AUTOMORPHISMS OF MANIFOLDS AND ALGEBRAIC K–THEORY FINALE

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Abstract. Let M be a closed topological n–manifold, and let $\mathcal{S}(M)$ be the moduli space of closed topological manifolds equipped with a homotopy equivalence to M. We give an algebraic description of $\mathcal{S}(M)$ in the h-cobordism stable range, assuming $n \geq 5$. (That is, we produce a highly connected map from $\mathcal{S}(M)$ to another space having an algebraic description.) The algebraic description is in terms of L–theory, Waldhausen's algebraic K–theory of spaces, and a natural transformation Ξ (constructed in our paper [WW2]) from L–theory to the Tate cohomology of $\mathbb{Z}_2$ acting on K–theory.

We develop a parallel theory for the moduli space $\mathcal{S}(\tau)$ of $\mathbb{R}^n$–bundles on M equipped with an "equivalence" to the tangent bundle τ of M. (The equivalence is a stable fiber homotopy equivalence of the corresponding spherical fibrations.) Results about moduli spaces of *smooth* manifolds can be obtained by combining the calculations of $\mathcal{S}(M)$ and $\mathcal{S}(\tau)$.

We have attempted to make this paper as self–contained as possible by summarizing results from the earlier papers in the series where necessary.

0. Introduction

Let M^n be a compact topological manifold. By a *structure* on M, we mean a pair (N, f) consisting of another topological manifold N^n and a homotopy equivalence

$$f : (N, \partial N) \longrightarrow (M, \partial M)$$

which restricts to a homeomorphism of the boundaries. One of our goals is to study and compute $\mathcal{S}(M)$, the *space* of structures on M. We refer to §1 for a precise definition; note however that we distinguish between $\mathcal{S}(M)$ and $\tilde{\mathcal{S}}(M)$, the space of *block structures* on M. The goal of classical surgery theory is the calculation of $\tilde{\mathcal{S}}(M)$, not $\mathcal{S}(M)$.

By a *structure* on τ^M, the tangent microbundle of M, we mean a pair (η, g) consisting of another n–dimensional microbundle η on M and a stable fiber homotopy equivalence $g : \eta \to \tau^M$ (over M) of the associated spherical fibrations. Our second goal is to study $\mathcal{S}(\tau^M)$, the *space* of structures on τ. Our third goal is to study and compute a certain map

$$\nabla : \mathcal{S}(M) \longrightarrow \mathcal{S}(\tau^M).$$

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This is defined as follows. By Spivak's result ([Spi], [Bro1], [Bro2]), a structure $f : N \xrightarrow{\simeq} M$ on M has a differential df which is a stable fiber homotopy equivalence between the spherical fibrations made from τ^N and $f^*\tau^M$, respectively. Use this to push the identity structure on τ^N forward to τ^M, and call the result $\nabla(N, f)$.

Summarizing the above, and writing τ for τ^M, we have:

0.1. Program. Compute $\nabla : \mathcal{S}(M) \longrightarrow \mathcal{S}(\tau)$.

0.2 Remark. When $\partial M = \emptyset$, an informal definition of $\mathcal{S}(M)$ is as follows:

$$\mathcal{S}(M) = \coprod_{\beta} G(N_\beta)/TOP(N_\beta)$$

where the coproduct is over the homeomorphism classes of closed manifolds homotopy equivalent to M, and N_β is a representative for the class β. We have written $G(N)$ for the space of homotopy automorphisms of N, and $TOP(N)$ for the space of homeomorphisms $N \to N$. Thus program 0.1 yields results about automorphisms of manifolds.

0.3. Remark. When M is a *smooth* manifold, we can similarly define spaces of *smooth* structures on M and τ (and we use a superscript D for the smoothness). In a relative sense, this makes no difference: the square

$$\begin{array}{ccc} \mathcal{S}^D(M) & \xrightarrow{\nabla} & \mathcal{S}^D(\tau) \\ \big\downarrow & & \big\downarrow \\ \mathcal{S}(M) & \xrightarrow{\nabla} & \mathcal{S}(\tau) \end{array}$$

with forgetful vertical arrows is homotopy cartesian (is a homotopy pullback square) if $\dim(M) \neq 4$. This follows from Morlet's sliced smoothing theory [Mor1], [Mor2], [Mor3], [KiSi], [BuLa1].

Program 0.1 refines the standard program of surgery theory, which we now recall. For greater uniformity we work with "decoration" h; then the n–th homotopy group of the block structure space $\tilde{\mathcal{S}}(M)$ is the set of structures on $M \times \mathbb{D}^n$ modulo h–cobordism over $M \times \mathbb{D}^n$. (For details on $\tilde{\mathcal{S}}(M)$, see §2.) Let $\tilde{\mathcal{S}}(\tau)$ be the space of *stable* structures on τ; a point in this space is a pair (η, g) consisting of a *stable* microbundle η on M and a stable fiber homotopy equivalence $g : \eta \to \tau$ of the associated spherical fibrations. A map ∇ from $\tilde{\mathcal{S}}(M)$ to $\tilde{\mathcal{S}}(\tau)$ can be defined as before, and the surgery program is:

0.4. Program. Compute the map $\nabla : \tilde{\mathcal{S}}(M) \longrightarrow \tilde{\mathcal{S}}(\tau)$.

This was done by Sullivan for simply connected M, and by Wall in the general case. To state their result, we need to say a few words about *assembly*.

0.5. Info. Let $\boldsymbol{F}$ be a functor from finite CW–spaces to spectra. Call $\boldsymbol{F}$ *homotopy invariant* if it respects homotopy equivalences, and call $\boldsymbol{F}$ *excisive* if it respects homotopy pushout squares (= homotopy cocartesian squares) and if $\boldsymbol{F}(\emptyset)$

is contractible. (Excisive implies homotopy invariant.) Given a homotopy invariant $\boldsymbol{F}$, there exists an excisive $\boldsymbol{F}^{\%}$ and a natural transformation $\alpha : \boldsymbol{F}^{\%}(Y) \to \boldsymbol{F}(Y)$ (variable Y) which is a homotopy equivalence if Y is a point. Together, $\boldsymbol{F}^{\%}$ and α are characterized by these properties up to natural homotopy equivalence. This existence and uniqueness statement presumably goes back to Quinn, [Qui1], [Qui2], [Qui3], who calls α the *assembly.* Note: $\boldsymbol{F}^{\%}(Y)$ is homotopy equivalent to $Y_+ \wedge \boldsymbol{F}(*)$, for all Y.

We shall also need assembly in the more general situation where $\boldsymbol{F}$ is a functor on the category of finite CW–spaces Y equipped with stable spherical fibrations γ. (The morphisms in this category are maps between spaces covered by stable maps between spherical fibrations.) Such an $\boldsymbol{F}$, with values in the category of spectra, may or may not be *homotopy invariant*, or *excisive*; the definitions are literally the same as before. If $\boldsymbol{F}$ is homotopy invariant, then it is the target of an essentially unique natural transformation $\alpha : \boldsymbol{F}^{\%}(Y,\gamma) \to \boldsymbol{F}(Y,\gamma)$, with excisive $\boldsymbol{F}^{\%}$, which is a homotopy equivalence if Y is a point. *Notation:* Often $\boldsymbol{F}$ is a functor with a long name, and then it is convenient to write the % superscript *after* the argument, as in $\boldsymbol{F}(Y)^{\%}$. We write

$$\boldsymbol{F}(Y)_{\%} := \text{homotopy fiber of } \alpha : \boldsymbol{F}(Y)^{\%} \longrightarrow \boldsymbol{F}(Y),$$
$$\boldsymbol{F}(Y,\gamma)_{\%} := \text{homotopy fiber of } \alpha : \boldsymbol{F}(Y,\gamma)^{\%} \longrightarrow \boldsymbol{F}(Y,\gamma),$$

as appropriate. We use "unbold" symbols for the corresponding (zero–th) infinite loop spaces, as in

$$F(Y)_{\%},\ F(Y,\gamma)_{\%},\ F(Y)^{\%},\ F(Y,\gamma)^{\%}.$$

0.6. Info. For a space Y with spherical fibration γ, let $\boldsymbol{L}(Y,\gamma,-k)$ be the k–fold loops on the *connected* L–theory spectrum, with decoration h, of the group ring $\mathbb{Z}[\pi_1(Y)]$ with the w–twisted involution ($w = w_1(\gamma)$). See [Ra1], [Ra2] or [WW2]. This may seem to depend on a choice of base point in Y, but it does not. See §4 for details. We are interested in the case $Y = M$, $k = n$, and $\gamma = \nu$ (the Spivak normal fibration of M). The Sullivan–Wall result, as formulated by Quinn [Qui1], [Qui2] and Ranicki[Ra1], is a commutative square

$$\begin{array}{ccc} \tilde{\mathcal{S}}(M) & \xrightarrow{\nabla} & \tilde{\mathcal{S}}(\tau) \\ \Big\downarrow{\scriptstyle\tilde\iota} & & \Big\downarrow{\scriptstyle\tilde\iota} \\ L(M,\nu,-n)_{\%} & \xrightarrow{\text{forget}} & L(M,\nu,-n)^{\%} \end{array}$$

where the vertical arrows $\tilde\iota$ are homotopy equivalences if $\dim(M) \geq 5$.

We now return to program 0.1. Below we describe homotopy invariant functors

$$(Y,\gamma) \mapsto \boldsymbol{LA}(Y,\gamma,-k)$$

(one for each $k \geq 0$) from spaces Y with spherical fibration γ to spectra. Our main result is that the map

$$\nabla : \mathcal{S}(M) \longrightarrow \mathcal{S}(\tau)$$

can be identified *in a certain range* with the forgetful map

$$LA(M,\nu,-n)_{\%} \longrightarrow LA(M,\nu,-n)^{\%}.$$

0.7. Theorem. *There is a commutative square*

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\nabla} & \mathcal{S}(\tau) \\ \big\downarrow{\scriptstyle\iota} & & \big\downarrow{\scriptstyle\iota} \\ LA(M,\nu,-n)_{\%} & \xrightarrow{\text{forget}} & LA(M,\nu,-n)^{\%} \end{array}$$

in which the vertical arrows ι are highly connected (details follow) if $n \geq 5$.

0.8. Details. An integer j is in the *topological h–cobordism stable range* for a compact manifold N^n if the (upper) stabilization maps

$$\mathfrak{H}(N) \to \mathfrak{H}(N \times \mathbb{D}^1) \to \mathfrak{H}(N \times \mathbb{D}^2) \to \cdots$$

of topological h–cobordism spaces are all j–connected. See [Wald2], [Ig]. By [Ig], this is the case when $j < n/3$ approximately, provided N is homeomorphic to a smooth manifold. Then j is also in the *smooth h–cobordism stable range* (defined using spaces of smooth concordances).

The right-hand vertical arrow in 0.7 is $(j+1)$–connected if j is in the smooth h–cobordism stable range for the disk $\mathbb{D}^n$ and $j \leq n-2$. The left–hand vertical arrow induces a bijection on π_0. Each component of $\mathcal{S}(M)$ determines a homeomorphism class of manifolds M' homotopy equivalent to M. If j is in the topological h-cobordism stable range for M', then the left–hand vertical arrow in 0.7 restricted to the component (and the corresponding image component) will be j–connected.

0.9. Description. Here we describe $\boldsymbol{LA}(Y,\gamma,-k)$. Denote by $\boldsymbol{A}(Y)$ the Waldhausen A–theory spectrum of Y, with (the usual) decoration h. The methods of [WW2] yield a map of spectra

$$\Xi : \boldsymbol{L}(Y,\gamma,-k) \longrightarrow \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y))$$

where $\widehat{\boldsymbol{H}}^{\blacktriangledown}$ denotes a Tate cohomology spectrum, and the $\mathbb{Z}_2$–action on $\boldsymbol{A}(Y)$ depends on γ and k. (It is the γ–twisted Spanier–Whitehead k–duality action, [Vo1].) Originally, one of the points of this construction was that it explained and strengthened the connection between L–theory and algebraic K–theory given by the long exact *Rothenberg* sequences; in fact, it leads to *higher* Rothenberg sequences involving higher K–groups. See the *Outline* of [WW2].

Recall that the Tate cohomology spectrum is the mapping cone of the *norm map*

$$\boldsymbol{H}_{\blacktriangledown}(\mathbb{Z}_2\,;\dots) \longrightarrow \boldsymbol{H}^{\blacktriangledown}(\mathbb{Z}_2\,;\dots)$$

from homotopy orbit spectrum to homotopy fixed point spectrum. (See [AdCoDw] or [GreMa] for a lucid and very general account.) *We define $\boldsymbol{LA}(Y,\gamma,-k)$ as the homotopy pullback of*

$$\boldsymbol{L}(Y,\gamma,-k) \xrightarrow{\Xi} \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y)) \hookleftarrow \boldsymbol{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y)).$$

The main theorem 0.7. has a relative version which we want to mention briefly because it explains the A in LA–theory. Let M^n be a compact topological manifold as before, and suppose that $\partial_+ M$ is a compact codimension zero submanifold of ∂M. Write $\partial_- M = \partial M \smallsetminus \mathrm{int}(\partial_+ M)$. A *structure* on $(M, \partial_+ M)$ is a homotopy equivalence of "triads"

$$f : (N, \partial_+ N, \partial_- N) \longrightarrow (M, \partial_+ M, \partial_- M)$$

where N^n is another compact topological manifold with a compact codimension zero submanifold $\partial_+ N$ of ∂N, and f restricts to a homeomorphism from $\partial_- N$ to $\partial_- M$. Write $\mathcal{S}(M, \partial_+ M)$ for the space of such structures. As in the absolute case, there is a map

$$\nabla : \mathcal{S}(M, \partial_+ M) \longrightarrow \mathcal{S}(\tau, \tau_+)$$

where τ is (still) the tangent bundle of M and τ_+ is the tangent bundle of $\partial_+ M$. The relative version of our main theorem "computes"this in a certain range, using a relative version of LA–theory. The result is complicated, and here we just focus on an "extreme"case:

0.10. Example. Take $M = P \times [0,1]$ and $\partial_+ M = P \times \{1\}$. This is extreme for two reasons:

(1) the block structure spaces $\tilde{\mathcal{S}}(M, \partial_+ M)$ and $\tilde{\mathcal{S}}(\tau, \tau_+)$ are contractible;
(2) after much cancellation, the appropriate relative LA–theory turns out to be just the A–theory of P.

Thus our main theorem takes the form of a commutative square

$$\begin{tikzcd}
\mathcal{S}(P \times [0,1]\,,\, P \times \{1\}) \arrow[r, "\nabla"] \arrow[d] & \mathcal{S}(\tau, \tau_+) \arrow[d] \\
A(P)_{\%} \arrow[r, "\text{forget}"] & A(P)^{\%}\ .
\end{tikzcd}$$

with highly connected vertical arrows. Notice, gentle reader: this is really the main result of Waldhausen's work relating h-cobordisms to A–theory. In particular, the space in the upper left–hand corner of the diagram is the space of h–cobordisms on P. (Actually, Waldhausen's theorems state that the vertical arrows in the diagram turn into homotopy equivalences under *stabilization*, [Wald1], [Wald2], [Wald3], [Wald4], [Wald5] and [Ig] shows that they are highly connected without stabilization provided P has a smooth structure.)

0.11. Remark. Theorem 0.7 *refines* the Sullivan–Wall–Quinn–Ranicki result, which means that the cube

$$\begin{tikzcd}
& \tilde{\mathcal{S}}(M) \arrow[rr] \arrow[dd] & & \tilde{\mathcal{S}}(\tau) \arrow[dd] \\
\mathcal{S}(M) \arrow[ur, "\subset"] \arrow[rr] \arrow[dd] & & \mathcal{S}(\tau) \arrow[ur, "\subset"] \arrow[dd] & \\
& L(M,\nu,-n)_{\%} \arrow[rr] & & L(M,\nu,-n)^{\%} \\
LA(M,\nu,-n)_{\%} \arrow[ur, "\text{forget}"] \arrow[rr] & & LA(M,\nu,-n)^{\%} \arrow[ur, "\text{forget}"] &
\end{tikzcd}$$

is commutative. Actually, for strict commutativity, some care has to be exercised in the construction of the upper face of the cube—make it *cofibrant.* A square of simplicial sets is cofibrant if it is isomorphic to a square of the form

$$\begin{array}{ccc} Y_1 \cap Y_2 & \xrightarrow{\subset} & Y_1 \\ {\scriptstyle\subset}\big\downarrow & & {\scriptstyle\subset}\big\downarrow \\ Y_2 & \xrightarrow{\subset} & Y\,. \end{array}$$

0.12. Credits. Localized at odd primes, $LA(M,\nu,-n)$ splits:

$$LA(M,\nu,-n)_{\%} \simeq L(M,\nu,-n)_{\%} \times A(M)^{(\pm)}_{\%}$$

where $A(M)^{(\pm)}_{\%}$ is the $(-1)^n$–eigenspace of the Spanier–Whitehead 0–duality involution [Vo1] on $A(M)_{\%}$ (equivalently, the $(+1)$–eigenspace of the n–duality involution). At odd primes therefore, the left–hand column of the square in Theorem 0.7 is a highly connected map

$$\mathcal{S}(M) \longrightarrow L(M,\nu,-n)_{\%} \times A(M)^{(\pm)}_{\%}.$$

The existence of the loop of such a map is well known (see [BuLa2, Cor.D] and [HJ, Thm.2.5]). Thus working at odd primes avoids the question of how L–theory and A–theory must be combined to get an "algebraic" description of $\mathcal{S}(M)$. Rationally, Burghelea and Fiedorowicz ([BuFie]) have given a highly connected map from $\mathcal{S}(M)$ to a version of hermitian A–theory. We understand that Fiedorowicz, Vogt and Schwänzl are also able to construct such a map at odd primes [FieSchwVo]. It would be interesting to directly compare their version of hermitian A–theory with our LA theory. See also [Gif] and [HS].

0.13. Remark. We have found it necessary to omit proofs of several technical statements in order to make this paper readable. Without exception, these statements are either variations on results in the literature, or straightforward calculations. They are labelled with a ♣ sign. We promise to deliver the proofs in due course.

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1. Structure Spaces

We begin by recalling the notion of *manifold modelled on* $\mathbb{R}^n \times \Delta^k$. This is a Hausdorff space N equipped with a (maximal) atlas with charts in $\mathbb{R}^n \times \Delta^k$, where the changes of charts take points in $\mathbb{R}^n \times d_i\Delta^k$ to ditto points. The *dimension* of such an N is $n+k$. Let s be a nonempty face of Δ^k; denote by $N(s)$ the subspace of N consisting of all points taken to $\mathbb{R}^n \times s$ by some (hence any) chart. We see that

$$s \mapsto N(s)$$

is a functor from $po[k]$, the poset of nonempty faces of Δ^k, to spaces.

