \mathcal{X} \to \mathcal{A}$.

Our first aim will be the construction of a functor from the category of cohomological systems of coefficients on $X.$ to the category of sheaves over the geometric realization of $X.$. This construction will be inverse to the construction in the introduction.

3.1. The geometric realization functor

We consider the following category $\widehat{\mathcal{X}}$:

objects : $(\sigma, \mathcal{U})$ with $\sigma \in \mathrm{Ob}(\mathcal{X}), \sigma \in X_n, n \geq 0$, and $\mathcal{U} \subseteq \Delta_n$ an open (non empty) connected subset.

morphisms : $(\sigma, \mathcal{U}) \overset{\alpha}{\to} (\sigma', \mathcal{U}')$ is a morphism $\alpha : \sigma \to \sigma'$ in $\mathcal{X}$ for which $\alpha_* : \Delta_n \to \Delta_{n'}$ maps $\mathcal{U}$ to $\mathcal{U}'$.

For every open subset $U \subseteq |X.|$ we consider the full subcategory $\widehat{\mathcal{X}}_U$ of $\widehat{\mathcal{X}}$ consisting of all objects $(\sigma, \mathcal{U})$ for which $\mathcal{U} \subsetneq [\sigma]^{-1}(U)$.

Let $pr : \widehat{\mathcal{X}} \to \mathcal{X}$ be the natural projection and pr_U its restriction to $\widehat{\mathcal{X}}_U$. We define

$$|F.|(U) := \varprojlim_{\widehat{\mathcal{X}}_U} (\mathcal{F} \circ pr_U) . \tag{3.1}$$

For an open subset $V \subseteq U$ we have the full subcategory $\widehat{\mathcal{X}}_V$ of $\widehat{\mathcal{X}}_U$ and therefore a morphism

$$\begin{array}{ccc} \varprojlim_{\widehat{\mathcal{X}}_U}(\mathcal{F} \circ pr_U) & \longrightarrow & \varprojlim_{\widehat{\mathcal{X}}_V}(\mathcal{F} \circ pr_V) \\ \| & & \| \\ |F.|(U) & \longrightarrow & |F.|(V) . \end{array}$$

This morphism is in an obvious way functorial if we consider $W \subseteq V \subseteq U$ and so we have defined a presheaf $|F.|$ on $|X.|$.

Proposition 3.1. *Let $F.$ be a cohomological system of coefficients on a simplicial set $X.$ with coefficients in the abelian category $\mathcal{A}$.*

(i) *The presheaf $|F.|$ is a sheaf.*

(ii) $|?|$ *defines a left exact functor from* $\mathrm{Func}(\mathcal{X}, \mathcal{A})$ *to* $SH(|X.|)$.

Proof. Property (ii) follows immediateley from the left exactness of the $\varprojlim$ - functor. To prove property (i) let $U = \bigcup U_i$ and $s_i \in |F.|(U_i)$ be a family of compatible sections. By definition it is given as a family $s_{(\sigma,\mathcal{U})} \in F_\sigma$ for all $\sigma \in X_n$ and all connected $\mathcal{U} \subseteq [\sigma]^{-1}(U_i)$ for some i with the compatibility property that $\alpha_*(s_{(\sigma,\mathcal{U})}) = s_{(\sigma',\mathcal{U}')}$ whenever $\alpha : \sigma \to \sigma'$ is such that $\alpha_*(\mathcal{U}) \subsetneq \mathcal{U}'$ (this condition implies $\mathcal{U} \subsetneq [\sigma]^{-1}(U_i \cap U_{i'})$). If $(\sigma, \mathcal{U}) \in \widehat{\mathcal{X}}_U$ there exists some i such that $\mathcal{U} \cap [\sigma]^{-1}(U_i) \neq \emptyset$ and we can take some open connected non empty $\mathcal{U}' \subsetneq \mathcal{U} \cap [\sigma]^{-1}(U_i)$ and define $s_{(\sigma,\mathcal{U})} := s_{(\sigma,\mathcal{U}')}$. This definition does not depend on the choice of $\mathcal{U}'$ because $\mathcal{U} = \bigcup_j \mathcal{U} \cap [\sigma]^{-1}(U_j)$ is connected. This gives a compatible system of elements $s_{(\sigma,\mathcal{U})} \in F_\sigma$ and hence a uniquely defined section $s \in |F.|(U)$ which restricts to the $s_i \in |F.|(U_i)$. □

3.2. The category of sheaves on $X.$

Let $F.$ be some cohomological system of coefficients on a simplicial set $X.$. In general the stalk of $|F.|$ at some point $x \in [\sigma](inn(\Delta_\sigma))$ is not F_σ. This can be seen easily if we take $X. = \Delta.[\underline{0}]$ the simplical point. Then $\mathcal{X} = \Delta$ and the category of cohomological systems of coefficients is the category of cosimplical objects of $\mathcal{A}$. We have $|X.| = \{x\}$ one point and

$$|F.|_x = \varprojlim_{\widehat{\mathcal{X}}} F. = \varprojlim_{\mathcal{X}} F.$$

which in general does not equal $F_{\underline{0}\to\underline{0}}$.

Thus it is natural to consider the following subcategory:

Definition 3.2. A cohomological system of coefficients $F.$ on a simplicial set $X.$ is called a *sheaf* if for all surjective $s : \sigma \twoheadrightarrow \tau$ the corresponding map $\mathcal{F}_\sigma \to \mathcal{F}_\tau$ is an isomorphism. We denote the full subcategory of $\mathrm{Func}(\mathcal{X}, \mathcal{A})$ consisting of all sheaves by $SH(X.)$.

Example 3.3. If $X. = \Delta.[\underline{0}]$ is the simplicial point, by taking the stalk in the unique nondegenerate simplex $\underline{0} \to \underline{0}$, we obviously have $SH(X.) \simeq \mathcal{A}$.

Lemma 3.4. *The category of sheaves $SH(X.)$ has the following properties:*

1. *Let* Sur *be the class of surjective morphisms in $\mathcal{X}$ and denote by* $\mathrm{Sur}^{-1}\mathcal{X}$ *the corresponding localized category. Then we have a natural equivalence*

$$SH(X.) \simeq \mathrm{Func}(\mathrm{Sur}^{-1}\mathcal{X}, \mathcal{A}).$$

In particular, $SH(X.)$ is an abelian category.

2. *Exactness, kernel and cokernel coincide in $SH(X.)$ and* $\mathrm{Func}(\mathcal{X}, \mathcal{A})$.
3. *If $F. \to G.$ is monomorphism of cohomological systems of coefficients and $G.$ is a sheaf, then $F.$ is also a sheaf.*
4. *If we denote the natural map $\mathcal{X} \to \mathrm{Sur}^{-1}\mathcal{X}$ by p_X, then the adjunction map*

$$id \longrightarrow p_{X,*} \circ p_X^* : SH(X.) \longrightarrow SH(X.)$$

is an isomorphism of functors.

Proof. Properties 1. and 2. are obvious, for property 3. use that for a surjective $s : \sigma \twoheadrightarrow \tau$ the map $F_\sigma \to F_\tau$ is automatically an epimorphism since s has a right inverse. Property 4. follows from the Yoneda lemma. □

In the following we give a more explicit description of the category $\mathrm{Sur}^{-1}\mathcal{X}$.

Let $\sigma, \tau \in X.$ be two simplices. Consider a relation $\sim$ on the set $\mathrm{Hom}_{\mathcal{X}}(\sigma, \tau)$. We define

$$\alpha \circ \delta_1 \sim \alpha \circ \delta_2 \tag{3.2}$$

if there is a diagram

$$\begin{array}{ccc} \sigma' & \xrightarrow{\alpha} & \tau \\ s\downarrow & & \\ \sigma & & \end{array} \tag{3.3}$$

with surjective s and δ_1, δ_2 two sections of s.

Lemma 3.5. *The relation $\sim$ has the following properties:*

1. *$\varphi \sim \varphi$ for every $\varphi : \sigma \to \tau$.*
2. *If $\varphi_1 \sim \varphi_2$, then $\varphi_2 \sim \varphi_1$ for every $\varphi_1, \varphi_2 : \sigma \to \tau$.*
3. *Let $\tilde{\sigma} \xrightarrow{\psi} \sigma \overset{\varphi_1,\varphi_2}{\rightrightarrows} \tau \xrightarrow{\lambda} \tilde{\tau}$ be morphisms and $\varphi_1 \sim \varphi_2$. Then*

$$\lambda \circ \varphi_1 \circ \psi \sim \lambda \circ \varphi_2 \circ \psi .$$

4. *Let $\varphi_1, \varphi_2 : \sigma \to \tau$ be two equivalent maps and σ nondegenerate. Then there exists a diagram (3.3) with injective α and two sections δ_1, δ_2 of s such that*

$$\varphi_1 = \alpha \circ \delta_1 , \quad \varphi_2 = \alpha \circ \delta_2 .$$

Proof. Properties 1. and 2. are obvious. Property 3. follows easily from the following fact: Every diagram

$$\begin{array}{ccc} & & \sigma' \\ & & \downarrow{\scriptstyle s\in \mathrm{Sur}} \\ \tilde{\sigma} & \xrightarrow{\lambda} & \sigma \end{array} \quad \text{has an extension to a commutative diagram} \quad \begin{array}{ccc} \tilde{\sigma}' & \xrightarrow{\lambda'} & \sigma' \\ \downarrow{\scriptstyle \tilde{s}\in \mathrm{Sur}} & & \downarrow{\scriptstyle s} \\ \tilde{\sigma} & \xrightarrow{\lambda} & \sigma . \end{array}$$

This continuation is constructed by the underlying fibre product diagram in Δ. A section δ of s induces a unique section $\tilde{\delta}$ of $\tilde{s}$ such that $\lambda' \circ \tilde{\delta} = \lambda \circ \delta$.

To show property 4. we first consider a diagram of type (3.3) and sections δ_1, δ_2 realizing the equivalence of φ_1 and φ_2. Then we decompose α in a surjective and injective map $\alpha = \alpha' \circ t$. Let $\sigma'' = \alpha^*(\tau)$ and $s' : \sigma'' \to \sigma$ be the unique surjective map. Then $\delta_1' = t \circ \delta_1$ and $\delta_2' = t \circ \delta_2$ are sections of s' and $\varphi_1 = \alpha' \circ \delta_1'$ and $\varphi_2 = \alpha' \circ \delta_2'$. □

Denote also by $\sim$ the minimal equivalence relation generated by $\sim$ and by $\mathcal{X}/\sim$ the category with the same objects as $\mathcal{X}$ and morphisms

$$\mathrm{Hom}_{\mathcal{X}/\sim}(\sigma, \tau) := \mathrm{Hom}_{\mathcal{X}}(\sigma, \tau)/\sim .$$

Proposition 3.6. *The natural functor $\mathcal{X} \to \mathrm{Sur}^{-1}\mathcal{X}$ induces an equivalence of categories:*

$$\mathcal{X}/\sim \xrightarrow{\sim} \mathrm{Sur}^{-1}\mathcal{X} .$$

Proof. Let $\mathcal{F} : \mathcal{X} \to \mathcal{T}$ be some functor which maps surjective maps to isomorphisms. Take some diagram 3.3 and δ_1, δ_2 two sections of s. Then we have in the category $\mathcal{T}$:

$$\mathcal{F}(s) \circ \mathcal{F}(\delta_i) = id, \quad \text{hence} \quad \mathcal{F}(\delta_i) = \mathcal{F}(s)^{-1} \quad \text{for } i = 1, 2 .$$

We get $\mathcal{F}(\delta_1) = \mathcal{F}(\delta_2)$ and $\mathcal{F}(\alpha \circ \delta_1) = \mathcal{F}(\alpha \circ \delta_2)$. This implies by Lemma 3.5 a factorisation of $\mathcal{F}$

$$\mathcal{X}/\sim \longrightarrow \mathcal{T} .$$

Furthermore, the image of a surjective morphism $s : \sigma \twoheadrightarrow \tau$ in $\mathcal{X}/\sim$ is an isomorphism. To verify this we can assume that $s = s_j$ is a standard surjective map. We will show that the image of ∂_{j+1} is an inverse map. We have to check that $\partial_{j+1} \circ s_j \sim id$. Let

$$\sigma' := s_j^*\sigma = s_j^* s_j^* \tau = s_{j+1}^* s_j^* \tau = s_{j+1}^* \sigma .$$

We consider the diagram

$$\begin{array}{ccc} \sigma' & \xrightarrow{s_j} & \sigma \\ s_{j+1}\downarrow & & \\ \sigma & & \end{array}$$

and the two sections $\partial_{j+1}, \partial_{j+2}$ of s_{j-1}. Then we have

$$id = s_j \circ \partial_{j+1} \sim s_j \circ \partial_{j+2} = \partial_{j+1} \circ s_j .$$

Thus $\mathcal{X}/\sim$ has the universal property of the localized category $\mathrm{Sur}^{-1}\mathcal{X}$, hence they are equivalent. □

Definition 3.7. Denote by $\mathcal{X}_0$ the full subcategory of $\mathcal{X}$ consisting of all nondegenerate simplices and denote by $\mathbf{i}_0$ the inclusion.

Corollary 3.8. *Assume that the boundary of nondegenerate simplices in $X.$ is nondegenerate. Then the composition*

$$\varphi_X : \mathcal{X}_0 \xrightarrow{\mathbf{i}_0} \mathcal{X} \xrightarrow{p_X} \mathrm{Sur}^{-1}\mathcal{X}$$

is an equivalence of categories.

Proof. Obviously every object σ of $\mathrm{Sur}^{-1}\mathcal{X}$ is isomorphic to some object σ_0 from $\mathcal{X}_0$, with σ_0 the uniquely defined nondegenerate simplex for which there exists a (also uniquely defined) surjective map $s_\sigma : \sigma \to \sigma_0$ in $\mathcal{X}$.

It remains to check that the functor is bijective on the sets of morphisms. By Proposition 3.6 we have to prove that the map

$$\mathrm{Hom}_{\mathcal{X}}(\sigma,\tau) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(\sigma,\tau)/\sim$$

is an isomorphism for all nondegenerate $\sigma,\tau \in X.$. But this follows easily from Lemma 3.5, 4. □

Corollary 3.9. *Let $\sigma \in X.$ be an arbitrary simplex. Then*

$$\mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{X}}(\sigma,\sigma) = \{id\}\,.$$

Proof. We can restrict to the case that σ is nondegenerate. Let $\alpha : \sigma \to \sigma$ be a morphism in $\mathcal{X}$. Then α has to be injective and hence it is the identity. □

Corollary 3.10. *The category $SH(X.)$ is a full subcategory of the category of functors* $\mathrm{Func}(\mathcal{X}_0, \mathcal{A})$.

Proof. Every σ of $\mathcal{X}/\sim$ is isomorphic to a nondegenerate simplex σ_0. Hence the full subcategory $\mathcal{X}_0/\sim$ of $\mathcal{X}/\sim$ is equivalent to $\mathcal{X}/\sim$. The corollary is an immediate consequence of the proposition. □

Corollary 3.11. *Let $X.$ be a simplicial set, such that an arbitrary boundary of a nondegenerate simplex is nondegenerate. Then we have an isomorphism*

$$p_X^* \simeq \mathbf{i}_{0,*} \circ \varphi_X^*\,.$$

In particular, p_X^* maps injective sheaves to injective cohomological systems of coefficients.

Proof. This is an easy direct calculation using the specific description of φ_X. □

Let us consider an example which shows that $\mathcal{X}_0$ and $\mathrm{Sur}^{-1}\mathcal{X}$ are in general not equivalent.

Example 3.12.

$$X. := \varinjlim \begin{pmatrix} \Delta.[1] & \overset{\partial_0}{\underset{\partial_2}{\overset{\partial_1}{\rightrightarrows}}} & \Delta.[2] \\ \downarrow s_0 & & \\ \Delta.[0] & & \end{pmatrix} = \begin{matrix}\text{two dimensional standard simplex} \\ \text{with a boundary contracted to a point.}\end{matrix}$$

Then $\mathcal{X}_0$ consists of two objects σ_0 and σ_2 with morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}_0}(\sigma_0,\sigma_2) &= \{\delta : \underline{0} \to \underline{2}\} \\ &= \{\delta_0, \delta_1, \delta_2\}\,. \end{aligned}$$

This implies that objects in $\mathrm{Func}(\mathcal{X}_0, \mathcal{A})$ are pairs (A_0, A_2) of objects of $\mathcal{A}$ together with three morphisms $\delta_{0*}, \delta_{1*}, \delta_{2*} : A_0 \to A_2$. But the image of $SH(X.)$ in the category of functors $\mathrm{Func}(\mathcal{X}_0, \mathcal{A})$ are the objects for which $\delta_{0*} = \delta_{1*} = \delta_{2*}$. This is easily seen considering the morphisms in $\mathcal{X}$:

$$\partial_0^0, \partial_1^0 : \sigma_0 \longrightarrow s_0^*\sigma_0\,, \quad \partial_0^1, \partial_1^1, \partial_2^1 : s^*\sigma_0 \longrightarrow \sigma_2$$

which satisfy the relations

$$\begin{aligned} \partial_0^1 \circ \partial_0^0 &= \partial_1^1 \circ \partial_0^0\,, \\ \partial_2^1 \circ \partial_0^0 &= \partial_0^1 \circ \partial_1^0\,, \\ \partial_1^1 \circ \partial_1^0 &= \partial_2^1 \circ \partial_1^0\,, \end{aligned} \qquad \begin{aligned} s_0 \circ \partial_0^0 &= id\,, \\ s_0 \circ \partial_1^0 &= id\,. \end{aligned}$$

But for a sheaf s_{0*} is an isomorphism. This implies that $\partial_{0*}^0 = \partial_{1*}^0$ is an isomorphism. So we get $\partial_{0*}^1 = \partial_{1*}^1 = \partial_{2*}^1$ and it follows immediately $\delta_{0*} = \delta_{1*} = \delta_{2*}$.

Definition 3.13. Let $X.$ be a simplicial set. A system (σ, x, V) of a simplex $\sigma \in X.$, a point $x \in inn(\Delta_\sigma)$ and an open subset $V \subseteq |X.|$ is called a *compatible data* if $[\sigma](x) \in V$. A morphism

$$(\sigma, x, V) \longrightarrow (\sigma', x', V')$$

of compatible data is given by the condition $V \subseteq V'$ and a map $\alpha : \sigma' \to \sigma$ in $\mathrm{Sur}^{-1}\mathcal{X}$ such that

$$\hat{\alpha}_*(x'),\ x \in [\sigma]^{-1}(V) \subseteq \Delta_\sigma$$

lye in the same connected component. Here $\widehat{\alpha}$ denotes some lift of α to $\mathcal{X}$. (In the following lemma we will see that the condition does not depend on the choice of $\hat{\alpha}$.) Composition of morphisms being the obvious one, this defines a category $Data(X.)$.

Lemma 3.14. *Let $X.$ be a simplicial set and $(\sigma, x, V) \in Data(X.)$. Let $\tau \in X.$ be a simplex and denote by V_τ the preimage of V under the map $[\tau]$. Now we consider the map*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}}(\sigma,\tau) &\longrightarrow \pi_0(V_\tau) \\ \alpha &\longmapsto [\alpha_*(x)]\,, \end{aligned} \tag{3.4}$$

where $[y]$ denotes the connected component corresponding to the point y. The left and the right-hand side obviously are functors with respect to $\tau \in \mathcal{X}$ and this map defines a morphism of functors.

1. *This map factorises to a map*

$$\lambda_\tau(\sigma, x, V) : \mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{X}}(\sigma, \tau) \longrightarrow \pi_0(V_\tau),$$

2. *The morphism of functors $\lambda(\sigma, x, V)$ depends only on $[\sigma](x)$ and not on the choice of x.*

3. *In a natural way the left and the right-hand side can be considered as functors* Hom *and π_0 on $Data(X.) \times \mathrm{Sur}^{-1}\mathcal{X}$ given by*

$$\begin{array}{rcccl} \mathrm{Hom} & : & ((\sigma, x, V), \tau) & \longmapsto & \mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{X}}(\sigma, \tau) \\ \pi_0 & : & ((\sigma, x, V), \tau) & \longmapsto & \pi_0(V_\tau). \end{array}$$

Then $\lambda : \mathrm{Hom} \to \pi_0$ is a morphism of functors.

Proof. First we show that the natural functor on $\mathcal{X}$ which maps τ to $\pi_0(V_\tau)$ is well defined on $\mathrm{Sur}^{-1}\mathcal{X}$. We consider some surjective $\tau \overset{s}{\twoheadrightarrow} \tau'$. The relation $V_\tau = s_*^{-1}(V_{\tau'})$ and the fact that the fibres of $s_* : \Delta_\tau \to \Delta_{\tau'}$ are connected imply that the natural map

$$s_* : \pi_0(V_\tau) \longrightarrow \pi_0(V_{\tau'})$$

is an isomorphism.

Now we prove property 1.. By Proposition 3.6 we have to check that equivalent morphisms have the same images. Let $\alpha \sim \alpha'$ given by the diagram

$$\begin{array}{ccc} \sigma' & \overset{\beta}{\longrightarrow} & \tau \\ s \downarrow & & \\ \sigma & & \end{array}$$

and sections δ, δ'. Then we have

$$\alpha_*(x) = \beta_*(\delta_*(x)) \in \beta_*(s_*^{-1}(x)) \ni \beta_*(\delta'_*(x)) = \alpha'_*(x).$$

But $\beta_*(s_*^{-1}(x)) \subseteq [\tau]^{-1}(x) \subseteq V_\tau$ is connected and hence $[\alpha_*(x)] = [\alpha'_*(x)]$.

Property 3. follows easily from the corresponding commutative diagram with respect to a lifting $\hat{\alpha} : \sigma' \to \sigma$ in $\mathcal{X}$ and the maps (3.4) instead of λ.

To show property 2. we consider the map $s : \sigma \twoheadrightarrow \sigma'$ to a nondegenerate σ'. Denote by x' the image of x under s_*. Then x' is the uniquely defined point in $inn(\Delta_{\sigma'})$ such that

$$[\sigma'](x') = [\sigma](x).$$

We apply property 3. to the morphism $(\sigma', x', V) \to (\sigma, x, V)$ given by s and get the assertion. □

Lemma 3.15. *Fix some $\sigma \in X.$ and $x \in inn(\Delta_\sigma)$. Denote also by x the image in $|X.|$ of x under the map $[\sigma]$. Then there exists a basis of neighbourhoods $\widetilde{V}$ of $x \in |X.|$ such that the morphism of functors $\lambda(\sigma, x, \widetilde{V})$ of Lemma 3.14 is an isomorphism.*

Proof. The proof consists of several steps.

Consider the topological standard simplices

$$\Delta_k = \left\{ y = (y_0, \dots, y_k) \in \mathbb{R}^{k+1}, \text{ where } \sum_{i=0}^{k} y_i = 1 \ y_i \geq 0 \text{ for all } i = 0, \dots, k \right.$$

with metric d induced by $\mathbb{R}^{k+1}$ with $d(y, z) = \sum_{i=0}^{k} |y_i - z_i|$. An easy calcul shows that for a map of ordered sets $\alpha : \underline{k} \to \underline{k'}$ the corresponding $\alpha_* : \Delta_k \to \Delta$ contracting:

(i) $d(y, z) \geq d(\alpha_*(y), \alpha_*(z))$ for all $y, z \in \Delta_k$.

For a subset $T \subseteq \Delta_k$ and $\varepsilon > 0$ we denote by $U_\varepsilon(T)$ the open set of points y with $d(y, T) < \varepsilon$.

From now on we fix some n, $x \in inn(\Delta_n)$ and some $\varepsilon' > 0$, such that

$$\varepsilon' < \min_{i=0}^{n} x_i = d(x, \partial\Delta_n). \tag{3.5}$$

The following properties are satisfied:

(ii) Consider some diagram

$$\begin{array}{ccc} \underline{k} & \overset{\delta}{\longleftarrow} & \underline{\tilde{k}} \\ & & \downarrow t \\ & & \underline{n} \end{array}$$

with injective δ and surjective t. Then

$$U_{\varepsilon'}(\delta_*(t_*^{-1}(x))) = \left\{ y \in \Delta_k \text{ with } \sum_{j \notin \mathrm{Im}(\delta)} y_j + \sum_{i=0}^{n} \left| x_i - \sum_{\tilde{j} : t(\tilde{j}) = i} y_{\delta(\tilde{j})} \right| < \varepsilon' \right.$$

The proof is an easy calculation using the triangle unequality.

(iii) Suppose to be given a diagram of the form

$$\begin{array}{ccc} \underline{k} & \overset{\delta}{\longleftarrow} & \underline{\tilde{k}} \\ \beta \downarrow & & \downarrow \tilde{\beta} \\ \underline{k'} & \overset{\delta'}{\longleftarrow} & \underline{\tilde{n}'} \\ & & \downarrow t \\ & & \underline{n} \end{array} \quad \text{such that} \quad \begin{array}{l} \text{upper square a fibre product,} \\ t \text{ surjective,} \\ \delta, \delta' \text{ injective,} \end{array}$$

where we allow $\tilde{k}$ to be -1, i.e., $\underline{\tilde{k}} = \emptyset$. Then the following relation is satisfied:

$$\beta_*^{-1}(U_{\varepsilon'}(\delta'_*(t_*^{-1}(x)))) = U_{\varepsilon'}\left(\delta_*\left((t \circ \tilde{\beta})_*^{-1}(x)\right)\right).$$

If $t \circ \tilde{\beta}$ is not surjective these sets are empty.

To see this we use (ii). The left-hand side of the equation is given by the set of points $y \in \Delta_k$ satisfying the relation

$$(3.6)\qquad \sum_{j:\beta(j)\notin \mathrm{Im}(\delta')} y_j + \sum_{i=0}^{n} \left| x_i - \sum_{\substack{j:\,\mathrm{ex.}\ \tilde{j}'\ \mathrm{with}\\ \beta(j)=\delta'(\tilde{j}'),\, t(\tilde{j}')=i}} y_j \right| < \varepsilon'$$

and the right-hand side by

$$(3.7)\qquad \sum_{j\notin \mathrm{Im}(\delta)} y_j + \sum_{i=0}^{n} \left| x_i - \sum_{\tilde{j}:\, t(\tilde{\beta}(j))=i} y_{\delta(\tilde{j})} \right| < \varepsilon' .$$

(If $t\circ\tilde{\beta}$ is not surjective we cannot apply (iii) to the right-hand side, but in this case $(t\circ\tilde{\beta})_*^{-1}(x) = \emptyset$, since x is an inner point of Δ_n and inequality (3.7) yields $x_i < \varepsilon'$ for $i \notin \mathrm{Im}\,(t\circ\tilde{\beta})$ which contradicts the condition (3.5).) Using the universal property of the fibre product is is easily seen that the conditions (3.6) and (3.7) are equivalent. Now we pass to the data given in the lemma.

According to Lemma 3.14 we can restrict ourself to the case that σ is nondegenerate. Furthermore, we fix $\varepsilon > 0$ satisfying the stronger inequality

$$(3.8)\qquad 2\varepsilon < \min_{i=0}^{n} x_i = d(x, \partial\Delta_n) .$$

For an arbitrary $\tau \in X.$ let

$$U_\tau := U_\varepsilon([\tau]^{-1}(x)) ..$$

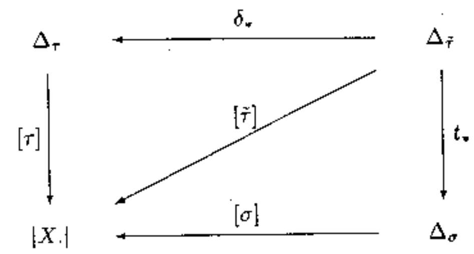
Then we have:

(iv) The following equation holds:

$$[\tau]^{-1}(x) = \bigcup_{\substack{\tau \overset{\delta}{\leftarrow} \tilde{\tau}\\ \downarrow t\\ \sigma}} \delta_*(t_*^{-1}(x)) ,$$

where the union is taken over all diagrams of this kind with injective δ and surjective t.

The relation "$\supseteq$" follows easily from the commutative diagram



The relation "$\subseteq$" is an easy consequence of the following two facts

a) $[\tau]^{-1}(x) \cap inn(\Delta_\tau) \neq \emptyset$ iff there exists $\tau \twoheadrightarrow \sigma$.

b) $\Delta_\tau = \bigcup_{\tau \overset{\delta}{\leftarrow} \tilde{\tau}} inn(\Delta_{\tilde{\tau}})$.

(v) Let $\beta : \tau \to \tau'$ be some morphism of $\mathcal{X}$. Then

$$\beta_*^{-1}(U_{\tau'}) = U_\tau .$$

Using property (i) and the fact that $\beta_*([\tau]^{-1}(x)) \subseteq [\tau']^{-1}(x)$ one can see that

$$\beta_*^{-1}(U_{\tau'}) \supseteq U_\tau .$$

On the other hand we have by property (v)

$$U_{\tau'} = \bigcup_{\substack{\tau' \overset{\delta'}{\leftarrow} \tilde{\tau}'\\ \downarrow t\\ \sigma}} U_\varepsilon(\delta'_*(t_*^{-1}(x))) .$$

Hence we have to show that for all such diagrams

$$\beta_*^{-1}(U_\varepsilon(\delta'_*(t_*^{-1}(x)))) \subseteq U_\tau .$$

This follows from property (iii) taking the fibre product of δ' and β and from prope (v).

We denote by U the image of $\coprod_{\tau\in X.} U_\tau$ under the natural map

$$\coprod_{\tau\in X.} \Delta_\tau \longrightarrow |X.| .$$

(vi) The set $U \ni x$ is open and $[\tau]^{-1}(U) = U_\tau$.

This is a consequence of property (v) and the definition of $|X.|$ and its topology.

(vii) If $V = U$ the morphism $\lambda(\sigma, x, V)$ is an isomorphism.

First we remark that by property (vi) our notations for U_τ are consistent.

We construct a morphism in the other direction. Using property (v) we have a description of U_τ as a union of open connected subsets:

$$U_\tau \;=\; \bigcup_{\substack{\tau \overset{\delta}{\leftarrow} \tau' \\ \downarrow t \\ \sigma}} U_\varepsilon\big(\delta_*(t_*^{-1}(x))\big)\,.$$

We now map $U_\varepsilon\big(\delta_*(t_*^{-1}(x))\big)$ to $\delta \circ s^{-1} \in \mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{X}}(\sigma, \tau)$. We have to check that if

$$U_\varepsilon\big(\delta_*(t_*^{-1}(x))\big) \cap U_\varepsilon\Big(\bar\delta_*\Big(\big(\bar t_*\big)^{-1}(x)\Big)\Big) \;\neq\; \emptyset\,,$$

then $\delta \circ t^{-1} = \bar\delta \circ \bar t^{-1}$. From this unequality we get

$$\big(\bar\delta_*\big)^{-1} U_{2\varepsilon}\big(\delta_*(t_*^{-1}(x))\big) \;\neq\; \emptyset\,.$$

Now we can apply property (iii) to the following diagram first without the upper left corner and then without the right lower corner

$$\begin{array}{ccccc} \sigma & \overset{\bar t}{\longleftarrow} & \bar\sigma' & \overset{\delta'}{\longleftarrow} & \eta \\ & & \bar\delta\downarrow & & \downarrow\bar\delta' \\ & & \tau & \overset{\delta}{\longleftarrow} & \sigma' \\ & & & & \downarrow t \\ & & & & \sigma, \end{array}$$

where η is the fibre product of δ and $\bar\delta$ and where $\varepsilon' = 2\varepsilon$. It follows that $\bar t \circ \delta'$ and $t \circ \bar t'$ are surjective. But σ is nondegenerate, therefore $\bar t \circ \delta' = t \circ \bar\delta'$. Furthermore, we see that δ' is an isomorphism in $\mathrm{Sur}^{-1}\mathcal{X}$ and we get the following equalities

$$\bar\delta\bar t^{-1} \;=\; \bar\delta\delta'\delta'^{-1}\bar t^{-1} \;=\; \delta\bar\delta'\bar\delta'^{-1}t^{-1} \;=\; \delta t^{-1}\,.$$

It is easy to check that these two maps are inverse to each other.

Now fix an arbitrary $V \ni x$. We will construct an open subset $\widetilde V \subseteq V$ containing x such that $\lambda(\sigma, x, \widetilde V\,)$ is an isomorphism.

Without loss of generality we can assume that $V \subseteq U$.

Let $\tau \in X_m$, $m \geq 0$, be a simplex. Let W_τ be the union of all open connected components of V, which lie in the image of $\mathrm{Hom}(\sigma, \tau)$ with respect to the map $\lambda_\tau(x, \sigma, V)$. For an arbitrary map $\alpha : \eta \to \tau$ one obviously has

$$\alpha_*(W_\eta) \subseteq W_\tau \text{ and if } \alpha = s \text{ is surjective then } s_*^{-1}(W_\tau) \;=\; W_\eta$$

by Lemma 3.14. The last equation is in general not true if s is not surjective. To achieve equality we have to modify the definition of W_η a little bit.

By induction on m we define

$$\widetilde V_\tau \;:=\; (W_\tau \cap inn(\Delta_\tau)) \cup \bigcup_{j=0}^{m} \partial_{j,*}\widetilde V_{\partial_j^*\tau}\,.$$

In the following we show that $\quad \alpha_*^{-1}(\widetilde V_\tau) \;=\; \widetilde V_\eta$

for every $\alpha : \eta \to \tau$. First we consider the case $\alpha = s_i$ and by induction we assu[me] that the assertion is true for all $\partial_j^*\tau \in X_{n-1}$. Using the standard relations betwe[en] the maps of ordered sets s_k and ∂_l we get

$$\begin{aligned} s_{i,*}^{-1}(\widetilde V_\tau) \;&=\; \big(s_{i,*}^{-1}(W_\tau) \cap s_{i,*}^{-1}(inn(\Delta_\tau))\big) \cup \bigcup_{j=0}^{m} s_{i,*}^{-1}\big(\partial_{j,*}\widetilde V_{\partial_j^*\tau}\big) \\ &=\; (W_\eta \cap inn(\Delta_\eta)) \cup \bigcup_{j=i,i+1} \partial_{j,*}\big(s_{i,*}^{-1}(W_\tau) \cap inn(\Delta_\tau)\big) \\ &\quad \cup \bigcup_{j<i} \partial_{j,*}\big(s_{i-1,*}^{-1}(\widetilde V_{\partial_j^*\tau})\big) \cup \bigcup_{j>i} \partial_{j+1,*}\big(s_{i,*}^{-1}(\widetilde V_{\partial_j^*\tau})\big) \cup \partial_{i,*}\big(\partial_{i,*}(\widetilde V_{\partial_i^*\partial_i^*\eta} \\ &=\; (W_\eta \cap inn(\Delta_\eta)) \cup \bigcup_{j=i,i+1} \partial_{j,*}(W_\tau \cap inn(\Delta_\tau)) \\ &\quad \cup \bigcup_{j\neq i,i+1} \partial_{j,*}(\widetilde V_{\partial_j^*\eta}) \cup \partial_{i,*}\big(\partial_{i,*}(\widetilde V_{\partial_i^*\partial_i^*\eta})\big) \\ &=\; (W_\eta \cap inn(\Delta_\eta)) \cup \bigcup_{j=i,i+1} \partial_{j,*}(W_\tau \cap inn(\Delta_\tau)) \cup \bigcup_{j\neq i,i+1} \partial_{j,*}(\widetilde V_{\partial_j^*\eta}) \\ &\quad \cup \bigcup_{j<i} \partial_{j,*}\big(\partial_{i,*}(\widetilde V_{\partial_j^*\partial_i^*\eta}) \cup \partial_{i-1,*}(\widetilde V_{\partial_j^*\partial_{i-1}^*\eta})\big) \\ &\quad \cup \bigcup_{j>i+1} \partial_{j,*}\big(\partial_{i,*}(\widetilde V_{\partial_j^*\partial_i^*\eta}) \cup \partial_{i+1,*}(\widetilde V_{\partial_j^*\partial_{i+1}^*\eta})\big) \cup \partial_{i,*}\big(\partial_{i,*}(\widetilde V_{\partial_i^*\partial_i^*\eta})\big) \\ &=\; (W_\eta \cap inn(\Delta_\eta)) \cup \bigcup_{j=1}^{m+1} \partial_{j,*}(\widetilde V_{\partial_j^*\eta}) \\ &=\; \widetilde V_\eta\,. \end{aligned}$$

Now we consider the case $\alpha = \partial_i$. By induction we may assume that the asser[tion] is true for all $\partial_j^*\tau$. We get

$$\begin{aligned} \partial_{i,*}^{-1}(\widetilde V_\tau) \;&=\; \bigcup_{j=0}^{m} \partial_{i,*}^{-1}(\partial_{j,*}\widetilde V_{\partial_j^*\tau}) \\ &=\; \bigcup_{j<i} \partial_{j,*}(\partial_{i-1,*}^{-1}\widetilde V_{\partial_j^*\tau}) \cup \widetilde V_\eta \cup \bigcup_{j>i} \partial_{j-1,*}(\partial_{i,*}^{-1}\widetilde V_{\partial_j^*\tau}) \\ &=\; \widetilde V_\eta \cup \bigcup_{j=0}^{m-1} \partial_{j,*}(\widetilde V_{\partial_j^*\eta}) \\ &=\; \widetilde V_\eta\,. \end{aligned}$$

Let us now check that the $\widetilde V_\tau$ are open. Suppose this to be true for all $\tau' \in X_k$ $k < m$. Using these compatibilities we see that

$$\partial_{i,*}\widetilde V_{\partial_i^*\tau} \cap \partial_{j,*}\Delta_{\partial_j^*\tau} \;\subseteq\; \partial_{j,*}\widetilde V_{\partial_j^*\tau}$$

for arbitrary i,j. Hence

$$\bigcup_{j=0}^{m} \partial_{j,*}\widetilde{V}_{\partial_j^*\tau} \subseteq \partial\Delta_\tau$$

is an open subset of the boundary of Δ_τ. By induction we get $\widetilde{V}_\tau \subseteq W_\tau$. It follows that

$$\widetilde{V}_\tau = W_\tau \cap \left(inn(\Delta_\tau) \cup \bigcup_{j=0}^{m} \partial_{j,*}\widetilde{V}_{\partial_j^*\tau} \right)$$

and $\widetilde{V}_\tau$ is an open subset of Δ_τ.

Hence we get an open subset $\widetilde{V} \subseteq V$ with $[\tau]^{-1}(V) = V_\tau$ and containing x. The map $\lambda_\tau(\sigma,x,\widetilde{V})$

$$\mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{X}}(\sigma,\tau) \longrightarrow \pi_0(\widetilde{V_\tau}) \xrightarrow{\sim} \pi_0(W_\tau)$$

is surjective by the construction of W_τ. Now we consider the commutative diagram of Lemma 3.14, 3. with respect to the data (σ,x,U) and $(\sigma,x,\widetilde{V})$ and we see by Step (vii) that $\lambda_\tau(\sigma,x,\widetilde{V})$ has to be injective. □

Denote by $\widehat{\mathrm{Sur}}$ the set of morphisms $(\tau,\mathcal{V}) \to (\tau',\mathcal{V}')$ of $\widehat{\mathcal{X}}_V$ such that the corresponding map $\tau \to \tau'$ is surjective.

Lemma 3.16. *Let $f : X. \to Y.$ be a morphism of simplical sets and (σ,y,V) be a compatible data on $Y.$. Then the functor*

$$\begin{array}{rrcl} (3.9) & \sigma\backslash Sur^{-1}f & \longrightarrow & \widehat{Sur}^{-1}\widehat{\mathcal{X}}_{|f|^{-1}V} \\ & (\tau,\sigma \xrightarrow{\alpha} f(\tau)) & \longmapsto & (\tau,[\alpha_*(y)]), \end{array}$$

where $[\alpha_*(y)]$ denotes the open connected component of $\alpha_*(y)$ in $(|f|^{-1}V)_\tau$, is well defined. It defines an equivalence of categories for $V = \widetilde{V}$ as in Lemma 3.15. The maps

$$(\sigma,y,V) \longmapsto \sigma\backslash Sur^{-1}f$$
$$(\sigma,y,V) \longmapsto \widehat{Sur}^{-1}\widehat{\mathcal{X}}_{|f|^{-1}V}$$

are in a natural way functors on $Data(Y.)$ and the functor given in (3.9) commutes with this structure.

Proof. By Lemma 3.14 the functor is well defined on objects.

Let α_1,α_2 be two liftings of $\alpha : \tau \to \tau' \in \mathrm{Sur}^{-1}\mathcal{X}$ to $\mathcal{X}$. Using the definition of the relation $\sim$ one easily gets that α_1 and α_2 have the same image. Hence the functor is well defined. The compatibility with respect to maps in $Data(Y.)$ is obvious.

It remains to show that the functor (3.9) is an equivalence of categories if $V = \widetilde{V}$ as in Lemma 3.15.

By Lemma 3.14 we have a compatible family of maps

$$\lambda_\tau(\sigma,y,\widetilde{V}) : \mathrm{Hom}_{\mathrm{Sur}^{-1}\mathcal{Y}}(\sigma,f(\tau)) \xrightarrow{\sim} \pi_0((|f|^{-1}V)_\tau).$$

We define an inverse functor of (3.9) by

$$(\tau,\mathcal{V}) \longmapsto \lambda_\tau(\sigma,y,V)^{-1}([\mathcal{V}]).$$

Then the composition is the identity functor on $\sigma\backslash\mathrm{Sur}^{-1}\mathcal{X}$ by the definition of λ. The composition on $\widehat{\mathrm{Sur}}^{-1}\widehat{\mathcal{X}}_V$ is given by

$$(\tau,\mathcal{V}) \longmapsto (\tau,[\mathcal{V}])$$

and the natural map

$$(\tau,\mathcal{V}) \longrightarrow (\tau,[\mathcal{V}])$$

given by the inclusion $\mathcal{V} \subseteq [\mathcal{V}]$ defines an isomorphism from the identity functor. □

The following proposition shows, that the geometric realization functor (3.1) satisfies nice properties if we restrict it to sheaves.

Proposition 3.17. *Let $X.$ be a simplicial set $X.$. Consider the functors $|\ |$ and stalk on $Data(X.)^{opp} \times SH(X.)$ with values in $\mathcal{A}$ given by*

$$\begin{array}{rcrcl} |\ | & : & ((\sigma,x,V),F.) & \longmapsto & |F.|(V) \\ stalk & : & ((\sigma,x,V),F.) & \longmapsto & F_\sigma. \end{array}$$

Then there exists a morphism of functors

$$|\ | \longrightarrow stalk$$

which is an isomorphism for all $(\sigma,x,V) \in Data(X.)$ with $V = \widetilde{V}$ as in Lemma 3.15. In particular,

$$|F.|_{|\sigma|(x)} \simeq F_\sigma.$$

Proof. Let $(\sigma,x,V) \in Data(X.)$ and $F.$ be a sheaf on $X.$.

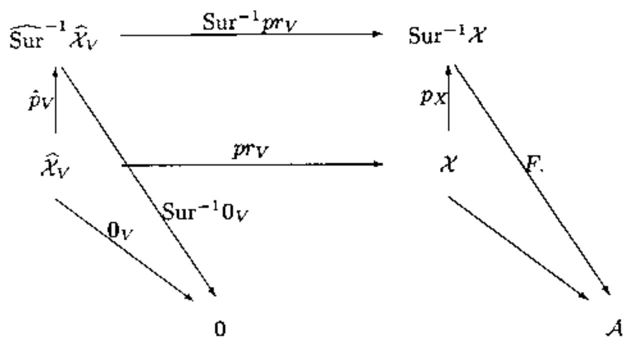
Analogously to Lemma 3.4, property 4, one easily checks that the natural functor

$$\widehat{\mathcal{X}}_V \xrightarrow{\hat{p}_V} \widehat{\mathrm{Sur}}^{-1}\widehat{\mathcal{X}}_V$$

satisfies

$$\hat{p}_{V,*} \circ \hat{p}_V^* \simeq id : \mathrm{Func}\left(\widehat{\mathrm{Sur}}^{-1}\widehat{\mathcal{X}}_V,\mathcal{A}\right) \longrightarrow \mathrm{Func}\left(\widehat{\mathrm{Sur}}^{-1}\widehat{\mathcal{X}}_V,\mathcal{A}\right).$$

Denote by $\mathbf{0}$ the category with one object and one morphism and identify the category of functors $\mathrm{Func}(\mathbf{0},\mathcal{A})$ with $\mathcal{A}$. Consider the commutative diagram:



We get

$$\begin{aligned} |F.|(V) &= \varprojlim_{\widehat{\mathcal{X}}_V} F. \circ p_{\mathcal{X}} \circ pr_V \\ &= 0_{V,*}(F. \circ \mathrm{Sur}^{-1}pr_V \circ \hat{p}_V) \\ &= (\mathrm{Sur}^{-1}0_V)_* \circ \hat{p}_{V,*} \circ \hat{p}_V^*(F. \circ \mathrm{Sur}^{-1}pr_V) \\ &\simeq \varprojlim_{\widehat{\mathrm{Sur}}^{-1}\widehat{\mathcal{X}}_V} F. \circ \mathrm{Sur}^{-1}pr_V\,. \end{aligned}$$

Now we use the functor in (3.9) and get a morphism

$$|F.|(V) \longrightarrow \varprojlim_{\sigma\backslash \mathrm{Sur}^{-1}\mathcal{X}} F. \circ pr_\sigma \simeq F_\sigma\,.$$

It is not difficult to see that this map is given by the projection of the projective limit over $\widehat{\mathcal{X}}_V$ to the component over $(\sigma, [x])$. Then an easy calculation shows that it defines a morphism of contravariant functors on $Data(X.)$

$$|\;| \longrightarrow \mathrm{stalk}\,.$$

The functoriality with respect to $F.$ is obvious.

The last part of the proposition follows from Lemma 3.15. □

Corollary 3.18. *The functor $|?| : SH(X.) \to SH(|X.|)$ is a full exact embedding. The image is given by the sheaves $\mathcal{F}$ on $|X.|$ satisfying the condition that $[\sigma]^*\mathcal{F}$ is a constant sheaf on $inn(\Delta_\sigma)$ for every simplex $\sigma \in X.$.*

Proof. From the Proposition 3.17 it follows immediately that the functor $|\;|$ is exact and injective on morphisms. To see that it is surjective on morphisms consider some morphism of sheaves $|F.| \to |G.|$. For every compatible data $(\sigma, x, \widetilde{V})$ with $\widetilde{V}$ as in Lemma 3.15 we define a map $F_\sigma \to G_\sigma$ by the commutative diagram

$$\begin{array}{ccc} |F.|(\widetilde{V}) & \xrightarrow{\sim} & F_\sigma \\ \downarrow & & \downarrow \\ |G.|(\widetilde{V}) & \xrightarrow{\sim} & G_\sigma\,. \end{array}$$

Let us check that this map does not depend on the choice of $x \in inn(\Delta_\sigma)$ and $\widetilde{V}$ Take another compatible data $(\sigma, x', \widetilde{V}')$. Choose a way

$$\gamma : [0,1] \longrightarrow inn(\Delta_\sigma)\,, \quad \text{with} \quad \gamma(0) = x\,, \;\; \gamma(1) = x'\,.$$

We can find a finite set of points $0 = t_0 < \cdots < t_k = 1$ such that there exists a set compatible data

$$\{(\sigma, \gamma(t_i), \widetilde{V}_i)\},\; i = 0, \ldots, k, \text{ with } \begin{cases} \widetilde{V}_i \text{ as in Lemma 3.15}\,, \\ \bigcup_{i=0}^{k} \widetilde{V}_i \supseteq \mathrm{Im}(\gamma)\,, \\ [\sigma]^{-1}\widetilde{V}_i \cap [\sigma]^{-1}\widetilde{V}_{i+1} \cap inn(\Delta_\sigma) \neq \emptyset\,, \\ \widetilde{V}_0 = V,\; \widetilde{V}_k = V'\,. \end{cases}$$

By induction on k we restrict to the case that $[\sigma]^{-1}V \cap [\sigma]^{-1}V'$ contains some poi $y \in inn(\Delta_\sigma)$. By Lemma 3.15 and Corollary 3.9 one sees that $V_\sigma = [\sigma]^{-1}V$ h to be connected. Let $W \subseteq V \cap V'$ be some open subset as in Lemma 3.15 wi $[\sigma](y) \in W$. Then we use the compatibility property of the proposition with respe to the morphisms

$$(\sigma, x, V) \longleftarrow (\sigma, y, W) \longrightarrow (\sigma, x', V')$$

in $Data(X.)$ given by inclusions and get the independence.

For a map between simplices $\alpha : \sigma \to \sigma'$ we take a compatible data $(\sigma, x, \widetilde{V})$ fir then we choose a compatible data $(\sigma', x', \widetilde{V}')$, such that x' can be connected wi $\alpha_*(x)$ inside $\widetilde{V}_{\sigma'}$ and $\widetilde{V}' \subseteq \widetilde{V}$. We consider the morphism

$$(\sigma', x', \widetilde{V}') \longrightarrow (\sigma, x, \widetilde{V})$$

of compatible data induced by α. By the proposition the commutative diagram

$$\begin{array}{ccc} |F.|(\widetilde{V}) & \longrightarrow & |F.|(\widetilde{V}') \\ \downarrow & & \downarrow \\ |G.|(\widetilde{V}) & \longrightarrow & |G.|(\widetilde{V}') \end{array} \quad \text{induces a commutative diagram} \quad \begin{array}{ccc} F_\sigma & \xrightarrow{\alpha} & F_{\sigma'} \\ \downarrow & & \downarrow \\ G_\sigma & \xrightarrow{\alpha} & G_{\sigma'}\,. \end{array}$$

Hence we get a well defined morphism of sheaves $F. \to G.$. It is not difficult to that this map induces the given map $|F.| \to |G.|$.

Let us now check that the sheaves $[\sigma]^*|F.|$ are constant on $inn(\Delta_\sigma)$ for every $\sigma \in$ Fix a $\sigma \in X.$. Let $U \subseteq inn(\Delta_\sigma)$ be a connected non-empty subset. We can fin base of open subsets $V \supseteq [\sigma](U)$ consisting of unions of open sets $\widetilde{V}_x$ for $x \in [\sigma]($ as in Lemma 3.15. We can assume that V is connected. From Proposition 3.17 get an isomorphism $|F.|(V) \simeq F_\sigma$ compatible with respect to inclusion. Therefore isomorphism $[\sigma]^{**}|F.|(U) \simeq F_\sigma$ is compatible with respect to inclusions $U \subseteq U'$, wh $[\sigma]^{**}$ denotes the preimage as a presheaf. It follows that the sheafification $[\sigma]^*|F.$ constant with fiber F_σ.

On the other hand, let $\mathcal{F}$ be a sheaf on $|X.|$ such that $[\sigma]^*\mathcal{F}$ is constant for $\sigma \in X.$. Then we define a sheaf $F.$ as follows: Let $T_\sigma = s_*^{-1}(inn(\Delta_{\sigma_0})) \subseteq \Delta_\sigma$, w

$s : \sigma \twoheadrightarrow \sigma_0$ the surjective map to a nondegenerate simplex σ_0. The sheaf $[\sigma]^*\mathcal{F}$ is constant on $T_\sigma \supsetneq inn(\Delta_\sigma)$ and we define

$$F_\sigma \;:=\; \Gamma(T_\sigma, [\sigma]^*\mathcal{F}).$$

Let $\alpha : \sigma \to \tau$ be a map in $\mathrm{Sur}^{-1}\mathcal{X}$. Take some compatible data (σ, x, V). We consider the map

$$\mathcal{F}(V) \overset{Ad}{\to} ([\tau]^*\mathcal{F})([\tau]^{-1}V) \to ([\tau]^*\mathcal{F})(\lambda_\tau(\sigma,x,V)(\alpha)\cap T_\tau) \overset{\sim}{\leftarrow} ([\tau]^*\mathcal{F})(T_\tau) \;=\; F_\tau\,,$$

where $\lambda_\tau(\sigma, x, V)$ is defined in Lemma 3.14. The lemma implies that this morphism is compatible with respect to maps of compatible data $(\sigma, x, V) \to (\sigma, x', V')$. Therefore we get a well defined morphism

$$F_\sigma \;=\; \Gamma(T_\sigma, [\sigma]^*\mathcal{F}) \overset{\sim}{\longrightarrow} ([\sigma]^*\mathcal{F})_x \;=\; \varinjlim_{V\ni[\sigma](x)} \mathcal{F}(V) \longrightarrow F_\tau$$

which is independend of the choice of x. It is not difficult to see that this construction is compatible with respect to compositions of α. We define a morphism of sheaves $\mathcal{F} \to |F.|$ by giving a compatible family of morphisms $\mathcal{F}(\widetilde{V}) \to |F.|(\widetilde{V})$ for all compatible data $(\sigma, x, \widetilde{V})$ with $\widetilde{V}$ as in Lemma 3.15. Remark that $[\sigma]^{-1}\widetilde{V} \subseteq T_\sigma$. Otherwise one would have a boundary $\tau = \partial_i^*\sigma$ so that $[\tau]^{-1}\widetilde{V} \neq \emptyset$ and there exist no surjections from σ and τ to the same nondegenerated simplex. By Lemma 3.15 this implies $\mathrm{Hom}(\tau, \sigma) \neq \emptyset$ what gives a contradiction. Therefore we can give the map $\mathcal{F}(\widetilde{V}) \to |F.|(\widetilde{V})$ by the composition

$$\mathcal{F}(\widetilde{V}) \overset{Ad}{\longrightarrow} ([\sigma]^*\mathcal{F})([\sigma]^{-1}\widetilde{V}) \overset{\sim}{\longleftarrow} ([\sigma]^*\mathcal{F})(T_\sigma) \;=\; F_\sigma \overset{\sim}{\longleftarrow} |F.|(\widetilde{V}).$$

This map is compatible with morphisms of compatible data by Proposition 3.17 and therefore defines a map of sheaves $\mathcal{F} \to |F.|$. It is an isomorphism, because it induces an isomorphism on the stalks. □

Let us now return to the question in the beginning of the subsection, namely, how to describe the stalks of $|F.|$ for a cohomological system of coefficients $F.$ on a simplicial set $X.$. The answer is given in the next lemma in combination with Proposition 3.17.

Remember that we denoted by p_X the natural projection from $\mathcal{X}$ to $\mathrm{Sur}^{-1}\mathcal{X}$, see Lemma 3.4. The natural inclusion

$$p_X^* \;:\; SH(X.) \longrightarrow \mathrm{Func}(\mathcal{X}, \mathcal{A})$$

has a right adjoint functor

$$p_{X,*} \;:\; \mathrm{Func}(\mathcal{X}, \mathcal{A}) \longrightarrow SH(X.)\,,$$

which is given by the right Kan-extension.

Lemma 3.19. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Func}(\mathcal{X}, \mathcal{A}) & & \\ p_{X,*}\downarrow & \searrow & \\ SH(X.) & \overset{|?|}{\longrightarrow} & SH(|X.|)\,. \end{array}$$

Proof. Let $F.$ be some cohomological system of coefficients. By the definition o the geometric realization we have to verify that for an arbitrary open $U \subseteq |X.|$ th morphism

$$\varprojlim_{\widehat{\mathcal{X}}_U} F.\circ pr_U \longrightarrow \varprojlim_{\widehat{\mathcal{X}}_U} p_{X,*}(F.)\circ pr_U$$

is an isomorphism. By the definition of $p_{X,*}(F.)$ we have

$$p_{X,*}(\mathcal{F})_\sigma \;=\; \varprojlim_{\sigma\backslash p_X} F.\circ pr$$

and the adjunction map is the map to the component $(\sigma, id_\sigma) \in \sigma\backslash p_X$. This map is monomorphism, because every morphism $\sigma \to \tau$ in $\mathrm{Sur}^{-1}\mathcal{X}$ has a preimage $\sigma \to \tau$ $\mathcal{X}$ by Proposition 3.6. We have to prove that the projection

$$\lambda \;:\; \varprojlim_{\widehat{\mathcal{X}}_U} F.\circ pr_U \longrightarrow (F.\circ pr_U)(\sigma, \mathcal{U}) \;=\; F_\sigma$$

has its image in $\varprojlim_{\sigma\backslash p_X} F.\circ pr$ for all $(\sigma, \mathcal{U})$ in $\widehat{\mathcal{X}_U}$.