We define similarly manifolds N modelled on $\mathbb{R}^n_{\curlywedge} \times \Delta^k$, where $\mathbb{R}^n_{\curlywedge}$ is closed upper half space. (Changes of charts take points in $\mathbb{R}^n_{\curlywedge} \times \Delta^k$ to points in $\mathbb{R}^n_{\curlywedge} \times \Delta^k$.) Denote by ∂N the subspace of N consisting of all points taken by some (hence any) chart to $\mathbb{R}^{n-1} \times \Delta^k$. Then ∂N is modelled on $\mathbb{R}^{n-1} \times \Delta^k$. Further, the rule

$$(s,i) \mapsto \begin{cases} \partial N(s) & \text{if } i = 0 \\ N(s) & \text{if } i = 1 \end{cases}$$

is a functor from the product poset $po[k] \times \{0,1\}$ to spaces.

Now, in order to define structure spaces, we need certain topological categories [Ad, p.70] associated with a fixed compact manifold M^n. (We pay no attention to the size of these categories, but the reader is encouraged to do so.) For simplicity, assume $\partial M = \emptyset$. Denote by $\mathbf{str}(M)$ the following *topological* category. Objects are homotopy equivalences $f : M' \to M$, where M' is another closed manifold. A morphism from $f : M' \to M$ to $g : M'' \to M$ is a homeomorphism $M' \to M''$ making the appropriate triangle commute (strictly). We want to allow continuous variation (in the compact–open topology) of all continuous maps in sight ; this puts a topology on the class of objects of $\mathbf{str}(M)$, and a compatible topology on the class of morphisms.

To construct the classifying space of $\mathbf{str}(M)$, make it into a simplicial category: replace object class and morphism class by their singular simplicial classes.

1.1. Definition. The structure space $\mathcal{S}(M)$ is the classifying space of the topological category $\mathbf{str}(M)$.

Generalizing this slightly, we have the structure space $\mathcal{S}_\lrcorner(N)$ of a manifold N modelled on $\mathbb{R}^n_{\curlywedge} \times \Delta^k$ (assume $\partial N = \emptyset$).

1.1. Definition [bis]. The block structure space is the geometric realization of the simplicial space without degeneracies (alias Δ–space)

$$[k] \mapsto \mathcal{S}_\lrcorner(M \times \Delta^k)\,.$$

If ∂M is nonempty, which we want to allow from now on, define $\mathbf{str}(M, \partial M)$ to have objects $f : (M', \partial M') \to (M, \partial M)$, where f is a homotopy equivalence of pairs. The classifying space is denoted by $\mathcal{S}(M, \partial M)$. Those objects f restricting to a homeomorphism of the boundaries, say $\partial M' \cong \partial M$, form a full subcategory $\mathbf{str}(M)$, with classifying space $\mathcal{S}(M)$. Proceed similarly to define the block structure spaces $\tilde{\mathcal{S}}(M, \partial M)$ and $\tilde{\mathcal{S}}(M)$.

As in [WW1] let $TOP(M)$ be the topological group of homeomorphisms $M \to M$ which agree with the identity near ∂M. A difficult theorem due to [McD] (see also [Seg3], [Math], [Thu]) asserts that the inclusion of $BTOP^\delta(M)$ in $BTOP(M)$ is a homology equivalence, where $TOP^\delta(M)$ is the underlying discrete group. (A map $f : X \to Y$ of connected spaces is a *homology equivalence* if it induces isomorphisms $f_* : H_*(X; J) \to H_*(Y; J)$ for any $\pi_1(Y)$–module J.) In the same spirit, we can make $\mathbf{str}(M)$ more discrete (if not entirely discrete) and see what happens.

Specifically, we want to keep the old topology on the class of objects, but we disallow continuous variation of the horizontal arrows in morphisms

$$
\begin{array}{ccc}
M' & \xrightarrow{\cong} & M'' \\
& \searrow \quad \swarrow & \\
& M &
\end{array}
$$

and we write $\mathcal{S}^\delta(M)$ for the classifying space. If ∂M is empty, for example, then $\mathcal{S}^\delta(M)$ classifies manifold fiber bundles with a flat connection (equivalently, discrete structure group in the sense of Steenrod) and with a fiber homotopy equivalence to a trivial bundle with fiber M. In general, the McDuff theorem implies:

1.2. Theorem. *The inclusion $\mathcal{S}^\delta(M) \hookrightarrow \mathcal{S}(M)$ is a homology equivalence.*

Note in passing that the inclusion of geometric realizations

$$\big|[k] \mapsto \mathcal{S}^\delta_\lrcorner(M \times \Delta^k)\big| \quad \subset \quad \big|[k] \mapsto \mathcal{S}_\lrcorner(M \times \Delta^k)\big| = \tilde{\mathcal{S}}(M)$$

is a homotopy equivalence. This is not a deep theorem.

Following [AnCoFePe] we introduce certain notions: A *control space* is a pair consisting of a locally compact Hausdorff space $\bar{E}$ and an open dense subset E of $\bar{E}$. Let $p : X \to E$ and $q : Y \to E$ be proper spaces over E (which means that X, Y are locally compact, and p, q are proper). A continuous proper map $f : X \to Y$ is a *controlled map* if it satisfies the following condition: Given $z \in \bar{E} \smallsetminus E$, and a neighborhood U of z in $\bar{E}$, there exists a smaller neighborhood U_0 of z in $\bar{E}$ such that $p(x) \in U_0$ implies $q(f(x)) \in U$, for all $x \in X$. It is straightforward to define controlled (proper) homotopies between controlled maps, and then controlled (proper) homotopy equivalences between proper spaces over E. Higher up in the hierarchy are the *morphisms between control spaces.* Such a morphism, say from $(E_1 \subset \bar{E}_1)$ to $(E_2 \subset \bar{E}_2)$, is a map $f : \bar{E}_1 \to \bar{E}_2$ such that $f^{-1}(E_2) = E_1$. The morphism f is a *homotopy equivalence* if there exists another morphism g, from $(E_2 \subset \bar{E}_2)$ to $(E_1 \subset \bar{E}_1)$, such that

$$(gf)_{|E_1} : E_1 \longrightarrow E_1 \ , \qquad (fg)_{|E_2} : E_2 \longrightarrow E_2$$

are controlled homotopy equivalences (between proper spaces over E_1 and E_2, respectively). Note: If f is a homotopy equivalence between control spaces, then f restricts to a *homeomorphism* from $\bar{E}_1 \smallsetminus E_1$ to $\bar{E}_2 \smallsetminus E_2$.

1.3. Example. Let V be a k–dimensional real vector space, $k < \infty$. Denote by $S(V)$ the orbit space of the action of $\mathbb{R}_\curlywedge$ on $V \smallsetminus \{0\}$. The union $\bar{V} := V \cup S(V)$ has a canonical topology such that there *exists* a homeomorphism from $\bar{V}$ to the cone on $S(V)$ extending the identity on $S(V)$. (The homeomorphism is not unique.)

Any compact manifold M^n (as above) and vector space V give rise to a control space $M \times V \subset M * S(V)$, where $*$ means a join.

We will need a space of *structures* on the control space $M \times V \subset M * S(V)$. Somewhat inconsistently, it will be denoted by $c\mathcal{S}(M \times V)$. For simplicity, assume $\partial M = \emptyset$. Define $c\,\mathbf{str}(M \times V)$ as the topological category with objects

$$f : (M' \times V \subset M' * S(V)) \longrightarrow (M \times V \subset M * S(V))$$

where f is a homotopy equivalence of control spaces restricting to the *identity* on $S(V)$. Morphisms are homeomorphisms of control spaces, over $(M \times V \subset M * S(V))$, and the topology comes from allowing continuous variation of all control space morphisms in sight. Let $c\mathcal{S}(M \times V)$ be the classifying space.

1.4. Remark. To be really consistent with [WW1], we should equip V with an inner product and define a space of *bounded* structures on $M \times V$, say $b\mathcal{S}(M \times V)$. But the controlled notions are equally useful ($b\mathcal{S}(M \times V) \simeq c\mathcal{S}(M \times V)$, see [HuTaWi]) and easier to handle.

Define $c\mathcal{S}^\delta(M \times V)$ by analogy with $\mathcal{S}^\delta(M)$, as the classifying space of a more discrete variant of $c\,\mathbf{str}(M \times V)$. (Keep the old topology on the class of objects.) We do not know whether the inclusion of $c\mathcal{S}^\delta(M \times V)$ in $c\mathcal{S}(M \times V)$ is a homology equivalence. But a slightly weaker statement can be proved. Let $\kappa = TOP^\delta(S(V))$ and $\lambda = TOP(S(V))$.

1.5. Theorem$^\clubsuit$. *The inclusion of homotopy orbit spaces*

$$\big(c\mathcal{S}^\delta(M \times V)\big)_{h\kappa} \quad \hookrightarrow \quad \big(c\mathcal{S}(M \times V)\big)_{h\lambda}$$

is a homology equivalence.

(Reason: we can prove a McDuff type theorem for the automorphism group of the control space $(M \times V \subset M * S(V))$, but not for the subgroup of those automorphisms restricting to the identity on $S(V)$. Note that the inclusion of classifying spaces $B\kappa \hookrightarrow B\lambda$ is a homology equivalence, again by [McD]. But note also that in a commutative square of pointed spaces where the vertical arrows are homology equivalences, the induced map of horizontal homotopy fibers need not be a homology equivalence. See [Ber].)

The block structure space $\tilde{\mathcal{S}}(M)$ comes with a filtration which is not easy to understand, and yet crucial. Our way to understand it is to trade block structures for controlled structures. The idea goes back to [WW1, §4], where it is applied to block *automorphisms*.

1.6. Reminder. Let X be a space with a filtration

$$\mathrm{Filt}_0(X) \subset \mathrm{Filt}_1(X) \subset \mathrm{Filt}_2(X) \subset \cdots$$

so that X equals the union of the $\mathrm{Filt}_i(X)$. Call a singular k–simplex $y : \Delta^k \to X$ *positive* if it maps the i–skeleton of Δ^k to $\mathrm{Filt}_i(X)$ for all i. The positive simplices form a simplicial subset

$${}^{pos}X \subset \mathrm{sing}(X)$$

where $\mathrm{sing}(X)$ is the singular simplicial set. Note that ${}^{pos}X$ is still filtered:

$$\mathrm{Filt}_i({}^{pos}X) := {}^{pos}X \cap \mathrm{sing}(\mathrm{Filt}_i(X))\,.$$

1.7. Example. Let $c\mathcal{S}(M\times\mathbb{R}^\infty)$ be the union of the spaces $c\mathcal{S}(M\times\mathbb{R}^i)$ for $i\ge 0$. (Take products with the identity structure on $\mathbb{R}^1$ to include $c\mathcal{S}(M\times\mathbb{R}^i)$ in $c\mathcal{S}(M\times\mathbb{R}^{i+1})$.) Then $c\mathcal{S}(M\times\mathbb{R}^\infty)$ is of course filtered by subspaces $c\mathcal{S}(M\times\mathbb{R}^i)$. We claim that

$$ {}^{pos}c\mathcal{S}(M\times\mathbb{R}^\infty)\simeq\tilde{\mathcal{S}}(M)\,. $$

For a more precise statement, we need simplicial machinery. Let $\mathfrak{Z}$ and $\mathfrak{Y}$ be the Δ–spaces (simplicial spaces without degeneracies) given by

$$ \begin{aligned} \mathfrak{Z}[k] &= {}^{pos}c\mathcal{S}_\lrcorner(M\times\Delta^k\times\mathbb{R}^\infty),\\ \mathfrak{Y}[k] &= \mathcal{S}_\lrcorner(M\times\Delta^k). \end{aligned} $$

Remember that $|\mathfrak{Y}|$ is $\tilde{\mathcal{S}}(M)$.

1.8. Lemma♣. *The inclusions*

$$ |\mathfrak{Y}|\hookrightarrow|\mathfrak{Z}|\hookleftarrow\mathfrak{Z}[0] $$

are homotopy equivalences.

1.9. Remark. A quick and consistent way to introduce *simple structures* is to use the Hilbert cube $\mathcal{Q}$. For example, a homotopy equivalence $f:X\to Y$ between compact ANR's is known to be simple if $f\times\mathrm{id}$ is homotopic to a homeomorphism from $X\times\mathcal{Q}$ to $Y\times\mathcal{Q}$. A controlled homotopy equivalence $f:X\to Y$ between proper ANR's over E (where $E\subset\bar{E}$ is some control space) is *simple*, by definition, if $f\times\mathrm{id}$ is controlled homotopic to a controlled homeomorphism from $X\times\mathcal{Q}$ to $Y\times\mathcal{Q}$. Thus there are "simple" versions of $\mathcal{S}(M)$, $\tilde{\mathcal{S}}(M)$ and $c\mathcal{S}(M\times V)$.

2. Assembly

In this section, all spaces are homotopy equivalent to CW–spaces, all pairs of spaces are homotopy equivalent to CW–pairs, and all spectra are CW–spectra.

A functor $\boldsymbol{F}$ from spaces to spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\boldsymbol{F}$ is *excisive* if $\boldsymbol{F}(\emptyset)$ is contractible and if $\boldsymbol{F}$ preserves homotopy pushout squares (alias homotopy cocartesian squares, see [Go1], [Go2]). The excision condition implies that $\boldsymbol{F}$ preserves finite coproducts, up to homotopy equivalence. Call $\boldsymbol{F}$ *strongly excisive* if it preserves arbitrary coproducts, up to homotopy equivalence.

2.1 Lemma. *For any homotopy invariant $\boldsymbol{F}$ from spaces to spectra, there exist a strongly excisive (and homotopy invariant) $\boldsymbol{F}^\%$ from spaces to spectra and a natural transformation*

$$ \alpha=\alpha_{\boldsymbol{F}}:\boldsymbol{F}^\%\longrightarrow\boldsymbol{F} $$

such that $\alpha:\boldsymbol{F}^\%(*)\to\boldsymbol{F}(*)$ is a homotopy equivalence. Moreover, $\boldsymbol{F}^\%$ and $\alpha_{\boldsymbol{F}}$ can be made to depend functorially on $\boldsymbol{F}$.

Proof. For a space X, let $\mathrm{simp}(X)$ be the category whose objects are maps $\Delta^n\to X$ where $n\ge 0$, and whose morphisms are commutative triangles

$$ \begin{array}{ccc} \Delta^m & \xrightarrow{f_*} & \Delta^n \\ & \searrow\ \ \swarrow & \\ & X & \end{array} $$

where f_* is the linear map induced by an monotone map f from $\{0,1,\dots,m\}$ to $\{0,1,\dots,n\}$. Then

$$\boldsymbol{F}_X : \mathrm{simp}(X) \longrightarrow \text{spectra} \quad ; \quad \big(\Delta^n \xrightarrow{g} X\big) \mapsto \boldsymbol{F}(\Delta^n)$$

is a covariant functor, and we let $\boldsymbol{F}^{\%}(X) := \text{hocolim } \boldsymbol{F}_X$. The natural transformation

$$\boldsymbol{F}_X\big(\Delta^n \xrightarrow{g} X\big) = \boldsymbol{F}(\Delta^n) \quad \xrightarrow{g_*} \quad \boldsymbol{F}(X)$$

induces $\alpha : \boldsymbol{F}^{\%}(X) \to \boldsymbol{F}(X)$. Clearly α is a homotopy equivalence when X is a point. For arbitrary X, we can understand the homotopy type of $\boldsymbol{F}^{\%}(X)$ by using the natural transformation

$$\boldsymbol{F}_X\big(\Delta^n \xrightarrow{g} X\big) = \boldsymbol{F}(\Delta^n) \quad \xrightarrow{(\text{const.})_*} \quad \boldsymbol{F}(*)$$

of functors on $\mathrm{simp}(X)$. It induces a homotopy equivalence of the homotopy direct limits, and since the right–hand functor is constant, its homotopy direct limit is

$$|\,\mathrm{simp}(X)|_+ \wedge \boldsymbol{F}(*)\,.$$

It is an exercise to show that $|\,\mathrm{simp}(X)| \simeq X$. Thus $\boldsymbol{F}^{\%}(X)$ is related to $X_+ \wedge \boldsymbol{F}(*)$ by a chain of natural homotopy equivalences. $\square$

2.2. Observation. *If $\boldsymbol{F}$ is already excisive, then*

$$\alpha : \boldsymbol{F}^{\%}(Y) \longrightarrow \boldsymbol{F}(Y)$$

is a homotopy equivalence for all finite Y, and if $\boldsymbol{F}$ is strongly excisive, then α is a homotopy equivalence for all Y.

Proof. By arguments going back to Eilenberg and Steenrod it is sufficient to verify that α is a homotopy equivalence for $Y =$ point. $\square$

We want to show that $\alpha = \alpha_{\boldsymbol{F}}$ is the "universal" approximation (from the left) of $\boldsymbol{F}$ by a strongly excisive homotopy invariant functor. Suppose therefore that

$$\beta : \boldsymbol{E} \longrightarrow \boldsymbol{F}$$

is another natural transformation with strongly excisive and homotopy invariant $\boldsymbol{E}$. The commutative square

$$\begin{array}{ccc} \boldsymbol{E}^{\%} & \xrightarrow{\alpha_{\boldsymbol{E}}} & \boldsymbol{E} \\ \big\downarrow{\scriptstyle \beta^{\%}} & & \big\downarrow{\scriptstyle \beta} \\ \boldsymbol{F}^{\%} & \xrightarrow{\alpha_{\boldsymbol{F}}} & \boldsymbol{F} \end{array}$$

in which the upper horizontal arrow is a homotopy equivalence by 2.2, shows that β essentially factors through $\alpha_{\boldsymbol{F}}$.

There is a variant of assembly which applies to functors defined on *pairs* of spaces. Let $\boldsymbol{F}$ be such a functor, from pairs (X,Y) to spectra. We call $\boldsymbol{F}$ *homotopy* *invariant* if it takes homotopy equivalences of pairs to homotopy equivalences. We call $\boldsymbol{F}$ *excisive* if it takes homotopy pushout squares of pairs to homotopy pushout squares. (A square of pairs

$$\begin{array}{ccc} (X_1, Y_1) & \longrightarrow & (X_2, Y_2) \\ \downarrow & & \downarrow \\ (X_3, Y_3) & \longrightarrow & (X_4, Y_4) \end{array}$$

is a homotopy pushout square if the two squares made from the X_i and the Y_i, respectively, are homotopy pushout squares.) Finally $\boldsymbol{F}$ is *strongly excisive* if it is excisive and respects arbitrary coproducts, up to homotopy equivalence.

2.3 Lemma♣. *For any homotopy invariant $\boldsymbol{F}$ from pairs of spaces to spectra, there exist a strongly excisive (and homotopy invariant) $\boldsymbol{F}^{\%}$ from pairs of spaces to spectra and a natural transformation*

$$\alpha = \alpha_{\boldsymbol{F}} : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}$$

such that

$$\alpha : \boldsymbol{F}^{\%}(\emptyset \subset *) \to \boldsymbol{F}(\emptyset \subset *)\,, \qquad \alpha : \boldsymbol{F}^{\%}(* \subset *) \longrightarrow \boldsymbol{F}(* \subset *)$$

are homotopy equivalences. Moreover, $\boldsymbol{F}^{\%}$ and $\alpha_{\boldsymbol{F}}$ can be made to depend functorially on $\boldsymbol{F}$.

2.4. Observation. *If $\boldsymbol{F}$ is already excisive, then*

$$\alpha : \boldsymbol{F}^{\%}(X, Y) \longrightarrow \boldsymbol{F}(X, Y)$$

is a homotopy equivalence for all homotopy finite (X, Y), and if $\boldsymbol{F}$ is strongly excisive, then α is a homotopy equivalence for all (X, Y).

2.5. Remark. Let $\boldsymbol{T}$ be a spectrum; then the functor

$$X \mapsto X_+ \wedge \boldsymbol{T}$$

is strongly excisive. Any strongly excisive functor $\boldsymbol{F}$ from spaces to spectra has this form, up to a chain of natural homotopy equivalences. We have verified this in the proof of 2.1 (see also 2.2). The appropriate $\boldsymbol{T}$ is of course $\boldsymbol{F}(*)$.

Let $f : \boldsymbol{T}_1 \to \boldsymbol{T}_2$ be a map of spectra. Then the functor

$$(X, Y) \mapsto \text{ homotopy pushout of } \Big(Y_+ \wedge \boldsymbol{T}_2 \xleftarrow{f_*} Y_+ \wedge \boldsymbol{T}_1 \hookrightarrow X_+ \wedge \boldsymbol{T}_1 \Big)$$

is strongly excisive. Any strongly excisive functor $\boldsymbol{F}$ from pairs of spaces to spectra has this form, up to a chain of natural homotopy equivalences. The appropriate $\boldsymbol{T}_1$ is $\boldsymbol{F}(\emptyset \subset *)$, the appropriate $\boldsymbol{T}_2$ is $\boldsymbol{F}(* \subset *)$, and the appropriate f is induced by the inclusion of $(\emptyset, *)$ in $(*, *)$.

It follows that a strongly excisive $\boldsymbol{F}$ defined on pairs need not take any collapse map $(X, Y) \to (X/Y, *)$ to a homotopy equivalence. It does, however, if $\boldsymbol{T}_2 = \boldsymbol{F}(* \subset *)$ is contractible; then $\boldsymbol{F}$ has the form

$$(X, Y) \mapsto (X/Y) \wedge \boldsymbol{T}_1$$

up to a chain of natural homotopy equivalences.

2.6. Variation♣. We can still do assembly when the functor $\boldsymbol{F}$ is defined on the category of spaces *over* a reference space Z. (For example, Z could be BG, the classifying space for stable spherical fibrations.) By abuse of notation, a map between spaces over Z is a *homotopy equivalence* if it becomes a homotopy equivalence when the reference maps to Z are omitted. A square of spaces over Z is a *homotopy pushout square* if it becomes a homotopy pushout square when the reference maps are omitted. For any homotopy invariant $\boldsymbol{F}$ defined on spaces over Z we have

$$\alpha : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}\,,$$

natural in $\boldsymbol{F}$, where $\boldsymbol{F}^{\%}$ is strongly excisive and

$$\alpha : \boldsymbol{F}^{\%}(* \to Z) \longrightarrow \boldsymbol{F}(* \to Z)$$

is a homotopy equivalence for any point $*$ in Z.

Further, we can still do assembly when the functor $\boldsymbol{F}$ is defined on the category of pairs of spaces over a reference space Z (assuming that $\boldsymbol{F}$ is homotopy invariant).

2.7. Example. Let $\mathcal{G}$ be a topological group with classifying space $B\mathcal{G}$, and suppose that $\mathcal{G}$ acts on a spectrum $\boldsymbol{T}$. For a space over $B\mathcal{G}$, say $f : X \to B\mathcal{G}$, let X^f be the pullback of

$$X \xrightarrow{f} B\mathcal{G} \leftarrow E\mathcal{G}.$$

The functor from spaces over $B\mathcal{G}$ to spectra given by

$$(f : X \to B\mathcal{G}) \quad \mapsto \quad X^f_+ \wedge_{\mathcal{G}} \boldsymbol{T}$$

is strongly excisive. (The example is “typical”, but we shall not go into details.)

2.8. Variation. Suppose that $\boldsymbol{F}$ is a functor from the category of *control spaces* to spectra (see sequel of 1.2). Say that $\boldsymbol{F}$ is *homotopy invariant* if it takes homotopy equivalences of control spaces (sequel of 1.2) to homotopy equivalences of spectra. A homotopy invariant $\boldsymbol{F}$ is *homotopy additive* if the following holds: For any control space $(Y \subset \bar{Y})$ in which Y is a coproduct $Y_1 \amalg Y_2$, the map

$$\boldsymbol{F}(Y_1 \subset \bar{Y}_1) \vee \boldsymbol{F}(Y_2 \subset \bar{Y}_2) \longrightarrow \boldsymbol{F}(Y \subset \bar{Y})$$

induced by the inclusions of the summands is a homotopy equivalence. Of course, $\bar{Y}_i$ is the closure of Y_i in $\bar{Y}$, for $i = 1, 2$.

A commutative square of control spaces and morphisms

$$\begin{array}{ccc} (Y_1 \subset \bar{Y}_1) & \longrightarrow & (Y_2 \subset \bar{Y}_2) \\ \big\downarrow & & \big\downarrow \\ (Y_3 \subset \bar{Y}_3) & \longrightarrow & (Y_4 \subset \bar{Y}_4) \end{array}$$

is a *homotopy pushout square of control spaces* if the induced map from the homotopy colimit of $(Y_3 \leftarrow Y_1 \to Y_2)$ to Y_4 is a controlled homotopy equivalence of spaces over Y_4, and the induced maps $(\bar{Y}_i \smallsetminus Y_i) \to (\bar{Y}_j \smallsetminus Y_j)$ are injections ($i < j$, but not $i = 2$ and $j = 3$). A homotopy invariant $\boldsymbol{F}$ (from control spaces to spectra) is *excisive* if it takes homotopy pushout squares to homotopy pushout squares.

2.8. [bis]♣. *For any homotopy invariant and homotopy additive $\boldsymbol{F}$ from control spaces to spectra, there exists*

$$\alpha : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}\,,$$

natural in $\boldsymbol{F}$, where $\boldsymbol{F}^{\%}$ is excisive and $\alpha : \boldsymbol{F}^{\%}(Y \subset \bar{Y}) \longrightarrow \boldsymbol{F}(Y \subset \bar{Y})$ is a homotopy equivalence whenever Y is discrete.

This is not directly useful unless we know what $\boldsymbol{F}$ does to control spaces $(Y \subset \bar{Y})$ with discrete Y. But sometimes we do. Suppose for example that there exists a chain of natural homotopy equivalences giving

$$\boldsymbol{F}(Y \subset \bar{Y}) \quad \simeq \quad \prod_{y \in Y} \boldsymbol{T}$$

(for discrete Y only), where $\boldsymbol{T}$ is some spectrum. Then the same is true for $\boldsymbol{F}^{\%}$ (same $\boldsymbol{T}$). Since $\boldsymbol{F}^{\%}$ is also excisive, it follows (by Eilenberg–Steenrod arguments) that

$$\boldsymbol{F}^{\%}(Y \subset \bar{Y}) \simeq \operatorname*{holim}_{\text{compact } C \subset Y} \big(Y/(Y \smallsetminus C)\big) \wedge \boldsymbol{T}$$

for *any* $(Y \subset \bar{Y})$. In this case $\pi_*(\boldsymbol{F}^{\%}(Y \subset \bar{Y}))$ is the *locally finite homology* of Y with coefficients in $\boldsymbol{T}$.

2.9. Preview. Let M^n be a compact manifold with Spivak normal bundle ν, as in the introduction. Recall that $\boldsymbol{LA}(M, \nu, -n)$ is defined as a certain homotopy pullback. Since the $_{\%}$–construction respects (natural) homotopy pullbacks, we find that the infinite loop space $LA(M, \nu, -n)_{\%}$ is the homotopy pullback of

$$L(M, \nu, -n)_{\%} \xrightarrow{\ \Xi\ } \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M))_{\%} \hookleftarrow H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M))_{\%}\,.$$

We can therefore construct the map $\iota : \mathcal{S}(M) \longrightarrow LA(M, \nu, -n)_{\%}$ from theorem 0.7 by lifting the composite map

$$\text{(2.}\mathbf{i}\text{)} \qquad \mathcal{S}(M) \hookrightarrow \tilde{\mathcal{S}}(M) \xrightarrow{\ \tilde{\iota}\ } L(M, \nu, -n)_{\%} \xrightarrow{\ \Xi\ } \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M))_{\%}$$

to $H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M))_{\%}$. This is what we will do in the next sections. (In §3, we construct a map from $\mathcal{S}(M)$ to $A(M)_{\%}$; in §4 and §5, we refine this to a map from $\mathcal{S}(M)$ to $H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M))_{\%}$, and in §6 we check compatibility with (2.**i**).)

3. Euler Characteristics

Let Y be a compact ENR. Think of $A(Y)$ as the K–theory of the category of retractive compact ENR's over Y (where the cofibrations are the injections). A small effort is required to make $A(Y)$ into a functor of the variable Y. We leave this to the reader. Any retractive compact ENR over Y determines a point in $A(Y)$; specifically, the retractive space

$$\mathbb{S}^0 \times Y \underset{s}{\overset{r}{\rightleftarrows}} Y$$

where r is the projection and s identifies Y with $\{1\} \times Y$, determines a point

$$\langle Y \rangle \in A(Y).$$

3.1. Definition. We call $\langle Y \rangle$ the *Euler characteristic* of Y. (Note that we are interested in the point $\langle Y \rangle$, not just in its connected component.)

A homotopy equivalence $f : X \longrightarrow Y$ between compact ENR's induces another homotopy equivalence $f_* : A(X) \longrightarrow A(Y)$. It also determines a path $\langle f \rangle$ in $A(Y)$ from $f_*\langle X \rangle$ to $\langle Y \rangle$. Namely, $f_*\langle X \rangle$ is the point in $A(Y)$ corresponding to the retractive space

$$\{-1\} \times X \cup \{1\} \times Y \rightleftarrows Y$$

where the retraction is equal to f on $\{-1\} \times X$. Now f gives a weak equivalence from this retractive space to

$$\mathbb{S}^0 \times Y \rightleftarrows Y.$$

The weak equivalence determines a path in $A(Y)$.

We could continue in this manner, looking e.g. at composable sequences of homotopy equivalences. Perhaps the best way to express the naturality properties of Euler characteristics is the following. Let $\mathcal{C}$ be a small category whose objects are compact ENR's and whose morphisms are homotopy equivalences (composition of morphisms being composition of maps). Form the homotopy inverse limit

$$A(\mathcal{C}) := \operatorname*{holim}_{Y} A(Y)$$

where Y runs over the objects of $\mathcal{C}$. *Euler characteristics define a point* $\langle \mathcal{C} \rangle \in A(\mathcal{C})$. (Use the very explicit description of homotopy inverse limits in [BK, XI.3.4].)

We refer to this type of naturality as *lax naturality.* The Euler characteristic $\langle Y \rangle$ is lax natural with respect to homotopy equivalences. To apply this concept, we need to know more about the space $A(\mathcal{C})$ above. Hence the following definition:

3.2. Definition. A *quasifibration* on a simplicial set X is a covariant functor

$$q: \text{ simplices of } X \longrightarrow \text{ spaces}$$

taking all morphisms between simplices to homotopy equivalences. (A morphism from an m–simplex x to an n–simplex y is a monotone map f from $\{0, 1, \dots, m\}$ to $\{0, 1, \dots, n\}$ such that $f^*(y) = x$.) A *quasisection* of q is a natural transformation from the functor

$$x \mapsto \Delta^{|x|}$$

to q. The space of quasisections is denoted by $\Gamma(q)$. Quasifibrations and quasisections can be pulled back under simplicial maps, just like fibrations and sections.

3.3. Example. Any fibration $p : E \to |X|$ over the geometric realization of X gives rise to a quasifibration q over X, by

$$q(x) = \text{pullback of } \left(E \to |X| \xleftarrow{cx} \Delta^{|x|}\right)$$

where cx is the characteristic map for x. Conversely, let q be a quasifibration over X as in 3.2; let E be the homotopy *direct* limit of q. Then E, which we call the total space of q, comes with a canonical projection map to $|X|$. This is not a fibration in general, but it is a *quasifibration in the sense of Dold–Thom*, [DoTho]. It follows that the space of quasisections of q is homotopy equivalent to the space of sections of the associated fibration.

Suppose that q_1 and q_2 are quasifibrations on X, and that $t : q_1 \to q_2$ is a natural transformation such that $t_x : q_1(x) \to q_2(x)$ is a homotopy equivalence for all simplices x in X. Then t induces a map over X between the total spaces of q_1, q_2 which is a homotopy equivalence and induces a fiber homotopy equivalence of the associated fibrations. We sometimes use this in order to "trivialize" a quasifibration. Thus if q_2 above is a constant functor with value Y, then $\Gamma(q_1) \simeq \Gamma(q_2) \simeq \mathrm{map}(|X|, Y)$.

3.4. Example. Let $\mathcal{C}$ be a small category, and let u be a covariant functor from $\mathcal{C}$ to spaces taking all morphisms to homotopy equivalences. This determines a quasifibration q_u on the nerve of $\mathcal{C}$:

$$(C_0 \to C_1 \to \cdots \to C_n) \longmapsto u(C_n)\,.$$

In this case we have

$$\Gamma(q_u) \cong \operatorname{holim} u$$

by inspection.

3.5. Sub–example. Let $\mathcal{C}$ be the category of all compact ENR's homotopy equivalent to Y (a compact ENR); as morphisms allow homotopy equivalences only. Let u be the functor sending X in $\mathcal{C}$ to $A(X)$. As in 3.4, this leads to a quasifibration on the nerve of $\mathcal{C}$ with fibers homotopy equivalent to $A(Y)$. Euler characteristics define a quasisection $\langle\mathcal{C}\rangle$ of the quasifibration. Observe that the (geometric realization of) the nerve of $\mathcal{C}$ is a classifying space for the *topological* monoid of homotopy automorphisms of Y. See [Fie], [DwKa].

Next we want to lift the Euler characteristic $\langle Y\rangle \in A(Y)$ to $A(Y)^{\%}$, the domain of the assembly map. To do so, we need some facts from controlled A–theory. Suppose that $E \subset \bar{E}$ is a control space, and also that E is an ENR. We form the category of proper retractive ENR's over E, where the morphisms are maps over E and relative to E. Such a morphism is a *cofibration* if it is injective, and a *controlled h–equivalence* if it is a controlled homotopy equivalence (when regarded as a controlled map between spaces over E). Taking controlled h–equivalences as weak equivalences, let $\boldsymbol{A}(E \subset \bar{E})$ be the K–theory spectrum of this category with cofibrations and weak equivalences. The following is a reformulation of a special case of the main theorem of [Vo2]; see also [PeWei], [AnCoFePe], [Vo3] [Vo4]. *Notation:* Y is a compact ENR which we sometimes identify with $Y \times \{0\}$.

3.6. Theorem. *The functor sending Y to the homotopy fiber of the inclusion map*

$$\boldsymbol{A}(Y) \hookrightarrow \boldsymbol{A}\big(Y \times [0,\infty) \subset Y \times [0,\infty]\big)$$

is homotopy invariant and excisive. Moreover, $\boldsymbol{A}\big(Y \times [0,\infty) \subset Y \times [0,\infty]\big)$ is contractible if Y is a point.

The contractibility statement in 3.6 can be proved and generalized as follows. The category of proper retractive spaces over $Y \times [0,\infty)$ has an endomorphism $\boldsymbol{t}$

induced by the shift map

$$Y \times [0,\infty) \to Y \times [0,\infty) \quad ; \quad (y,s) \mapsto (y,s+1)\,.$$

It has another endomorphism

$$\boldsymbol{u} := \boldsymbol{t} \amalg \boldsymbol{t}^2 \amalg \boldsymbol{t}^3 \amalg \cdots .$$

Call a morphism f in the category a *microequivalence* if $\boldsymbol{u}(f)$ is a controlled h–equivalence. (For example, an isomorphism is a microequivalence.) Form the K–theory of the category, allowing only microequivalences as weak equivalences; call it $\boldsymbol{P}(Y)$. Since microequivalences are controlled h-equivalences, we have

$$\boldsymbol{P}(Y) \subset \boldsymbol{A}\big(Y \times [0,\infty) \subset Y \times [0,\infty]\big)\,;$$

the inclusion is an equality if Y is a point.

3.7. Lemma. *The spectrum $\boldsymbol{P}(Y)$ is contractible.*

Proof by Eilenberg–swindle: Note that $\boldsymbol{t}$ and $\boldsymbol{u}$ define self–maps of $\boldsymbol{P}(Y)$. Write $[\boldsymbol{t}]$ and $[\boldsymbol{u}]$ for their homotopy classes. Then

$$[\mathrm{id}] = [\boldsymbol{t}] = [\boldsymbol{u}] - [\boldsymbol{tu}] = [\boldsymbol{u}] - [\boldsymbol{t}][\boldsymbol{u}] = [\boldsymbol{u}] - [\boldsymbol{u}] = [*]. \quad \square$$

We now *decree* that $\boldsymbol{A}(Y)^{\%}$ is the homotopy pullback of

$$\boldsymbol{A}(Y) \hookrightarrow \boldsymbol{A}\big(Y \times [0,\infty) \subset Y \times [0,\infty]\big) \hookleftarrow \boldsymbol{P}(Y)\,.$$

To justify this, we note that

(1) the spectrum $\boldsymbol{A}(Y)^{\%}$ so defined comes with a forgetful map to $\boldsymbol{A}(Y)$;
(2) this forgetful map is a homotopy equivalence when Y is a point;
(3) the functor $Y \mapsto \boldsymbol{A}(Y)^{\%}$ so defined is indeed homotopy invariant and excisive, by 3.7 and 3.6.