Let $\sigma \overset{\varphi}{\to} \tau$ be an arbitrary morphism in $\mathrm{Sur}^{-1}\mathcal{X}$ and φ_1, φ_2 two liftings of φ to . All we have to verify is that $\varphi_{1,*}\circ\lambda = \varphi_{2,*}\circ\lambda$. We can assume that

$$\varphi_1 \;=\; \alpha\circ\delta_1 \;\sim\; \alpha\circ\delta_2 \;=\; \varphi_2$$

as in (3.3) and (3.2). Let $\mathcal{U}' = s_*^{-1}\mathcal{U}$ and $\mathcal{U}''$ be some open connected subset of $[\tau]^{-1}$ containing $\alpha_*\mathcal{U}'$. Then we consider the diagram

$$\begin{array}{cccccc} & & (F.\circ pr_U)(\sigma,\mathcal{U}) & & = & F_\sigma \\ \varprojlim_{\widehat{\mathcal{X}}_U} F.\circ pr_U & \begin{array}{c}\nearrow\\ \searrow\end{array} & \delta_{1,*}\downarrow\downarrow\delta_{2,*} & & & \varphi_{1,*}\downarrow\downarrow\varphi_{2,*} \\ & & (F.\circ pr_U)(\sigma',\mathcal{U}') & \overset{\alpha_*}{\longrightarrow} & (F.\circ pr_U)(\tau,\mathcal{U}'') \;= & F_\tau \end{array}$$

with commutative triangles and commutative squares with respect to the index 1 a 2. The assertion follows immediately from this diagram.

Let us give a more explicit description of the functor $p_{X,*}$. Fix a nondegerate simp $\sigma \in X_n$ and a cohomological system of coefficients $F.$. We want to calculate $(p_{X,*}F.$ Let $(\mathcal{X}/\sigma)_{surj}$ be the full subcategory of the category $\mathcal{X}/\sigma$ given by the objects $\tau \overset{\alpha}{-}$ with $\alpha = s$ a surjective map. We have a functor

$$\begin{array}{ccc} (\mathcal{X}/\sigma)_{surj} & \longrightarrow & \sigma\backslash p_X \\ \tau \overset{s}{\twoheadrightarrow} \sigma & \longmapsto & \sigma \overset{s^{-1}}{\longrightarrow} \tau \end{array}$$

trivially on morphisms, which is an embedding. Applying Proposition 3.6 it is difficult to see that $(\mathcal{X}/\sigma)_{surj}$ is a cofinite subcategory of $\sigma\backslash p_X$.

Denote by $\mathcal{T}$ the subcategory of $(\mathcal{X}/\sigma)_{surj}$ given by the objects $\sigma \in X_n$ and all with $i = 0, \ldots, n$. The sets of morphisms we define by

$$\mathrm{Hom}_{\mathcal{T}}(\sigma, s_i^*\sigma) \;=\; \mathrm{Hom}_{(\mathcal{X}/\sigma)_{surj}}(\sigma, s_i^*\sigma) \;=\; \{\delta_i, \delta_{i+1}\}$$

and all the other to be trivial. Let us check that $\mathcal{T}$ is a cofinal subcategory of $(\mathcal{X}/\sigma)_{surj}$.

Let $\tau \twoheadrightarrow \sigma$ be some object of $(\mathcal{X}/\sigma)_{surj}$. Consider some injective morphism from $\delta : s_i^*\sigma \to \tau$. We call the morphisms $\delta\delta_i$ and $\delta\delta_{i+1}$ elementary equivalent. Consider the minimal distributive relation generated by it. Comparing the images of the morphisms $\delta : \sigma \to \tau$ it is not difficult to see that all morphisms are equivalent. The cofinality follows immediately.

Combining these considerations we get a description of $(p_{X,*}F.)_\sigma$ for nondegenerate $\sigma \in X_n$ by the following exact sequence

$$(3.10) \qquad 0 \longrightarrow (p_{X,*}F.)_\sigma \longrightarrow F_\sigma \xrightarrow{\prod_{i=0}^{n} \delta_{i,*} - \delta_{i+1,*}} \prod_{i=0}^{n} F_{s_i^*\sigma}\,.$$

The first arrow is induced by the adjunction map $p_X^* p_{X,*}F. \to F.$ at the simplex σ.

4. The inverse image

Let $f. : X. \to Y.$ be a morphism of simplicial sets. Consider the natural diagram

$$(4.1) \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ \mathrm{Sur}^{-1}\mathcal{X} & \xrightarrow{\mathrm{Sur}^{-1}f} & \mathrm{Sur}^{-1}\mathcal{Y}, \end{array}$$

where p_X, resp. p_Y, is the structure morphism of the localized category $\mathrm{Sur}^{-1}\mathcal{X}$, resp. $\mathrm{Sur}^{-1}\mathcal{Y}$.

Define

$$\begin{array}{rcll} f^* & := & (\mathrm{Sur}^{-1}f)^*_{cat} : & SH(Y.) \longrightarrow SH(X.) \\ f^* & := & f^*_{cat} : & \mathrm{Func}(\mathcal{Y},\mathcal{A}) \longrightarrow \mathrm{Func}(\mathcal{X},\mathcal{A}) \end{array}$$

as the inverse image functor. These definitions obviously agree if we consider a sheaf as a cohomological system of coefficients.

Lemma 4.1. *The following diagram is commutative:*

$$\begin{array}{ccc} SH(Y.) & \xrightarrow{f^*} & SH(X.) \\ \downarrow |?| & & \downarrow |?| \\ SH(|Y.|) & \xrightarrow{|f|^*} & SH(|Y.|)\,. \end{array}$$

Proof. Let $F.$ be some sheaf on $Y.$ We will construct a morphism of sheaves

$$|f|^*|F.| \longrightarrow |f^*F.|$$

compatible with respect to the morphism of sheaves $F. \to G.$. It is sufficient to give a compatible family with respect to inclusion of maps

$$|f|^*|F.|(\tilde V) \longrightarrow |f^*F.|(\tilde V)$$

for all compatible data $(\sigma, x, \tilde V)$ on $X.$.

We define this map by the following commutative diagram

$$\begin{array}{ccccccccc} |f|^*|F.|(\tilde V) & & & & \longrightarrow & & & & |f^*F.|(\tilde V) \\ \downarrow & & & & & & & & \downarrow \wr \\ (|f|^*|F.|)_x & \simeq & |F.|_{f(x)} & \simeq & F_{f(\sigma)} & \simeq & (f^*F)_\sigma & \simeq & |f^*F|_x \end{array}$$

with the usual morphisms. Now let $(\sigma', x', \tilde V')$ be another such data with $\tilde V' \subseteq \tilde V$.

By Lemma 3.15 there exists a unique map $\alpha : \sigma \to \sigma'$ in $\mathrm{Sur}^{-1}\mathcal{X}$ such that $\alpha_*(x)$ and x' can be connected inside $[\sigma']^{-1}(\tilde V)$. This gives us a map of compatible data

$$(\sigma', x', \tilde V') \longrightarrow (\sigma, x, \tilde V)\,.$$

By definition, bearing in mind Proposition 3.17, the compatibility with respect to the inclusion $\tilde V' \subseteq \tilde V$ will be assured if we verify the commutativity of the following diagram:

$$(4.2) \qquad \begin{array}{ccccc} (|f|^*|F.|)_x & \longrightarrow & |f|^*|F.|(\tilde V) & \longrightarrow & (|f|^*|F.|)_{x'} \\ \wr & & & & \wr \\ |F.|_{f(x)} & & & & |F.|_{f(x')} \\ \wr & & & & \wr \\ F_{f(\sigma)} & & & & F_{f(\sigma')} \\ \| & & & & \| \\ (f^*F.)_\sigma & & \xrightarrow{\alpha_*} & & (f^*F.)_{\sigma'}\,. \end{array}$$

Obviously we can change $|f^*||F.|$ to the presheaf

$$V \longmapsto \varinjlim_{W \supseteq |f|(V)} |F.|(W)\,, \quad \text{with } V \subseteq |X.| \text{ open.}$$

Hence it is sufficient to prove the commutativity of the diagram with $|F.|(W)$ instead of $|f|^*|F.|(\tilde V)$ for all open $W \supseteq |f|(\tilde V)$. The relations $[f(\sigma')] = |f| \circ [\sigma']$ yield

$$[f(\sigma')]^{-1}(W) \supseteq [f(\sigma')]^{-1}(|f|(\tilde V)) = [\sigma']^{-1}(\tilde V)\,.$$

Thus $\alpha_*(x)$ and x' can be connected inside $[f(\sigma')]^{-1}(W)$ and we can apply once more Proposition 3.17 with respect to the morphism of compatible data

$$(f(\sigma'), x', W) \longrightarrow (f(\sigma), x, W)$$

on $Y.$. This shows the commutativity of diagram (4.2).

Hence we have a well defined morphism of sheaves

$$|f^*||F.| \longrightarrow |f^*F.|\,.$$

This is an isomorphism because it induces an isomorphism of the stalks. □

Lemma 4.2. *Let $X. \xrightarrow{f} Y. \xrightarrow{g} Z.$ be morphisms of simplicial sets. Then we have a natural functorial isomorphism $f^* \circ g^* \simeq (g \circ f)^*$.*

Proof. Obvious. □

5. The direct image

Similar to the definition of the inverse image, we define the direct image functor f_* using the diagram (4.1):

$$\begin{array}{lllcl} f_* & := & (\mathrm{Sur}^{-1} f)_{cat,*} : & SH(X.) & \longrightarrow & SH(Y.) \\ f_* & := & f_{cat,*} : & \mathrm{Func}(\mathcal{X}, \mathcal{A}) & \longrightarrow & \mathrm{Func}(\mathcal{Y}, \mathcal{A}) . \end{array}$$

Lemma 5.1. *The functor f^* is left adjoint to the functor f_*.*

Proof. Obvious. □

Let $\tau \in Y.$ be a simplex. For every object $(\sigma, \tau \xrightarrow{\alpha} f(\sigma))$ of the category $\tau \backslash f_{cat}$ there exists a unique σ' $(= \alpha^*(\sigma))$ such that $f(\sigma') = \tau$ and such that there exists a unique map $(= \alpha)$

$$(\sigma', id) \to (\sigma, \tau \xrightarrow{\alpha} f(\sigma)) .$$

Hence we can give the following alternative definition of $f_* F.$ for a cohomological system of coefficients $F.$:

$$(f_* F.)_\tau = \prod_{\sigma \in f^{-1}(\tau)} F_\sigma . \tag{5.1}$$

If $\beta : \tau \to \tau'$ is a map in $\mathcal{Y}$ we obviously have a map

$$\beta^* : f^{-1}(\tau') \longrightarrow f^{-1}(\tau) \text{ inducing a map } \beta_* : (f_* F.)_\tau \longrightarrow (f_* F.)_{\tau'} .$$

Lemma 5.2. *Let $f : X. \to Y.$ be a morphism of simplicial sets. Then the following diagrams are commutative:*

$$\begin{array}{ccc} SH(X.) & \xrightarrow{f_*} & SH(Y.) \\ \downarrow |?| & & \downarrow |?| \\ SH(|X.|) & \xrightarrow{|f|_*} & SH(|Y.|) \end{array} \qquad \begin{array}{ccc} \mathrm{Func}(\mathcal{X}, \mathcal{A}) & \xrightarrow{f_*} & \mathrm{Func}(\mathcal{Y}, \mathcal{A}) \\ \downarrow |?| & & \downarrow |?| \\ SH(|X.|) & \xrightarrow{|f|_*} & SH(|Y.|) . \end{array}$$

Proof. Let $F.$ be a sheaf on $X.$. It is sufficient to construct an isomorphism

$$(|f_* F.|)(\tilde{V}) \longrightarrow (|f|_* |F.|)(\tilde{V}) = |F.|(|f|^{-1}(\tilde{V})) ,$$

compatible with respect to maps of compatible data, for all compatible data $(\sigma, y,$ [...] on $Y.$ with $\tilde{V}$ as in Lemma 3.15. By Lemma 3.16 we have an equivalence of categor[...]

$$\widehat{\mathrm{Sur}}^{-1} \hat{\mathcal{X}}_{|f|^{-1}(\tilde{V})} \xrightarrow{\sim} \sigma \backslash \mathrm{Sur}^{-1} f$$

compatible with respect to maps in $Data(Y.)$. We get an isomorphism

$$\begin{array}{rcl} |f_* F.|(\tilde{V}) \xrightarrow{\sim} (f_* F.)_\sigma & = & \varprojlim_{\sigma \backslash \mathrm{Sur}^{-1} f} F. \circ pr \\ & \simeq & \varprojlim_{\widehat{\mathrm{Sur}}^{-1} \hat{\mathcal{X}}_{|f|^{-1}(\tilde{V})}} F. \circ pr \\ & \simeq & \varprojlim_{\hat{\mathcal{X}}_{|f|^{-1}(\tilde{V})}} F. \circ pr \\ & = & |F.|(|f|^{-1}(\tilde{V})) , \end{array}$$

which is compatible with respect to morphisms of compatible data. Lemma 3.19 a[...] the commutativity of the first diagram assure that the second diagram commutes.

Lemma 5.3. *Let $X. \xrightarrow{f} Y. \xrightarrow{g} Z.$ be morphisms of simplicial sets. Then we hav[e a] natural isomorphism of functors $g_* \circ f_* \simeq (g \circ f)_*$.*

Proof. Obvious.

5.1. Global sections

Let $F.$ be a cohomological system of coefficients or a sheaf on the simplicial set [...] We define the global sections of $F.$ as follows:

$$\Gamma(X., F.) := \varprojlim_{\mathcal{X}} F. . \tag{5.2}$$

Obviously Γ defines a left exact functor from the category of cohomological syste[ms] of coefficients to $\mathcal{A}$.

Example 5.4. (i) Let $X. = \Delta.[\underline{0}]$ be the simplicial point and $F.$ a cohomologi[cal] system of coefficients on $X.$. We get $\mathcal{X} = \Delta$ and it is well known that the subcateg[ory] of Δ given by the objects $\underline{0}$ and $\underline{1}$ and injective maps between them is cofinal. Theref[ore]

$$\Gamma(X., F.) = \ker(\delta_{0,*} - \delta_{1,*}) : F_{\underline{0}} \longrightarrow F_{\underline{1}} .$$

(ii) Let $F.$ be a sheaf on a simplicial set $X.$ and denote by f the map from $X.$ [to] $\Delta.[\underline{0}]$. Then

$$\Gamma(X., F.) \simeq f_* F.$$

with respect to the identification in Example 3.3.

Lemma 5.5. *Let $F.$ be a cohomological system of coefficients on the simplicial [set] $X.$. Then the adjunction map $p_X^* p_{X,*} F. \to F.$ induces an isomorphism*

$$\Gamma(X., p_X^* p_{X,*} F.) \xrightarrow{\sim} \Gamma(X., F.) .$$

Proof. It can be easily proved by definition or using Example 5.4 (ii) that

$$\Gamma(X., p_X^* p_{X.,*} F.) \;\simeq\; f_* p_{X.,*} F. \;\simeq\; \varprojlim_{\mathcal{X}} F. \,. \qquad \square$$

Lemma 5.6. *Let X. be a simplicial set. Then the following diagram of functors is commutative*

$$\begin{array}{ccc} \mathrm{Func}(\mathcal{X},\mathcal{A}) & \xrightarrow{|?|} & SH(|X.|) \\ & {\scriptstyle\Gamma(X.,?)}\searrow & \downarrow {\scriptstyle\Gamma(|X.|,?)} \\ & & \mathcal{A} \end{array} \tag{5.3}$$

Proof. This is an immediate consequence of the definition of the geometric realization 3.1 or of Lemma 5.2 with $Y. = \Delta.[\underline{0}]$ in combination with Lemma 5.5 and Lemma 3.19. $\square$

5.2. Cohomology

Let $F.$ be a cohomological system of coefficients on a simplicial set $X.$. Remember (see for example [9]) that the cohomology of $F.$ denoted by $H^*(X., F.)$ is defined as follows:

Let $f : X. \to \Delta.[\underline{0}]$ be the map to the point. Then we denote by $C^*(X., F.)$ the cosimplicial object $f_* F.$ in $\mathcal{A}$. According to (5.1) it is given by

$$\underline{n} \longmapsto C^n(X., F.) \;:=\; \prod_{\sigma \in X_n} F_\sigma$$

the morphisms being the usual ones if $\underline{n} \to \underline{n}'$ is a map of odered sets. By $C^*(X., F.)$ we also denote the associated cohomology complex of $F.$

$$\ldots \xrightarrow{d^{n-1}} \prod_{\sigma\in X_{n-1}} F_\sigma \xrightarrow{d^n} \prod_{\sigma\in X_n} F_\sigma \xrightarrow{d^{n+1}} \prod_{\sigma\in X_{n+1}} F_\sigma \xrightarrow{d^{n+2}} \ldots \text{ with } d^n = \sum_{i=0}^{i=n} (-1)^i \partial_{i*}$$

and by $H^*(X., F.)$ its cohomology. If $F.$ is a sheaf the cohomology is defined analogously.

Obviously the correspondence $F. \mapsto H^*(X., F.)$ defines a ∂-functor on the category of cohomological systems of coefficients (resp. the category of sheaves).

Proposition 5.7. *Let X. be a simplicial set. Then there exists an isomorphism of* ∂*-functors*

$$H^*(X., ?) \;\simeq\; R^*\Gamma(X., ?)$$

on the category of cohomological systems of coefficients. A sufficiently large class of acyclic objects is given by the following construction:

Let $\mathcal{X}^{triv}$ *be the subcategory of* $\mathcal{X}$ *consisting of the same objects and only identity morphisms and let* $\mathbf{i}$ *be the inclusion. Let* $N : \mathcal{X}^{triv} \to \mathcal{A}$ *be an arbitrary functor, i. e., a family of objects* $N(\sigma) \in \mathcal{A}$ *for each* $\sigma \in X.$ *. Then the functor*

$$\mathbf{i}_{cat,*}N \;:\; \mathcal{X} \longrightarrow \mathcal{A}$$

is acyclic.

Proof. By Example 5.4 (i) we have $\Gamma(X., ?) \simeq H^0(X., ?)$. The homology of homological systems of coefficients on a simplicial set is a derived functor by [9], Appendix 2, § 3 and § 4. This is verified by constructing a sufficiently large class of acyclic objects. Changing the abelian category of coefficients to the dual category we can change homological systems to cohomological, homology to cohomology and the assertion follows immediately.

We can prove it also easily directly using the isomorphisms

$$C^*(X., \mathbf{i}_{cat,*}N) \;\simeq\; f_*\mathbf{i}_{cat,*}N \;\simeq\; (f\circ \mathbf{i})_{cat,*}N \;\simeq\; \prod_{\sigma\in X.} C^*(\Delta.[\sigma], \underline{N(\sigma)})$$

bearing in mind that $|\Delta.[\underline{n}]| = \Delta_n$ is contractible for all n. (Here $\underline{N(\sigma)}$ denotes the constant cohomological system of coefficients with fibre $N(\sigma)$.) $\square$

The exact functor

$$C^*(X., ?) \;:\; \mathrm{Func}(\mathcal{X},\mathcal{A}) \longrightarrow K^{\geq 0}(\mathcal{A})$$

extends to a functor

$$C^*(X., ?) \;:\; D^+(\mathrm{Func}(\mathcal{X},\mathcal{A})) \longrightarrow D^+(\mathcal{A})$$

via double complexes. Proposition 5.7 gives us immediately

$$\mathbf{R}\Gamma \;\simeq\; C^*(X., ?)\,.$$

The following example shows that Proposition 5.7 is in general not true for th category of sheaves.

Example 5.8. Let $X.$ be the two dimensional sphere as in Example 3.12. Denot by σ_0 the nondegenerate 0-simplex and by σ_2 the nondegenerate 2-simplex. Assum that $\mathcal{A}$ has sufficiently many injective objects. Let I be an injective object and defin N as in Proposition 5.7 by

$$N(\sigma) \;=\; \begin{cases} I, & \text{if } \sigma = \sigma_0\,, \\ 0, & \text{otherwise}. \end{cases}$$

Then N is injective and the sheaf $p_{X.,*}\mathbf{i}_{cat,*}N$ is also injective. Using the explic description of the sheaf category in 3.12 it is not difficult to check that $p_{X.,*}\mathbf{i}_{cat,*}N$ the constant sheaf on $X.$ with stalk N and hence not acyclic for Γ (see Propositi 5.12).

Definition 5.9. We denote the extension of $C^*(X, ?)$ from the category of sheav to the derived category using the total product complex of a double complex by

$$\overline{\mathbf{R}}\Gamma \;:\; D^+(X.) \longrightarrow D^+(\mathcal{A})\,.$$

Lemma 5.10. *Let X. have the property that an arbitrary boundary of an arbitrary nondegenerate simplex is nondegenerate and let $\mathcal{A}$ have sufficienlty many injective objects. Then the natural morphism*

$$\mathbf{R}\Gamma \longrightarrow \overline{\mathbf{R}}\Gamma$$

is an isomorphism of functors on $D^+(X.)$.

Proof. The proof follows immediately from Proposition 5.7 and Lemma 3.11. □

Further we want to give another description of the cohomology in the case that $F.$ is a sheaf:

We define a "reduced" complex:

$$C^n_{\mathrm{red}}(X., F.) := \prod_{\substack{\sigma \in X_n \\ \text{nondegenerate}}} F_\sigma \text{ and } d^n_{\mathrm{red}} := pr_n(F.) \circ d \circ i_n(F.),$$

where $i_n(F.)$ is the inclusion to and $pr_n(F.)$ the projection of $C^n(X., F.)$.

Lemma 5.11. *Let F. be a sheaf on X. . Then*

(i) *$C^*_{\mathrm{red}}(X., F.)$ is a complex and a functor in F. .*

(ii) *$i(F.)$ and $pr(F.)$ are morphisms of complexes depending on F. in a functorial way.*

(iii) *The morphisms of functors* $C^*_{\mathrm{red}}(X., ?) \underset{pr}{\overset{i}{\rightleftarrows}} C^*(X., ?)$ *are inverse to each other up to homotopy.*

(iv) *The extension of $C^*_{\mathrm{red}}(X, ?)$ to $D^+(X.)$ via the maps i and pr is isomorphic to $\overline{\mathbf{R}}\Gamma$.*

Proof. Let $C^n_{\deg}(X., F.) := \ker(pr_n(F.))$. One can see that the map

$$s_{i*} : C^n(X., F.) \longrightarrow C^{n-1}(X., F.), \quad i = 0, \dots, n-1,$$

is an epimorphism and that it factors through the projection $\prod_{\sigma \in s_i^*(X_{n-1})} F_\sigma$, which induces an isomorphism because $s_i^* : X_{n-1} \to s_i^*(X_{n-1}) \subseteq X_n$ is bijective, and $F.$ is a sheaf. Hence we get another description

$$C^n_{\mathrm{red}}(X., F.) = \bigcap_{i=0}^{n-1} \ker(s_{i*}) \subseteq C^n(X., F.).$$

The properties (i) - (iii) are consequences of general facts about (co)simplical objects in abelian categories. (See for instance [6], Section 3, in particular equation (3.21), Satz 3.22. and its proof.) Property (iv) follows from (iii). □

Proposition 5.12. *The geometric realization functor $|?|$ induces an isomorphism of ∂ functors*

$$H^*(X., ?) \simeq H^*(|X.|, |?|)$$

on the category of sheaves.

Proof. We prove in detail the similar statement about the cohomology with com[p] support, Proposition 6.12. The proofs have the same structure, here we mention [o] the main points, using the notations of the proof of 6.12. We consider the diag[r] of excision sequences with respect to the decompositions $X_i = X_{i-1} \sqcup (X_i \setminus X_i$ $i = 1, 2, \dots$, and the usual cohomology. Furthermore,

$$H^k(j_!j^*\mathcal{F}) \simeq \prod_{\substack{\sigma\in X_i \\ \text{nondeg.}}} H^k(j_{\sigma,!}j^*_\sigma\mathcal{F}) \simeq \prod_{\substack{\sigma\in X_i \\ \text{nondeg.}}} H^k([\sigma](\Delta_\sigma), j'_{\sigma,!}j^*_\sigma\mathcal{F}) \simeq \prod_{\substack{\sigma\in X_i \\ \text{nondeg.}}} H^k_c(inn(\Delta_\sigma), j^*_\sigma\mathcal{F}),$$

where $j : X_i \setminus X_{i-1} \to X_i$, $j_\sigma : inn(\Delta_\sigma) \to X_i$ and $j'_\sigma : [\sigma](inn(\Delta_\sigma)) \to [\sigma](\Delta_\sigma)$ the open inclusions.

The analogous statement to Lemma 6.13 is the well known fact that the restric[t]

$$H^n(X) \longrightarrow H^n(X_{n+k}) \longrightarrow H^n(X_{n+1}), \quad \text{for} \quad k \geq 1,$$

are isomorphisms and that

$$H^{n+k}(X_n) = 0 \quad \text{for} \quad k \geq 1.$$

The analogous statement to Lemma 6.14 is proved by the same method.

6. The direct image with compact support

In this section we assume the abelian category $\mathcal{A} = \mathrm{R-Mod}$ to be the catego[ry of] modules over a ring R.

Remember that a simplicial set $X.$ is called locally finite, if for every $\sigma \in X.$ th[e set] of all nondegenerate simplices $\tau \in X.$ for which there exists a map $\sigma \to \tau$ is f[inite]. This amounts to say that $|X.|$ is locally compact.

A morphism $f : X. \to Y.$ between simplicial sets is called locally finite if the simp[licial] set $X. \times_{Y.} \Delta.[\underline{n}]$ is locally finite for every map $\Delta.[\underline{n}] \to Y.$ with arbitrary n. T[his is] equivalent to say that all fibres of $|X.| \to |Y.|$ are locally finite.

One of the conditions of the topological Verdier duality is that the spaces X a[nd Y] are locally compact or at least that $f : X \to Y$ has locally compact fibres. S[o it is] sufficient for our purposes to construct the direct image with compact support for locally finite morphisms f.

Let $f : X. \to Y.$ be a locally finite morphism between simplicial sets and let $F.$ [be a] cohomological system of coefficients on $X.$. We define the direct image with co[mpact] support $f_!F.$ to be the cohomological system of coefficients on $Y.$ given by

$$(6.1) \qquad (f_!F.)_\tau := \bigoplus_{\substack{\sigma\in X. \\ f(\sigma)=\tau}} F_\sigma \subseteq \prod_{\substack{\sigma\in X. \\ f(\sigma)=\tau}} F_\sigma = (f_*F.)_\tau \quad \text{with} \quad \tau \in Y..$$

The structure map for $\tau \to \tau'$ is induced by the corresponding map $(f_* F.)_\tau \to (f_* F.)_{\tau'}$. This make sence because for every morphism $\alpha : \tau \to \tau'$ of simplices in $Y.$ and every $\sigma \in X.$ such that $f(\sigma) = \tau$ the set of $\sigma' \in X.$ with $f(\sigma') = \tau'$ and $\alpha^*(\sigma') = \sigma$ is finite.
This defines an exact functor

$$f_! \; : \; \mathrm{Func}(\mathcal{X}, \mathcal{A}) \longrightarrow \mathrm{Func}(\mathcal{Y}, \mathcal{A}) .$$

To define $f_!$ on the category of sheaves we need the following lemma.

Lemma 6.1. *The adjunction map defines an isomorphism*

$$p_{Y.*} f_! \longrightarrow p_{Y.*} f_! p_X^* p_{X.*} .$$

Proof. Let $F.$ be a cohomological system of coefficients and let $\tau \in Y_n$ be a nondegenerate simplex. By the relation (3.10) we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (p_{Y.*} f_! p_X^* p_{X.*} F.)_\tau & \xrightarrow{ad_Y} & \bigoplus\limits_{\substack{\sigma\in X. \\ f(\sigma)=\tau}} (p_X^* p_{X.*} F.)_\sigma & \xrightarrow{\bigoplus_{i=0}^n \delta_i - \delta_{i+1}} & \bigoplus\limits_{\substack{\sigma'\in X. \\ 0\le i\le n \\ f(\sigma')=s_i^*\tau}} (p_X^* p_{X.*} F.)_{\sigma'} \\
 & & {\scriptstyle ad_X}\downarrow & & {\scriptstyle ad_X}\downarrow & & {\scriptstyle ad_X}\downarrow \\
0 & \longrightarrow & (p_{Y.*} f_! F.)_\tau & \xrightarrow{ad_Y} & \bigoplus\limits_{\substack{\sigma\in X. \\ f(\sigma)=\tau}} F_\sigma & \xrightarrow{\bigoplus_{i=0}^n \delta_i - \delta_{i+1}} & \bigoplus\limits_{\substack{\sigma'\in X. \\ 0\le i\le n \\ f(\sigma')=s_i^*\tau}} F_{\sigma'} \\
 & & & & \downarrow {\scriptstyle \bigoplus_{i=0}^n \delta_i - \delta_{i+1}} & \swarrow {\scriptstyle p} & \\
 & & & & \bigoplus\limits_{\substack{\sigma\in X. \\ 0\le i\le n \\ f(\sigma)=\tau}} F_{s_i^*\sigma} & &
\end{array}$$

with exact rows and columns, where ad_X, resp. ad_Y, denotes the morphims given by adjunction and p is the natural projection. Diagram chase shows that the left downarrow is an isomorphism. □

Now we are able to define $f_!$ on the category of sheaves by

$$f_! \;:=\; p_{Y.*} f_! p_X^* \;\subseteq\; p_{Y.*} f_* p_X^* \;\simeq f_* .$$

Lemma 6.2. *For the composition $X. \xrightarrow{f} Y. \xrightarrow{g} Z.$ we have the isomorphism of functors $g_! \circ f_! \simeq (g \circ f)_!$, induced by the isomorphism $g_* \circ f_* \simeq (g \circ f)_*$.*

Proof. For cohomological systems of coefficients this property can be seen directly from the definition of the direct image with compact support. On the category of

sheaves this follows easily from Lemma 6.1.

Lemma 6.3. *Let $f. : X. \to Y.$ be a morphism of locally finite simplicial sets. T the following diagram is commutative*

$$(6.2) \qquad \begin{array}{ccc}
SH(X.) & \xrightarrow{|?|} & SH(|X.|) \\
\downarrow f_! & & \downarrow |f|_! \\
SH(Y.) & \xrightarrow{|?|} & SH(|Y.|) .
\end{array}$$

Proof. We have proved the analogous assertion for the f_*-functor in Lemma We compare $|f|_!(|F.|)$ and $|f._!(F.)|$ as subobjects of $|f|_*(|F.|) \simeq |f._*(F.)|$.
Let $(\tau, y, \widetilde{V})$ be some compatible data on $Y.$ with $\widetilde{V}$ as in Lemma 3.15 and τ ı degenerate. By Lemma 3.17 we have to compare

$$\begin{array}{l} \{s \in |F.|(f^{-1}\widetilde{V}) \mid \mathrm{supp}(s) \to \widetilde{V} \text{ proper}\} \\ (f_! F.)_\tau \;=\; (p_{Y,*} f_! p_X^* F.)_\tau \end{array} \quad \subseteq \; |F.|(f^{-1}\widetilde{V}) \;=\; \varprojlim_{\tau\backslash \mathrm{Sur}^{-1} f} F. .$$

Denote by $s_{(\sigma, \tau \to f(\sigma))}$ the image of s under the map from the projective limit to component F_σ given by the object $(\sigma, \tau \to f(\sigma)) \in \tau\backslash \mathrm{Sur}^{-1} f$. By Lemma 3.17 we that the map $\mathrm{supp}(s) \to V$ is proper, iff for all $\tau \xrightarrow{\alpha} \tau'$ in $\mathrm{Sur}^{-1}\mathcal{Y}$ the set of σ' ı $f(\sigma') = \tau'$ and $s_{(\sigma', \alpha)} \neq 0$ is finite. So every s gives an element

$$s_\alpha \in \bigoplus_{\substack{\sigma' \in X. \\ f(\sigma') = \tau'}} F_{\sigma'} \;=\; (f_! p_X^* F.)_{\tau'}$$

for every $\tau \xrightarrow{\alpha} \tau'$. It can be seen easily that this set of elements for the differeı determines an element in $(p_{Y,*} f_! p_X^* F.)_\tau$ and every such element gives an s.

6.1. Global sections with compact support

Let $F.$ be a cohomological system of coefficients or a sheaf on a simplicial set We define the global sections with compact support of $F.$ by

$$\Gamma_c(X., F.) \;:=\; \varprojlim_{x \in \mathcal{X}} F. \cap \varinjlim_{T. \subseteq X.} \prod_{\sigma \in T.} F_\sigma \subseteq \prod_{\sigma \in X.} F_\sigma ,$$

where the direct limit is taken over the finite simplicial subsets $T.$ of $X.$.
It is easily seen that $\Gamma_c(X., ?)$ is left exact.

Example 6.4. For an arbitrary locally finite $X.$ consider the map $f. : X. \to \Delta$ to the simplicial point. Then we have, for an arbitrary sheaf $F.$, an isomorphism

$$f_! F. \;\simeq\; \Gamma_c(X., F.)$$

with respect to the identification $SH(\Delta.[\underline{0}]) \simeq \mathcal{A}$. This follows immediately fron analogous statement for global sections Example 5.4 (ii).

Lemma 6.5. *For a cohomological system of coefficients $F.$ on a simplicial set $X.$ the adjunction map $p_X^* p_{X,*} F. \to F.$ defines an isomorphism*

$$\Gamma_c(X., p_X^* p_{X,*} F.) \xrightarrow{\sim} \Gamma_c(X., F.)\,.$$

Proof. The proof follows from the definition and Lemma 5.5 by some standard categorial considerations. □

Lemma 6.6. *Let $F.$ be a cohomological system of coefficients on the simplicial set $X.$. Then the isomorphism of Lemma 5.6 induces an isomorphism of the subobjects*

$$\Gamma_c(X., F.) \simeq \Gamma_c(|X.|, |F.|)\,.$$

Proof. By Lemma 6.5 and Lemma 3.19 we can restrict to the case that $F.$ is a sheaf. The assertion now follows from Proposition 3.17 using the fact that a subset $K \subseteq |X.|$ is compact iff it is closed and a subset of $|T.|$ for a finite simplicial subset $T. \subseteq X.$. □

6.2. Cohomology with compact support

We define the cohomology with compact support of a cohomological system of coefficients $F.$ on a locally finite simplicial set $X.$ similarly to ordinary cohomology defined in Subsection 5.2.:

If $f : X. \to \Delta.[\underline{0}]$ is the map to the point, we consider the cosimplicial abelian group $C_c^\cdot(X., F.) := f_! F.$ given by

$$\underline{n} \longmapsto C_c^n(X., F.) := (f_! F.)_{\underline{n}} = \bigoplus_{\sigma \in X_n} F_\sigma\,. \tag{6.3}$$

By $C_c^\cdot(X., F.)$ we also denote the corresponding complex in $\mathcal{A}$, the cohomology of which we call cohomology with compact support $H_c^\cdot(X., F.)$.

Obviously the correspondence $F. \mapsto H_c^\cdot(X., F.)$ defines a ∂-functor on the category of cohomological systems of coefficients (resp. the category of sheaves).

The exact functor

$$C_c^\cdot(X., ?) : \mathrm{Func}(\mathcal{X}, \mathcal{A}) \longrightarrow K^{\geq 0}(\mathcal{A})$$

extends to a functor

$$\begin{array}{rcl} C^\cdot(X., ?) : D^+(\mathrm{Func}(\mathcal{X}, \mathcal{A})) & \longrightarrow & D^+(\mathcal{A}) \\ D^+(X.) & \longrightarrow & D^+(\mathcal{A}) \end{array}$$

via double complexes and the direct sum total complex.

Definition 6.7. We denote the extension of $C^\cdot(X, ?)$ from the category of sheaves to the derived category using the total sum complex of a double complex by

$$\overline{\mathbf{R}}\Gamma_c : D^+(X.) \longrightarrow D^+(\mathcal{A})\,.$$

Proposition 6.8. *Let $X.$ be a locally finite simplicial set and assume that the boundary of a nondegenerate simplex is nondegenerate. Then the natural morphism*

$$\mathbf{R}\Gamma_c \longrightarrow \overline{\mathbf{R}}\Gamma_c$$

is an isomorphism on $D^+(X.)$.

Proof. We have $\Gamma_c(X., \ ) \simeq H_c^0(X., \ )$ by Example 6.4. It remains to construct a sufficiently large class of acyclic objects for the cohomology with compact support. Let $F.$ be a sheaf on $X.$. Let $I(F.)$ be the sheaf on $X.$ given by the following equation (we use Corollary 3.8)

$$I(F.)_\sigma := \prod_{\sigma \xrightarrow{\alpha} \tau \in \mathcal{X}_0} F_\tau = \bigoplus_{\sigma \xrightarrow{\alpha} \tau \in \mathcal{X}_0} F_\tau$$

for every nondegenerate $\sigma \in X.$, the structure map for $\sigma \to \sigma'$ given in the obvious way. We have a monomorphism $F. \to I(F.)$ given by the maps $F_\sigma \xrightarrow{\alpha} F_\tau$. Let us verify that $I(F.)$ is acyclic for the cohomology with compact support. Obviously

$$I(F.) = \bigoplus_{\tau \in \mathcal{X}_0} I(\tau)\,, \quad \text{with} \quad I(\tau)_\sigma = \prod_{\sigma \xrightarrow{\alpha} \tau \in \mathcal{X}_0} F_\tau\,.$$

But $C_c^\cdot(X., I(\tau)) \simeq C^\cdot(\Delta.[\tau], \underline{F_\tau})$ and the assertion follows easily. □

Remark 6.9. The boundary condition in Proposition 6.8 is necessary as the Example 5.8 shows.

Furthermore, we want to give a "reduced" definition of the cohomology with compact support in the case that $F.$ is a sheaf

$$C_{c,\mathrm{red}}^n(X., F.) := \bigoplus_{\substack{\sigma \in X_n \\ \text{nondegenerate}}} F_\sigma \quad \text{and} \quad d_{\mathrm{red}}^n := pr_n(F.) \circ d \circ i_n(F.)$$

where $i_n(F.)$ is the canonical inclusion and $pr_n(F.)$ the canonical projection from $C_c^n(X., F.)$.

Lemma 6.10. *Let $F.$ be a cohomological system of coefficients on $X.$. Then*

(i) *$C_{c,\mathrm{red}}^\cdot(X., F.)$ is a complex and a functor in $F.$.*

(ii) *$i(F.)$ and $pr(F.)$ are morphisms of complexes depending of $F.$ in a functorial way.*

(iii) *The functormorphisms $C_{n,\mathrm{red}}^\cdot(X., ?) \underset{pr}{\overset{i}{\rightleftarrows}} C_c^\cdot(X., ?)$ are homotopically inverse.*

(iv) *The extension of $C_c^\cdot(X., ?)$ to $D^+(X.)$ via the maps i and pr is isomorphic to $\overline{\mathbf{R}}\Gamma_c$.*

Proof. The proof is the same as for Lemma 5.11.

Lemma 6.11. *Let $X.$ be a locally finite simplicial set of finite dimension. Then the natural extension of $C_c^\cdot(X., ?)$ to $K^-(X.)$ factors to the derived category and gives* *functor $D^-(X.) \to D^-(\mathcal{A})$ which is isomorphic (via the natural maps) to the continuation of $C^{\cdot}_{c,\mathrm{red}}(X.,?)$.*

Proof. Obvious. □

Proposition 6.12. *Let $X.$ be a locally finite simplicial set. On the category of sheaves $SH(X.)$ there exists an isomorphism of ∂-functors*

$$H_c^\bullet(X., F.) \simeq H_c^\bullet(|X.|, |F.|).$$

Proof. The proof will be similar to the constructions in ([15], p. 1, Ch. 3). Let $X := |X.|$ and $X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots \subseteq X$ be the filtration by the n-skeletons $X^n := sk_n X$. Let $F.$ be a sheaf on $X.$. Let $U^n := X \setminus X^n$ be the open complement and $\mathcal{F} := |F.|$. The idea is to consider the excision-sequences for the following sets:

$$X = X^0 \sqcup U^0, \quad U^0 = (X^1 \setminus X^0) \sqcup U^1, \quad \ldots \quad U^{n-1} = (X^n \setminus X^{n-1}) \sqcup U^n, \quad \ldots .$$

We get the following commutative diagram with exact columns (we omit the sheaf $\mathcal{F}$ as sheaf of coefficients):

$$\tag{6.4}\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & H_c^n(X^n \setminus X^{n-1}) & & H_c^n(X^{n+1} \setminus X^n) & & H_c^n(X^{n+2} \setminus X^{n+1}) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \leftarrow & H_c^{n+1}(U^n) & \leftarrow & H_c^{n+1}(U^{n+1}) & \leftarrow & H_c^{n+1}(U^{n+2}) & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \leftarrow & H_c^{n+1}(U^{n-1}) & \leftarrow & H_c^{n+1}(U^n) & \leftarrow & H_c^{n+1}(U^{n+1}) & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & H_c^{n+1}(X^n \setminus X^{n-1}) & & H_c^{n+1}(X^{n+1} \setminus X^n) & & H_c^{n+1}(X^{n+2} \setminus X^{n+1}) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \leftarrow & H_c^{n+2}(U^n) & \leftarrow & H_c^{n+2}(U^{n+1}) & \leftarrow & H_c^{n+2}(U^{n+2}) & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \leftarrow & H_c^{n+2}(U^{n-1}) & \leftarrow & H_c^{n+2}(U^n) & \leftarrow & H_c^{n+2}(U^{n+1}) & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & H_c^{n+2}(X^n \setminus X^{n-1}) & & H_c^{n+2}(X^{n+1} \setminus X^n) & & H_c^{n+2}(X^{n+2} \setminus X^{n+1}) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

On the other hand we get

$$X^n \setminus X^{n-1} = \coprod_{\substack{\sigma \in X_n \\ \text{nondegenerate}}} [\sigma](inn(\Delta_\sigma)).$$

Each summand is homeomorphic to the n-dimensional euclidean space $\mathbb{R}^n$. The sheaf $\mathcal{F} = |F.|$ is constant on $inn(\Delta_\sigma)$ with stalk F_σ by Lemma 3.17. Hence we get

$$\tag{6.5}\begin{aligned} H_c^i(X^n \setminus X^{n-1}, \mathcal{F}) &\simeq \bigoplus_{\substack{\sigma \in X_n \\ \text{nondegenerate}}} H^i(inn(\Delta_\sigma), F_\sigma) \\ &\simeq \begin{cases} 0 & \text{for } i \neq n, \\ \bigoplus_{\substack{\sigma \in X_n \\ \text{nondegenerate}}} F_\sigma & \text{for } i = n, \end{cases} \end{aligned}$$

(see for example [12], III.8. in the case $F_\sigma = R$ is a ring, the same proof can be extended without changes to the case of a module over a ring.)

Writing the zeros to the diagram (6.4) we have the following sequences of isomorphisms with arrows "extension by zero":

$$\tag{6.6} \ldots \xrightarrow{\sim} H_c^n(U^{n+k}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} H_c^n(U^{n+1}) \xrightarrow{\sim} H_c^n(U^n)$$

and

$$\tag{6.7} H_c^n(U^{n-2}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} H_c^n(U^0) \xrightarrow{\sim} H_c^n(X).$$

Further we need:

Lemma 6.13. *Let X be a locally compact CW-complex, X^k the k-skeleton of X and U^k the open complement, $\mathcal{F}$ a sheaf on X constant on the inner of each cell. Then $\varprojlim_{k \geq n} H_c^n(U^k, \mathcal{F}) = 0$, where the $\varprojlim$ is taken by the extension by zero maps*

$$H_c^n(U^{k+1}, \mathcal{F}) = H_c^n(U^k, u_!^k \mathcal{F}) \longrightarrow H_c^n(U^k, \mathcal{F}),$$

where $u^k : U^{k+1} \to U^k$ denotes the inclusion.

Proof. Let I_k be the set of k-cells of X and B_i^k with $i \in I_k$ the inner of this k-cell. As topological space we have $B_i^k \simeq \mathbb{R}^k$. Let

$$\alpha = (\ldots, \alpha_k, \ldots, \alpha_n) \in \varprojlim_{k \geq n} H_c^n(U^k, \mathcal{F}) \text{ with } \alpha_k \in H_c^n(U^k, \mathcal{F}) \text{ and } \alpha \neq 0.$$

We fix some k with $\alpha_k \neq 0$. We can write

$$U^k = \coprod_{l \geq k, i \in I_l} B_i^l.$$

Let us consider the following set

$$\mathcal{M} = \left\{ M \subseteq U^k, \text{ with } \begin{array}{ll} 1. & M \cap B_i^l = \emptyset \text{ or } B_i^l \text{ for all } l \geq k \text{ and } i \in I_l \\ 2. & 0 \neq \alpha_k \in H_c^n(M, \mathcal{F}) \end{array} \right\}.$$

$\mathcal{M}$ is not empty because $U^k \in \mathcal{M}$. If $Z_i \in \mathcal{M}$ is a downward directed family and $Z := \bigcap Z_i$ then Z satisfies property 1. and from [12], 3.9.3 we get

$$H^m(Z, \mathcal{F}) = \varinjlim H^m(Z_i, \mathcal{F}).$$

Hence Z has property 2. and so $Z \in \mathcal{M}$. By Zorn's lemma $\mathcal{M}$ has a minimal element Z.

Further let $x \in \overline{Z} \cap X^0 \subseteq X$ and $B_i^l \subseteq Z$ be a cell of maximal dimension with $x \in \partial B_i^l \subseteq X$. Such a B_i^l exists because X is locally compact. Then $B_i^l \subseteq Z$ is an open subset and we consider a part of the excision sequence:

$$H_c^n(B_i^l, \mathcal{F}) \longrightarrow H_c^n(Z, \mathcal{F}) \longrightarrow H_c^n(Z \setminus B_i^l, \mathcal{F}).$$

The set $Z \setminus B_i^l$ satisfies property 2. and hence $0 = \alpha_k \in H_c^n(Z \setminus B_i^l \mathcal{F})$ by the minimality of Z. Thus $\alpha \in im(H_c^n(B_i^l, \mathcal{F}) \to H_c^n(Z, \mathcal{F}))$. But $\mathcal{F}$ is constant on B_i^l and therefore we get $l = n$ since $\alpha \neq 0$ in $H_c^n(Z, \mathcal{F})$.

This construction we can repeat for all $x \in \overline{Z} \cap X^0 \subseteq X$. So Z is the union of cells of dimension less than or equal to n. This means $Z \subseteq X^n \cap U^k = U^k \setminus U^n$. Hence $n \geq k$ and we get a contradiction between the excision sequence

$$\begin{array}{ccccc} H_c^n(U^n, \mathcal{F}) & \longrightarrow & H_c^n(U^k, \mathcal{F}) & \longrightarrow & H_c^n(U^k \setminus U^n, \mathcal{F}) \\ \alpha_n & \longmapsto & \alpha_k & \longmapsto & 0 \end{array}$$

and $0 \neq \alpha \in H_c^n(Z, \mathcal{F})$. Therefore $\alpha_k = 0$ for all $k \geq n$, hence $\alpha = 0$. □

Let us continue the proof of the proposition. By equation (6.6) and this lemma we get $H_c^{n+k}(U^n, \mathcal{F}) = 0$ for all n and k.

Combining our results diagram (6.4) yields the following diagram with exact columns:

$$\begin{array}{cccc}
\bigoplus\limits_{\substack{\sigma \in X_n \\ \text{nondeg.}}} F_\sigma & & & \\
\downarrow & & & \\
H_c^{n+1}(U^n) & 0 & & \\
\downarrow \quad \backslash\backslash & \downarrow & & \\
H_c^{n+1}(X) & H_c^{n+1}(U^n) & & \\
\downarrow & \downarrow & & \\
0 & \bigoplus\limits_{\substack{\sigma \in X_{n+1} \\ \text{nondeg.}}} F_\sigma & & \\
& \downarrow & & \\
& H_c^{n+2}(U^{n+1}) & 0 & \\
& \downarrow \quad \backslash\backslash & \downarrow & \\
& H_c^{n+2}(X) & H_c^{n+2}(U^{n+1}) & \\
& \downarrow & \downarrow & \\
& 0 & \bigoplus\limits_{\substack{\sigma \in X_{n+2} \\ \text{nondeg.}}} F_\sigma. &
\end{array}$$

From this diagram we get

$$\begin{aligned} H_c^{n+1}(U^n) &\simeq \ker\left(\bigoplus_{\substack{\sigma \in X_{n+1} \\ \text{nondeg.}}} F_\sigma \longrightarrow H_c^{n+2}(U^{n+1}) \right) \\ &\simeq \ker\left(\bigoplus_{\substack{\sigma \in X_{n+1} \\ \text{nondeg.}}} F_\sigma \longrightarrow \bigoplus_{\substack{\sigma \in X_{n+2} \\ \text{nondeg.}}} F_\sigma \right) \end{aligned}$$

and

$$H_c^{n+1}(X) \simeq \operatorname{coker}\left(\bigoplus_{\substack{\sigma \in X_n \\ \text{nondeg.}}} F_\sigma \longrightarrow H_c^{n+1}(U^n) \right).$$

This combines to

$$H_c^{n+1}(X, \mathcal{F}) \simeq H^{n+1}\left(\cdots \xrightarrow{\tilde{d}_{\text{red}}^n} \bigoplus_{\substack{\sigma \in X_n \\ \text{nondeg.}}} F_\sigma \xrightarrow{\tilde{d}_{\text{red}}^{n+1}} \bigoplus_{\substack{\sigma \in X_{n+1} \\ \text{nondeg.}}} F_\sigma \xrightarrow{\tilde{d}_{\text{red}}^{n+2}} \bigoplus_{\substack{\sigma \in X_{n+2} \\ \text{nondeg.}}} F_\sigma \xrightarrow{\tilde{d}_{\text{red}}^{n+3}} \cdots \right)$$

The proposition follows from Lemma 6.10 if we show that $\tilde{d}_{\text{red}}^n = d_{\text{red}}^n$. This is t[illegible] assertion of the following lemma:

Lemma 6.14. *Let $F.$ be a sheaf on the simplicial set $X.$. Then the following diagr[illegible] is commutative:*

$$\begin{array}{ccccc}
\bigoplus\limits_{\substack{\tau \in X_n \\ \text{nondeg.}}} F_\tau & & \xrightarrow{d_{\text{red}}^{n+1}} & & \bigoplus\limits_{\substack{\sigma \in X_{n+1} \\ \text{nondeg.}}} F_\sigma \\
\downarrow \wr & & & & \downarrow \wr \\
H_c^n(X^n \setminus X^{n-1}, |F.|) & \xrightarrow{\partial} & H_c^{n+1}(U^n, |F.|) & \longrightarrow & H_c^{n+1}(X^{n+1} \setminus X^n, |F.|).
\end{array}$$

Proof. Let $\sigma \in X_{n+1}$ be a nondegenerate simplex, $f. : \Delta.[\underline{n+1}] \to X.$ the ass[illegible] ated morphism and $f : \Delta_{n+1} \to |X.| = X$ its geometric realization. Since f is pro[illegible] we can apply the functoriality of the excision sequences for f^* (see [12], Ch. 3,7 and get the following commutative diagram:

$$\begin{array}{ccccc}
\bigoplus\limits_{\substack{\tau = \partial_i^* \sigma \in X_n \\ \text{nondeg.}}} F_\tau & & & & F_\sigma \\
\| & & & & \| \\
H_c^n(\partial\Delta_{n+1} \setminus f^{-1}(X^{n-1})) & \xrightarrow{\partial} & H_c^{n+1}(\Delta_{n+1} \setminus \partial\Delta_{n+1}) & = & H_c^{n+1}(\Delta_{n+1} \setminus \partial\Delta_{n+1} \\
\uparrow f^* & & \uparrow f^* & & \uparrow f^* \\
H_c^n(X^n \setminus X^{n-1}) & \xrightarrow{\partial} & H_c^{n+1}(U^n) & \longrightarrow & H_c^{n+1}(X^{n+1} \setminus X^n).
\end{array}$$

For simplicity we do not write $|F.|$ and $f^*|F.| \simeq |f.^*F.|$ (by Lemma 4.1) as coefficients. The left and right f^* are the projections of direct sums. So all we have to prove is that the induced horizontal map is d_{red}^{n+1} . On the standard simplicial set $\Delta.[\underline{k}]$ we have for every sheaf $F.$ a morphism into the constant sheaf $\underline{F_{id}}$ with fiber F_{id} ($id \in \Delta_k[\underline{k}]$ denotes the identical map), induced by

$$F_{\alpha:\underline{i}\to\underline{k}} \xrightarrow{\alpha^*} F_{id},$$

(see also Corollary 7.7). Because of the functoriality of our maps in $F.$ we can restrict to the case that the sheaf of coefficients is constant. But in this case the topological cohomology with compact support can be defined for instance in terms of the Alexander-Spanier cochains and the lemma follows from [15], p. 1, Ch. 5, in particular [15], Lemma 5.3 and Theorem 5.11 with $a_{i,j} = (-1)^k$ if $e_i = \partial_k^* e_j$. □

7. Tensor product and inner homomorphisms

In this section we assume our abelian category of coefficients $\mathcal{A}$ to be the category of modules over a noetherian commutative ring R. Then all the abelian groups of morphisms of modules, systems of coefficients and functors have a R-mod structure. We will write $\mathcal{H}om$ instead of Hom if we consider this additional structure.

7.1. The construction of the tensor product and the inner Hom-functor

We want to continue the inner Hom structure $\underline{\text{Hom}}(M,N) := \mathcal{H}om(M,N)$ on R-Mod to the category of cohomological systems of coefficients and sheaves over a simplicial set $X.$. Remember that both categories are functor categories $\text{Func}(\mathcal{X}, \text{R-Mod})$ and $\text{Func}(\text{Sur}^{-1}\mathcal{X}, \text{R-Mod})$. The main constructions can be done for arbitrary categories of functors $\text{Func}(\mathcal{T}, \text{R-Mod})$, but we restrict ourself to these cases and write $\check{\mathcal{X}}$ for $\mathcal{X}$ and $\text{Sur}^{-1}\mathcal{X}$, respectively.

The "inner" tensor product on R-Mod can be extended to the cohomological systems of coefficients (sheaves) by

$$(F.\otimes G.)_\sigma := F_\sigma \otimes G_\sigma \tag{7.1}$$

with the obvious induced morphisms for $\alpha : \sigma \to \sigma'$. It is easily seen that this construction is functorial in $F.$ and $G.$.

Remark 7.1. If $F.$ and $G.$ are sheaves, then $F.\otimes G.$ (as cohomological systems of coefficients) is a sheaf too.