Returning to the Euler characteristic $\langle Y \rangle$, we note that

$$\langle Y \rangle \in A(Y) \cap P(Y) \quad \subset \quad A\big(Y \times [0,\infty) \subset Y \times [0,\infty]\big)\,.$$

It follows that $\langle Y \rangle$ has a canonical lift $\langle\!\langle Y \rangle\!\rangle$ to $A(Y)^{\%}$, which we call the *microcharacteristic* of Y. More remarkable is the following: Let $f : X \to Y$ be a homeomorphism between compact ENR's. Then the entire path $\langle f \rangle$ from $f_*\langle X \rangle$ to $\langle Y \rangle$ is contained in $A(Y) \cap P(Y)$. It follows that $\langle f \rangle$ has a canonical lift $\langle\!\langle f \rangle\!\rangle$ to a path from $f_*\langle\!\langle X \rangle\!\rangle$ to $\langle\!\langle Y \rangle\!\rangle$. In other words:

3.8. Proposition. *The microcharacteristic $\langle\!\langle Y \rangle\!\rangle \in A(Y)^{\%}$ is lax natural with respect to homeomorphisms.*

Actually, the same argument shows that the microcharacteristic is lax natural with respect to *cell–like* maps [Lac1], [Lac2], [Lac3]. We will not use this fact.

Back to structure spaces: Let M be a manifold, closed until further notice. Here is some general nonsense related to our definition of structure spaces. Let $\mathcal{C}$ be a simplicial category. Form the ordinary category $\mathcal{C}_\flat$ with objects (j, C) where C is an object in $\mathcal{C}[j]$; a morphism in $\mathcal{C}_\flat$, from (j, C) to (k, D), is a pair (e, g) where e is a monotone map from $[j] = \{0, 1, \ldots, j\}$ to $[k]$ and g is a morphism in $\mathcal{C}[j]$ from C to e^*D. Then the classifying space of $\mathcal{C}$ is homotopy equivalent to that of $\mathcal{C}_\flat$ (by a chain of natural homotopy equivalences). This is a special case of the homotopy colimit theorem [Tho].

Using this, we may think of $\mathcal{S}^\delta(M)$ (see 1.2) as the classifying space $|\mathcal{I}|$ of the following discrete category $\mathcal{I}$. Objects are of the form $f : M' \times \Delta^j \to M \times \Delta^j$ where f is a map over Δ^j and a homotopy equivalence. A morphism

$$\big(f_1 : M' \times \Delta^j \to M \times \Delta^j\big) \quad \longrightarrow \quad \big(f_2 : M'' \times \Delta^k \to M \times \Delta^k\big)$$

is a pair (e, g) where $e : [j] \to [k]$ is monotone (inducing $e_* : \Delta^j \to \Delta^k$) and $g : M' \to M''$ is a homeomorphism making the appropriate triangle commute.

Define a diagram of functors from $\mathcal{I}$ to spaces

$$\begin{tikzcd} F_1^\% \arrow[d] & F_2^\% \arrow[l] \arrow[r] \arrow[d] & F_3^\% \arrow[d] \\ F_1 & F_2 \arrow[l] \arrow[r] & F_3 \end{tikzcd}$$

as follows. The functors send an object $f : M' \times \Delta^j \to M \times \Delta^j$ to

$$\begin{array}{rl} A(M')^\% & (F_1^\%) \\ A(M' \times \Delta^j)^\% & (F_2^\%) \\ A(M)^\% & (F_3^\%) \\ A(M') & (F_1) \\ A(M' \times \Delta^j) & (F_2) \\ A(M) & (F_3) \end{array}$$

and the natural transformations are obvious. The upper row of the diagram trivializes the quasifibration $q^\%$ on $|\mathcal{I}|$ determined by $F_1^\%$, showing that the quasisection $s^\%$ of $q^\%$ determined by the microcharacteristics of the various M' is a map $u^\%$ from $|\mathcal{I}|$ to $A(M)^\%$. The lower row trivializes the quasifibration q on $|\mathcal{I}|$ determined by F_1, showing that the quasisection s of q determined by the Euler characteristics of the various M' is a map u from $|\mathcal{I}|$ to $A(M)$. Lax naturality of the Euler characteristic for homotopy equivalences shows also that u is homotopic to the constant map with value $\langle M \rangle$. Together, $u^\%$ and the nullhomotopy of $u = \alpha u^\%$ determine a map from $|\mathcal{I}| \simeq \mathcal{S}^\delta(M)$ to the homotopy fiber of the assembly

$$\alpha : A(M)^\% \longrightarrow A(M)$$

over the point $\langle M \rangle$. A translation argument (using the infinite loop space structures) identifies this homotopy fiber with $A(M)_\%$, so we have

(3.i)
$$\mathcal{S}^\delta(M) \longrightarrow A(M)_\%.$$

Since $A(M)_{\%}$ is an infinite loop space, its fundamental group is abelian. Now theorem 1.2 and an obstruction theory argument show that the map (3.**i**) factors (uniquely, and up to a preferred homotopy) through a map

$$\mathcal{S}(M) \longrightarrow A(M)_{\%}. \tag{3.ii}$$

Better perhaps, we can say that the homotopy pushout of

$$\mathcal{S}(M) \hookleftarrow \mathcal{S}^{\delta}(M) \hookrightarrow A(M)_{\%}$$

is still an acceptable model of $A(M)_{\%}$ (it is homotopy equivalent to the previous model), and contains $\mathcal{S}(M)$.

Next topic: Euler characteristics and microcharacteristics for control spaces. (See §1, sequel of 1.2.)

3.9. Observation. *The functor*

$$(Y \subset \bar{Y}) \mapsto \boldsymbol{A}(Y \subset \bar{Y})$$

takes homotopy equivalences of control spaces to homotopy equivalences of spectra.

Let $\langle Y \rangle$ be the point in $A(Y \subset \bar{Y})$ determined by the proper retractive space

$$\mathbb{S}^0 \times Y \underset{s}{\overset{r}{\rightleftarrows}} Y\,.$$

3.10. Definition. We call $\langle Y \rangle$ the controlled Euler characteristic of Y. It is lax natural with respect to homotopy equivalences of control spaces.

For microcharacteristics, we need versions of A–theory with excision properties.

3.11. Theorem$^{\clubsuit}$. *The functor sending* $(Y \subset \bar{Y})$ *to the homotopy fiber of the inclusion*

$$\boldsymbol{A}(Y \subset \bar{Y}) \hookrightarrow \boldsymbol{A}\big(Y \times [0,\infty) \subset \bar{Y} \times [0,\infty]\big)$$

is homotopy invariant and excisive (see 2.8*). Moreover,* $\boldsymbol{A}\big(Y \times [0,\infty) \subset \bar{Y} \times [0,\infty]\big)$ *is contractible if* Y *is discrete.*

3.12. Notation. It is only a mild abuse of notation to write $\boldsymbol{A}(Y \subset \bar{Y})^{\%}$ for the homotopy fiber in 3.11, and $\boldsymbol{A}(Y \subset \bar{Y})_{\%}$ for the homotopy fiber of the forgetful map from $\boldsymbol{A}(Y \subset \bar{Y})^{\%}$ to $\boldsymbol{A}(Y \subset \bar{Y})$. Indeed, the last sentence of 3.11 states that $\boldsymbol{A}(Y \subset \bar{Y})^{\%}$ maps by a homotopy equivalence to $\boldsymbol{A}(Y \subset \bar{Y})$ if Y is discrete ; further, a surprisingly difficult theorem of Carlsson [Car] shows that $\boldsymbol{A}(Y \subset \bar{Y})$ maps by a homotopy equivalence to $\Pi_{y\in Y}\boldsymbol{A}(*)$, still for discrete Y. Compare 2.8 and sequel.

We abbreviate

$$c\boldsymbol{A}(M \times V) := \boldsymbol{A}(M \times V \subset M * S(V))$$

(also with decorations %). At this point we must worry a little about ambiguous notation: $c\boldsymbol{A}(M \times V)^{\%}$ may be regarded as a functor in the variable M alone, and then the superscript % has a meaning defined in §2 which might conflict with the "mild abuse" introduced just above. To see that there is no conflict, we have to check agreement only in the case where M is a point, which amounts to proving:

3.13. Lemma♣. $\boldsymbol{A}\big(V\times[0,\infty)\subset\bar V\times[0,\infty]\big)$ *is contractible.*

Proceeding as in 3.7, 3.8, we obtain a microcharacteristic

$$\langle\!\langle Y\rangle\!\rangle\in A(Y\subset\bar Y)^{\%}$$

which is lax natural for homeomorphisms of control spaces. As in the no–control setting, this leads to a map

$$c\mathcal{S}^{\delta}(M\times V)\longrightarrow cA(M\times V)_{\%}\,. \tag{3.iii}$$

The map is equivariant for the actions of $\kappa=TOP^{\delta}(S(V))$. Nevertheless, it appears to fall short of expectations in two respects. For one thing, we really want a map from $c\mathcal{S}(M{\times}V)$ to $cA(M{\times}V)_{\%}$. Theorem 1.5 seems too weak to bridge the gap. For another thing, we would really like to have a map from $c\mathcal{S}(M{\times}V)$ to $cA(M\times V)_{\%}$ which is *natural* in the variable V. Still more: We would like to think of the map as a natural transformation between *continuous* functors in the variable V. Continuity is important because the hyperplane test uses it [WW1, §3]. A digression on the meaning of *continuity* is in order.

3.14. Digression. Let $\mathcal{J}$ be the category of finite dimensional vector spaces V with a positive definite inner product (as in [WW1, Def.1.11]. Morphisms are injective linear maps respecting the inner product. Each morphism set has a topology making it homeomorphic to a Stiefel manifold. We call a covariant functor F from $\mathcal{J}$ to pointed spaces *continuous* if the evaluation maps

$$\mathrm{mor}(V,W)\times F(V)\longrightarrow F(W)\quad;\quad(f,x)\mapsto f_*(x)$$

are continuous for arbitrary V,W in $\mathcal{J}$. (For the purposes of this digression, a pointed space is understood to be a well–pointed space homotopy equivalent to a CW–space.)

A natural transformation $t:F\to G$ between continuous covariant functors from $\mathcal{J}$ to pointed spaces is an *equivalence* if t_V from $F(V)$ to $G(V)$ is a homotopy equivalence for every V. In practice, when an equivalence from F to G exists and has been specified, then we make no further attempt to distinguish between F and G. Keeping this in mind, we can approach the classification of continuous functors from $\mathcal{J}$ to pointed spaces as follows. We replace $\mathcal{J}$ by an equivalent subcategory, namely, the full subcategory with objects $\mathbb{R}^i$ for $i\geq 0$. A continuous functor F from this subcategory to pointed spaces is nothing but a diagram of pointed spaces and pointed maps

$$X_0\xrightarrow{\iota_0}X_1\xrightarrow{\iota_1}X_2\xrightarrow{\iota_2}X_3\to\cdots \tag{3.iv}$$

where each X_n comes with a continuous action of the orthogonal group $O(n)$, and each ι_n is an $O(n)$–map. Note: $O(n)\subset O(n+1)$. (Let $X_n=F(\mathbb{R}^n)$ and let ι_n be the map induced by the standard inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$.) Now write

$$Y_n:=(X_n)_{hO(n)}=\mathcal{E}O(n)\times_{O(n)}X_n\,.$$

Then we have a commutative diagram

(3.**v**)
$$
\begin{tikzcd}
Y_0 \arrow[r] \arrow[d, "p_0"'] & Y_1 \arrow[r] \arrow[d, "p_1"'] & Y_2 \arrow[r] \arrow[d, "p_2"'] & Y_3 \arrow[r] \arrow[d, "p_3"'] & \cdots \\
BO(0) \arrow[r, "\subset"] & BO(1) \arrow[r, "\subset"] & BO(2) \arrow[r, "\subset"] & BO(3) \arrow[r] & \cdots
\end{tikzcd}
$$

where the arrows in the upper row are induced by the maps ι_n, and the vertical arrows p_n are the projection maps, each with a distinguished section. From (3.**v**), we can recover (3.**iv**) up to equivalence. In fact, X_n is contained in the homotopy pullback of

$$\mathcal{E}(n) \to BO(n) \xleftarrow{p_n} Y_n$$

as an $O(n)$–subspace, and the inclusion is a homotopy equivalence. The conclusion is that we may think of a continuous functor F from $\mathcal{J}$ to pointed spaces as a diagram like (3.**v**). Remember that Y_n is determined by F as the homotopy orbit space of $O(n)$ acting on $F(\mathbb{R}^n)$.

Sometimes it is more appropriate to work with $TOP(\mathbb{S}^{n-1})$ instead of $O(n)$. In this connection, note the following. A commutative diagram of the form

(3.**vi**)
$$
\begin{tikzcd}
Z_0 \arrow[r] \arrow[d, "q_0"'] & Z_1 \arrow[r] \arrow[d, "q_1"'] & Z_2 \arrow[r] \arrow[d, "q_2"'] & \cdots \\
BTOP(\mathbb{S}^{-1}) \arrow[r, "\subset"] & BTOP(\mathbb{S}^{0}) \arrow[r, "\subset"] & BTOP(\mathbb{S}^{1}) \arrow[r] & \cdots
\end{tikzcd}
$$

(where the maps q_n are equipped with compatible sections) gives rise to one of the form (3.**v**): let Y_n be the homotopy pullback of

$$BO(n) \hookrightarrow BTOP(\mathbb{S}^{n-1}) \xleftarrow{q_n} Z_n$$

and so on. Furthermore, a commutative diagram of the form

(3.**vii**)
$$
\begin{tikzcd}
Z^\sharp_0 \arrow[r] \arrow[d, "r_0"'] & Z^\sharp_1 \arrow[r] \arrow[d, "r_1"'] & Z^\sharp_2 \arrow[r] \arrow[d, "r_2"'] & \cdots \\
BTOP^\delta(\mathbb{S}^{-1}) \arrow[r, "\subset"] & BTOP^\delta(\mathbb{S}^{0}) \arrow[r, "\subset"] & BTOP^\delta(\mathbb{S}^{1}) \arrow[r, "\subset"] & \cdots
\end{tikzcd}
$$

(where the maps r_n are equipped with compatible sections) gives rise to one of the form (3.**vi**): let Z_n be the homotopy pushout of

$$BTOP(\mathbb{S}^{n-1}) \xleftarrow{\supset} BTOP^\delta(\mathbb{S}^{n-1}) \xrightarrow{\text{section}} Z^\sharp_n$$

where "section" refers to the distinguished section of r_n. The inclusion of Z_n in $Z^\sharp_n$ is a homology equivalence.

First example: Let $Z^\sharp_n := (c\mathcal{S}^\delta(M\times\mathbb{R}^n))_{h\kappa}$ as in theorem 1.5, where $\kappa = \kappa(n)$ is $TOP^\delta(\mathbb{S}^{n-1})$. Let r_n be the projection. Then, as a consequence of 1.5, we find that Z_n is homotopy equivalent to $(c\mathcal{S}(M\times\mathbb{R}^n))_{h\lambda}$, again as in 1.5, where $\lambda = \lambda(n)$ is $TOP(\mathbb{S}^{n-1})$. Therefore Y_n is homotopy equivalent to $(c\mathcal{S}(M\times\mathbb{R}^n))_{hO(n)}$, and finally

X_n is homotopy equivalent to $c\mathcal{S}(M\times\mathbb{R}^n)$. Thus the *continuous* functor F on $\mathcal{J}$ constructed (ultimately) from the spaces $Z_n^\sharp$ and the maps r_n is $V\mapsto c\mathcal{S}(M\times V)$. Details are left to the reader.

Second example: Let $Z_n^\sharp := (cA(M\times V)_\%)_{h\kappa}$, where $\kappa=\kappa(n)$ is $TOP^\delta(\mathbb{S}^{n-1})$. Let r_n be the projection. It is a fact that elements of $TOP(\mathbb{S}^{n-1})$ isotopic to the identity act on $cA(M\times V)_\%$ by automorphisms homotopic to the identity. (See the last sentence of this section for an explanation.) It follows (using the McDuff theorem for $\mathbb{S}^{n-1}$ and a spectral sequence argument) that Z_n has the form $(cA(M\times V)_\%)_{h\lambda}$, for a suitable *continuous* action of $\lambda=\lambda(n)=TOP(\mathbb{S}^{n-1})$ on $(cA(M\times V))_\%$. Then Y_n must be homotopy equivalent to $(cA(M\times V)_\%)_{hO(n)}$, and X_n must be homotopy equivalent to $cA(M\times V)_\%$. Thus the *continuous* functor F on $\mathcal{J}$ constructed (ultimately) from the spaces $Z_n^\sharp$ and the maps r_n is $V\mapsto cA(M\times V)$.

Third example: What prompted the digression was a "morphism" from "first example" to "second example" (display (3.**iii**), sequel of 3.13) and our inability to deal with it. Concluding the digression, we can say that the (3.**iii**) induces a natural transformation of continuous functors, from $\big(V\mapsto c\mathcal{S}(M\times V)\big)$ to $\big(V\mapsto cA(M\times V)_\%\big)$. $\square$

So far relative Euler characteristics have not been mentioned. The relative Euler characteristic of a pair (X,Y) of compact ENR's is an element $\langle X,Y\rangle$ in $A(X,Y)$. Here $\boldsymbol{A}(X,Y)$ is defined as the K–theory of retractive pairs of compact ENR's over the pair (X,Y). The microcharacteristic $\langle\!\langle X,Y\rangle\!\rangle$ is an element of $A(X,Y)^\%$, and the superscript % must be interpreted as in 2.3. The additivity theorem [Wald3] shows that the "obvious" map

$$\boldsymbol{A}(X,Y)\longrightarrow \boldsymbol{A}(X)\times\boldsymbol{A}(Y)\simeq\boldsymbol{A}(X)\vee\boldsymbol{A}(Y)$$

is a homotopy equivalence. With superscripts % added, or with subscripts % added, it is still a homotopy equivalence.

By a straightforward generalization of the absolute case, we have an Euler characteristic type map

$$\mathcal{S}(M,\partial M)\longrightarrow A(M,\partial M)_\%$$

for any compact manifold M with boundary. This fits into a commutative diagram

$$\text{(3.viii)}\qquad \begin{array}{ccc} \mathcal{S}(M,\partial M) & \longrightarrow & A(M,\partial M)_\% \\ \big\downarrow{\scriptstyle\text{forget}} & & \big\downarrow{\scriptstyle\text{forget}} \\ \mathcal{S}(\partial M) & \longrightarrow & A(\partial M)_\%\,. \end{array}$$

Now the easiest way to define an Euler characteristic type map for $\mathcal{S}(M)$ (when $\partial M\neq\emptyset$) is to define it as the induced map between vertical homotopy fibers in diagram (3.**viii**):

$$\mathcal{S}(M)\longrightarrow A(M)_\%\,.$$

(Again, the additivity theorem shows that the inclusion of $A(M)_\%$ in the right–hand vertical homotopy fiber in (3.**viii**) is a homotopy equivalence.)

In an even more relative spirit we now assume that ∂M is the union of two codimension zero compact submanifolds $\partial_0 M$ and $\partial_1 M$ with intersection equal to $\partial\partial_0 M = \partial\partial_1 M$. Allowing structures of the form

$$f : (N, \partial_0 N, \partial_1 N) \longrightarrow (M, \partial_0 M, \partial_1 M)$$

where f is a homotopy equivalence of triads restricting to a homeomorphism from $\partial_1 N$ to $\partial_1 M$, we obtain a relative structure space $\mathcal{S}(M, \partial_0 M)$. Relative Euler characteristics give rise, with the trick just explained, to a map

$$\mathcal{S}(M, \partial_0 M) \longrightarrow A(M, \partial_0 M)_{\%}$$

Applying this insight to $\mathcal{S}(M \times [0,1], M \times \{1\})$, which is the h–cobordism space $\mathfrak{H}(M)$, we get

$$\mathfrak{H}(M) \longrightarrow A(M \times [0,1])_{\%} \simeq A(M)_{\%} \tag{3.ix}$$

by discarding the boundary component of the relative Euler characteristic. This is the Waldhausen map, mentioned at the end of the introduction. Waldhausen's own description of it [Wald1] is quite similar to ours, except that he uses the technology of *simple maps* where we use the McDuff theorem (via 1.2). We think it is much the same thing. There is another construction of the Waldhausen map which we shall need in §6. It requires some preparation. In the commutative square

$$\begin{array}{ccc} A(M)_{\%} & \longrightarrow & cA(M \times [0,\infty))_{\%} \\ \big\downarrow & & \big\downarrow \\ cA(M \times (-\infty, 0])_{\%} & \longrightarrow & cA(M \times \mathbb{R})_{\%} \end{array}$$

(all maps induced by inclusions of control spaces), the upper right–hand and lower left–hand corner are contractible by the Eilenberg swindle. Controlled A–theory as in [Vo2] yields:

3.15. Lemma$^{\clubsuit}$. *The resulting map from $A(M)_{\%}$ to $\Omega(cA(M\times\mathbb{R})_{\%})$ is a homotopy equivalence.*

Notice that there are two "obvious" maps from $A(M)_{\%}$ to $cA(M \times \mathbb{R})_{\%}$. One of these is induced by the inclusion of control spaces, and we have just seen that it is nullhomotopic. The other, denoted $\times\langle\!\langle\mathbb{R}\rangle\!\rangle$, is given by external smash product with the retractive space

$$\mathbb{S}^0 \times \mathbb{R} \underset{s}{\overset{r}{\rightleftarrows}} \mathbb{R}\,.$$

(See [Vo1, §1] for the definition of external smash products.) It is also nullhomotopic. Informal explanation: The class of $\langle\!\langle\mathbb{R}\rangle\!\rangle$ in $\pi_0(A(\mathbb{R} \subset \bar{\mathbb{R}})^{\%})$ is zero, because the whole group is zero. See §6 (before 6.1) for more details and a specific nullhomotopy of $\times\langle\!\langle\mathbb{R}\rangle\!\rangle$, which we need in the following. *(End of preparation.)*

Euler characteristics give us the vertical arrows in a commutative diagram

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\times\mathbb{R}} & c\mathcal{S}(M\times\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ A(M)_\% & \xrightarrow{\times\langle\!\langle\mathbb{R}\rangle\!\rangle} & cA(M\times\mathbb{R})_\% \,. \end{array} \tag{3.x}$$

The upper horizontal homotopy fiber in (3.**x**) is homotopy equivalent to $\mathfrak{H}(M)$ (details left to the reader ; see [WW1, §1, §5] for similar statements). But $\times\langle\!\langle\mathbb{R}\rangle\!\rangle$ is nullhomotopic, so that the map between horizontal homotopy fibers in (3.**x**) becomes

$$\mathfrak{H}(M) \quad\longrightarrow\quad A(M)_\% \times \Omega(cA(M\times\mathbb{R})_\%)$$

and projecting to the second coordinate we have

$$\mathfrak{H}(M) \quad\longrightarrow\quad \Omega(cA(M\times\mathbb{R})_\%) \simeq A(M)_\% \,. \tag{3.xi}$$

3.16. Proposition$^\clubsuit$. *The maps* (3.**ix**) *and* (3.**xi**) *are homotopic.*

We can play the same game with control. The space of controlled h–cobordisms on $M\times\mathbb{R}^i$ is $c\mathfrak{H}(M\times\mathbb{R}^i) := c\mathcal{S}(M\times[0,1]\times\mathbb{R}^i, M\times\{0\}\times\mathbb{R}^i)$. Using relative Euler characteristics (and discarding certain components) we obtain directly a "Waldhausen" map from $c\mathfrak{H}(M\times\mathbb{R}^i)$ to $cA(M\times\mathbb{R}^i)_\%$. The fibration sequence up to homotopy

$$c\mathfrak{H}(M\times\mathbb{R}^i) \longrightarrow c\mathcal{S}(M\times\mathbb{R}^i) \longrightarrow c\mathcal{S}(M\times\mathbb{R}^{i+1})$$

gives us a way to construct a homotopic map using absolute Euler characteristics only.

Note that the Waldhausen map from $c\mathfrak{H}(M\times\mathbb{R}^i)$ to $cA(M\times\mathbb{R}^i)_\%$ is a (usually nonconnected) delooping of another Waldhausen map, from $c\mathfrak{H}(M\times\mathbb{D}^1\times\mathbb{R}^{i-1})$ to $cA(M\times\mathbb{D}^1\times\mathbb{R}^{i-1})_\%$, if $i>0$. To see that $\Omega c\mathfrak{H}(M\times\mathbb{R}^i)$ is homotopy equivalent to $c\mathfrak{H}(M\times\mathbb{D}^1\times\mathbb{R}^{i-1})$, for example, use the commutative square

$$\begin{array}{ccc} c\mathfrak{H}(M\times\mathbb{D}^1\times\mathbb{R}^{i-1}) & \longrightarrow & c\mathfrak{H}(M\times[-1,+\infty)\times\mathbb{R}^{i-1}) \\ \big\downarrow & & \big\downarrow \\ c\mathfrak{H}(M\times(-\infty,1]\times\mathbb{R}^{i-1}) & \longrightarrow & c\mathfrak{H}(M\times\mathbb{R}\times\mathbb{R}^{i-1}) \end{array}$$

and the observation that upper right–hand and lower left–hand corner are contractible. This method is superior to "pushing methods" used in [WW1, §1, §5], because it has a counterpart in controlled A–theory. Granting that the square

$$\begin{array}{ccc} c\mathfrak{H}(M\times\mathbb{R}^i) & \xrightarrow{\text{upper stabilization}} & c\mathfrak{H}(M\times\mathbb{D}^1\times\mathbb{R}^i) \\ {\scriptstyle\text{Waldhausen}}\big\downarrow & & {\scriptstyle\text{Waldhausen}}\big\downarrow \\ cA(M\times\mathbb{R}^i)_\% & \xrightarrow{\simeq} & cA(M\times\mathbb{D}^1\times\mathbb{R}^i)_\% \end{array}$$

commutes up to a preferred homotopy (implicit in ??? below) we conclude that

$$\text{(3.}\mathbf{xii}\text{)} \qquad \left\{ c\mathfrak{H}(M \times \mathbb{R}^i) \xrightarrow{\text{Waldhausen}} cA(M \times \mathbb{R}^i)_{\%} \quad \middle| \quad i \geq 0 \right\}$$

is a map of spectra. The induced map of (-1)–connected covers is a homotopy equivalence

$$\Omega\, \mathbf{Wh}(M) \longrightarrow \boldsymbol{A}(M)_{\%}$$

(Waldhausen's theorem ; $\mathbf{Wh}$ is short for $\mathbf{Wh}^{TOP}$). It is however known that it is not necessary to pass to (-1)–connected covers ; the map of spectra (3.**xii**) is a homotopy equivalence. We shall not use this fact.

4. Spanier–Whitehead Duality and Poincaré Duality

Our goal in this section is to get a good understanding of Spanier–Whitehead duality and Poincaré duality in a controlled setting. We start with an axiomatic description of Spanier–Whitehead duality.

Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Suppose that the weak equivalences in $\mathcal{C}$ satisfy the saturation and extension axioms. Suppose further that the category $\mathcal{Y}$ of compact pointed CW–spaces and pointed cellular maps *acts* on $\mathcal{C}$. The action consists of a bi–exact functor [Wald3, p.342]

$$\wedge : \mathcal{Y} \times \mathcal{C} \longrightarrow \mathcal{C}$$

and natural isomorphisms with suitable coherence properties (details below),

$$\begin{aligned} S^0 \wedge C &\cong C \qquad \text{(unit)} \\ (X \wedge Y) \wedge C &\cong X \wedge (Y \wedge C) \qquad \text{(associativity)}. \end{aligned}$$

Think of $\mathcal{Y}$ as just another category with cofibrations and weak equivalences: the cofibrations are the cellular injections, and the weak equivalences are the homology equivalences. The action functor from $\mathcal{Y} \times \mathcal{C}$ to $\mathcal{C}$ is required to be *exact*: it must take cofibrations to ditto and weak equivalences to ditto. Furthermore, we require

$$X \wedge * \cong *, \qquad * \wedge C \cong *$$

for X in $\mathcal{Y}$ and C in $\mathcal{C}$.

4.1. Example. Let R be a ring (associative, with unit). Let $\mathcal{C}$ be the category of finitely generated chain complexes of projective left R–modules. Define the action by

$$X \wedge C := W(X) \otimes_{\mathbb{Z}} C$$

where $W(X)$ is the reduced cellular chain complex of X.

4.2. Example. Take $\mathcal{C} = \mathcal{Y}$ and let $X \wedge Y$ have the usual meaning.

4.3. Details. When an action of $\mathcal{Y}$ on $\mathcal{C}$ with the above properties is given, we have *two* canonical isomorphisms between $(X \wedge Y \wedge Z) \wedge C$ and $X \wedge (Y \wedge (Z \wedge C))$. They are required to be the same. Next, there are two canonical isomorphisms between $X \wedge (S^0 \wedge C)$ and $X \wedge C$. They are also required to be the same. Finally, there are two canonical isomorphisms between $S^0 \wedge (X \wedge C)$ and $X \wedge C$, which we require to be the same.

Note that $\mathcal{C}$ has a *cylinder functor* (see [Wald3]). The mapping cylinder of a morphism $f : C \to D$ in $\mathcal{C}$ is the pushout of

$$[0,1]_+ \wedge C \hookleftarrow C \xrightarrow{f} D$$

(For the left–hand arrow, which is a cofibration, identify C with $\{1\}_+ \wedge C$ and use the inclusion $\{1\} \subset [0,1]$.) Strictly speaking, the cylinder of f is only determined up to unique isomorphism. The associated suspension functor Σ can be defined directly by $\Sigma X = S^1 \wedge X$.

By a *weak morphism* from an object C in $\mathcal{C}$ to another object D in $\mathcal{C}$, we mean a diagram

$$C \longrightarrow D_1 \xleftarrow{e} D$$

where e is a weak equivalence. The weak morphisms from C to D are the objects of a category $\mathcal{M}(C,D)$; a morphism in $\mathcal{M}(C,D)$, from

$$C \longrightarrow D_1 \xleftarrow{e} D \quad \text{to} \quad C \longrightarrow D_2 \xleftarrow{f} D\,,$$

is a morphism $D_1 \to D_2$ in $\mathcal{C}$ making the appropriate diagram commute. We denote by $[C,D]$ the set of connected components of the nerve of $\mathcal{M}(C,D)$, and we write

$$[[C,D]] := \lim_n [\Sigma^n C, \Sigma^n D]\,.$$

4.4. Definition. A Spanier–Whitehead product (SW–product) on $\mathcal{C}$ is a covariant functor

$$(C,D) \mapsto C \odot D$$

from $\mathcal{C} \times \mathcal{C}$ to pointed spaces which takes pairs of weak equivalences to homotopy equivalences, and which is *symmetric, bilinear*, and *nondegenerate* (explanations follow).

(1) *Symmetry* means that the functor comes with a natural homeomorphism

$$C \odot D \cong D \odot C$$

(natural in both variables) whose square is the identity on $C \odot D$.

(2) *Bilinearity* means (in the presence of symmetry) that, for fixed but arbitrary D, the functor

$$C \mapsto C \odot D$$

takes any *pushout* square in $\mathcal{C}$ where the horizontal arrows are cofibrations to a homotopy pullback square of spaces. (We assume for convenience that all pointed spaces in sight are homotopy equivalent to pointed CW–spaces.) Bilinearity also means that $* \odot D$ is contractible.

(3) The functor in 4.4 is *nondegenerate* if, for every C in $\mathcal{C}$, there exist $n \geq 0$ and B and a class $[\eta] \in \pi_n(B \odot C)$ such that a certain map (slant product)

$$\backslash[\eta] : [[C, D]] \longrightarrow \pi_n(B \odot D)$$

is a bijection for arbitrary D. (We call such an $[\eta]$ an *n–duality.*)

The map $\backslash[\eta]$ takes the class of a weak morphism

$$C \xrightarrow{g} D' \xleftarrow{e} D$$

(where e is a weak equivalence) to $e_*^{-1}(g_*[\eta])$. This defines $\backslash[\eta]$ on $[C, D]$ only; extend to $[[C, D]]$ using the natural isomorphisms (which are a consequence of bilinearity)

$$\pi_{n+k}(B \odot \Sigma^k(\text{—})) \cong \pi_n(B \odot (\text{—}))\,.$$

4.5. Example. Let R be a ring with involution (involutory antiautomorphism). Let $\mathcal{C}$ be the category of finitely generated chain complexes of projective left R-modules, graded over the integers. As cofibrations take injective chain maps which split in each dimension, and as weak equivalences take the chain homotopy equivalences. For C and D in $\mathcal{C}$, let $C \odot D$ be the simplicial abelian group made from the chain complex of abelian groups $C^t \otimes_R D$ by the Dold–Kan method. See [Cur].

4.6 Example. Let $\mathcal{C} = \mathcal{Y}$ as in 4.2. Let $C \odot D := Q(C \wedge D)$, where $Q = \Omega^\infty \Sigma^\infty$.

When $\mathcal{C}$ is equipped with an SW product, the following are defined:

(1) a duality involution on $\boldsymbol{K}(\mathcal{C})$ (more precisely, on something homotopy equivalent to $\boldsymbol{K}(\mathcal{C})$);
(2) a quadratic L–theory spectrum $\boldsymbol{L}(\mathcal{C})$;
(3) a symmetric L–theory spectrum $\boldsymbol{L}^\blacktriangledown(\mathcal{C})$;
(4) maps of spectra

$$\begin{aligned}(1+T) : \boldsymbol{L}(\mathcal{C}) &\to \boldsymbol{L}^\blacktriangledown(\mathcal{C})\,,\\ \Xi : \boldsymbol{L}^\blacktriangledown(\mathcal{C}) &\longrightarrow \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{K}(\mathcal{C}))\,.\end{aligned}$$

We begin with the description of (1), which is technically (not conceptually) the most demanding item. To simplify, let us throw in a hypothesis:

4.7. Hypothesis. *The suspension map $[C, D] \to [\Sigma C, \Sigma D]$ is a bijection for all C, D in $\mathcal{C}$.*

Then one finds $[C, D] \cong [[C, D]]$ for all C, D in $\mathcal{C}$, and the "nondegenerate" property of $\odot$ simplifies to the following:

$$\forall C, \exists B, \exists \eta \in B \odot C : \quad \backslash[\eta] \textit{ from } [C, D] \textit{ to } \pi_0(B \odot D) \textit{ is a bijection, } \forall D.$$

Moreover, we can add a uniqueness statement to the existence statement. For this, let $w\mathcal{C}$ be the category of weak equivalences. Let $w^!\mathcal{C}$ be the *topological* category whose objects are triples (B, C, η) with B, C in $\mathcal{C}$, and $\eta \in B \odot C$ such that slant product with the class of η is a natural bijection. (Note that η can vary continuously—this is where the topology comes from.) A morphism from (B, C, η) to (B', C', ζ) consists of $f : B \to B'$ and $g : C \to C'$ such that $\zeta = (f \odot g)\eta$.

4.8. Lemma. *The forgetful functor $w^!\mathcal{C} \to w\mathcal{C}$ sending (B, C, η) to C induces a homotopy equivalence of the classifying spaces.*

Now recall Waldhausen's simplicial category $S_\bullet\mathcal{C}$ constructed from $\mathcal{C}$. Each $S_m\mathcal{C}$ is a category with cofibrations and weak equivalences whose objects are certain diagrams $\mathfrak{C}$ in $\mathcal{C}$, indexed by the poset of pairs (i, j) (where $0 \le i \le j \le m$) to $\mathcal{C}$. Up to isomorphism, such a diagram is determined by its top row which can be an arbitrary string of cofibrations

$$* = \mathfrak{C}(0,0) \to \mathfrak{C}(0,1) \to \mathfrak{C}(0,2) \to \mathfrak{C}(0,3) \to \cdots \to \mathfrak{C}(0,m)\,.$$

The rest of the diagram consists of chosen subquotients, i.e.

$$\mathfrak{C}(i,j) \cong \mathfrak{C}(0,j)/\mathfrak{C}(0,i)\,.$$

For two objects $\mathfrak{C}, \mathfrak{D}$ in $S_m\mathcal{C}$, we define

$$\mathfrak{C} \odot \mathfrak{D} := \operatorname*{holim}_{\substack{i+q\ge m\\ j+p\ge m}} \mathfrak{C}(i,j) \odot \mathfrak{D}(p,q)\,.$$

4.9. Lemma$^\clubsuit$. *The functor $\odot$ on $S_m\mathcal{C} \times S_m\mathcal{C}$ is an SW–product.*

The best way to understand the SW product in 4.9 is to ask how it depends on m. We abbreviate $[m] := \{0, 1, \ldots, m\}$. Note that the category with objects $[m]$ for $m \ge 0$ and monotone maps as morphisms has an automorphism (conjugation) of order two which takes $f : [k] \to [m]$ to $\bar f = r_m f r_k$ where r_k and r_m are the order reversing bijections of $[k]$ and $[m]$, respectively. Recall also that $[m] \mapsto S_m\mathcal{C}$ is a *simplicial* category. Then, for $\mathfrak{C}$ and $\mathfrak{D}$ in $S_m\mathcal{C}$, and a monotone $f : [k] \to [m]$, we have $f^*\mathfrak{C}$ and $\bar f^*\mathfrak{D}$ in $S_k\mathcal{C}$. For example,

$$(f^*\mathfrak{C})(i,j) = \mathfrak{C}(f(i), f(j))\,, \qquad (\bar f^*\mathfrak{D})(p,q) = \mathfrak{D}(\bar f(p), \bar f(q))\,,$$

and if $i + q \ge k$ and $j + p \ge k$, then $f(i) + \bar f(q) \ge m$ and $f(j) + \bar f(p) \ge m$. It follows that we have a forgetful map

$$\mathfrak{C} \odot \mathfrak{D} \longrightarrow f^*\mathfrak{C} \odot \bar f^*\mathfrak{D}\,.$$

4.9. Lemma [bis]$^\clubsuit$. *The forgetful map ? : $\mathfrak{C} \odot \mathfrak{D} \to f^*\mathfrak{C} \odot \bar f^*\mathfrak{D}$ takes n–dualities to n–dualities, for all n.*

In particular, if $[\eta] \in \pi_n(\mathfrak{C} \odot \mathfrak{D})$ is an n–duality, then $\mathfrak{C}(i,j)$ is n–dual in $\mathcal{C}$ to $\mathfrak{D}(m-j, m-i)$ for $0 \le i \le j \le m$. (Proof: take $k = 1$, and note that $S_1\mathcal{C}$ is isomorphic to $\mathcal{C}$.)