Definition 7.2. Let $F.$ and $G.$ be two cohomological systems of coefficients (sheaves) on $X.$. We define a third cohomological system (sheaf), *the inner homomorphisms between $F.$ and $G.$* as follows:

$$\begin{aligned}\underline{\text{Hom}}(F.,G.)_\sigma &:= \mathcal{H}om_{\text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A})}(F.\circ pr_\sigma, G.\circ pr_\sigma)\\ &= \mathcal{H}om_{\text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A})}(pr_\sigma^* F., pr_\sigma^* G.)\end{aligned} \tag{7.2}$$

where pr_σ is the natural map from $\sigma\backslash\check{\mathcal{X}}$ to $\check{\mathcal{X}}$. For a morphism $\alpha : \sigma \to \sigma'$ we get a commutative diagram

$$\begin{array}{ccc} \sigma\backslash\check{\mathcal{X}} & & \\ \uparrow & \searrow pr_\sigma & \\ \alpha^* & & \check{\mathcal{X}} \\ | & \nearrow pr_{\sigma'} & \\ \sigma'\backslash\check{\mathcal{X}} & & \end{array} \quad \text{which induces} \quad \begin{array}{ccc} \text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A}) & & \\ | & \nwarrow pr_\sigma^* & \\ (\alpha^*)^* & & \text{Func}(\check{\mathcal{X}},\mathcal{A}) \\ \downarrow & \swarrow pr_{\sigma'}^* & \\ \text{Func}(\sigma'\backslash\check{\mathcal{X}},\mathcal{A}) & & \end{array} \tag{7.3}$$

Hence we get a morphism of R-modules

$$\mathcal{H}om_{\text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A})}(pr_\sigma^* F., pr_\sigma^* G.) \xrightarrow{\alpha_*} \mathcal{H}om_{\text{Func}(\sigma'\backslash\check{\mathcal{X}},\mathcal{A})}(pr_{\sigma'}^* F., pr_{\sigma'}^* G.). \tag{7.4}$$

One can see that $(\alpha\circ\beta)_* = \alpha_*\circ\beta_*$ for a morphism $\beta : \sigma' \to \sigma''$. Hence $\underline{\text{Hom}}(F.,G.)$ is defined by a cohomological system of coefficients (sheaf).

Proposition 7.3. *Suppose $F.$, $G.$ and $H.$ are cohomological systems of coefficients (sheaves). There exist isomorphisms functorial in all arguments*

$$\begin{aligned}\text{Hom}(F.\otimes G., H.) &\simeq \text{Hom}(F., \underline{\text{Hom}}(G., H.)),\\ \underline{\text{Hom}}(F.\otimes G., H.) &\simeq \underline{\text{Hom}}(F., \underline{\text{Hom}}(G., H.)).\end{aligned}$$

Proof. Let $\varphi : F.\otimes G. \to H.$ be a morphism. This is nothing but a family of maps of R-modules

$$\varphi(\sigma) : F_\sigma \otimes G_\sigma \longrightarrow H_\sigma$$

for every $\sigma \in \check{\mathcal{X}}$ making the diagram

$$\begin{array}{ccc} F_\sigma\otimes G_\sigma & \xrightarrow{\varphi(\sigma)} & H_\sigma \\ \downarrow & & \downarrow \\ F_{\sigma'}\otimes G_{\sigma'} & \xrightarrow{\varphi(\sigma')} & H_{\sigma'} \end{array} \tag{7.5}$$

commutative for every $\alpha : \sigma \to \sigma'$.

On the other hand let $\psi : F. \to \underline{\text{Hom}}(G., H.)$ be a morphism. This is a family of maps of R-modules

$$\psi(\sigma) : F_\sigma \longrightarrow \mathcal{H}om_{\text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A})}(G.\circ pr_\sigma, H.\circ pr_\sigma)$$

for every $\sigma \in \check{\mathcal{X}}$ making the diagram

$$\begin{array}{ccc} F_\sigma & \xrightarrow{\psi(\sigma)} & \mathcal{H}om_{\text{Func}(\sigma\backslash\check{\mathcal{X}},\mathcal{A})}(G.\circ pr_\sigma, H.\circ pr_\sigma) \\ \downarrow & & \downarrow \\ F_{\sigma'} & \xrightarrow{\psi(\sigma')} & \mathcal{H}om_{\text{Func}(\sigma'\backslash\check{\mathcal{X}},\mathcal{A})}(G.\circ pr_{\sigma'}, H.\circ pr_{\sigma'}) \end{array}$$

commutative for every $\alpha : \sigma \to \sigma'$. By (7.4) this condition is equivalent to the following two facts:

a) For every $\alpha : \sigma \to \sigma'$ and $(\sigma' \xrightarrow{\beta} \tau) \in \sigma' \backslash \check{\mathcal{X}}$ the diagram

$$\begin{array}{ccc} F_\sigma \otimes G_\tau & \xrightarrow{\psi(\sigma)(\sigma \xrightarrow{\alpha} \sigma' \xrightarrow{\beta} \tau)} & H_\tau \\ \downarrow & & \| \\ F_{\sigma'} \otimes G_\tau & \xrightarrow{\psi(\sigma')(\sigma' \xrightarrow{\beta} \tau)} & H_\tau \end{array} \tag{7.6}$$

is commutative.

b) $\psi(\sigma)$ is really a morphism into the homorphisms in the functor category, which signifies that for all commutative diagrams

$$\begin{array}{ccc} \sigma & \xrightarrow{\beta} & \tau \\ \| & & \downarrow \gamma \\ \sigma & \xrightarrow{\beta'} & \tau' \end{array} \quad \text{the diagram} \quad \begin{array}{ccc} F_\sigma \otimes G_\tau & \xrightarrow{\psi(\sigma)(\beta)} & H_\tau \\ id \otimes \gamma_* \downarrow & & \downarrow \gamma_* \\ F_\sigma \otimes G_{\tau'} & \xrightarrow{\psi(\sigma)(\beta')} & H_{\tau'} \end{array} \tag{7.7}$$

is commutative.

For a given φ we define for all $\sigma \xrightarrow{\beta} \tau$

$$\psi(\sigma)(\beta) := \varphi(\tau) \circ (\beta_* \otimes id) : F_\sigma \otimes G_\tau \longrightarrow H_\tau . \tag{7.8}$$

Then the property (7.6) follows trivially by construction and (7.7) follows easily from (7.5). The corresponding $\varphi \mapsto \psi$ is linear.

In the other direction let ψ be given. We define φ by the equation

$$\varphi(\sigma) := \psi(\sigma)(id) : F_\sigma \otimes G_\sigma \longrightarrow H_\sigma . \tag{7.9}$$

The condition (7.5) can be derived from (7.6) with $\beta = id$ and (7.7) with $\beta = id, \beta' = \gamma = \alpha$. The map $\psi \mapsto \varphi$ is linear too.

The correspondences (7.8) and (7.9) are inverse to each other (to see this use condition (7.6) in one direction) and define the asserted first isomorphism of the proposition. The second isomorphism follows from the first and from Yoneda lemma by applying $\mathrm{Hom}(E., ?)$ to the left and the right-hand side. □

Corollary 7.4. *Let $F.$ and $G.$ be sheaves on $X.$. Then*

$$\underline{\mathrm{Hom}}(F., G.) \simeq p_{\check{X}.*} \underline{\mathrm{Hom}}\,(p^*_{\check{X}} F., p^*_{\check{X}} G.) .$$

Proof. Standard abstract nonsense using Proposition 7.3, both versions, and Remark 7.1. □

Using the adjunction morphism on every component one easily gets a generalization of Proposition 7.3 to complexes:

Corollary 7.5. *Let $F^{\cdot}_{\cdot}, G^{\cdot}_{\cdot}, H^{\cdot}_{\cdot}$ be complexes of cohomological systems of coefficients (sheaves). Then there exists a canonical isomorphism*

$$\mathrm{Hom}(F^{\cdot}_{\cdot} \otimes G^{\cdot}_{\cdot}, H^{\cdot}_{\cdot}) \simeq \mathrm{Hom}(F^{\cdot}_{\cdot}, \underline{\mathrm{Hom}}^{\cdot}(G^{\cdot}_{\cdot}, H^{\cdot}_{\cdot})) .$$

This isomorphism is compatible with the homotopy relation.

By adjunction we associate to

$$\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}_{\cdot}, G^{\cdot}_{\cdot}) \xrightarrow{id} \underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}_{\cdot}, G^{\cdot}_{\cdot})$$

a canonical morphism, the value map

$$\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}_{\cdot}, G^{\cdot}_{\cdot}) \otimes F^{\cdot}_{\cdot} \longrightarrow G^{\cdot}_{\cdot} . \tag{7.10}$$

Obviously this is a combination of the value maps on the components. On a component this map can be given more explicitly.

Using the proof of Proposition 7.3, in particular equation 7.8, the following lemma is easily seen:

Lemma 7.6. *The value map*

$$F. \otimes \underline{\mathrm{Hom}}(F., G.) \longrightarrow G.$$

is given on every σ by the diagram

$$\begin{array}{ccc} F_\sigma \otimes \underline{\mathrm{Hom}}(F., G.)_\sigma & \longrightarrow & G_\sigma \\ \downarrow & & \uparrow \text{can.} \\ F_\sigma \otimes \mathcal{H}\mathrm{om}_{\mathrm{Func}(\sigma \backslash \check{\mathcal{X}}, \mathcal{A})}(F. \circ pr_\sigma, G. \circ pr_\sigma) & \xrightarrow{\text{at } \sigma \xrightarrow{id} \sigma \in \sigma \backslash \check{\mathcal{X}}} & F_\sigma \otimes \mathcal{H}\mathrm{om}(F_\sigma, G_\sigma) . \end{array}$$

Furthermore, we will give an important example where it is possible to calculate the inner Hom explicitly:

Corollary 7.7. *Let $X. = \Delta.[\underline{n}]$ be a standard simplex and M be some R-module. Let $F.$ be an arbitrary cohomological system of coefficients (sheaf) on $\Delta.[\underline{n}]$ and $\underline{M}$ be the constant sheaf with stalk M. Then there exists a functorial isomorphism*

$$\underline{\mathrm{Hom}}(F., \underline{M}) \simeq \underline{\mathcal{H}\mathrm{om}(F_{id}, M)} ,$$

where $id \in \Delta_n[\underline{n}]$ is the identical map.

Proof. For an arbitrary sheaf $G.$ the canonical map

$$\begin{array}{ccc} Hom(G.\underline{M}) & \longrightarrow & Hom(G_{id}, M) \\ \varphi. & \mapsto & \varphi_{id} , \end{array}$$

is an isomorphism because id is a final object in the category associated to $\Delta.[\underline{n}]$. Now let $H.$ be an arbitrary cohomological system of coefficients (sheaf). We get

$$\begin{aligned} \mathrm{Hom}(H. \underline{\mathrm{Hom}}(F., \underline{M})) &\simeq \mathrm{Hom}(H. \otimes F., \underline{M}) \\ &\simeq \mathrm{Hom}((H. \otimes F.)_{id}, M) \\ &\simeq \mathrm{Hom}(H_{id}, \mathcal{H}\mathrm{om}(F_{id}, M)) \\ &\simeq \mathrm{Hom}(H., \underline{\mathcal{H}\mathrm{om}(F_{id}, M)}) \end{aligned}$$

The corollary follows from Yoneda lemma. □

Corollary 7.8. *Let $F.$ and $G.$ be cohomological systems of coefficients (sheaves) on a simplicial set $X.$. Then the map*

$$(F. \overset{\varphi}{\to} G.) \longmapsto (\dots, pr_\sigma^*(\varphi), \dots) \in \prod_{\sigma \in \mathcal{X}} \underline{\mathrm{Hom}}(F., G.)_\sigma$$

defines a functorial isomorphism $\mathrm{Hom}(F., G.) \simeq \Gamma(X., \underline{\mathrm{Hom}}(F., G.))$.

Proof. Let $\overline{F}. := \underline{R}.$ be the constant cohomological system (sheaf) of coefficients with stalk R and $\overline{G}. := F.$, $\overline{H}. := G.$. Then by Proposition 7.3 applied to $\overline{F}.$, $\overline{G}.$, $\overline{H}.$ and by the universal property of $\varprojlim$ we get

$$\begin{aligned}
\mathrm{Hom}(F., G.) &\simeq \mathrm{Hom}(\underline{R}. \otimes F., G.)\\
&\simeq \mathrm{Hom}(\underline{R}, \underline{\mathrm{Hom}}(F., G.))\\
(&\simeq \mathrm{Hom}(p_X^* \underline{R}, p_X^* \underline{\mathrm{Hom}}(F, G.)) \text{ in the case of sheaves})\\
&\simeq \varprojlim_{\mathcal{X}} \underline{\mathrm{Hom}}(F., G.)\\
&\simeq \Gamma(X., \underline{\mathrm{Hom}}(F., G.)) .
\end{aligned}$$

This construction is functorial and with the help of the calculations in the proof of Proposition 7.3 it is easily seen that this is the same morphism as in the assertion. □

Corollary 7.9. *Let $f : Y. \to X.$ be a morphism of simplicial sets. For cohomological systems (sheaves) $F.$, $G.$ on $X.$ and $H.$ on $Y.$ there exist functorial isomorphisms*

$$\begin{aligned}
f^*(F. \otimes G.) &\simeq f^*F. \otimes f^*G.,\\
\underline{\mathrm{Hom}}(G., f_*H.) &\simeq f_* \underline{\mathrm{Hom}}(f^*G., H.) .
\end{aligned}$$

Proof. The first isomorphism is obvious. To get the second we apply $\mathrm{Hom}(F., ?)$ to both sides and use Proposition 7.3, Lemma 5.1 and the first isomorphism:

$$\begin{aligned}
\mathrm{Hom}(F., \underline{\mathrm{Hom}}(G., f_*H.)) &\simeq \mathrm{Hom}(F. \otimes G., f_*H.)\\
&\simeq \mathrm{Hom}(f^*(F. \otimes G.), H.)\\
&\simeq \mathrm{Hom}(f^*F., \underline{\mathrm{Hom}}(f^*G., H.))\\
&\simeq \mathrm{Hom}(F., f_* \underline{\mathrm{Hom}}(f^*G., H.)) .
\end{aligned}$$ □

Lemma 7.10. *Let $F.$, $G.$ be sheaves on $X.$. Then there exist functorial isomorphisms*

$$\begin{aligned}
|F. \otimes G.| &\simeq |F.| \otimes |G.|,\\
|\underline{\mathrm{Hom}}(F., G.)| &\simeq \underline{\mathrm{Hom}}(|F.|, |G.|) .
\end{aligned}$$

Proof. Let $(\sigma, x, \widetilde{V})$ be a compatible data with $\widetilde{V}$ as in Lemma 3.15. Let us calculate $(|F.| \otimes |G.|)(\widetilde{V})$. By definition, Lemma 3.15 and Proposition 3.17 a section s is given by a compatible family of

$$s_i \in |F.|(\widetilde{V}_i) \otimes |G.|(\widetilde{V}_i) = F_{\sigma_i} \otimes G_{\sigma_i} = |F. \otimes G.|(\widetilde{V}_i)$$

for a system of compatible data $(\sigma_i, x_i, \widetilde{V}_i)$ and with respect to inclusions $\widetilde{V}_i \subseteq \widetilde{V}_j$. Hence we get an isomorphism

$$|F. \otimes G.|(\widetilde{V}) \simeq (|F.| \otimes |G.|)(\widetilde{V}) .$$

The analogous statement for the inner Hom follows from similar considerations. It is not difficult to see that these isomorphisms are compatible with respect to maps of compatible data and hence induce isomorphisms of sheaves. □

7.2. Flat resolutions

Let $P.$ be a cohomological system of coefficients (resp. a sheaf) on the simplicial set $X.$. Obviously $P.$ is flat (i.e., the functor $? \otimes P.$ is exact) iff P_σ is a flat R-module for all $\sigma \in X.$. In this subsection we construct flat resolutions and give another description of $D^-(X.)$.

Let $\mathbf{i} : (\mathrm{Sur}^{-1}\mathcal{X})^{triv} \to \mathrm{Sur}^{-1}\mathcal{X}$ be the inclusion of the subcategory with the same objects and identity morphisms only. Denote by $_*\mathbf{i}$ the left Kan-extension of $\mathbf{i}$ with values in R-Mod. Let $F.$ be a sheaf on $X.$. For every $\sigma \in X.$ we choose an epimorphism $P(\sigma) \twoheadrightarrow F_\sigma$ with $P(\sigma)$ a free R-module. This can be considered as an epimorphism

$$P \longrightarrow \mathbf{i}^*F.$$

where P is given by $\sigma \mapsto P(\sigma)$. Obviously $_*\mathbf{i}\,P$ is a projective sheaf with fibres

$$(_*\mathbf{i}\,P)_\sigma = \bigoplus_{\tau \to \sigma} P(\tau) .$$

Therefore the map

$$_*\mathbf{i}\,P \twoheadrightarrow {}_*\mathbf{i}\,\mathbf{i}^*F. \twoheadrightarrow F.$$

shows that $SH(X.)$ has enough projective and flat objects.

Denote by $P^-(X.)$ the full subcategory of $D^-(X.)$ of flat complexes. Then we have

Lemma 7.11. *The embedding $P^-(X.) \to D^-(X.)$ is an equivalence of categories.*

Let $F^\cdot$, $G^\cdot$ be two complexes of sheaves on $X.$. Let us consider the value map

$$\underline{\mathrm{Hom}}^\cdot(F^\cdot, G^\cdot) \otimes F^\cdot \longrightarrow G^\cdot$$

defined in (7.10). Assume that $F^\cdot$ is a complex with bounded to the right cohomology and $G^\cdot$ is a complex with bounded to the left cohomology. Changing if necessary $G^\cdot$ to a quasiisomorphic bounded to the left complex of injective sheaves and $F^\cdot$ to a quasiisomorphic bounded to the right complex of flat sheaves one easily gets a extension of the value map to the derived category

$$\mathbf{R}\underline{\mathrm{Hom}}^\cdot(F^\cdot, G^\cdot) \overset{L}{\otimes} F^\cdot \longrightarrow G^\cdot \tag{7.11}$$

with $F^{\cdot} \in \mathcal{D}^-(X.)$ and $F^{\cdot} \in \mathcal{D}^+(X.)$.

Taking analogous resolutions and using that $\underline{\mathrm{Hom}}(P., I.)$ is injective if $P.$ is flat and $I.$ is injective (follows from Proposition 7.3), we get the adjunction formula on the derived category from Corollary 7.5:

Lemma 7.12. *Let $F^{\cdot}, G^{\cdot} \in \mathcal{D}^-(X.)$ and $H^{\cdot} \in \mathcal{D}^+(X.)$. There exists an isomorphism*

$$\mathrm{Hom}\left(F^{\cdot} \overset{L}{\otimes} G^{\cdot}, H^{\cdot}\right) \simeq \mathrm{Hom}\left(F^{\cdot}, \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}(G^{\cdot}, H^{\cdot})\right)$$

functorial in all arguments.

Lemma 7.13. *Let $F^{\cdot}, G^{\cdot} \in D^-(X.)$. Then there exists a functorial isomorphism*

$$\left|F^{\cdot} \overset{L}{\otimes} G^{\cdot}\right| \simeq |F^{\cdot}| \overset{L}{\otimes} |G^{\cdot}|\,.$$

Proof. By Lemma 7.11 we can assume that $F^{\cdot}$ is a bounded to the right complex of flat sheaves. By Lemma 3.17 $|F^{\cdot}|$ is also a bounded to the left complex of flat sheaves and the isomorphism follows from Lemma 7.10. □

7.3. Constructible sheaves

Definition 7.14. A sheaf $F.$ of R-modules on a simplicial set $X.$ is called *a constructible sheaf* if all stalks F_σ with $\sigma \in X.$ are finite R-modules.

Obviously constructible sheaves define a full abelian subcategory of $SH(X.)$. (Remember that R is noetherian.)

Lemma 7.15. *Let $F.$ and $G.$ be constructible sheaves. Then*
(i) *$F. \otimes G.$ is constructible.*
(ii) *If $X.$ is locally finite, then $\underline{\mathrm{Hom}}(F., G.)$ is constructible.*

Proof. (i) is obvious.
Let σ be a nondegenerate simplex. By the definition of the inner Hom 7.2 we have

$$\underline{\mathrm{Hom}}(F., G.)_\sigma = \mathcal{H}om_{\mathrm{Func}(\sigma\backslash \mathrm{Sur}^{-1}\mathcal{X}, \mathrm{R-Mod})}(F. \circ pr_\sigma, G. \circ pr_\sigma)\,.$$

Using the natural functor $\mathcal{X}_0 \to \mathrm{Sur}^{-1}\mathcal{X}$ we get a homomorphism

$$(7.12) \qquad \begin{array}{c} \mathcal{H}om_{\mathrm{Func}(\sigma\backslash \mathrm{Sur}^{-1}\mathcal{X}, \mathrm{R-Mod})}(F. \circ pr_\sigma, G. \circ pr_\sigma) \\ \downarrow \\ \mathcal{H}om_{\mathrm{Func}(\sigma\backslash \mathcal{X}_0, \mathrm{R-Mod})}(F. \circ pr'_\sigma, G. \circ pr'_\sigma) \end{array}$$

where pr'_σ denotes the natural map $\sigma\backslash\mathcal{X}_0 \to \mathrm{Sur}^{-1}\mathcal{X}$. By Lemma 3.6 we see that the homomorphism (7.12) is injective. It remains to prove that the lower term is a finite R-module. But this follows trivially from the fact that the category $\sigma\backslash\mathcal{X}_0$ is finite. □

8. Verdier duality

Let $f : X. \to Y.$ be a map of locally finite simplicial sets with fibers of bounc finite dimension. The general problem is to construct a functor

$$f^! : D^+(X.) \longrightarrow D^+(Y.)$$

which has the same properties as the topological $f^!$-functor. In particular, we w: it to satisfy a duality of Verdier type. We will restrict to the case of absolute Verc duality, i. e., $Y. = \Delta.[\underline{0}]$. One should also add the condition, that a boundary of arbitrary nondegenerate simplex in $X.$ is nondegenerate. Actually we only need (condidition only from paragraph 8.3. on, but our construction of $f^!$ is not the ri one if we omit this assumption.

8.1. The functor $f^!$ for a map to a point

Let $X.$ be a locally finite simplicial set of finite dimension and $f : X. \to \Delta.[\underline{0}]$ map to the point. For every natural number n we have a morphism of simplicial s

$$(8.1) \qquad \begin{array}{rcl} [n] : \coprod_{\sigma \in X_n} \Delta.[\sigma] & \longrightarrow & X. \\ (\sigma, \underline{i} \overset{\alpha}{\to} \underline{n}) & \longmapsto & \alpha^*(\sigma)\,. \end{array}$$

First we construct a functor

$$(8.2) \qquad f^! : Sh(\Delta.[\underline{0}]) = \mathcal{A} \longrightarrow K_+(X.)\,.$$

Let A be an object in $\mathcal{A}$. We define

$$(8.3) \qquad f^!(A)_n := [n]_*(\underline{A}) = \prod_{\sigma \in X_n} [\sigma]_* \underline{A}\,,$$

where $\underline{A}$ denotes the constant sheaf with stalk A. For a morphism $\beta : \sigma \to \tau$ in $\mathcal{A}$ have a commutative diagram

$$(8.4) \qquad \begin{array}{ccc} \Delta.[\sigma] & \searrow^{[\sigma]} & \\ \downarrow \beta. & & X. \\ \Delta.[\tau] & \nearrow_{[\tau]} & \end{array} \quad \text{inducing} \quad \begin{array}{ccc} [\tau]_* \underline{A} & \overset{Ad}{\longrightarrow} & [\tau]_* \beta_* \beta^* \underline{A} \\ & \searrow^{\beta^*} & \| \\ & & [\sigma]_* \underline{A}\,, \end{array}$$

where Ad denotes the adjunction morphism given in Lemma 5.1. Furthermore we l used Lemma 5.3 and the obvious relation $\beta^* \underline{A} = \underline{A}$. This construction is obvio functorial in A and β.

Lemma 8.1. *If $\beta = s$ is a surjective map the induced s^* in (8.4) is an isomorph*

Proof. It is sufficient to check that $s_* \underline{A} = \underline{A}$. One way to see this is to pass to geometric realization, but here we give a simplicial proof. Let $\tau \in X_m$ and $\alpha : \underline{k}$ -

be a simplex in $\Delta.[\tau]$. Then we get

$$(s_*\underline{A})_{\underline{k}\xrightarrow{\alpha}\underline{m}} = \varprojlim_{\underline{k}\xrightarrow{\alpha}\underline{m}\backslash s_*} \underline{A} = \prod_{\pi_0(\underline{k}\xrightarrow{\alpha}\underline{m}\backslash s_*)} A .$$

The objects of the category $\underline{k} \xrightarrow{\alpha} \underline{m}\backslash s_*$ are given by commutative diagrams

$$\begin{array}{ccc} \underline{j} & \xleftarrow{\gamma} & \underline{k} \\ \beta\downarrow & & \downarrow\alpha \\ \underline{n} & \xrightarrow{s} & \underline{m} \end{array}$$

where β is a map of ordered sets and γ is a morphism in $\mathrm{Sur}^{-1}\mathcal{Y}$ with $Y = \Delta.[\tau] = \Delta.[\underline{m}]$. The morphisms are morphisms $\beta \to \beta'$ in $\mathrm{Sur}^{-1}\mathcal{X}$ with $X. = \Delta.[\sigma] = \Delta.[\underline{n}]$ making all these diagrams commutative. Let us consider the object given by $j = n$, $\beta = id$ and $\gamma = s^{-1} \circ \alpha$. It is not difficult to see that every object has a morphism to this object, which is given by β. Hence the category $\underline{k} \xrightarrow{\alpha} \underline{m}\backslash s_*$ is non empty and connected and therefore $s_*(\underline{A})_{\underline{k}\xrightarrow{\alpha}\underline{n}} = A$. By definition it is easily seen that the maps between the stalks are the identical. □

For a fixed map of ordered sets $\beta : \underline{m} \to \underline{n}$ and $\sigma \in X_m$ the set $\{\tau \in X_n \,|\, \beta^*\tau = \sigma\}$ is finite, since $X.$ is locally finite. Hence the following definition makes sense:
For $\beta : \underline{m} \to \underline{n}$ let β^* be defined by the diagram

(8.5)
$$\begin{array}{ccc} f^!(A)_n & \xrightarrow{\beta^*} & f^!(A)_m \\ \| & & \| \\ \prod_{\tau\in X_n} [\tau]_*\underline{A} & \dashrightarrow & \prod_{\sigma\in X_m} [\sigma]_*\underline{A} \\ \downarrow & & \downarrow \\ \prod_{\substack{\tau\in X_n \\ \beta^*\tau=\sigma}} [\tau]_*\underline{A} & \xrightarrow{\sum\beta^*} & [\sigma]_*\underline{A} . \end{array}$$

The construction is functorial in A and β and therefore $f^!(A).$ is a cosimplicial object in $\mathcal{A}$. If we consider the associated (homological) complex we get a functor

$$f^! \; : \; \mathcal{A} \; = \; SH(\Delta[\underline{0}]) \;\longrightarrow\; K^-(X.) .$$

This functor is exact, since the direct image functor is exact on constant sheaves and the product functor on sheaves is exact. With the help of the total product complex we extend $f^!$ to

(8.6)
$$K^+(\mathcal{A}) \; = \; K^+(\Delta.\underline{0}) \;\longrightarrow\; K(X.) .$$

Let us now consider the reduced complex:

Lemma 8.2. *The surjective map $s_j : \underline{m} \to \underline{m-1}$ induces the following isomorphis φ*

$$\begin{array}{ccc} f^!(A)_{m-1} & \xrightarrow{s_j^*} & f^!(A)_m \\ \| & & \| \\ \prod_{\tau\in X_{m-1}} [\tau]_*\underline{A} & \longrightarrow & \prod_{\sigma\in X_m} [\sigma]_*\underline{A} \\ & \searrow^{\varphi} & \cup \\ & & \prod_{\sigma\in Im(s_j^*)\subseteq X_m} [\sigma]_*\underline{A} . \end{array}$$

Proof. By Definition 8.4 we see that the map φ exists. For a fixed $\sigma \in Im(s_j^*) \subseteq X.$ there is exactly one $\tau \in X_{m-1}$ with $s_j^*(\tau) = \sigma$, since $\partial_j^* s_j^* = id$. The assertion follow from Lemma 8.1.

This lemma allows us to specify the reduced complex associated to the cosimplici object $f^!(A)$.

(8.7)
$$\begin{aligned} f^!_{\mathrm{red}}(A)_n &= f^!(A)_n / \sum_j \mathrm{Im}(s_j^* : f^!(A)_{n-1} \longrightarrow f^!(A)_n) \\ &= \prod_{\substack{\sigma\in X_n \\ \text{nondeg.}}} [\sigma]_*\underline{A} \\ &= \bigoplus_{\substack{\sigma\in X_n \\ \text{nondeg.}}} [\sigma]_*\underline{A} \end{aligned}$$

with the induced differential. The canonical projection from $f^!(A)$ to $f^!_{\mathrm{red}}(A)$ defines functorial homotopy equivalence by a theorem of Eilenberg - MacLane (see [6], Sectic 3.). $X.$ has finite dimension and so we get an exact additive functor to the catego of bounded complexes

(8.8)
$$f^!_{\mathrm{red}} \; : \; \mathcal{A} \; = \; SH(\Delta[\underline{0}]) \;\longrightarrow\; K^b(X.) .$$

This functor can be extended to an exact functor

$$f^!_{\mathrm{red}} \; : \; K^+(\mathcal{A}) \; = \; K^+(\Delta[\underline{0}]) \;\longrightarrow\; K^+(X.)$$

and to the derived categories

$$f^!_{\mathrm{red}} \; : \; D^+(\mathcal{A}) \;\longrightarrow\; D^+(X.) .$$

Lemma 8.3. *The functors*

$$f^!, f^!_{\mathrm{red}} \; : \; K^+(\mathcal{A}) \;\longrightarrow\; K(X.)$$

are homotopic. (The maps are the natural inclusion and projection.)

Hence their extensions to the homotopy category $K^+(\mathcal{A})$ are isomorphic and we g

Lemma 8.4. *$f^!: K^+(\mathcal{A}) \to K(X.)$ preserves quasiisomorphisms. The natural projection and inclusion maps induce an isomorphism of functors*

$$f^! \simeq f^!_{\mathrm{red}} : D^+(\mathcal{A}) \longrightarrow D^+(X.)\,.$$

Lemma 8.5. *For all $F^\cdot_. \in D^-(X.)$ there exists a natural isomorphism*

$$f^!(F^\cdot_.) \simeq f^!(R) \overset{L}{\otimes} F^\cdot_.\,.$$

Proof. By Lemma 7.11 we can assume that $F^\cdot_.$ is a complex of flat sheaves. The assertion follows immediately from the definition of $f^!_{\mathrm{red}}$ given in (8.7). □

8.2. Absolute Verdier duality

Proposition 8.6. *Let $X.$ be a locally finite simplicial set of finite dimension and f be the canonical map to the simplicial point $\Delta.[\underline{0}]$. Let $\mathcal{A}$ be the category of modules over some noetherian ring R as category of coefficents and $\mathcal{H}\mathrm{om}$ the inner homomorphisms in $\mathcal{A}$. Then the following properties hold:*

(i) *Let $F^\cdot_. \in K^+(X.)$ and $M^\cdot \in K^+(\mathcal{A})$. There is a functorial isomorphism in $K(\mathcal{A})$*

$$\underline{\mathrm{Hom}}^\cdot(\overline{\mathbf{R}}\Gamma_c(X., F^\cdot_.), M^\cdot) \simeq \mathcal{H}\mathrm{om}^\cdot(F^\cdot_., f^!(M^\cdot))\,.$$

(ii) *Let $F^\cdot_. \in K^-(X.)$ and $M^\cdot \in K^+(\mathcal{A})$. There is a functorial isomorphism in $K(\mathcal{A})$*

$$\underline{\mathrm{Hom}}^\cdot(C^\cdot_c(X., F^\cdot_.), M^\cdot) \simeq \mathcal{H}\mathrm{om}^\cdot(F^\cdot_., f^!(M^\cdot))\,.$$

Proof. The proofs of (i) and (ii) are exactly the same. We will consider them simultaniously.

Let us calculate the left and the right complex in a fixed degree n.

$$\underline{\mathrm{Hom}}^n(\overline{\mathbf{R}}\Gamma_c(X., F^\cdot_.), M^\cdot) = \prod_{m+i=n} \underline{\mathrm{Hom}}(\overline{\mathbf{R}}\Gamma_c(X., F^\cdot_.)^{-m}, M^i)$$

with the differential $D^n(\varphi) = d_M \circ \varphi - (-1)^n \varphi \circ d_{\overline{\mathbf{R}}\Gamma_c}$. By the definition of $\overline{\mathbf{R}}\Gamma_c(X., F^\cdot_.)$ we get

$$\underline{\mathrm{Hom}}^n(\overline{\mathbf{R}}\Gamma_c(X., F^\cdot_.), M^\cdot) = \prod_{i+j+k=n} \mathcal{H}\mathrm{om}\left(\bigoplus_{\sigma \in X_{-k}} F^{-j}_\sigma, M^i\right) \tag{8.9}$$

with differential D

$$\begin{aligned} D^n(\varphi) &= d_M \circ \varphi - (-1)^n(\varphi \circ d_{C^\cdot_c} + (-1)^{-k}\varphi \circ d_F) \\ &= d_M \circ \varphi + (-1)^i \varphi \circ ((-1)^{j+1} d_F) + (-1)^{i+j}\varphi \circ ((-1)^{k+1} d_{C^\cdot_c})\,. \end{aligned} \tag{8.10}$$

Therefore this is just the total (product) complex of the triple complex

$$(i,j,k) \longmapsto \mathcal{H}\mathrm{om}\left(\bigoplus_{\sigma \in X_{-k}} F^{-j}_\sigma, M^i\right) \tag{8.11}$$

where we consider $j \mapsto F^{-j}$ and $k \mapsto \bigoplus_{\sigma \in X_{-k}} F^{fixed}_\sigma$ as the dual homological complexes. (The sign convention is the standard one ([4], §1.1) with the order of indices $i < j < k$.)

On the right-hand side we get

$$\mathcal{H}\mathrm{om}^n(F^\cdot_., f^!(M^\cdot)) = \prod_{j+m=n} \mathcal{H}\mathrm{om}(F^{-j}_., f^!(M^\cdot)^m)$$

with the differential $D^n(\varphi) = d_{f^!(M^\cdot)} \circ \varphi - (-1)^n \varphi \circ d_{F^\cdot}$. By definition of $f^!$ we furthermore have

$$\begin{aligned} \mathcal{H}\mathrm{om}^n(F^\cdot_., f^!(M^\cdot)) &= \prod_{j+m=n} \mathcal{H}\mathrm{om}\left(F^{-j}_., \prod_{i+k=m} \prod_{\sigma \in X_{-k}} [\sigma]_* \underline{M^i}\right) \\ &= \prod_{i+j+k=n} \mathcal{H}\mathrm{om}\left(F^{-j}_., \prod_{\sigma \in X_{-k}} [\sigma]_* \underline{M^i}\right) \end{aligned} \tag{8.12}$$

and the differential has the following form

$$\begin{aligned} D^n(\varphi) &= (-1)^{k+1} d_{f^!_.} \circ \varphi + (-1)^k d_M \circ \varphi - (-1)^n \varphi \circ d_F \\ &= ((-1)^{k+1} d_{f^!_.}) \circ \varphi + (-1)^k d_M \circ \varphi + (-1)^{i+k} \varphi \circ ((-1)^{j+1} d_F)\,. \end{aligned} \tag{8.13}$$

We need the following

Lemma 8.7. *Let $F.$ be a sheaf of R-modules on $X.$ and M be some R-module. For every $m \geq 0$ let λ_m be the (functorial) composite of the morphisms*

$$\begin{array}{ccc} \mathcal{H}\mathrm{om}\left(F., \prod_{\sigma \in X_m} [\sigma]_* \underline{M}\right) & \xrightarrow{Ad} & \prod_{\sigma \in X_m} \mathcal{H}\mathrm{om}([\sigma]^* F., \underline{M}) \\ & & \downarrow \textit{stalk in } id \in \Delta.[\underline{m}] \\ \mathcal{H}\mathrm{om}\left(\bigoplus_{\sigma \in X_m} F_\sigma, M\right) & \xleftarrow{\sim} & \prod_{\sigma \in X_m} \underline{\mathrm{Hom}}(([\sigma]^* F.)_{id}, M)\,. \end{array}$$

Then λ_m is an isomorphism of R-modules and for every $\alpha : \underline{m} \to \underline{n}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}\mathrm{om}\left(F., \prod_{\sigma \in X_m} [\sigma]_* \underline{M}\right) & \xrightarrow{\lambda_m} & \mathcal{H}\mathrm{om}\left(\bigoplus_{\sigma \in X_m} F_\sigma, M\right) \\ \uparrow \alpha^*_{(f^!)} \circ ? & & \uparrow ? \circ \alpha_{C_c,*} \\ \mathcal{H}\mathrm{om}\left(F., \prod_{\sigma \in X_n} [\sigma]_* \underline{M}\right) & \xrightarrow{\lambda_n} & \mathcal{H}\mathrm{om}\left(\bigoplus_{\sigma \in X_n} F_\sigma, M\right). \end{array}$$

In other words $\lambda = \{\lambda_m\}$ defines an isomorphism of cosimplicial R modules and in particular an isomorphism of the associated complexes.

Proof. For the first part we verify that the stalk map in the definition of λ_n is an isomorphism. But this follows easily from the same considerations as in the proof of Corollary 7.7.

For the second part let us take some map

$$\varphi : F^{\cdot} \longrightarrow f_n^!(M) = \prod_{\sigma\in X_n} [\sigma]_* \underline{M}.$$

For the components of φ we will write $\varphi(\sigma) = i_\sigma \circ pr_\sigma \circ \varphi$, where i_σ and pr_σ are the inclusion and the projection map between $f_n^!(M)$ and $[\sigma]_*\underline{M}$. We have to prove that

$$\lambda_m(\alpha^* \circ \varphi) = \lambda_n(\varphi) \circ \alpha_* : C_c^m(X_\cdot, F_\cdot) = \bigoplus_{\tau\in X_m} F_\tau \longrightarrow M.$$

It is sufficient to verify this on every $F_{\tau_0} \subset C_c^m(X_\cdot, F_\cdot)$. By the construction of α_* on the cohomology with compact support, α^* on $f_\cdot^!(M)$ and λ we see that

$$\begin{aligned} \alpha_*(F_{\tau_0}) &\subseteq \bigoplus_{\substack{\sigma\in X_n\\ \alpha^*\sigma=\tau_0}} F_\sigma, \\ \lambda_m(\alpha^*\circ\varphi)|_{F_{\tau_0}} &= \lambda_m((\alpha^*\circ\varphi)(\tau_0))|_{F_{\tau_0}}, \\ (\alpha^*\circ\varphi)(\tau_0) &= \sum_{\substack{\sigma\in X_n\\ \alpha^*\sigma=\tau_0}} \alpha^*\circ\varphi(\sigma). \end{aligned}$$

Therefore it is sufficiently to consider the case

$$\varphi = \prod_{\sigma\in X_n} \varphi(\sigma) = \prod_{\substack{\sigma\in X_n\\ \alpha^*\sigma=\tau_0}} \varphi(\sigma).$$

But the set of such σ is finite, since $X_\cdot$ is locally finite. Using the fact that the assertion is additive in φ we may assume that $\varphi = \varphi(\sigma_0)$ for some $\sigma_0 \in X_n$ with $\alpha^*\sigma_0 = \tau_0$. Passing from $\prod_{\sigma\in X_n} [\sigma]_*\underline{M}$ to $[\sigma_0]_*\underline{M}$ and from $\bigoplus_{\sigma\in X_n} F_\sigma$ to F_{σ_0}, all we have to prove is the following:

Let $\sigma := \sigma_0$, $\tau := \tau_0$ and $\psi = pr_\sigma \circ \varphi$. Then the commutative diagram

$$\begin{array}{ccc} F^{\cdot} & \xrightarrow{\psi} & [\sigma]_*\underline{M} \\ \| & & \downarrow \alpha^* \\ F^{\cdot} & \xrightarrow{\alpha^*\circ\psi} & [\tau]_*\underline{M} \end{array} \quad \text{induces a commutative diagram} \quad \begin{array}{ccc} F_\sigma & \xrightarrow{Ad(\psi)_{id:\underline{n}\to\underline{n}}} & M \\ \alpha_*\uparrow & & \| \\ F_\tau & \xrightarrow{Ad(\alpha^*\circ\psi)_{id:\underline{m}\to\underline{m}}} & M. \end{array}$$

This can be seen using the following commutative diagram. Bearing in mind that $[\tau] = [\sigma] \circ \alpha_*$ with the natural $\alpha_* : \Delta.[\underline{m}] \to \Delta.[\underline{n}]$, we have

$$\begin{array}{ccccc} F_\tau & = & [\tau]^*(F_\cdot)_{id:\underline{m}\to\underline{m}} & = & ((\alpha_*)^*[\sigma]^*F_\cdot)_{id} \\ \downarrow & & \downarrow Ad(\alpha^*\circ\psi)_{id:\underline{m}\to\underline{m}} & (*) & \downarrow (\alpha_*)^*Ad(\psi)_{id} \\ M & = & \underline{M}_{id:\underline{m}\to\underline{m}} & = & ((\alpha_*)^*\underline{M})_{id} \end{array}$$

$$\begin{array}{cccccc} = & [\sigma]^*(F_\cdot)_\alpha & \xrightarrow{\alpha_*} & [\sigma]^*(F_\cdot)_{id} & = & F_\sigma \\ & \downarrow Ad(\psi)_\alpha & & \downarrow Ad(\psi)_{id} & & \downarrow \\ = & \underline{M}_\alpha & \xrightarrow{\alpha_*=id} & \underline{M}_id & = & M. \end{array}$$

The square $(*)$ is commutative by the definition of α^* given in (8.4). This proves the lemma. □

Let us continue the proof of the theorem.

By (8.12), (8.13) and this lemma we see that

$$\mathcal{H}om^n(F^{\cdot}, f^!(M^{\cdot})) = \prod_{i+j+k=n} \mathcal{H}om\left(\bigoplus_{\sigma\in X_{-k}} F_\sigma^{-j}, M^i\right)$$

with the differential

$$D^n(\varphi) = \varphi \circ \left((-1)^{k+1} d_{f^!\cdot}\right) + (-1)^k d_M \circ \varphi + (-1)^{i+k} \varphi \circ \left((-1)^{j+1} d_F\right).$$

This is just the total product complex of the triple complex

$$(k,i,j) \longmapsto \mathcal{H}om\left(\bigoplus_{\sigma\in X_{-k}} F_\sigma^{-j}, M^i\right)$$

where we consider $j \mapsto F^{-j}$ and $k \mapsto \bigoplus_{\sigma\in X_{-k}} F_\sigma^{fixed}$ as dual homological complexes the order of indexes being $k < i < j$. This differs from the left-hand side (see (8.9), (8.10) and (8.11)) only in the order of the indices i, j, k. But between them we hav a natural isomorphism (using Deligne's sign convention, see [4], 1.1.4.2) of the tot (product) complexes of triple complexes

$$\begin{array}{ccc} \underline{\mathrm{Hom}}^n(\overline{\mathbf{R}}\Gamma_c(X_\cdot, F^{\cdot}), M^{\cdot}) & \dashrightarrow & \mathcal{H}om^n(F^{\cdot}, f^!(M^{\cdot})) \\ \| & & \| \\ \underbrace{\prod_{i+j+k=n} \mathcal{H}om\left(\bigoplus_{\sigma\in X_{-k}} F_\sigma^{-j}, M^i\right)}_{\text{with order } i<j<k} & \longrightarrow & \underbrace{\prod_{i+j+k=n} \mathcal{H}om\left(\bigoplus_{\sigma\in X_{-k}} F_\sigma^{-j}, M^i\right)}_{\text{with order } k<i<j} \\ \{\varphi_{i,j,k}\} & \longmapsto & \{((-1)^{ik+jk}\varphi_{i,j,k})\}. \end{array}$$

(This can also be proved directly by showing that this sign convention defines isomorphism of complexes.) The constructed isomorphism is obviously functorial $F^{\cdot}$ and $M^{\cdot}$. This proves the proposition.

Theorem 8.8. *Let $X_\cdot$ be a locally finite simplicial set of finite dimension and the canonical map to the simplicial point $\Delta.[\underline{0}]$. Let $F^{\cdot} \in D^b(X_\cdot)$ and $M^{\cdot} \in D^+($ Then we have a functorial isomorphism in $D(\mathcal{A})$*

$$\mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}(\overline{\mathbf{R}}\Gamma_c(X_\cdot, F^{\cdot}), M^{\cdot}) \simeq \mathbf{R}\Gamma(X_\cdot, \mathbf{R}\mathcal{H}om^{\cdot}(F^{\cdot}, f^!(M^{\cdot}))).$$

Proof. Without loss of generality we can suppose that $F^{\cdot}$ is bounded and $M^{\cdot}$ is a bounded to the left complex. Let us prove the theorem in several steps

Step 1. $M^{\cdot}$ can be changed to a quasiisomorphic bounded to the left complex $I^{\cdot}$ of injective R-modules.

Step 2. $F^{\cdot}$ can be changed to a bounded to the right quasiisomorphic complex of flat sheaves $P^{\cdot} \in D^b(X.)$.

Proof. See Lemma 7.11. □

Step 3. The complex $f^!(I^{\cdot})$ is a complex of injective sheaves on $X.$. It is homotopic to $f^!_{\text{red}}(I^{\cdot})$ which is injective and bounded to the left.

Proof. By definition of $f^!$ (see 8.3), of $f^!_{\text{red}}$ (see 8.7) and their extension to complexes we have to prove that the sheaf $[\sigma]_* \underline{I^m}$ is injective for all simplices $\sigma \in X_n$ and $m \in \mathbb{Z}$. But the direct image functor has an exact right adjoint by 5.1. Hence it maps injective sheaves to injective ones. It remains to verify that the constant sheaf $\underline{I^m}$ is injective on the simplicial set $\Delta.[\underline{n}]$. But the corresponding category $\mathrm{Sur}^{-1}\mathcal{Y}$ to $Y := \Delta.[\underline{n}]$ has the final object $id : \underline{n} \to \underline{n}$ and for an arbitrary sheaf F on Y we get the functorial isomorphism

$$\mathrm{Hom}_{\Delta.[\underline{n}]}(F, \underline{I^m}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(F_{id}, \underline{I^m}_{id}) = \mathrm{Hom}_{\mathcal{A}}(F_{id}, I^m).$$

Hence this functor is exact in F. The second part follows from Lemma 8.3 □

Step 4. The sheaves $\underline{\mathrm{Hom}}\,(P^m, f^!(I^{\cdot})^n)$ and $\underline{\mathrm{Hom}}\,(P^m, f^!_{\text{red}}(I^{\cdot})^n)$ are injective for arbitrary $m, n \in \mathbb{N}$. ($P^{\cdot}$ and $I^{\cdot}$ as in Step 1. and 2.)

Proof. From Step 3. we see that it suffices the injectivity of the sheaf $\underline{\mathrm{Hom}}(P, I)$ for an injective sheaf I and a flat sheaf P. This follows from 7.3. □

Step 5. The complex $\underline{\mathrm{Hom}}^{\cdot}(P^{\cdot}, f^!(I^{\cdot}))$ is a complex of injective sheaves, the homotopic complex $\underline{\mathrm{Hom}}^{\cdot}(P^{\cdot}, f^!_{\text{red}}(I^{\cdot}))$ is a bounded to the left complex of injective sheaves.

Proof. By definition we have

$$\underline{\mathrm{Hom}}^n(P^{\cdot}, f^!(I^{\cdot})) = \prod_i \underline{\mathrm{Hom}}\,(P^i, f^!(I^{\cdot})^{n+i}),$$

and similarly for $f^!_{\text{red}}$ which is zero for $n << 0$. We get the assertion from Step 4. and from the fact that the product of injective objects is injective. □

Now we can prove the theorem. Let us consider the left-hand side

$$\begin{aligned}
\mathbf{R}\,\underline{\mathrm{Hom}}\,(\overline{\mathbf{R}}\,\Gamma_c(X., F^{\cdot}), M^{\cdot}) &= \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\,(C^{\cdot}_c(X., F^{\cdot}), M^{\cdot}) && \text{by definition of } \overline{\mathbf{R}}\,\Gamma_c \\
&\simeq \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\,(C^{\cdot}_c(X., P^{\cdot}), I^{\cdot}) && \text{by Step 1. and 2., Lemma 6.11} \\
&\simeq \underline{\mathrm{Hom}}^{\cdot}\,(C^{\cdot}_c(X., P^{\cdot}), I^{\cdot}).
\end{aligned}$$

For the right-hand side we get

$$\begin{aligned}
\mathbf{R}\,\Gamma(X., \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\,(F^{\cdot}, f^!(M^{\cdot}))) &\simeq \mathbf{R}\,\Gamma(X., \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\,(P^{\cdot}, f^!(I^{\cdot}))) && \text{by Step 1. and} \\
&\simeq \mathbf{R}\Gamma(X., \underline{\mathrm{Hom}}^{\cdot}(P^{\cdot}, f^!(I^{\cdot}))) && \text{by Step 3.} \\
&\simeq \Gamma(X., \underline{\mathrm{Hom}}^{\cdot}(P^{\cdot}, f^!(I^{\cdot}))) && \text{by Step 5.} \\
&\simeq \mathcal{H}om^{\cdot}\,(P^{\cdot}, f^!(I^{\cdot})) && \text{by Corollary 7}
\end{aligned}$$

Combining these calculations and applying Proposition 8.6 (ii) changing $F^{\cdot}$ to $P^{\cdot}$, $M^{\cdot}$ to $I^{\cdot}$, the theorem follows.

8.3. The dualizing functor

Let $X.$ be a locally finite simplicial set of finite dimension. Let f be the map to simplicial point $\Delta.[\underline{0}]$ and $\mathcal{A} = R$-Mod the category of modules over the ring R.

In this paragraph we study the properties of the dualizing sheaf (complex)

$$\mathcal{D}^{\cdot}_X := f^!(R)$$

and the associated dualizing map:

$$\begin{aligned}
\mathcal{D}_X \;:\; D^b(X.) &\longrightarrow D^b(X.) \\
\mathcal{F}^{\cdot} &\longmapsto \mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\,(\mathcal{F}^{\cdot}, \mathcal{D}^{\cdot}_X).
\end{aligned}$$

Moreover, let $D^b_c(X.)$ be the full subcategory consisting of complexes with construct cohomology. We will see that $\mathcal{D}_X$ preserves this subcategory (see 8.18). This is main theorem:

Theorem 8.9. [Verdier-duality.] *Let $X.$ be a locally finite simplicial set of f dimension and suppose that the boundary of a nondegenerate simplex is nondegene Let $\mathcal{D}^b_c(X.)$ be the subcategory of $\mathcal{D}^b(X.)$ of complexes with constructible cohomo Then the contravariant functor $\mathcal{D}_X$ maps $\mathcal{D}^b_c(X.)$ to $\mathcal{D}^b_c(X.)$ and defines an antis uality of $D^b_c(X.)$.*

The proof will be given later in §8.6..

Remark 8.10. The Example 8.17 shows that Theorem 8.9 is not true if we the condition that the boundary of a nondegenerate simplex is nondegenerate. problem is that the given definition of $f^!$ in § 8.1 works well only under this assump It seems possible to give a definition in general, but the constructions get more more complicated. The idea is the following:

One has to construct a functor

$$\overline{\mathbf{R}}\,g_* \;:\; D^+(Y.) \longrightarrow D^+(X)$$

for an arbitrary morphism $g : Y. \to X.$ which is compatible with $\mathbf{R}|g|_*$ with re to the geometric realization. This functor should come from an explicitly given functor

$$SH(Y.) \longrightarrow K^+(X.)$$

as it was the case in § 5.2 for $X. = \Delta.[\underline{0}]$.

The "right" definition of $f^!$ (f the map from $X.$ to the point) seems to be as follows: Take the old Definition 8.3 and change $[n]_*$ to $\overline{\mathbf{R}}[n]_*$. It is easily seen that $[n]$ is exact under the nondegeneracy assumption. (The fibres of the geometric realization of $[n]$ are finite unions of points.) Then one should have an isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}^{\cdot}\left(\overline{\mathbf{R}}\Gamma_c(X.,F^{\cdot}),M^{\cdot}\right) \simeq \overline{\mathbf{R}}\Gamma(X.,\mathbf{R}\,\mathcal{H}om^{\cdot}(F^{\cdot},f^!(M^{\cdot})))$$

analogous to Theorem 8.8. By Lemma 5.10 the two formulations agree, if $X.$ has the nondegeneracy property. Theorem 8.9 should be satisfied. The difficulty is to find a manageable definition of $\overline{\mathbf{R}}\,g_*$.

For further considerations we fix an injective resolution

$$I^{\cdot} = 0 \longrightarrow I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} \ldots$$

of the ring R.

8.4. The dualizing sheaf

Let us analyze the main object of duality, the dualizing sheaf

$$\mathcal{D}_X := f^!(R) \simeq f^!_{\mathrm{red}}(R) \in D^b(X.)\,.$$

Let $\eta \in X_m$ be a nondegenerate simplex. We want to specify the complex of $R-$modules $\mathcal{D}_{X,\sigma}$. By the definitions of $f^!$ (see § 8.1) and the direct image (see Section 5) we get

$$\mathcal{D}^n_{X,\eta} = \prod_{\substack{\sigma\in X_{-n}\\ \text{nondeg.}}} ([\sigma]_*\underline{R})_\eta = \prod_{\substack{\sigma\in X_{-n}\\ \text{nondeg.}}} R^{\pi_0\left(\eta\backslash\mathrm{Sur}^{-1}[\sigma]\right)}.$$

Obviuosly, the sheaf structure map $\mathcal{D}^n_{X,\eta} \to \mathcal{D}^n_{X,\eta'}$ for an arbitrary $\eta \to \eta'$ is given by the induced map

$$\eta'\backslash\mathrm{Sur}^{-1}[\sigma] \longrightarrow \eta\backslash\mathrm{Sur}^{-1}[\sigma]\,.$$

The differential comes from the cosimplicial structure defined by:
If $\beta : \underline{-m} \to \underline{-n}$ and $\beta : \sigma \to \tau$ we have a functor

$$\eta\backslash\mathrm{Sur}^{-1}[\sigma] \longrightarrow \eta\backslash\mathrm{Sur}^{-1}[\tau]$$

defined in the obvious way by $\beta_* : \Delta.[\underline{-m}] \to \Delta.[\underline{-n}]$. Taking these maps together we get $\mathcal{D}^n_{X,\eta} \to \mathcal{D}^m_{X,\eta}$ (see (8.5)).

Lemma 8.11. *Let σ, η be simplices on a simplicial set $X.$. Then the map*

$$\begin{array}{ccc} \mathrm{Hom}_{Sur^{-1}\mathcal{X}}(\eta,\sigma) & \longrightarrow & \pi_0\left(\eta\backslash Sur^{-1}[\sigma]\right) \\ \eta \xrightarrow{\alpha} \sigma & \longmapsto & \left(id \in \Delta.[\sigma], \eta \xrightarrow{\alpha} \sigma = [\sigma](id)\right) \end{array}$$

is an isomorphism.