Now for involutions: $w^!\mathcal{C}$ is a category with an involution sending (B, C, η) to (C, B, η'), where η' is the image of η under the flip $B \odot C \cong C \odot B$. Exactly the same thing can be said of $w^!S_m\mathcal{C}$ for $m \ge 0$, but we need to say more.

4.10. Observation. *The rule* $[m] \mapsto w^! S_m \mathcal{C}$ *defines a simplicial category with antisimplicial involution* $\boldsymbol{\iota}$.

Explanation: For monotone $f : [k] \to [m]$, we define f^* from $w^! S_m \mathcal{C}$ to $w^! S_k \mathcal{C}$ by the formula

$$(\mathfrak{C}, \mathfrak{D}, \eta) \mapsto (f^* \mathfrak{C}, \bar{f}^* \mathfrak{D}, ?(\eta)) .$$

Then $\boldsymbol{\iota}$, defined as above in each degree, commutes with the simplicial operators *up to conjugation.* $\square$

Conclusion: the space $\Omega \big| k \mapsto w^! S_k \mathcal{C} \big|$ *is an infinite loop space with involution (e.g. by* [Se2]*), and it maps forgetfully to* $K(C) = \Omega \big| k \mapsto w S_k \mathcal{C} \big|$. *The forgetful map is a homotopy equivalence and (e.g. by* [Se2]*) an infinite loop space map.*

This leaves the question: what if $\mathcal{C}$ does not satisfy hypothesis 4.7 ? We then force it by creating a suitable *stable* category $\mathcal{C}_\omega$. The objects of $\mathcal{C}_\omega$ are the same as those of $\mathcal{C}$, and a morphism from C to D in $\mathcal{C}_\omega$ is an element of

$$\operatorname*{colim}_n \operatorname{mor}_{\mathcal{C}}(\Sigma^n C, \Sigma^n D) .$$

Such a morphism is a cofibration {weak equivalence} if it can be represented by a cofibration {weak equivalence} $f : \Sigma^n C \to \Sigma^n D$, for some n. The next lemma is an easy consequence of [Wald3, 1.6.2].

4.11. Lemma. *The inclusion* $\mathcal{C} \subset \mathcal{C}_\omega$ *induces a homotopy equivalence from* $\boldsymbol{K}(\mathcal{C})$ *to* $\boldsymbol{K}(\mathcal{C}_\omega)$.

It is not difficult to extend the action of $\mathcal{Y}$ on $\mathcal{C}$ to one of $\mathcal{Y}$ on $\mathcal{C}_\omega$. By contrast, it is a little hairy to extend or lift the $\odot$ product from $\mathcal{C}$ to $\mathcal{C}_\omega$. (For better distinction, write $\odot_\omega$ for the new product.) Let $\mathcal{I}$ be the category generated by the diagram of inclusion maps

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d] & E^1_u \arrow[d] & & & \\
E^1_\ell \arrow[r] & S^1 \arrow[r] \arrow[d] & E^2_u \arrow[d] & & \\
& E^2_\ell \arrow[r] & S^2 \arrow[r] \arrow[d] & E^3_u \arrow[d] & \\
& & E^3_\ell \arrow[r] & S^3 \arrow[r] \arrow[d] & \cdots \\
& & & \cdots &
\end{tikzcd}$$

where E^n_u and E^n_ℓ are the upper and lower hemispheres of S^n, respectively. Let $C \odot_\omega D$ be the space of *almost* natural transformations from the identity on $\mathcal{I} \times \mathcal{I}$ to the functor

$$(X, Y) \mapsto (X \wedge C) \odot (Y \wedge D) .$$

Here an *almost* natural transformation need only be defined for X and Y in $\mathcal{I}$ whose (naive) dimension is sufficiently large. Thanks to this little precaution, the rule $(C, D) \mapsto C \odot_\omega D$ is a functor on $\mathcal{C}_\omega \times \mathcal{C}_\omega$.

4.12. Lemma♣. *The functor $(C, D) \mapsto C \odot_\omega D$ is an SW–product on $\mathcal{C}_\omega$. Furthermore, $\mathcal{C}_\omega$ satisfies hypothesis 4.7.*

We turn to the construction of $\boldsymbol{L}^\blacktriangledown(\mathcal{C})$ and $\boldsymbol{L}(\mathcal{C})$. Assume for convenience that hypothesis 4.7 holds—if it does not, apply the procedure sketched in 4.11 and sequel.

By a *symmetric Poincaré object* in $\mathcal{C}$, we mean an object C together with a $\mathbb{Z}_2$–map ϕ from $\mathcal{E}\mathbb{Z}_2$ to a nondegenerate component of $C \odot C$. The symmetric Poincaré objects form a set (or class) $sp(\mathcal{C})$.

Let $\mathcal{C}[m]$ be the category of covariant functors from the poset $po[m]$ (set of nonempty faces of Δ^m) to $\mathcal{C}$. A morphism $f : C \to D$ in $\mathcal{C}[m]$ is a *cofibration* / *weak equivalence* if $f_s : C(s) \to D(s)$ is a cofibration / weak equivalence for every face s of Δ^m.

4.13. Lemma♣. *The formula*

$$C \odot D := \operatorname*{holim}_{s \in po[m]} \quad C(s) \odot D(s)$$

defines an SW product on $\mathcal{C}[m]$. For a monotone $f : [k] \to [m]$, the (obvious) forgetful map from $C \odot D$ to $f^*C \odot f^*D$ takes n–dualities to n–dualities, for all n.

4.14. Definition. $L^\blacktriangledown(\mathcal{C})$ is the realization of the incomplete simplicial set

$$[m] \mapsto sp(\mathcal{C}[m]) \,.$$

(*Incomplete* means: without degeneracy operators.) Using the coproduct in $\mathcal{C}$, we can define a structure of infinite loop space on $L^\blacktriangledown(\mathcal{C})$. That is, $L^\blacktriangledown(\mathcal{C})$ is the underlying space of a Γ–space [Se2]. The associated (-1)–connected spectrum is $\boldsymbol{L}^\blacktriangledown(\mathcal{C})$.

A symmetric Poincaré object in $sp(\mathcal{C})$ determines a homotopy fixed point in the involutive model of $K(\mathcal{C})$ (one might say, the self–dual Euler characteristic ; compare §5 below, sequel of 5.4). The homotopy fixed point is an element in $H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{K}(\mathcal{C}))$. Now replace $\mathcal{C}$ by $\mathcal{C}[m]$, arbitrary m, to obtain a simplicial map from an incomplete simplicial set to a simplicial space,

$$\Big([m] \mapsto sp(\mathcal{C}[m])\Big) \quad \longrightarrow \quad \Big([m] \mapsto H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{K}(\mathcal{C}[m]))\Big) \,.$$

Taking realizations on both sides gives

$$\Xi : L^\blacktriangledown(\mathcal{C}) \longrightarrow \widehat{H}^\blacktriangledown(\mathbb{Z}_2; K(\mathcal{C}))$$

because the simplicial spectrum $\big(m \mapsto \boldsymbol{K}(\mathcal{C}[m])$ is an *augmented $\mathbb{Z}_2$–resolution* of $\boldsymbol{K}(\mathcal{C})$ (see [WW2, §2]). In fact, the additivity theorem of algebraic K–theory implies that

$$\boldsymbol{K}(\mathcal{C}[m]) \simeq \bigvee_{s \in po[m]} \boldsymbol{K}(\mathcal{C})$$

and the involution can be unravelled as in [WW2, 4.6]. The map Ξ is a map of infinite loop spaces, so it could be written in the form

$$\Xi : \boldsymbol{L}^{\blacktriangledown}(\mathcal{C}) \longrightarrow \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}_2; K(\mathcal{C}))\,.$$

Now for quadratic L–theory: A quadratic Poincaré object in $\mathcal{C}$ is a symmetric Poincaré object (C, ϕ) together with a $\mathbb{Z}_2$–nullhomotopy of the composition

$$\mathcal{E}\mathbb{Z}_2 \xrightarrow{\ \phi\ } C \odot C \hookrightarrow \operatorname*{hocolim}_p \quad (S^p \wedge C) \odot (S^p \wedge C)\,.$$

(Note that $C \cong \mathbb{S}^0 \wedge C$, and we use the standard inclusions $\mathbb{S}^p \hookrightarrow \mathbb{S}^{p+1}$.) This definition will be justified in a moment. Construct a space $L(\mathcal{C})$ like $L^{\blacktriangledown}(\mathcal{C})$, replacing symmetric Poincaré objects by quadratic ones. $L(\mathcal{C})$ is an infinite loop space, with spectrum $\boldsymbol{L}(\mathcal{C})$. Tradition forces us to write

$$(1+T) : \boldsymbol{L}(\mathcal{C}) \longrightarrow \boldsymbol{L}^{\blacktriangledown}(\mathcal{C})$$

for the forgetful map. Occasionally we write Ξ instead of $\Xi \cdot (1+T)$.

Justification. Bilinearity of the $\odot$ product shows that $C \odot C$ is an infinite loop space: loosely speaking, it is the $(0,0)$–space of an Ω–bispectrum

$$C \odot^{\infty} C := \{(\mathbb{S}^i \wedge C) \odot (\mathbb{S}^j \wedge C)\}\,.$$

$\mathbb{Z}/2$ acts on this because $\odot$ is symmetric, but the action interchanges vertical and horizontal suspension. We note that

$$C \odot^{\infty} C \to (\mathbb{S}^1 \wedge C) \odot^{\infty} (\mathbb{S}^1 \wedge C) \to (\mathbb{S}^2 \wedge C) \odot^{\infty} (\mathbb{S}^2 \wedge C) \to (\mathbb{S}^3 \wedge C) \odot^{\infty} (\mathbb{S}^3 \wedge C) \to \cdots$$

is an augmented $\mathbb{Z}_2$–resolution (as a filtered bispectrum ; convert all maps to cofibrations). Therefore symmetric Poincaré structures ϕ on C live in $H^{\blacktriangledown}(\mathbb{Z}_2 : C \odot^{\infty} C)$, and quadratic Poincaré structures live in the homotopy fiber of

$$H^{\blacktriangledown}(\mathbb{Z}_2 : C \odot^{\infty} C) \longrightarrow \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2 : C \odot^{\infty} C)$$

which is the infinite loop space $H_{\blacktriangledown}(\mathbb{Z}_2; C \odot^{\infty} C)$ of the homotopy orbit spectrum of $\mathbb{Z}/2$ acting on $C \odot^{\infty} C$. See 0.9. Quadratic structures are symmetric structures that have been lifted across the norm map. This is in agreement with [Wa1] and [Ran2].

Enough of the general theory; here are some examples, essentially variations on 4.2. *Notation*: Recall that the external smash product of retractive spaces

$$Y \underset{s}{\overset{r}{\rightleftarrows}} A\,, \qquad Z \underset{s'}{\overset{r'}{\rightleftarrows}} B$$

is the retractive space $Y_A \wedge_B Z$ over $A \times B$ given by

$$Y_A \wedge_B Z = \text{ pushout of } \big(Y \times Z \overset{\supset}{\longleftarrow} Y \times B \cup Z \times A \longrightarrow A \times B\big)\,.$$

4.15. Example/Proposition♣. Fix B, an ENR. Let $\mathcal{C}$ be the category of compact retractive ENR's over B. $\mathcal{Y}$ acts on $\mathcal{C}$ by $(Y,X)\mapsto Y_*\wedge_B X$ for $Y\in\mathcal{Y}$ and $X\in\mathcal{C}$. Let $P(X,Y)$ be the homotopy pullback of

$$X_B\wedge_B Y \xrightarrow{\text{retraction}} B\times B \xleftarrow{\text{diagonal}} B\,,$$

so that $P(X,Y)$ is a retractive space over B. Let

$$X\odot Y := Q(P(X,Y)/B)\,.$$

This defines a Spanier–Whitehead product on $\mathcal{C}$. The proof is given, in a somewhat different language, in [WW2]. (This description of duality in $\mathcal{C}$ is unfortunately dual to the one in [Vo1].)

4.15. Example [bis]. The construction $P(X,Y)$ does not generalize well to controlled situations, so here is a way to avoid it. Given

$$X \underset{s}{\overset{r}{\rightleftarrows}} B\,, \qquad Y \underset{s'}{\overset{r'}{\rightleftarrows}} B$$

in $\mathcal{C}$, let $\mathfrak{P}(X,Y)$ be the category whose objects are finite retractive spaces

$$Z \underset{t}{\overset{q}{\rightleftarrows}} B$$

together with a map $f: Z\to X_B\wedge_B Y$ extending the diagonal $\Delta: B\to B\times B$ and a homotopy $\{h_t: Z\to B\times B\}$ rel B from $(r\wedge r')f$ to Δq. Morphisms in $\mathfrak{P}(X,Y)$ are retractive maps over B respecting the extra structure. Now let

$$X\odot Y := \underset{Z \text{ in } \mathfrak{P}(X,Y)}{\operatorname{hocolim}}\ Q(Z/B)\,.$$

To compare this with the previous definition, note that an object in $\mathfrak{P}(X,Y)$ is the same as a retractive space over B together with a retractive map to $P(X,Y)$. Morally speaking, $P(X,Y)$ is the terminal object of $\mathfrak{P}(X,Y)$, but technically speaking, it does not belong to $\mathfrak{P}(X,Y)$ because of its size.

4.16. Example/Proposition♣. Let $B\subset\bar{B}$ be a control space (cf. sequel of 1.2). The category $\mathcal{C}$ of proper retractive $ENR's$ over B is a category with cofibrations and weak equivalences (cf. sequel of 3.5). Given X,Y in $\mathcal{C}$, define $\mathfrak{P}(X,Y)$ much as before, substituting *controlled* homotopies for homotopies where possible. Let

$$X\odot Y := \underset{Z \text{ in } \mathfrak{P}(X,Y)}{\operatorname{hocolim}}\ Q(\bar{Z}/\bar{B})$$

where $\bar{Z}$ is the strict pullback of

$$Z\cup\{\infty\} \xrightarrow{\text{retraction}} B\cup\{\infty\} \xrightarrow{\text{collapse}} \bar{B}\,.$$

(In general, $\bar{Z}/\bar{B}$ is not homotopy equivalent to a CW–space, but we can define $Q(\bar{Z}/\bar{B})$ as the simplicial set of stable maps from $\mathbb{S}^0$ to $\bar{Z}/\bar{B}$.) Then $\odot$ is an SW–product on $\mathcal{C}$.

4.17. Example. With B and $\mathcal{C}$ as in 4.15, suppose that $\gamma : E \to B$ is a spherical fibration on B with a preferred section (and with fibers homotopy equivalent to $\mathbb{S}^k$). We may vary 4.15 by letting

$$X \odot_\gamma Y := \underset{Z \text{ in } \mathfrak{P}(X,Y)}{\text{hocolim}} \; \Omega^k Q(Z \times_B E/E)$$

(noting that E is $B \times_B E$). Again, this is an SW–product on $\mathcal{C}$, and the proof (in a different language) can be found in [WW2].

4.18. Example/Proposition♣. With $B \subset \bar{B}$ and $\mathcal{C}$ as in 4.16, suppose that $\gamma : E \to B$ is a spherical fibration with preferred section (fibers homotopy equivalent to $\mathbb{S}^k$). Vary 4.16 by letting

$$X \odot_\gamma Y := \underset{Z \text{ in } \mathfrak{P}(X,Y)}{\text{hocolim}} \; \Omega^k Q(\bar{Z} \times_{\bar{B}} \bar{E}/\bar{E})$$

(where the meaning of the closure bars should be clear although $\gamma : E \to B$ is not assumed to be proper). This is an SW–product.

The SW–products in 4.15–4.18 behave well under pushforward. Suppose for example that $f : B \to B'$ is a map between CW–spaces. The pushforward f_* is a functor from the category of retractive spaces over B to the category of retractive spaces over B', given by $f_* X = X \amalg_B B'$. Pushforward also defines a map from $X \odot Y$ to $f_* X \odot f_* Y$ where we assume that X, Y are retractive CW–spaces over B, and the $\odot$ products are defined as in 4.15.

4.19. Proposition♣. *The pushforward map takes n–dualities to n–dualities, for all n.*

For the twisted version of this statement, we would assume that B' comes with a spherical fibration γ as in 4.17. We would use this as in 4.17 to define an SW–product on the category of finite retractive CW–spaces over B', and we would use $f^*\gamma$ to define an SW–product on the category of finite retractive CW–spaces over B.

For the twisted and *controlled* version of the statement, we would assume that $B \subset \bar{B}$ and $B' \subset \bar{B}'$ are control spaces, that $f : \bar{B} \to \bar{B}'$ is a morphism of control spaces (continuous map such that $f^{-1}(B') = B$), and again that B' comes equipped with γ as in 4.18, so that B is equipped with $f^*\gamma$.

5. Poincaré Duality

We are now ready to discuss Poincaré duality, and the point of view we shall take is that it is some form of Spanier–Whitehead self duality. In the setting without control, this looks as follows.

5.1. Definitions. We say that a compact CW–space Y (or a compact ENR) is a Poincaré space of formal dimension n if there exist a spherical fibration $\nu : E \to Y$ with section $Y \to E$ (and fibers $\simeq \mathbb{S}^k$), and a map $\rho : \mathbb{S}^{n+k} \to E/Y$ such that the composition

$$\mathbb{S}^n \to \Omega^k(E/Y) \xrightarrow{\text{diagonal}} (\mathbb{S}^0 \times Y) \odot_\nu (\mathbb{S}^0 \times Y)$$

(left–hand arrow adjoint to ρ) is an n–duality.

Explanation: We think of $Z := \mathbb{S}^0 \times Y$ as an object in the category $\mathcal{C}$ of retractive spaces over Y, and we define an SW–product $\odot_\nu$ on $\mathcal{C}$ as in 4.17, taking $\gamma := \nu$. Then the diagonal map $Z \to Z_Y \wedge_Y Z$ makes Z into an object of $\mathfrak{P}(Z, Z)$, so that $\Omega^k(Z \times_Y E/E) \cong E/Y$ is contained in $Z \odot Z$.

Suppose next that $Y \subset \bar{Y}$ is a control space. We call it a Poincaré control space of formal dimension n if there exists a spherical fibration $\nu : E \to Y$ with section $Y \to E$ (and fibers $\simeq \mathbb{S}^k$), and a map $\rho : \mathbb{S}^{n+k} \to \bar{E}/\bar{Y}$ such that the composition

$$\mathbb{S}^n \to \Omega^k(\bar{E}/\bar{Y}) \xrightarrow{\text{diagonal}} (\mathbb{S}^0 \times Y) \odot_\nu (\mathbb{S}^0 \times Y)$$

(left–hand arrow adjoint to ρ) is nondegenerate.

To make these definitions really satisfactory, we have to show that the fibration ν and reduction ρ are unique up to contractible choice. It is enough to consider the controlled case. Suppose therefore that $Y \subset \bar{Y}$ is a Poincaré control space of formal dimension n, and that

$$\begin{aligned} \nu : E \to Y\,, \quad & \rho : \mathbb{S}^{n+k} \to \bar{E}/\bar{Y} \\ \gamma : F \to Y\,, \quad & \theta : \mathbb{S}^{n+k} \to \bar{F}/\bar{Y} \end{aligned}$$

are two pairs having the properties spelled out in 5.1. Assume that ν and γ behave well under iterated fiberwise suspension, in the sense that they remain spherical fibrations. Let W be the space of pairs (f, h) where f is a stable fiber homotopy equivalence from ν to γ, respecting the preferred sections, and h is a stable homotopy from $f_*(\rho)$ to θ.

5.2.Proposition. *W is contractible.*

Sketch proof. Write $\operatorname{smap}(\nu, \gamma)$ for the space of fiber preserving and section preserving stable maps, not necessarily of degree one in each fiber, from ν to γ (where stable means: defined after many fiberwise suspensions). We will show that

$$\text{(5.}\mathbf{i}\text{)} \qquad \operatorname{smap}(\nu, \gamma) \longrightarrow \Omega^{n+k} Q(\bar{F}/\bar{Y}) \quad ; \quad f \mapsto f_*(\rho)$$

is a homotopy equivalence. (Then 5.2 follows with very little extra work.) To show this, we examine domain and codomain in (5.**i**) separately.

Domain: Let $\mathcal{C}$ be the category of proper retractive $ENR's$ over Y. We are going to pretend that $\nu : E \to Y$ and $\gamma : F \to Y$ (with the preferred sections) belong to $\mathcal{C}$. In general this cannot be arranged, so the honest solution would be to replace $\mathcal{C}$ by a slightly larger category containing all retractive spaces over Y which are controlled h-equivalent to retractive spaces in $\mathcal{C}$. Pretending this, we have

$$\pi_i(\operatorname{smap}(\nu, \gamma)) \cong [[\Sigma^i_Y E, F]]$$

for all i (notation explained just before 4.4). The proof is in two steps. The first step consists in showing that, for any two objects

$$C \underset{s}{\overset{r}{\rightleftarrows}} Y \quad , \quad D \underset{s'}{\overset{r'}{\rightleftarrows}} Y$$

in $\mathcal{C}$, the set $[C,D]$ can be identified with the set of controlled homotopy classes of pairs (g,w) where $g : C \to D$ is a controlled map relative to Y, and w is a controlled homotopy from r to $r'f$. This is not really difficult. In the second step, assume that $r' : D \to Y$ is a fibration, and use this to show that $[C,D]$ can be identified with the set of homotopy classes of retractive maps from C to D. (This is not difficult either, but the result is surprising. It shows that $[C,D]$ does not depend on the compactification $\bar{Y}$ of Y (provided $r' : D \to Y$ is a fibration).) Finish by stabilizing.

Codomain: Let $\odot$ be the *untwisted* SW–product in $\mathcal{C}$ (use 4.16, not 4.18). Verify that $\Omega^{n+k}Q(\bar{F}/\bar{Y})$ is homotopy equivalent to $\Omega^n\big((\mathbb{S}^0 \times Y) \odot F\big)$.

To finish the proof, remember that we assumed that

$$\mathbb{S}^n \xrightarrow{\text{adjoint of } \rho} \Omega^k(\bar{E}/\bar{Y}) \xrightarrow{\text{diagonal}} (\mathbb{S}^0 \times Y) \odot_\nu (\mathbb{S}^0 \times Y)$$

is nondegenerate. This is equivalent to assuming that

$$\mathbb{S}^{n+k} \xrightarrow{\rho} \bar{E}/\bar{Y} \xrightarrow{\iota} Q(\bar{E}/\bar{Y}) \simeq (\mathbb{S}^0 \times Y) \odot E$$

(untwisted SW–product) is nondegenerate, where ι is the inclusion. Then in particular

$$\backslash[\iota\rho] : [[E,F]] \longrightarrow \pi_{n+k}\big((\mathbb{S}^0 \times Y) \odot F\big) \simeq \pi^s_{n+k}(\bar{F}/\bar{Y})$$

is a bijection, and more generally

$$\backslash[\Sigma^i(\iota\rho)] : [[\Sigma^i_Y E, F]] \longrightarrow \pi^s_{n+k+i}(\bar{F}/\bar{Y})$$

is a bijection for all i. Translating back, we conclude that (5.**i**) induces isomorphisms on homotopy groups. Then the homotopy fiber of (5.**i**) over θ is contractible, giving us a contractible choice of stable fiber preserving and section preserving maps from ν to γ. We would like to know that this choice is a stable fiber homotopy equivalence. It is, of course, because we can find a fiber homotopy inverse by interchanging ν and γ. $\square$

5.3. Observation. *If the control spaces $B \subset \bar{B}$ and $B' \subset \bar{B}'$ are homotopy equivalent, and $B \subset B'$ is Poincaré of formal dimension n, then the same holds for $B' \subset \bar{B}'$.*

5.4. Proposition♣. *Any control space of the form $B \subset \bar{B}$, where B is an n–manifold, is a Poincaré control space of formal dimension n. In fact, suppose that $\nu : E \to B$ is a spherical fibration with section (fibers $\simeq \mathbb{S}^k$), and that $\rho : \mathbb{S}^{n+k} \to \bar{E}/\bar{B}$ has degree 1. Then*

$$\mathbb{S}^n \xrightarrow{\text{adjoint of } \rho} \Omega^k(\bar{E}/\bar{B}) \xrightarrow{\text{diagonal}} (\mathbb{S}^0 \times B) \odot_\nu (\mathbb{S}^0 \times B)$$

is nondegenerate.

In general, whether $B \subset \bar{B}$ is a Poincaré control space depends on the size of $\bar{B} \smallsetminus B$ (big size, slim chance).

Poincaré spaces and Poincaré control spaces have *self–dual* Euler characteristics. Let Y be a Poincaré space (and a compact ENR) of formal dimension n, with Spivak normal fibration $\nu : E \to Y$ and reduction $\rho : \mathbb{S}^{n+k} \to E/Y$. (We assume that ν has a preferred section and fibers $\simeq \mathbb{S}^k$.) Then $\mathbb{S}^0 \times Y$ is a retractive space over Y, and the composite map η given by

$$\mathbb{S}^0 \longrightarrow \Omega^{n+k}(E/Y) \xrightarrow{\text{diagonal}} \Omega^n((\mathbb{S}^0 \times Y) \odot_\nu (\mathbb{S}^0 \times Y))$$

(left–hand arrow adjoint to ρ, as in 5.1) is nondegenerate. The SW–product $(X, X') \mapsto \Omega^n(X \odot_\nu X')$ on the category of retractive ENR's over Y determines an involution on a suitable model of $A(Y)$; notice that this depends on ν and on n. The pair consisting of the retractive space $\mathbb{S}^0 \times Y$ and η determines a point $\langle Y \rangle$ in this model of $A(Y)$; by inspection, the point is fixed under the involution. Fixed points are homotopy fixed points, so we may write

$$\langle Y \rangle \in H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}(Y))\,.$$

More generally, if $Y \subset \bar{Y}$ is a Poincaré control space of formal dimension n, still equipped with normal fibration $\nu : E \to Y$ and reduction $\rho : \mathbb{S}^{n+k} \to \bar{E}/\bar{Y}$, then we have the *controlled* self–dual Euler characteristic

$$\langle Y \rangle \in H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}(Y \subset \bar{Y}))\,.$$

Returning to the no–control situation, suppose that Y is a closed n–manifold, with Spivak normal fibration ν and reduction ρ as above. Then we have a self–dual microcharacteristic

$$\langle\!\langle Y \rangle\!\rangle \in H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}(Y)^\%)$$

for a suitable involution on $\boldsymbol{A}(Y)^\%$. Here the only surprise is that Poincaré spaces do not qualify in general. To understand why, recall (from around 3.8) that we constructed the microcharacteristic $\langle\!\langle Y \rangle\!\rangle$ in $A(Y)^\%$ using an Eilenberg swindle which involved the retractive space $(Y \times \mathbb{N}) \amalg (Y \times [0, \infty))$ over $Y \times [0, \infty)$. In order to make the argument work "with self–duality", we have to know that this retractive space is controlled self-dual ; the control comes from the compactification

$$Y \times [0, \infty) \subset Y \times [0, \infty]$$

and duality refers to the SW–product $(X, X') \mapsto \Omega^n(X \odot_\nu X')$. (Feel free to think of ν as a spherical fibration on $Y \times [0, \infty)$.) Using 4.19, we see that it is enough to show that $Y \times \mathbb{N}$ with compactification $Y \times (\mathbb{N} \cup \{\infty\})$ is a Poincaré control space of formal dimension n and with Spivak normal fibration ν. By 5.4, this is indeed the case if Y is a closed n–manifold, but it is not true for an arbitrary Poincaré space Y.

Finally suppose that $Y \subset \bar{Y}$ is a control space, and that Y is an n–manifold. Assuming that Y is equipped with Spivak normal fibration $\nu : E \to Y$ and reduction $\rho : \mathbb{S}^{n+k} \to E/Y$ as usual, and taking care to pick up the self–duality, we have (as in 3.13 and sequel) the controlled self–dual microcharacteristic

$$\langle\!\langle Y \rangle\!\rangle \in H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}^\%(Y \subset \bar{Y}))\,.$$

5.5. Remark. We will often encounter manifolds Y^n equipped with a proper map $f : Y \to V$, where V is a real vector space. Then we define $\bar{Y}$ as the closure of the graph of f in $(Y \cup \{\infty\}) \times \bar{V}$ (and we identify Y with the graph of f, so that $Y \subset \bar{Y}$.) In this situation it is more natural to define "reduction" as a map with domain $\mathbb{S}^p \wedge V^c$ (some large p), where V^c is the one–point compactification. The codomain is E/Y, where $\nu : E \to Y$ plays the role of the normal bundle of Y in $\mathbb{R}^p \oplus V$. The appropriate involution on $\boldsymbol{A}(Y \subset \bar{Y})$ and %–decorated relatives is the one made from the SW–product $(X, X') \mapsto \Omega^V \Omega^n(X \odot_\nu X')$. The same remark applies when a Poincare control space comes with a morphism of control spaces to $V \subset \bar{V}$.

What are the naturality properties of these new Euler characteristics ? It seems best to redefine e.g. *Poincaré spaces* as triples (Y, ν, ρ). A *homotopy equivalence* from a Poincaré space (Y, ν, ρ) to another Poincare space (Y', ν', ρ') would then be a pair $f : Y \xrightarrow{\simeq} Y'$, $\phi : \Sigma^i_Y \nu \to f^*(\nu')$ such that $(f, \phi)_*(\Sigma^i \rho) = \rho'$. We assume that ν and ν' come with sections which are cofibrations, and that ν has fibers $\simeq \mathbb{S}^k$, whereas ν' has fibers $\simeq \mathbb{S}^{k+i}$. Similar conventions will be adopted for controlled Poincaré spaces: everybody, manifolds not excepted, comes with Spivak normal fibration and reduction. Then lax naturality of the new Euler characteristics is obvious. In the summary which follows, and elsewhere, we write informally Y to mean (Y, ν, ρ), and so on.

5.6. Summary.

(1) Self dual Euler characteristic $\langle Y \rangle \in H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y))$: defined for Poincaré spaces Y, lax natural for homotopy equivalences.
(2) Self dual microcharacteristic $\langle\langle Y \rangle\rangle \in H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y)^{\%})$: defined for closed manifolds Y, lax natural for homeomorphisms.
(3) Controlled self dual Euler characteristic $\langle Y \subset \bar{Y} \rangle \in H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y \subset \bar{Y}))$: defined for Poincaré control spaces $(Y \subset \bar{Y})$, lax natural for homotopy equivalences between Poincaré control spaces.
(4) Controlled microcharacteristic $\langle\langle Y \subset \bar{Y} \rangle\rangle \in H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(Y \subset \bar{Y}))$: defined for control spaces $Y \subset \bar{Y}$ such that Y is a manifold, lax natural for homeomorphisms between such control spaces.

In all cases, the appropriate involution on the appropriate A–theory spectrum can be described in terms of the Spivak normal fibration of Y and the (formal) dimension of Y.

Remember now that we used Euler characteristics and microcharacteristics in section 3 to construct maps from structure spaces to A–theory:

$$\mathcal{S}(M) \to A(M)_{\%}\,, \qquad c\mathcal{S}(M \times V) \to cA(M \times V)_{\%}$$

where M is a compact manifold. Using self–dual Euler characteristics and microcharacteristics, we can refine these maps to get

$$\chi : \mathcal{S}(M) \to H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%})\,, \qquad \chi : c\mathcal{S}(M \times V) \to H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times V)_{\%})$$

(with the conventions of remark 5.5). The details are omitted. What about naturality in V ? Consider a direct sum of vector spaces, $W = U \oplus V$. Taking products with the identity structure on U gives a map $c\mathcal{S}(M\times V)\to c\mathcal{S}(M\times W)$. We have a compatible map

$$c\boldsymbol{A}(M\times V)_{\%}\longrightarrow c\boldsymbol{A}(M\times W)_{\%}$$

given by external smash product with the retractive space $\mathbb{S}^0\times U$ over U. Is it a $\mathbb{Z}_2$–map ? More than a verification is required ; we have to relate the appropriate SW–products. The SW–products are given by

$$\begin{array}{ll}(R,R')\mapsto \Omega^V\Omega^n(R\odot_\nu R') & \text{for retractive spaces } R,\ R' \text{ over } M\times V\\ (T,T')\mapsto \Omega^W\Omega^n(T\odot_\nu T') & \text{for retractive spaces } T,\ T' \text{ over } M\times W\,.\end{array}$$

We have to "relate" these in the case $T=R\times U$, $T'=R'\times U$. This is easily done: take products with the diagonal map $U\to U\times U$. With these conventions, the following commutes:

$$\begin{array}{ccc} c\mathcal{S}(M\times V) & \xrightarrow{\times U} & c\mathcal{S}(M\times W)\\ \big\downarrow & & \big\downarrow\\ H^{\blacktriangledown}(\mathbb{Z}_2;c\boldsymbol{A}(M\times V)_{\%}) & \xrightarrow{\times\langle\langle U\rangle\rangle} & H^{\blacktriangledown}(\mathbb{Z}_2;c\boldsymbol{A}(M\times W)_{\%})\ . \end{array}$$

At this point we can specialize, taking $V=\mathbb{R}^i$ and $W=\mathbb{R}^j$, say. Then we can "unspecialize" again, to the tune of 3.14. Consequently we can say that

$$\chi: c\mathcal{S}(M\times V)\to H^{\blacktriangledown}(\mathbb{Z}_2;c\boldsymbol{A}(M\times V)_{\%})$$

is a natural transformation between *continuous* functors in the variable V.

6. Structures Vs. Block Structures

The goal is to sketch the construction of a highly connected map

$$\iota:\mathcal{S}(M)\longrightarrow LA(M,\nu,-n)_{\%}$$

as advertised in Theorem 0.7.

Let $D(V):=c\mathcal{S}(M\times V)$ and $E(V):=H^{\blacktriangledown}(\mathbb{Z}_2;cA(M\times V))$. We are interested in the commutative square

$$\begin{array}{ccc} D(0) & \xrightarrow{\subset} & D(\mathbb{R}^\infty)\\ \chi\big\downarrow & & \chi\big\downarrow\\ E(0) & \xrightarrow{\subset} & E(\mathbb{R}^\infty)\end{array}$$

where $D(\mathbb{R}^\infty):=\operatorname{holim}_i D(\mathbb{R}^i)$, $E(\mathbb{R}^\infty):=\operatorname{holim}_i E(\mathbb{R}^i)$. Since $D(\mathbb{R}^\infty)$ and $E(\mathbb{R}^\infty)$ are filtered by subspaces homotopy equivalent to $D(\mathbb{R}^i)$ and $E(\mathbb{R}^i)$, respectively, we can write down an even more interesting commutative square

$$\text{(6.}\mathbf{i}\text{)}\qquad \begin{array}{ccc} D(0) & \longrightarrow & {}^{pos}D(\mathbb{R}^\infty)\\ \chi\big\downarrow & & \chi\big\downarrow\\ E(0) & \longrightarrow & {}^{pos}E(\mathbb{R}^\infty)\end{array}$$

(notation as in 1.6). General nonsense, to be developed in this section, will allow us to identify the bottom line of (6.**i)** with the standard inclusion

$$H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M)_{\%}) \longrightarrow \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M)_{\%})\,.$$

(The appropriate involution on $\boldsymbol{A}(M)_{\%}$ is the one determined by the normal bundle ν and the integer n, as in the sequel of 5.4. We ought to write $\boldsymbol{A}(M,\nu,-n)_{\%}$ for this spectrum, indicating that ν has "formal" fiber dimension $-n$, but of course we will not.) Recall now that ${}^{pos}D(\mathbb{R}^\infty)$ is homotopy equivalent to the block structure space of M by 1.7 and 1.8, and therefore also to $L(M,\nu,-n)_{\%}$ by Sullivan, Wall, Quinn, Ranicki (if $n\ge 5$). With these identifications, (6.**i)** takes the form

(6.**ii)**

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\ \subset\ } & L(M,\nu,-n)_{\%} \\ {\scriptstyle\chi}\big\downarrow & & {\scriptstyle\chi}\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M)_{\%}) & \xrightarrow{\ \subset\ } & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M)_{\%})\,. \end{array}$$

General nonsense will further enable us to show that (6.**i)**, or for that matter (6.**ii)**, is a homotopy pullback square in a certain range ; i.e., the map determined by (6.**i)** from $D(0)$ to the homotopy pullback of

$${}^{pos}D(\mathbb{R}^\infty) \longrightarrow {}^{pos}E(\mathbb{R}^\infty) \longleftarrow E(0)$$

is highly connected. Then our mission (for this section) will be accomplished if we can show that the right–hand χ in (6.**ii)** is homotopic to

$$\Xi_{\%} : L(M,\nu,-n)_{\%} \longrightarrow H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M)_{\%})$$

(see 0.9). This may appear to be too obvious, since both χ and $\Xi_{\%}$ were constructed using self dual Euler characteristics. But it turns out to be a difficult point.