Proof. Let $\sigma \in X_n$. An object of $\eta\backslash\mathrm{Sur}^{-1}[\sigma]$ is given by a pair

$$\left(\underline{i} \xrightarrow{\beta} \underline{n},\, \eta \xrightarrow{\gamma} \beta^*\sigma\right)$$

with $\gamma \in \mathrm{Sur}^{-1}\mathcal{X}$ and β a map of ordered sets. β gives a morphism $\beta^*\sigma \to \sigma$ and we get a morphism

$$(\beta,\varphi) \longrightarrow (id, \beta\circ\varphi)\,.$$

This shows that the map in the lemma is surjective. For an arbitrary morphism

$$(\beta,\gamma) \longrightarrow (\beta',\gamma') \quad \text{given by} \quad \varphi : \beta \longrightarrow \beta'$$

we have a commutative diagram

$$\begin{array}{cccc} & \beta^*\sigma & \xrightarrow{\beta} & \sigma \\ \eta \;\lessgtr & \downarrow\gamma & & \| \\ & \beta'^*\sigma' & \xrightarrow{\beta'} & \sigma \end{array}$$

and hence the map is an isomorphism. □

For a locally finite set $X.$ the set of morphisms $\mathrm{Hom}(\eta,\sigma)$ is finite for all σ and not empty only for a finite number of nondegenerate σ. Hence we get

Lemma 8.12. *The complex $f^!_{\mathrm{red}}(R)_\eta$ is a bounded from $-\dim(X.)$ to $-\dim(\eta)$ complex of finitely generated free R-modules for every $\eta \in X.$. Hence $\mathcal{D}_X \in \mathcal{D}^b_c(X.)$.*

Lemma 8.13. *There exists a functorial isomorphism*

$$f^!(F^{\cdot}) \simeq \mathcal{D}_X \overset{L}{\underset{R}{\otimes}} F^{\cdot}$$

for $F^{\cdot} \in D^+(X.)$.

Proof. From the definition of $f^!_{\mathrm{red}}$ (see equation (8.7), where we can obviously change the product to a direct sum) we get immediately

$$f^!_{\mathrm{red}}(F^{\cdot}) \simeq f^!_{\mathrm{red}}(R) \underset{R}{\otimes} F^{\cdot}$$

and we can pass to the derived category since $f^!_{\mathrm{red}}(R)$ is a bounded complex of flat sheaves by Lemma 8.12. □

8.5. An explicit description of the dual of a sheaf complex

By the definition of $f^!$ and $f^!_{\mathrm{red}}$ (see § 8.1) we have the following explicit description of the dualizing sheaf complex: $\mathcal{D}_X$ is quasiisomorphic to the total product complex of the double complex

$$(8.14) \qquad i, j \longmapsto \prod_{\substack{\sigma\in X_{-i}\\ \text{nondeg.}}} [\sigma]_*\underline{I^j}\,.$$

The differential of the total complex on the (i,j)-component is given by

$$D = ((-1)^{i+1} d^{\cdot}_{-i}) + (-1)^i d^j_I ,$$

where $d^{\cdot}$ denotes the (homological) differential coming from the cosimplicial structure defined in (8.5).

Lemma 8.14. *Let I be an injective module and $\sigma \in X_\cdot$ some simplex. Then the sheaf $[\sigma]_* \underline{I}$ is $\underline{\mathrm{Hom}}$ acyclic, i. e., the functor $\underline{\mathrm{Hom}}(?, [\sigma]_* \underline{I})$ is exact.*

Proof. From Corollary 7.9 and Corollary 7.7 we get for an arbitrary sheaf $F_\cdot$ on $X_\cdot$

$$\begin{aligned} \underline{\mathrm{Hom}}(F_\cdot, [\sigma]_* \underline{I}) &\simeq [\sigma]_* \underline{\mathrm{Hom}}([\sigma]^* F_\cdot, \underline{I}) \\ &\simeq [\sigma]_* \underline{\mathcal{H}\mathrm{om}(F_\sigma, I)} . \end{aligned}$$

This proves the assertion since all the components on the right-hand side are exact functors ($[\sigma]_*$ for constant sheaves only). □

By this lemma we can give an explicit description of $\mathcal{D}_X(F^\cdot)$ for a sheaf complex $F^\cdot$ with differential d_F. The explicit description of the dualizing sheaf complex 8.14 consists of inner Hom acyclic sheaves, since $\underline{\mathrm{Hom}}$ commutes with products. Hence we can write

$$\mathcal{D}_X(F^\cdot) = \underline{\mathrm{Hom}}^\cdot(F^\cdot, \mathcal{D}_X^\cdot)$$

and we get the total (product) complex of the triple complex

$$i, j, k \longmapsto \underline{\mathrm{Hom}}\left(F^{-k}, \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{I^j} \right)$$

with differential on this component

$$D = ((-1)^i d^{\cdot}_{-i}) + (-1)^i d^j_I + (-1)^{i+j} \left((-1)^{k+1} d_F^{-k-1} \right) .$$

Using the Corollaries 7.9 and 7.7 we get for the components

$$(8.15) \qquad \begin{aligned} \underline{\mathrm{Hom}}\left(F^{-k}, \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{I^j} \right) &\simeq \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \underline{\mathrm{Hom}}\left(F^{-k}, [\sigma]_* \underline{I_j}\right) \\ &\simeq \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathrm{Hom}}\left([\sigma]^* F^{-k}, \underline{I_j}\right) \\ &\simeq \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}\mathrm{om}\left(F_\sigma^{-k}, I_j\right)} . \end{aligned}$$

We have to understand what happens to the differentials under these isomorphisms. The differentials coming from $F^\cdot$ and $I^\cdot$ still act in the natural way. The differential coming from $d^{\cdot}$ is described in the following

Lemma 8.15. *Let $F_\cdot$ be a sheaf of R-modules on $X_\cdot$ and M some R-module Then the isomorphism*

$$\underline{\mathrm{Hom}}\left(F_\cdot, \prod_{\substack{\sigma \in X_n \\ \text{nondeg.}}} [\sigma]_* \underline{M} \right) \xrightarrow[\sim]{\lambda_n} \prod_{\substack{\sigma \in X_n \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}\mathrm{om}(F_\sigma, M)}$$

constructed in 8.15 defines a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}\left(F_\cdot, \prod_{\substack{\tau \in X_m \\ \text{nondeg.}}} [\tau]_* \underline{M} \right) & \xrightarrow{\lambda_m} & \prod_{\substack{\tau \in X_m \\ \text{nondeg.}}} [\tau]_* \underline{\mathcal{H}\mathrm{om}(F_\tau, M)} \\ \Big\uparrow \alpha^*_{(f^!)} \circ ? & & \Big\uparrow \alpha' \\ \underline{\mathrm{Hom}}\left(F_\cdot, \prod_{\substack{\sigma \in X_n \\ \text{nondeg.}}} [\sigma]_* \underline{M} \right) & \xrightarrow{\lambda_n} & \prod_{\substack{\sigma \in X_n \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}\mathrm{om}(F_\sigma, M)} \end{array}$$

for all $\alpha : \underline{m} \to \underline{n}$. The map α' is induced by the maps

$$[\sigma]_* \underline{\mathcal{H}\mathrm{om}(F_\sigma, M)} \longrightarrow [\tau]_* \underline{\mathcal{H}\mathrm{om}(F_\tau, M)}$$

for all $\tau \xrightarrow{\alpha} \sigma$, i. e., $\alpha^* \sigma = \tau$, coming from the adjunction morphism

$$\begin{aligned} \underline{\mathcal{H}\mathrm{om}(F_\sigma, M)} &\longrightarrow \alpha_* \underline{\mathcal{H}\mathrm{om}(F_\sigma, M)} \\ &\longrightarrow \alpha_* \underline{\mathcal{H}\mathrm{om}(F_\tau, M)} \end{aligned}$$

after applying $[\sigma]_*$. (Here α_* denotes the natural map $\Delta_\cdot[\underline{m}] \to \Delta_\cdot[\underline{n}]$.)

In other words $\lambda = \{\lambda_n\}$ defines an isomorphism of cosimplicial sheaves and in particular an isomorphism of the associated complexes.

Proof. This is essentially the inner Hom analog of Lemma 8.7. The proof can be done using the same reduction process.

An alternative proof of the assertion is to apply the functor $\mathcal{H}\mathrm{om}(G_\cdot, ?)$ with an arbitrary sheaf $G_\cdot$ to the diagram 8.15 and to show that we get a commutative diagram of functors in $G_\cdot$. Bearing in mind the construction of λ_n via Corollaries 7.9 and 7.7 one can easily get a reduction to Lemma 8.7 with $G_\cdot \otimes F_\cdot$ instead of $F_\cdot$. We omit the details. □

Accumulating all our information we now get the following explicit description of the dual of $F^\cdot$:

Lemma 8.16. *Let $K_f(X_\cdot) \subset K(X_\cdot)$ be the full subcategory of complexes with* *bounded cohomology. We consider the functor*

$$\begin{array}{cccc} D_X : K_f(X.) \longrightarrow & K(K(K(X.))) & \xrightarrow{Tot} & K(X.) \\ F^\cdot \longmapsto & \left(i,\, j,\, k \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om\,(F_\sigma^{-k}, I_j)} \right) & & \end{array}$$

where:

(i) *The differential in i is given in Lemma* 8.15.

(ii) *The differential in j is induced by the differential on $I^\cdot$.*

(iii) *The differential in k is induced by the differential on $F^\cdot$.*

(iv) *The order on the set of indices is given by $i < j < k$. Tot denotes functor mapping a triple complex to its associated total complex with respect to this order. Then D_X is exact and has image in $K_f(X.)$. It induces a commutative diagram*

$$\begin{array}{ccc} K_f(X.) & \xrightarrow{D_X} & K_f(X.) \\ \downarrow & & \downarrow \\ \mathcal{D}^b(X.) & \xrightarrow{\mathcal{D}_X} & \mathcal{D}^b(X.)\,. \end{array}$$

Proof. We remark that $\mathbf{R}\,\underline{\mathrm{Hom}}(F^\cdot, \mathcal{I}^\cdot)$ with $\mathcal{I}^\cdot \in \mathcal{D}^+(X.)$ is a bounded to the left complex of inner Hom acyclic sheaves and that $F^\cdot \in \mathcal{D}^-(X.)$ (not necessary bounded to the right) is isomorphic to $\underline{\mathrm{Hom}}(F^\cdot, I^\cdot)$. This follows from the convergence of the spectral sequence of the associated double complex.

Hence the diagram of the lemma is commutative and D_X has image in $K_f(X.)$. The exactness of D_X is obvious. □

Example 8.17. Let $X.$ be the 2-dimensional standard simplex with boundary contracted to a point. Suppose R is an injective R-module and the injective resolution is given by R. Let η be the 0-dimensional simplex and τ the nondegenerate 2-dimensional simplex. In Example 3.12 it was shown that $\mathrm{Sur}^{-1}\mathcal{X}$ is equivalent to the category consisting of the two objects η and τ and one morphism ($\delta_0 = \delta_1 = \delta_2$) from η to τ. Let $F.$ be a constructible sheaf given by a map $\varphi_F : F_\eta \to F_\tau$. By Lemma 8.16 we get the following explicit description of $\mathcal{D}_X(F.)$:

$$D_X(F.)^n \;=\; \prod_{i+j+k=n} \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om(F_\sigma^{-k}, I^j)}$$

$$= \begin{cases} [\eta]_* \underline{\mathcal{H}om(F_\eta, R)} & \text{if } n = 0, \\ [\tau]_* \underline{\mathcal{H}om(F_\tau, R)} & \text{if } n = -2, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore

$$D_X(F.) \;=\; [\eta]_* \underline{\mathcal{H}om(F_\eta, R)} \oplus [\tau]_* \underline{\mathcal{H}om(F_\tau, R)}[2]\,.$$

We see that the dual sheaf complex has "forgotten" the structure map φ_F. Hence the double dual of $F.$ cannot give $F.$. It can be verified that the double dual of $F.$ is the complex

$$D_X(D_X(F.)) \;=\; [\eta]_* \underline{F_\eta} \oplus [\eta]_* \underline{F_\tau}[2] \oplus [\tau]_* \underline{F_\tau}[4]\,.$$

Lemma 8.18. *The functor $D_X : K_f(X.) \to K_f(X.)$ preserves the full subcategory $K_{f,c}(X.)$ of complexes with constructible cohomology. In particular, $\mathcal{D}_X$ maps $\mathcal{D}^b_c(X.)$ to $\mathcal{D}^b_c(X.)$.*

Proof. The functor D_X is exact, hence it is sufficient to prove that $D_X(F.)$ has constructible cohomology for a constructible sheaf $F.$.

$D_X(F.)$ is the total product complex of the double complex

$$i,\, j \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om(F_\sigma, I_j)}\,.$$

We must verify that the stalk complex in every η

$$i,\, j \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \mathcal{H}om(F_\sigma, I_j)^{\pi_0(\eta \backslash \mathrm{Sur}^{-1}[\sigma])}$$

has finitely generated cohomology. Let us consider the associated spectral sequence for the cohomology of the total complex. We have

$$E_1^{i,j} \;=\; \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \underline{\mathrm{Ext}^j(F_\sigma, R)}^{\pi_0(\eta \backslash \mathrm{Sur}^{-1}[\sigma])}\,.$$

Taking a free finitely generated resolution of F_σ one easily sees that $\mathrm{Ext}^j(F_\sigma, R)$ is finitely generated. By Lemma 8.11 we see that $E_1^{i,j}$ are finitely generated R-modules. The spectral sequence is concentrated in $-\dim(X.) \le i \le 0$, hence is convergent. It follows that the cohomology is also finitely generated. □

8.6. Proof of the duality theorem

Let us now give the proof of the duality Theorem 8.9.

Let $F^\cdot \in \mathcal{D}^b(X.)$. We consider the value map

$$F^\cdot \overset{L}{\otimes} \mathbf{R}\,\underline{\mathrm{Hom}}^\cdot(F^\cdot, \mathcal{D}^\cdot_X) \longrightarrow \mathcal{D}^\cdot_X$$

defined in 7.11. By adjunction, see Lemma 7.12, this defines a functorial morphism

$$F^\cdot \longrightarrow \mathbf{R}\,\underline{\mathrm{Hom}}^\cdot(\mathbf{R}\,\underline{\mathrm{Hom}}^\cdot(F^\cdot, \mathcal{D}^\cdot_X), \mathcal{D}^\cdot_X)\,.$$

An alternative definition of this morphism would be the execution of this construction on the complex level, thus getting

$$F^\cdot \longrightarrow \underline{\mathrm{Hom}}^\cdot(\underline{\mathrm{Hom}}^\cdot(F^\cdot, \mathcal{D}^\cdot_X), \mathcal{D}^\cdot_X)$$

and to pass to the derived category by taking an injective resolution of $\mathcal{D}_X^{\cdot}$. This construction has the advantage that we do not need an flat resolution of $F^{\cdot}$ to pass to complexes.

This map defines a functor morphism $id \to \mathcal{D}_X \circ \mathcal{D}_X$. We will show that this morphism is an isomorphism on $\mathcal{D}_c^b(X.)$.

First we lift the problem from the derived category to the category of complexes.

Let $F^{\cdot} \in K_f(X.)$. Applying Lemma 8.16 all we need to prove is that the value morphism of complexes

$$F^{\cdot} \longrightarrow D_X(D_X(F^{\cdot}))$$

is a quasiisomorphism. But the left and the right-hand side are exact functors in $F.$, hence it is sufficient to consider the case that $F^{\cdot} = F.$ is a constructible sheaf.

Let us calculate the right-hand side with the help of Lemma 8.16:

$$\begin{aligned}
& D_X(D_X(F.))^n \\
= & \prod_{i+j+r=n} \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om(\underline{\mathrm{Hom}}^{-r}(F^{\cdot}, \mathcal{D}_X^{\cdot})_\sigma, I^j)} \\
= & \prod_{i+j+r=n} \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om\left(\prod_{(-l)+(-k)=-r} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\tau]_* \underline{\mathcal{H}om(F_\tau, I^{-k})}_\sigma, I^j \right)} \\
= & \prod_{i+j+r=n} \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om\left(\prod_{(-l)+(-k)=-r} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} \mathcal{H}om(F_\tau, I^{-k})^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}, I^j \right)} \\
= & \prod_{i+j+k+l=n} \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om\left(\mathcal{H}om(F_\tau, I^{-k})^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}, I^j \right)}
\end{aligned}$$

with a differential induced on a fixed i, j, k, l-component by

$$\begin{aligned}
D & = (-1)^i d^!_{-i} + (-1)^i d^j_I + (-1)^{i+j}(-1)^{r+1} d^{-r-1}_{\underline{\mathrm{Hom}}\ldots} \\
& = (-1)^i d^!_{-i} + (-1)^i d^j_I + (-1)^{i+j}(-1)^{k+l+1}\left((-1)^{l-1} d^!_{l+1} + (-1)^{-l} d_I^{-k-1}\right) \\
& = (-1)^i d^!_{-i} + (-1)^i d^j_I + (-1)^{i+j}(-1)^{k+1} d_I^{-k-1} + (-1)^{i+j+k} d^!_{l+1}.
\end{aligned}$$

Hence we get the total complex of the quadruple complex

$$i, j, k, l \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om(\mathcal{H}om(F_\tau, I^{-k}), I^j)^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}}.$$

The morphism $F. \to D(D(F.))^0$ is induced by the maps for $i = -l$, $k = -j$:

$$F. \longrightarrow \prod_{\substack{\sigma \in X_l \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om(\mathcal{H}om(F_\tau, I^j), I^j)^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}},$$

which goes to the components of the product with $\sigma = \tau$. It is associated to the map

$$[\sigma]^* F. \longrightarrow \underline{F_\sigma} \longrightarrow \underline{\mathcal{H}om(\mathcal{H}om(F_\sigma, I^j), I^j)}.$$

(Remark that $\sigma \backslash \mathrm{Sur}^{-1}[\sigma]$ is connected by Lemma 8.11 and Corollary 3.9.)

Fix now i and l; the double complex

$$j, k \longmapsto \mathcal{H}om(\mathcal{H}om(F_\tau, I^{-k}), I^j)$$

is just the double dual for the finite R-module F_τ. The natural map

$$F_\tau \longrightarrow \mathcal{H}om^{\cdot}(\mathcal{H}om^{\cdot}(F_\tau, I^{\cdot}), I^{\cdot})$$

gives a quasiisomorphism. Therefore the associated map of constant sheaves is a quasiisomorphism too. Applying the exact on constant sheaves functor $[\sigma]_*$ and taking the product over the σ and τ we get a quasiisomorphism

$$\begin{gathered}
\prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{F_\tau^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}} \\
\downarrow \\
\mathrm{Tot}\left(j, k \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{\mathcal{H}om\left(\mathcal{H}om\left(F_\tau^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}, I^{-k} \right), I^j \right)} \right)
\end{gathered}$$

(where the first complex is concentrated in the zero component.) We have a natural map from $F.$ to the first sheaf ($i = -l, \ldots$) which is compatible with the map to the second. Using the specific description of the differentials in i and l we see that this map induces a map of double complexes. Passing to the total complexes we get by some standard spectral sequence arguments a quasiisomorphism too.

It remains to calculate the cohomology of the first double complex

$$i, l \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} [\sigma]_* \underline{F_\tau^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right)}}.$$

It is sufficient to verify that for all nondegenerate simplices η the natural map from F_η into the total complex of the double complex

$$\text{(8.16)} \qquad i, l \longmapsto \prod_{\substack{\sigma \in X_{-i} \\ \text{nondeg.}}} \prod_{\substack{\tau \in X_l \\ \text{nondeg.}}} F_\tau^{\pi_0\left(\sigma \backslash \mathrm{Sur}^{-1}[\tau]\right) \times \pi_0\left(\eta \backslash \mathrm{Sur}^{-1}[\sigma]\right)}$$

is a quasiisomorphism. Let us fix some $\eta \in X_s$.

Remember that the differentials in i and l are induced by the cosimplicial structure: If $\alpha : \underline{-i} \to \underline{-i}'$ is the map of ordered sets and $\alpha : \sigma \to \sigma'$ the various maps with the same underlying map of ordered sets, then the natural functors

$$\begin{aligned}
\sigma' \backslash \mathrm{Sur}^{-1}[\tau] & \xrightarrow{\alpha^*} \sigma \backslash \mathrm{Sur}^{-1}[\tau] \\
\eta \backslash \mathrm{Sur}^{-1}[\sigma] & \xrightarrow{\alpha_*} \eta \backslash \mathrm{Sur}^{-1}[\sigma']
\end{aligned}$$

induce the cosimplicial structure map in the i-direction.

For $\beta : \underline{l} \to \underline{l}'$ with maps $\beta : \tau \to \tau'$ the cosimplicial structure map is induced by the functor

$$\sigma\backslash \mathrm{Sur}^{-1}[\tau] \xrightarrow{\beta_*} \sigma\backslash \mathrm{Sur}^{-1}[\tau']$$

and the map $F_\tau \to F_{\tau'}$.

By Lemma 8.11 we can replace π_0 by a specific set:

$$(8.17) \qquad i, l \longmapsto \prod_{\substack{\sigma\in X_{-i}\\ \text{nondeg.}}} \prod_{\substack{\tau\in X_l\\ \text{nondeg.}}} F_\tau^{\{\eta\to\sigma\to\tau\}}.$$

The cosimplicial structures in i and l are induced by composition, since the diagrams

$$\begin{array}{ccc} \sigma'\backslash \mathrm{Sur}^{-1}[\tau] & \xrightarrow{\alpha^*} & \sigma\backslash \mathrm{Sur}^{-1}[\tau] \\ \uparrow & & \uparrow \\ \{\sigma'\to\tau\} & \xrightarrow{\alpha^*} & \{\sigma\to\tau\} \end{array}, \qquad \begin{array}{ccc} \eta\backslash \mathrm{Sur}^{-1}[\sigma] & \xrightarrow{\alpha_*} & \eta\backslash \mathrm{Sur}^{-1}[\sigma'] \\ \uparrow & & \uparrow \\ \{\eta\to\sigma\} & \xrightarrow{\alpha_*} & \{\eta\to\sigma'\} \end{array}$$

and their analog for τ and β are commmutative.

Let us now fix l. Consider the complex 8.17 as complex in i. It is the product of complexes for the various $\tau \in X_l$. Let us fix some nondegenerate $\tau \in X_l$ and consider the complex with this τ:

$$i \longmapsto \prod_{\substack{\sigma\in X_{-i}\\ \text{nondeg.}}} F_\tau^{\{\eta\to\sigma\to\tau\}}.$$

Looking at the cosimplicial structure maps which are given by commutative diagrams

$$\begin{array}{ccccc} \eta & \longrightarrow & \sigma & \longrightarrow & \tau \\ \| & & \downarrow & & \| \\ \eta & \longrightarrow & \sigma' & \longrightarrow & \tau \end{array}$$

we see that this complex is the product of complexes with a fixed composition map $\eta \to \sigma \to \tau$.

So let us fix some $\gamma : \eta \to \tau$, we get the complex

$$(8.18) \qquad C^{\cdot}(F_\tau) := \left(\cdots \longrightarrow \prod_{\substack{\eta\xrightarrow{a}\sigma\xrightarrow{b}\tau \\ \text{with } \sigma\in X_{-i} \\ \text{and } b\circ a=\gamma}} F_\tau \longrightarrow \prod_{\substack{\eta\xrightarrow{a}\sigma\xrightarrow{b}\tau \\ \text{with } \sigma\in X_{-i-1} \\ \text{and } b\circ a=\gamma}} F_\tau \longrightarrow \cdots \right)$$

with differential given by the cosimplicial structure.

Lemma 8.19. *The complex given in 8.18 is exact for $\gamma \neq id$. For $\gamma = id$ the complex is $F_\eta[s]$ $(\eta \in X_s)$.*

Let us first prove the duality theorem with the help of this lemma.

Returning to the double complex 8.16 we have for every $l \neq s$ that the complex i is exact and for $l = s$ it has cohomology at $i = -s$ given by the natural morphis from F_η to the component with $l = s$, $i = -s$. Hence the total complex has coh mology only in the zero-component and the map from F_η to the zero component is quasiisomorphism. This proves the Verdier duality theorem.

It remains to prove the lemma.

Proof. The complex $C^{\cdot}(F_\tau)$ is induced by a cosimplicial structure which gives homological complex. Let us return to the associated homological complex $C_{\cdot}(F_\tau$ This will be easier for notation.

Let us consider the index sets of the complex. Define

$$\begin{aligned} Y_j &:= \left\{\eta \xrightarrow{a} \sigma \xrightarrow{b} \tau\right\} && \text{with } \sigma \in X_{s+j+1} \text{ and } b\circ a = \gamma \\ &= \left\{\underline{s} \xrightarrow{a} \underline{s+j+1} \xrightarrow{b} \underline{l}\right\} && \text{with } b \circ a = \gamma, \end{aligned}$$

with $i \geq -1$.

For further calculations we allow $\underline{-1}$ as the empty set, i.e., we are working in t category of finite ordered sets with empty set $\check{\Delta}$. We consider in the following on injective maps of ordered sets.

Let $0 \leq \gamma(0) < \gamma(1) < \cdots < \gamma(s) \leq l$ be the image of $\gamma : \underline{s} \to \underline{l}$. Let $\gamma(-1) :=$ - and $\gamma(s+1) := l+1$. We consider the maps

$$\begin{array}{cccc} \beta_k : & \underline{\gamma(k)-\gamma(k-1)-2} & \longrightarrow & \underline{l} \\ & i & \longmapsto & i+\gamma(k-1) \end{array}$$

for $k = 0, \ldots, s+1$.

Let $\check{\Delta}^{(k)} := \check{\Delta}.[\gamma(k) - \gamma(k-1) - 2]$ (we consider augmented simplicial sets). R mark that the category $\check{\Delta}$ has a fibre product and it makes sense to define maps

$$\begin{array}{cccc} \varphi_{j,k} : & Y_j & \longrightarrow & \check{\Delta}^{(k)} \\ & x = (\underline{s} \xrightarrow{a} \underline{s+j+1} \xrightarrow{b} \underline{l}) & \longmapsto & \varphi_{j,k}(x) \end{array}$$

$$\text{with} \quad \begin{array}{ccc} \underline{i} & \xrightarrow{\varphi_{j,k}(x)} & \underline{\gamma(k)-\gamma(k-1)-2} \\ \downarrow & & \beta_k \downarrow \\ \underline{s+j+1} & \xrightarrow{b} & \underline{l} \end{array} \quad \text{a fibre product}.$$

Gluing these maps for various k together we get a map of sets

$$\varphi_j : Y_j \longrightarrow \coprod_{\substack{l_0,\ldots,l_{s+1} \\ -1\leq l_k \\ \text{for } k=0,\ldots,s+1 \\ \sum_k (l_k+1)=j+1}} \check{\Delta}^{(0)}_{l_0} \times \cdots \times \check{\Delta}^{(s+1)}_{l_{s+1}}.$$

It can be seen that this map is bijective.

For an augmented simplicial set $\check{Z}$ we denote by $C.(\check{Z}.,M)$ the augmented homology complex with coefficients in a R-module M, i.e., we add the -1-th component to the standard homology complex:

$$\dots \longrightarrow \bigoplus_{\sigma\in\check{Z}_0} M \longrightarrow \bigoplus_{\sigma\in\check{Z}_{-1}} M \longrightarrow 0\,.$$

If we consider the nondegenerate part we write $C^{\mathrm{red}}_{\cdot}(\check{Z}.,M)$.

The maps φ_j for $j=-1,\dots$ induce an isomorphism

$$C.(F_\tau)[s+1] \;\simeq\; F_\tau \otimes \bigotimes_{i=0}^{s+1} C^{\mathrm{red}}_{\cdot}(\check{\Delta}^{(i)},R)\,. \tag{8.19}$$

Let us verify that this gives an isomorphism of complexes:

First of all we may assume that $F_\tau = R$. We consider the differentials on the j-th components of the complexes.

Fix some

$$\underline{s} \xrightarrow{a} \underline{s+j+1} \xrightarrow{b} \underline{l} \in Y_j\,.$$

Let

$$\left(\underline{l_0} \xrightarrow{c_0} \underline{\gamma(0)-\gamma(-1)-2}, \dots, \underline{l_{s+1}} \xrightarrow{c_{s+1}} \underline{\gamma(s+1)-\gamma(s)-2}\right)$$

be the image under φ_j, i.e., the fiberproduct with the β_k, $k=0,\dots,s+1$. In particular, we see that

$$a(k) - a(k-1) \;=\; l_k + 2 \quad \text{for all} \quad k = 0,\dots,s+1$$

with $a(-1) := -1$, $a(s+1) := s+j+2$.

Let us calculate the image of the differential on these components in $C_j[s+1]$, resp. $\left(\check{\Delta}^{(0)}_{l_0} \times \dots \times \check{\Delta}^{(s+1)}_{l_{s+1}}\right)_j$.

The commutative diagrams

$$\begin{array}{ccccc} \underline{s} & \xrightarrow{a} & \underline{s+j+1} & \xrightarrow{b} & \underline{l} \\ \| & & \partial_i\uparrow & & \| \\ \underline{s} & \longrightarrow & \underline{s+j} & \longrightarrow & \underline{l} \end{array} \tag{8.20}$$

with $0 \le i \le s+j+1$, determine a component of the differential, the identical homorphism

$$R_{(\underline{s}\xrightarrow{a}\underline{s+j+1}\xrightarrow{b}\underline{l})} \longrightarrow R_{(\underline{s}\to\underline{s+j}\to\underline{l})}$$

multiplied by $(-1)^i$.

The differential on the right-hand side is a combination of the differentials on the $C^{\mathrm{red}}(\check{\Delta}^{(i)},R)$ multiplied by $+1$ or -1. For all k_0 with $0 \le k_0 \le s+1$ the diagrams

$$\begin{array}{ccc} \underline{l_{k_0}} & \xrightarrow{c_{k_0}} & \underline{\gamma(k_0)-\gamma(k_0-1)-2} \\ \delta_{i'}\uparrow & & \| \\ \underline{l_{k_0}-1} & \to & \underline{\gamma(k_0)-\gamma(k_0-1)-2} \end{array}\;,\quad \begin{array}{ccc} \underline{l_k} & \xrightarrow{c_k} & \underline{\gamma(k)-\gamma(k-1)-2} \\ \| & & \| \\ \underline{l_k} & \to & \underline{\gamma(k)-\gamma(k-1)-2} \end{array} \quad \text{for } k \ne k_0 \tag{8.21}$$

with $0 \le i' \le l_{k_0}$ give a component of the differrential, the identical homomorphism

$$\begin{array}{c} R_{\left(\underline{l_0}\xrightarrow{c_0}\underline{\gamma(0)-\gamma(-1)-2},\dots,\underline{l_{k_0}}\xrightarrow{c_{k_0}}\underline{\gamma(k_0)-\gamma(k_0-1)-2},\dots,\underline{l_{s+1}}\xrightarrow{c_{s+1}}\underline{\gamma(s+1)-\gamma(s)-2}\right)} \\ \downarrow \\ R_{\left(\underline{l_0}\xrightarrow{c_0}\underline{\gamma(0)-\gamma(-1)-2},\dots,\underline{l_{k_0}-1}\to\underline{\gamma(k_0)-\gamma(k_0-1)-2},\dots,\underline{l_{s+1}}\xrightarrow{c_{s+1}}\underline{\gamma(s+1)-\gamma(s)-2}\right)} \end{array}$$

multipliied by $(-1)^{l_0+\dots+l_{k-1}+i'}$.

For a given diagram (8.20) we can apply to it the fibreproduct with β_k for all $k=0,\dots,s+1$. It can be seen that we get a system of diagrams of type (8.21) where k_0 is given by the relation $\gamma(k_0) > b(i) > \gamma(k_0-1)$ (or the equivalent relation $a(k_0) > i > a(k_0-1)$) and i' is given by $i' = i - a(k_0-1)+1$.

On the other hand for a given family of diagrams of type (8.21) we have a uniquely defined diagram of type (8.20) given by $i = i' + a(k_0-1) - 1$ mapped to this one.

This shows that both differentials map to the same components with respect to φ_{j-1}. It remains to compare the signs, on the left we have $(-1)^i$ and on the right $(-1)^{l_0+\dots+l_{k_0-1}+i'}$, but

$$\begin{aligned} l_0 + \dots + l_{k_0-1} + i' &= (a(0)-a(-1)-2) + \dots + (a(k_0-1)-a(k_0-2)-2) + i' \\ &= -1 + a(k_0-1) - 2k_0 + i' \\ &= i - 2k_0\,. \end{aligned}$$

This shows that the signs coincide, hence the relation (8.19) is satisfied.

The complex $C^{\mathrm{red}}_{\cdot}(\check{\Delta}.[\underline{m}],R)$ is exact for $m \ge 0$. This follows from the analogous result for the cohomology complex (which is a complex of finitely generated free R-modules) by applying $\mathcal{H}om(?,R)$. (The -1-th component cancels the (co)homology on the 0-th component, if we consider the non-augmented case.) For $m=-1$ we get the complex $R.[1]$. $C^{\mathrm{red}}_{\cdot}(\check{\Delta}.[\underline{m}],R)$ is a finite complex of flat R-modules. Furthermore all images of the differentials are flat R-modules, this follows from the exactness of the complex for $m \ge 0$ (and is trivial for $m=-1$) by induction using the argument that if

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is an exact sequence of R-modules and P, P'' are flat, then P' is also flat. This gives us the possibility to apply successively the Kuenneth-formula to calculate the homology of a product of such complexes given by $m_0,\dots,m_t$. We get that the complex

$$\bigotimes_{k=0}^{t} C^{\mathrm{red}}_{\cdot}(\check{\Delta}.[\underline{m_k}],R) \;=\; \begin{cases} R[s+1] & \text{if } m_0 = \dots = m_s = -1\,, \\ \text{exact}\,, & \text{otherwise}\,. \end{cases}$$

It continues to be a finite complex of flat R-modules and by the same argument as before we see that all images of the differentials are flat R-modules. Applying Kuenneth-formula we get for an arbitrary R-module M that

$$M \otimes \bigotimes_{k=0}^{t} C^{\mathrm{red}}_{\cdot}(\check{\Delta}.[\underline{m_k}],R) \;=\; \begin{cases} M[s+1] & \text{if } m_0 = \dots = m_s = -1\,, \\ \text{exact}, & \text{otherwise}\,. \end{cases}$$

Applying this result to the isomorphism (8.19) we see that $C^{\cdot}(F_\tau)$ is exact, except for the case

$$\gamma(k) - \gamma(k-1) - 2 = -1 \quad \text{for all} \quad k = 0, \dots, s+1,$$

i.e., $\gamma = id$. In the last case it is given by F_τ. This proves Lemma 8.19. □

Concluding, the Verdier-duality theorem 8.9 is proved. □

8.7. Comparison with the topological case

The main point of this paragraph is to show the following proposition:

Proposition 8.20. *Let $X.$ be a locally finite simplicial set of finite dimension and $f : X. \to \Delta.[\underline{0}]$ the map to the point. Assume that $X.$ has the property that the boundary of an arbitrary nondegenerate simplex is nondegenerate. Then the following diagram is commutative:*

$$\begin{array}{ccc} D^+(X.) & \xrightarrow{\;|\,|\;} & D^+(|X.|) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle |f|^!} \\ \text{R-Mod} & = & \text{R-Mod}\,. \end{array}$$

In particular, $|\mathcal{D}_{X}|$ is a dualizing sheaf complex of the topological space $X.$.

Before we pass to the proof of the proposition, we need certain preparations.

We use the usual properties of the topological Verdier duality (see [12] and [10]).

For an arbitrary locally compact topological space of finite dimension Y we denote by $\mathcal{D}_Y^{\cdot}$ a dualizing sheaf complex. The next lemma seems to be well known, but we could not find a reference.

Lemma 8.21. *Let X be a locally compact topological space of dimension not greater than n, $Z \subseteq X$ a closed subset of dimension not greater than $n-1$ and $U = X \setminus Z$ the open complement. Denote by $i : Z \to X$ and $j : U \to X$ the inclusions. For $G^{\cdot} = \mathcal{D}_X^{\cdot}$ we consider the morphism*

$$\mathbf{R}j_*\mathcal{D}_U \longrightarrow i_*\mathcal{D}_Z[1]$$

which is induced from the distinguished triangle

$$\longrightarrow i_*\mathbf{R}i^!G^{\cdot} \longrightarrow \mathbf{R}j_*j^*G^{\cdot} \longrightarrow i_*\mathbf{R}i^![1] \longrightarrow .$$

Applying to this map H^{-n} and considering the sections over an open set $V \subseteq X$ we get a map

$$\begin{array}{ccc} \mathcal{H}om(H_c^n(V\cap U,\underline{R}),R) & & \mathcal{H}om(H_c^{n-1}(V\cap Z,\underline{R}),R) \\ \wr\| & & \wr\| \\ j_*H^{-n}(\mathcal{D}_U)(V) & \longrightarrow & i_*H^{n-1}(\mathcal{D}_Z)(V)\,. \end{array}$$

Then this map is induced by the map

$$H_c^{n-1}(V\cap Z,\underline{R}) \longrightarrow H_c^n(V\cap U,\underline{R})$$

coming from the excision sequence with respect to the decomposion $V = (V\cap Z)\cup(V\cap U)$ by applying H_c^*.

Proof. Let $F^{\cdot}, G^{\cdot} \in D^+(X)$ be two complexes. Let us first check that the following diagram

$$(8.22)\qquad \begin{array}{ccc} \mathcal{H}om(F^{\cdot},\mathbf{R}j_*j^*G^{\cdot}) & \longrightarrow & \mathcal{H}om(F^{\cdot}, i_*\mathbf{R}i^!G^{\cdot}[1]) \\ \wr\| & & \wr\| \\ \mathcal{H}om(j_!j^*F^{\cdot},G^{\cdot}) & \longrightarrow & \mathcal{H}om(i_*i^*F^{\cdot}[-1],G^{\cdot}) \end{array}$$

given by the adjointness of functors and by the usual distinguished triangles is commutative. To see this take a morphism $\varphi : F^{\cdot} \to \mathbf{R}j_*j^*G^{\cdot}$ and consider the diagram with distinguished triangles as rows:

$$\begin{array}{ccccccc} j_!j^*F^{\cdot} & \longrightarrow & F^{\cdot} & \longrightarrow & i_*i^*F^{\cdot} & \longrightarrow & j_!j^*F[1] \\ & & \downarrow{\scriptstyle\varphi} & & & & \\ G^{\cdot} & \longrightarrow & \mathbf{R}j_*j^*G^{\cdot} & \longrightarrow & i_*\mathbf{R}i^!G^{\cdot}[1] & \longrightarrow & G[1]\,. \end{array}$$

We have a (unique) morphism $j_!j^*F^{\cdot} \to G^{\cdot}$ making this diagram commutative and we have also a (unique) morphism $i_*i^*F^{\cdot} \to i_*\mathbf{R}i^!G^{\cdot}[1]$ making this diagram commutative. The images of φ in the diagram (8.22) are obviously related to them. But

$$\mathrm{Hom}(j_!j^*F^{\cdot}, i_*\mathbf{R}i^!G^{\cdot}) = 0, \quad \text{since} \quad i^*j_! = 0.$$

Using some elementary calculations in triangulated categories (or see [3], Prop. 1.1.9. we get that φ has a unique extension to a morphism of triangles. The commutativity of diagram (8.22) follows immediately.

We apply this to $G^{\cdot} = \mathcal{D}_X$ and $F^{\cdot} = j_{V,!}\underline{R}[n]$ where $j_V : V \to X$ is the open inclusion. We get the following commutative diagram

$$\begin{array}{ccc} j_*H^{-n}(\mathcal{D}_U^{\cdot})(V) & \longrightarrow & i_*H^{-n+1}(\mathcal{D}_Z^{\cdot})(V) \\ \wr\| & & \wr\| \\ \mathcal{H}om(j_{V,!}\underline{R}[n], \mathbf{R}j_*j^*\mathcal{D}_X^{\cdot}) & \longrightarrow & \mathcal{H}om(j_{V,!}\underline{R}[n], i_*\mathbf{R}i^!\mathcal{D}_X[1]) \\ \wr\| & & \wr\| \\ \mathcal{H}om(j_!j^*j_{V,!}\underline{R}[n],\mathcal{D}_X^{\cdot}) & \longrightarrow & \mathcal{H}om(i_*i^*j_{V,!}\underline{R}[n-1],\mathcal{D}_X^{\cdot}) \\ \wr\| & & \wr\| \\ \mathcal{H}om(\mathbf{R}\Gamma_c(X, j_!j^*j_{V,!}\underline{R}[n]),R) & \longrightarrow & \mathcal{H}om(\mathbf{R}\Gamma_c(X, i_*i^*j_{V,!}\underline{R}[n-1]),R) \\ \wr\| & & \wr\| \\ \mathcal{H}om(H_c^n(X, j_!j^*j_{V,!}\underline{R}),R) & \longrightarrow & \mathcal{H}om(H_c^{n-1}(X, i_*i^*j_{V,!}\underline{R}),R) \end{array}$$

from which we easily get the assertion.

Lemma 8.22. *Let $X.$ be a locally finite simplicial set of dimension n. Let*

$$U = U_n := \bigcup_{\substack{\sigma\in X_n \\ \text{nondeg.}}} [\sigma](inn(\Delta_\sigma))$$

be the complement to the $n-1$*-skeleton. Denote by* $j_\sigma : inn(\Delta_\sigma) \to \Delta_\sigma$ *the inclusion. For every* $\sigma \in X_n$ *we consider the isomorphism* $\mathcal{D}_{inn(\Delta_\sigma)} \simeq \underline{R}[n]$, *which is given by the fact that for open subsets* $V \subseteq inn(\Delta_\sigma)$ *with* V *homeomorphic to* $\mathbb{R}^n$ *we have*

$$R \simeq \mathcal{H}om(R,R) \simeq \mathcal{H}om(H^n_c(V,\underline{R}),R) \simeq H^{-n}(\mathcal{D}_{inn(\Delta_\sigma)})(V).$$

Then the composition of the morphisms

$$\begin{array}{ccccc} \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R}[n] & & & & \mathbf{R}\,j_*\mathcal{D}_U \\ \downarrow & & & & \downarrow \wr \\ \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\mathbf{R}\,j_{\sigma,*}\underline{R}[n] & \xrightarrow{\sim} & \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\mathbf{R}\,j_{\sigma,*}\mathcal{D}_{inn(\Delta_\sigma)} & \xrightarrow{\sim} & \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} \mathbf{R}\big([\sigma]|_{inn(\Delta_\sigma)}\big)_*\mathcal{D}_{inn(\Delta_\sigma)} \end{array}$$

is an isomorphism.

Proof. Obviously we have $j_{\sigma,*}\underline{R} = \underline{R}$. So it is sufficient to check that $\mathbf{R}^k j_{\sigma,*}\underline{R} = 0$ for $k > 0$. Let $i_\sigma : \partial\Delta_\sigma \to \Delta_\sigma$ be the inclusion of the boundary. From the distinguished triangle

$$\longrightarrow i_{\sigma,*}\mathbf{R}i^!_\sigma\underline{R} \longrightarrow \underline{R} \longrightarrow \mathbf{R}\,j_{\sigma,*}j^*_\sigma\underline{R} \longrightarrow i_{\sigma,*}\mathbf{R}i^!_\sigma\underline{R}[1] \longrightarrow$$

we get an isomorphism

$$\mathbf{R}^k j_{\sigma,*}\underline{R} \xrightarrow{\sim} i_{\sigma,*}\mathbf{R}^{k+1}i^!_\sigma\underline{R}.$$

From the exact sequence

$$0 \longrightarrow j_{\sigma,!}\underline{R} \longrightarrow \underline{R} \longrightarrow i_{\sigma,*}\underline{R} \longrightarrow 0$$

we get an isomorphism

$$\mathbf{R}^{k+1}i^!_\sigma j_!\underline{R} \xrightarrow{\sim} \mathbf{R}^{k+1}i^!_\sigma\underline{R}.$$

But the complex $j_{\sigma,!}\underline{R}[n] \simeq j_{\sigma,!}\mathcal{D}_{inn(\Delta_\sigma)}$ is a dualizing complex on Δ_σ and therefore $\mathbf{R}i^!_\sigma j_{\sigma,!}\underline{R}[n]$ has nonvanishing cohomology only in degree $-n+1$. □

Lemma 8.23. *Let* $\mathcal{T}$ *be a triangulated category and let*

$$\begin{array}{ccccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & A[1] \\ \downarrow & & & & \downarrow & & \downarrow \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & A'[1] \end{array}$$

be a commutative diagram with distinguished triangles as rows. Furthermore assume that $\mathrm{Hom}(A, C'[-1]) = 0$. *Fix some* $\varphi_0 : B \to B'$ *making this diagram commutative. Then the map*

$$\begin{array}{ccc} \mathrm{Hom}(C,A') & \longrightarrow & \mathrm{Hom}(B,B') \\ \lambda & \longmapsto & a'\lambda b + \varphi_0 \end{array}$$

induces a bijection between $\mathrm{Hom}(C,A')$ *and the maps* $\varphi : B \to B'$ *making this diagram commutative.*

Proof. Easy diagram chase. □

Now we pass to the proof of Proposition 8.20.

The proof will be done by induction over the dimension n of the simplicial set $X_.$. Denote by $i : Z = |Z_.| \to X = |X_.|$ the closed inclusion of the $n-1$-skeleton and by $j : U \to X$ the open complement.

We will show that there exists an isomorphism

$$|\mathcal{D}_{X.}| \sim |f^!_{\text{red}}(R)| \xrightarrow{\sim} \mathcal{D}_{|X.|}$$

such that the diagram

$$\text{(8.23)}\qquad \begin{array}{ccc} \mathcal{D}_X & \longrightarrow & j_*\mathcal{D}_U \\ \uparrow & & \uparrow \\ |\mathcal{D}_{X.}| & \longrightarrow & \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R}[n] \end{array}$$

is commutative. Here the right arrow was defined in Lemma 8.22, the upper arrow i given by adjunction and the lower arrow comes from the definition of $f^!_{\text{red}}$.

The case $n = 0$ is obvious. Assume the assertion we have proved up to $n-1$. W consider the following diagram with distinguished triangles as rows

$$\text{(8.24)}\qquad \begin{array}{ccccccccc} \longrightarrow & i_*\mathcal{D}_Z & \longrightarrow & \mathcal{D}_X & \longrightarrow & \mathbf{R}j_*\mathcal{D}_U & \longrightarrow & i_*\mathcal{D}_Z[1] \\ & \uparrow\wr & & & & \uparrow\wr & & \uparrow\wr \\ \longrightarrow & i_*|\mathcal{D}_{Z.}| & \longrightarrow & |\mathcal{D}_{X.}| & \longrightarrow & \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R}[n] & \longrightarrow & i_*|\mathcal{D}_{Z.}|[1] \end{array}$$

where the morphisms come from the induction assumption with respect to $Z.$ ar Lemma 8.22. The lower sequence is given by the exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & f^!_{\text{red},n}(R) & = & \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*f^!_{Z.,\text{red},n-1}(R) & \simeq & f^!_{\text{red},n-1}(R) & \longrightarrow & 0 & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

(By $f_{Z.}$ we denoted the map from $Z.$ to the simplicial point.) Let us verify that t diagram (8.24) is commutative. It is sufficient to check that the diagram

$$\begin{array}{ccc} H^{-n}(\mathbf{R}j_*\mathcal{D}_U^{\cdot}) & \longrightarrow & i_*H^{-n+1}(\mathcal{D}_Z^{\cdot}) \\ \uparrow\wr & & \uparrow\wr \\ \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R} & \longrightarrow & i_*\left|H^{-n+1}(\mathcal{D}_{Z.}^{\cdot})\right| \end{array}$$

is commutative. Denote by $j_Z : U_Z \to Z$ the open inclusion from the complement of the $n-2$-skeleton to Z. By the induction assumption (8.23) with respect to Z, it is sufficient to prove that the diagram

$$\begin{array}{ccccc} H^{-n}(\mathbf{R}\,j_*\mathcal{D}_U^{\cdot}) & \longrightarrow & i_*H^{-n+1}(\mathcal{D}_Z^{\cdot}) & \longrightarrow & i_*j_{Z,*}H^{-n+1}(\mathcal{D}_{U_Z}^{\cdot}) \\ \uparrow\wr & & & & \uparrow\wr \\ \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} [\sigma]_*\underline{R} & \longrightarrow & i_*\left|H^{-n+1}(\mathcal{D}_{Z.}^{\cdot})\right| & \longrightarrow & \prod\limits_{\substack{\sigma\in X_{n-1}\\ \text{nondeg.}}} [\sigma]_*\underline{R} \end{array}$$

is commutative. Let $V \subseteq X$ be an open set. Using Lemma 8.21 we have to check the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}om\left(H_c^n(U\cap V,\underline{R}),R\right) & \longrightarrow & \mathcal{H}om\left(H_c^{n-1}(U_Z\cap V,\underline{R}),R\right) \\ \uparrow\wr & & \uparrow\wr \\ \prod\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} R^{\pi_0(|\sigma|^{-1}(V))} & \longrightarrow & \prod\limits_{\substack{\sigma\in X_{n-1}\\ \text{nondeg.}}} R^{\pi_0(|\sigma|^{-1}(V))}\,. \end{array} \tag{8.25}$$

The upper arrow is induced by the composition of a extension by zero morphism and the boundary morphism of the excision sequence

$$H_c^n(U\cap V,\underline{R}) \longleftarrow H_c^{n-1}(Z\cap V,\underline{R}) \leftarrow H_c^{n-1}(U_Z\cap V,\underline{R})\,.$$

By some elementary calculations we see that this is just the boundary morphism of the excision sequence with respect to the decomposition $X\setminus X^{n-2} = U\cup U_Z$, where X^{n-2} denotes the $n-2$-skeleton of X. The lower arrow is dual to the morphism

$$\begin{array}{ccccc} \bigoplus\limits_{\substack{\sigma\in X_n\\ \text{nondeg.}}} \underline{R}[\pi_0(|\sigma|^{-1}(V))] & \longleftarrow & \bigoplus\limits_{\substack{\tau\in X_{n-1}\\ \text{nondeg.}}} \underline{R}[\pi_0(|\tau|^{-1}(V))] & & \\ \sum\limits_{\substack{0\le i\le n\\ \sigma\in X_n\\ \text{nondeg}\\ \partial_i^*\sigma=\tau}} (-1)^i r\partial_{i,*}(W) & \longleftarrow & rW\,, & & r\in R,\ W\subseteq|\tau|^{-1}(V)\text{connected}\,, \end{array}$$

by the definition of $f_{\text{red}}^{!}$. The left and right arrows are described in Lemma 8.22. The commutativity of diagram (8.25) in the case $V = X$ is obviously the statement of Lemma 6.14 with n changed to $n-1$. The case of general V can be handled by the same procedure as the proof of Lemma 6.14.

Therefore diagram (8.25) is commutative. Using the relations

$$\begin{aligned} \operatorname{Hom}(i_*|\mathcal{D}_{Z.}^{\cdot}|, \mathbf{R}\,j_*\mathcal{D}_U^{\cdot}[-1]) &= 0\,, \quad \text{since} \quad j^*i_* = 0\,, \\ \operatorname{Hom}\left(\prod_{\substack{\sigma\in X_n\\ \text{nondeg.}}} \sigma]_*\underline{R}[n], i_*\mathcal{D}_Z^{\cdot}[-1]\right) &= 0\,, \quad \text{since} \quad H^{-n}(\mathcal{D}_Z^{\cdot}[-1]) = 0\,, \end{aligned}$$

we get from Lemma 8.23 a unique morphism

$$|\mathcal{D}_{X.}^{\cdot}| \longrightarrow \mathcal{D}_X^{\cdot}$$

making diagram (8.24) commutative. This morphism is an isomorphism and satifies trivially the induction property (8.23).

Proposition 8.20 follows now from the natural isomorphisms

$$\left|f^{\cdot}(F^{\cdot})\right| \simeq |\mathcal{D}_{X.}^{\cdot}| \overset{L}{\otimes} |F^{\cdot}| \simeq \mathcal{D}_X^{\cdot} \overset{L}{\otimes} |F^{\cdot}| \simeq f^{!}(|F^{\cdot}|)$$

given in Lemma 8.5 and Lemma 7.13.

References

[1] Beilinson, A. A.: Residues and Adeles, Functional Anal. Appl. **14** (1980), 34–35

[2] Beilinson, A. A.: Letter to A. N. Parshin, Dec. 1985

[3] Beilinson, A. A., Bernstein, J., and Deligne, J.: Faisceaux Pervers, Astérisque **100** (1982)

[4] Deligne, P.: Cohomologie a Support Propres, In: Volume **3** of Théorie des Topos et Cohomologie Etale des Schémas (SGA4), Lect. Notes in Math., Vol. **305**, Exposee 17, pp. 250–480, Springer–Verlag, Berlin–Heidelberg–New York, 1973

[5] Deligne, P., and Lusztig, G.: Duality for Representations of a Reductive Group over a Finite Field, J. Algebra **74** (1982), 284–291

[6] Dold, A., and Puppe, D.: Homologie Nicht–Additiver Funktoren. Anwendungen, Ann. Inst. Fourier (Grenoble) **11** (1961), 201–312

[7] Fimmel, T.: Simplicial Duality and Deligne–Lusztig Duality for Representations of Algebraic Groups over Finite Fields, Preprint

[8] Fimmel, T., and Parshin, A. N.: An introduction into Higher Adelic Theory, (In preparation)

[9] Gabriel, P., and Zisman, M.: Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. **35**, Springer–Verlag, Berlin–New York, 1967

[10] Gel'fand, S. I., and Manin, Yu. I.: Methods in Homological algebra, Nauka, Moscow, 1988, (in Russian)

[11] Hartshorne, R.: Residues and Duality, Lect. Notes in Math., Vol. **20**, Springer–Verlag, Berlin–Heidelberg–New York, 1966

[12] Iversen, B.: Cohomology of Sheaves, Springer–Verlag, Berlin–Heidelberg–New York, 1986

[13] Kan, D. M.: Adjoint Functors, Trans. Amer. Math. Soc. **7** (1958), 294–329

[14] MacLane, S.: Categories for the Working Mathematician, Graduate Texts in Mathematics, V. **5**, Springer–Verlag, Berlin–Heidelberg–New York, 1971

[15] Massey, W. S.: Homology and Cohomology Theory, Marcel Dekker Inc., 1978

[16] Verdier, J.–L.: Dualité dans la Cohomologie des Espaces Localement Compacts, Sem. Bourbaki 1965/66, Exp. **300**, W. A. Benjamin Inc., New York, 1966

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A Counterexample to Completeness Properties for Indefini Sturm – Liouville Problems

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Abstract. In this note we construct an odd weight function w on $(-1,1)$ with $w(x)>0$ for x such that the eigenfunctions of the indefinite Sturm – Liouville problem $-f''=\lambda wf$ with bound conditions $f(-1)=f(1)=0$ do not form a Riesz basis of $L^2_{|w|}(-1,1)$.

1. Introduction

In the recent years a number of papers appeared dealing with completeness pr erties of the eigenfunctions of indefinite Sturm – Liouville problems. The simpl problems of this kind have the form

$$(1.1)\qquad -f'' = \lambda w f \quad \text{on} \quad (-1,1),$$

$$(1.2)\qquad f(-1) = f(1) = 0$$

where $w\in L^1(-1,1)$ is a real weight function. If w is positive on $(-1,1)$, it classical result that the eigenfunctions form an orthonormal basis of the Hilbert sp $L^2_w(-1,1)$. However, if the function w changes its sign at some points in $(-1,1)$, wl we will call "turning points", then it is a nontrivial problem to decide whether eigenfunctions form a Riesz basis of the Hilbert space $L^2_{|w|}(-1,1)$ equipped with scalar product

$$(f,g) := \int_{-1}^{1} f\,\bar g\,|w|\,dx \quad \left(f,g\in L^2_{|w|}(-1,1)\right).$$

By the definition of "Riesz basis" (see e.g. [GK, Chapter VI, §2]) this means the eigenfunctions form an orthonormal basis of $L^2_{|w|}(-1,1)$ with respect to s scalar product equivalent to $(\cdot,\cdot)$. Starting from a paper of BEALS [B] some autl

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