We begin with the analysis of E. In fact, it is convenient to analyse the functor F given by $V\mapsto cA(M\times V)$ first. F is a (continuous) functor from $\mathcal{J}$ to pointed spaces with $\mathbb{Z}_2$–action, and we may write

$$E(V)=\mathrm{map}_{\mathbb{Z}_2}(\mathcal{E}\mathbb{Z}_2,F(V))\,.$$

A remarkable property of F is that the map $F(e):F(V)\to F(V\oplus\mathbb{R})$ induced by the inclusion $e:V\hookrightarrow V\oplus\mathbb{R}$ is nullhomotopic for arbitrary V. To give the idea, we construct a nullhomotopy for

$$\times\langle\mathbb{R}\rangle : c\boldsymbol{A}(M\times V)\longrightarrow c\boldsymbol{A}(M\times(V\oplus\mathbb{R}))$$

(note the missing subscripts %). The nullhomotopy does not respect the $\mathbb{Z}_2$–actions, and to describe it we use unadulterated models of A–theory. Let

$$\boldsymbol{X}(V)\subset c\boldsymbol{A}(M\times(V\oplus\mathbb{R}))$$

be the K–theory of a certain full subcategory of the category of proper retractive ENR's over $M \times (V \oplus \mathbb{R})$. An object belongs to the subcategory if the retraction map r is such that

$$r^{-1}(M \times V \times [0, \infty))$$

is an ENR. Additivity theorem and Eilenberg swindle show that $\boldsymbol{X}(V)$ is contractible. Clearly the map $\times\langle\mathbb{R}\rangle$ has image contained in $\boldsymbol{X}(V)$. This gives the required nullhomotopy, well defined up to contractible choice.

We leave it to the reader to refine the argument and to factorize $F(e) : F(V) \to F(V \oplus \mathbb{R})$ (non–equivariantly) through a contractible space $F^\sharp(V)$, depending naturally on V. In more detail: $F(e)$ should equal ji, where i and j are natural transformations:

$$F(V) \xrightarrow{i} F^\sharp(V) \xrightarrow{j} F(V \oplus \mathbb{R})\,.$$

(For the time being, a factorization through a functor with contractible values is an acceptable substitute for a natural nullhomotopy.) We can ask whether the nullhomotopy substitute respects the $\mathbb{Z}_2$–actions. It fails dramatically.

6.1. Proposition♣. *The commutative square*

$$\begin{array}{ccc} F(V) & \xrightarrow{\quad i \quad} & F^\sharp(V) \\ {\scriptstyle iT}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ F^\sharp(V) & \xrightarrow{\quad Tj \quad} & F(V \oplus \mathbb{R}) \end{array}$$

(where $T \in \mathbb{Z}_2$ is the generator) is a homotopy pullback square.

Isolating these properties, let us suppose that G is a continuous functor from $\mathcal{J}$ to pointed $\mathbb{Z}_2$–spaces, and that there exists another continuous functor $G^\sharp$ from $\mathcal{J}$ to pointed *contractible* spaces and a natural factorization

$$G(V) \xrightarrow{i} G^\sharp(V) \xrightarrow{j} G(V \oplus \mathbb{R})$$

of the map $G(V) \to G(V \oplus \mathbb{R})$ induced by inclusion, such that

$$\begin{array}{ccc} G(V) & \xrightarrow{\quad i \quad} & G^\sharp(V) \\ {\scriptstyle iT}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ G^\sharp(V) & \xrightarrow{\quad Tj \quad} & G(V \oplus \mathbb{R}) \end{array}$$

is a homotopy pullback square.

6.2. Proposition. *In this situation, there exists a spectrum $\boldsymbol{\Theta}$ with action of $\mathbb{Z}_2$ such that G is equivalent (details below) to the functor*

$$V \mapsto Q(V^c \wedge \boldsymbol{\Theta})\,.$$

Here $T \in \mathbb{Z}_2$ acts by $v \mapsto -v$ on V and on the one–point compactification V^c, and diagonally on $V^c \wedge \boldsymbol{\Theta}$.

Explanation. Two continuous functors from $\mathcal{J}$ to spaces are *equivalent* if they are related by a chain of equivalences (see 3.14). Here $\mathbb{Z}_2$ acts on each of the two functors, so we require it to act on each of the functors in the chain, and require the equivalences to commute with the actions.

Proof. We would like to say that $G(V) \to G(V \oplus \mathbb{R})$ is naturally nullhomotopic (naturally in V, that is). Our assumptions do not imply this, except if G is *cofibrant* [We]. We say that a continuous functor G_1 from $\mathcal{J}$ to spaces is *cofibrant* if, for any diagram of continuous functors

$$G_1 \xrightarrow{f} G_2 \xleftarrow{e} G_3$$

where e is an equivalence, there exists a natural transformation

$$\hat{f} : G_1 \longrightarrow G_3$$

and a natural homotopy h from $e \cdot \hat{f}$ to f. It is shown in [We] that for an arbitrary G_1, there exists a cofibrant $G_1^{\diamond}$ and an equivalence $G_1^{\diamond} \to G_1$. The whole construction is natural in G_1, so if $\mathbb{Z}_2$ acts on G_1, it acts compatibly on $G_1^{\diamond}$.

We may therefore assume that G is cofibrant. Then $i : G(V) \to G^{\sharp}(V)$ is naturally nullhomotopic (exercise), and composing such a nullhomotopy with j from $G^{\sharp}(V)$ to $G(V \oplus \mathbb{R})$, we have a specific natural nullhomotopy $\{h_s \mid s \in [0, \infty]\}$ of the map $G(V) \to G(V \oplus \mathbb{R})$. Then

$$\sigma : (\mathbb{R})^c \wedge G(V) \longrightarrow G(V \oplus \mathbb{R}) \quad ; \quad (s, x) \mapsto \begin{cases} h_s(x) & s \geq 0 \\ Th_sT(x) & s \leq 0 \\ * & s = \infty \end{cases}$$

is a natural transformation. It respects the $\mathbb{Z}_2$ actions, where $T \in \mathbb{Z}_2$ acts on $\mathbb{R}$ and $(\mathbb{R})^c$ by the flip, and diagonally on $(\mathbb{R})^c \wedge G(V)$. Last not least, our assumptions imply that the adjoint of σ is a homotopy equivalence from $G(V)$ to $\Omega G(V \oplus \mathbb{R})$.

We can use this to "improve" G to a continuous functor $\boldsymbol{G}$ from $\mathcal{J}$ to Ω–spectra with $\mathbb{Z}_2$–action. In detail, define the n–th term of $\boldsymbol{G}(V)$ as

$$\operatorname*{hocolim}_i \Omega^i(\mathbb{S}^n \wedge G(V \oplus \mathbb{R}^i))$$

where we use σ to define the maps in the direct system. $\mathbb{Z}_2$ acts on each of the terms in the direct system via the given action on G and the sign change or flip action on the functor $\Omega^i = \mathrm{map}((\mathbb{R}^i)^c, —)$. Then $\boldsymbol{G}(V)$ is an Ω–spectrum with action of $\mathbb{Z}_2$, depending naturally and continuously on V, and moreover the composition of $\boldsymbol{G}$ with the zero–th term functor is equivalent to G, as a continuous functor from $\mathcal{J}$ to $\mathbb{Z}_2$–spaces. Using σ once again, we find that we still have a natural equivariant map

$$\boldsymbol{\sigma} : (\mathbb{R})^c \wedge \boldsymbol{G}(V) \longrightarrow \boldsymbol{G}(V \oplus \mathbb{R})$$

which is a homotopy equivalence. We use it to define a natural equivariant homotopy equivalence

$$V^c \wedge \boldsymbol{G}(0) \longrightarrow \boldsymbol{G}(V)$$

in such a way that for every orthogonal sum decomposition $V = L \oplus L^{\perp}$ where $\dim(L) = 1$, the following commutes:

$$\begin{array}{ccc} L^c \wedge \boldsymbol{G}(0) & \longrightarrow & L^c \wedge \boldsymbol{G}(L^{\perp}) \\ {\scriptstyle\subset}\big\downarrow & & {\scriptstyle\boldsymbol{\sigma}}\big\downarrow \\ V^c \wedge \boldsymbol{G}(0) & \longrightarrow & \boldsymbol{G}(L \oplus L^{\perp}0 = \boldsymbol{G}(V)\,. \end{array}$$

Taking $\Theta = \boldsymbol{G}(0)$ completes the proof. $\square$

6.3. Remark. It follows from 6.2 and proof that each space $G(V)$ has a preferred structure of infinite loop space, naturally in V. In our example, $G(V) = F(V) = cA(M \times V)$, we have such a natural infinite loop space structure to begin with, and one would hope that it is "the same". Suppose therefore, inspired by the example, that each $G(V)$ is (the underlying space of) a grouplike Γ–space in the sense of [Se2], depending naturally on V. (*Grouplike* means that the abelian monoid $\pi_0(G(V))$ is a group for all V). Impose the same condition on $G^\sharp$, and suppose that

$$\begin{gathered} G(V) \xrightarrow{i} G^\sharp(V) \xrightarrow{j} G(V \oplus \mathbb{R}) \\ T : G(V) \longrightarrow G(V) \end{gathered}$$

are Γ–maps. Repeat the proof of 6.2, taking care not to lose the Γ–structures, and conclude that Θ is a spectrum made from Γ–spaces, such that all the structure maps $\Theta_n \to \Omega\Theta_{n+1}$ are Γ–maps. Taking zero–th spaces, we recover G as a functor to Γ–spaces. Speaking loosely, the two natural infinite loop space structures to be compared are encoded in what is now a "bispectrum" $\boldsymbol{\Theta}$ with $\mathbb{Z}_2$–action, as the vertical and horizontal suspension directions, respectively. In this sense they are "the same".

6.4. Corollary. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} E(0) & \xrightarrow{\ \subset\ } & {}^{pos}E(\mathbb{R}^\infty) \\ {\scriptstyle=}\big\downarrow & & {\scriptstyle\simeq}\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) & \xrightarrow{\ \subset\ } & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) \end{array}$$

(see diagram (6.**i**) *earlier in this section).*

Proof. Using the definitions and 6.2 we may pretend

$$E(V) = H^{\blacktriangledown}(\mathbb{Z}_2; V^c \wedge \boldsymbol{\Theta})$$

for some spectrum $\boldsymbol{\Theta}$ with action of $\mathbb{Z}_2$. Note: $\mathbb{Z}_2$ acts diagonally on $V^c \wedge \boldsymbol{\Theta}$. By 6.3 we can further identify the (-1)–connected cover $\boldsymbol{\Theta}^{[0]}$ of $\boldsymbol{\Theta}$ with $\boldsymbol{A}(M)_{\%}$. So we should prove

$$(6.\mathbf{iii}) \qquad {}^{pos}E(\mathbb{R}^\infty) \simeq H^{\blacktriangledown}(\mathbb{Z}_2; V^c \wedge \boldsymbol{\Theta}^{[0]})\,,$$

relative to $E(0) = H^\blacktriangledown(\mathbb{Z}_2;\mathbf{\Theta}) \simeq H^\blacktriangledown(\mathbb{Z}_2;\mathbf{\Theta}^{[0]})$. To begin, observe that there is a homotopy commutative diagram of filtration preserving maps

$$\begin{array}{ccc} {}^{pos}E(\mathbb{R}^\infty) & \longrightarrow & \bigcup_i H^\blacktriangledown(\mathbb{Z}_2;((\mathbb{R}^i)^c\wedge\mathbf{\Theta}^{[0]})) \\ {\scriptstyle\subset}\big\downarrow & & \big\downarrow \\ E(\mathbb{R}^\infty) & \xrightarrow{=} & \bigcup_i H^\blacktriangledown(\mathbb{Z}_2;((\mathbb{R}^i)^c\wedge\mathbf{\Theta})) \end{array}$$

where the top horizontal arrow is obtained by obstruction theory, and is a homotopy equivalence by a check on homotopy groups. We now recall some easy results from [WW2,§1,§2]. Let $\boldsymbol{X}$ be a spectrum with $\mathbb{Z}_2$–action, $\boldsymbol{X} = \cup_i\boldsymbol{X}(i)$ where the $\boldsymbol{X}(i)$ are $\mathbb{Z}_2$–invariant subspectra, $i \geq 0$, and $\boldsymbol{X}(i) \subset \boldsymbol{X}(i+1)$ (by a cofibration, say) for $i \geq 0$. The filtered spectrum $\boldsymbol{X}$ is an *augmented resolution* of $\boldsymbol{X}(0)$ if

(1) each inclusion $\boldsymbol{X}(i) \hookrightarrow \boldsymbol{X}(i+1)$ is nullhomotopic (no equivariance required for the nullhomotopy) ;
(2) $\boldsymbol{X}(i+1)/\boldsymbol{X}(i)$ is *induced* for $i \geq 0$, which means there exists a spectrum $\boldsymbol{Y}(i)$ (no action) and a homotopy equivalence from $(\mathbb{Z}_2)_+\wedge\boldsymbol{Y}(i)$ to $\boldsymbol{X}(i+1)/\boldsymbol{X}(i)$ which is a $\mathbb{Z}_2$–map.

Then, by [WW2, 2.10, 1.9, 1.10 and sequel] we have

$$\bigcup_i \boldsymbol{H}^\blacktriangledown(\mathbb{Z}_2;\boldsymbol{X}(i)) \simeq \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}_2;\boldsymbol{X}(0))$$

and therefore, for the corresponding infinite loop spaces,

$$\bigcup_i H^\blacktriangledown(\mathbb{Z}_2;\boldsymbol{X}(i)) \simeq \widehat{H}^\blacktriangledown(\mathbb{Z}_2;\boldsymbol{X}(0))\,.$$

This homotopy equivalence is relative to $H^\blacktriangledown(\mathbb{Z}_2;\boldsymbol{X}(i))$. Now clearly $\boldsymbol{X}(i) = (\mathbb{R}^i)^c\wedge\mathbf{\Theta}^{[0]}$ defines an augmented resolution, which proves (6.**iii**). $\square$

The next assignment is to prove that the square (6.**i**) (beginning of this section) is highly–connected, i.e., that it is a homotopy pullback square in a certain range. We begin with a slightly weaker statement. *Terminology:* A commutative square of spaces

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \big\downarrow & & \big\downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

is *j–connected* if it induces a j–connected map from X_1 to the homotopy limit of $(X_3 \to X_4 \leftarrow X_2)$.

6.5. Proposition. *The square*

$$\begin{array}{ccc} \Omega D(0) & \xrightarrow{\subset} & \Omega\big({}^{pos}D(\mathbb{R}^\infty)\big) \\ {\scriptstyle\Omega\chi}\big\downarrow & & {\scriptstyle\Omega\chi}\big\downarrow \\ \Omega E(0) & \xrightarrow{\subset} & \Omega\big({}^{pos}E(\mathbb{R}^\infty)\big) \end{array}$$

is $(j-1)$–connected if j is in the topological h–cobordism stable range for M.

Proof. Write Y_k for the homotopy fiber of $D(\mathbb{R}^k) \to D(\mathbb{R}^{k+1})$, and Z_k for the homotopy fiber of $E(\mathbb{R}^k) \to E(\mathbb{R}^{k+1})$. Write also

$$\begin{aligned} Y_k^{[k]} &:= (k-1)\text{–connected cover of } Y_k \\ Z_k^{[k]} &:= (k-1)\text{–connected cover of } Z_k \end{aligned}$$

and note that these can be identified with the homotopy fibers of

$$\begin{aligned} {}^{pos}D(\mathbb{R}^k) &\longrightarrow {}^{pos}D(\mathbb{R}^{k+1})\,, \\ {}^{pos}E(\mathbb{R}^k) &\longrightarrow {}^{pos}E(\mathbb{R}^{k+1})\,, \end{aligned}$$

respectively. An induction argument shows that it is sufficient to prove

$$\chi : Y_k^{[k]} \longrightarrow Z_k^{[k]} \text{ is } j\text{–connected}, \ \forall k. \tag{6.iv}$$

To prove this, we use the fact that D and E are continuous functors on $\mathcal{J}$. Using [WW1, 3.8] or, better, [We], we then have suspension maps $\Sigma Y_k \to Y_{k+1}$ and $\Sigma Z_k \to Z_{k+1}$ for all i, making $\{Y_k\}$ and $\{Z_k\}$ into spectra and making $\{\chi : Y_k \to Z_k\}$ into a map of spectra. It is a consequence of 6.1 that $\{Z_k\}$ is an Ω–spectrum, and it follows from 6.2 that it has $\boldsymbol{A}(M)_{\%}$ as a (-1)–connected cover. Recall from the last part of §3 that

$$Y_k \simeq c\mathfrak{H}(M \times \mathbb{R}^k)\,.$$

Therefore (6.**iv**) will follow if we can show that $\chi : Y_k \longrightarrow Z_k$ is the Waldhausen map (end of §3). To this end we have the commutative diagram

$$\begin{array}{ccccc}
Y_k & \longrightarrow & D(\mathbb{R}^k) & \longrightarrow & D(\mathbb{R}^{k+1}) \\
{\scriptstyle\chi}\big\downarrow & & {\scriptstyle\chi}\big\downarrow & & {\scriptstyle\chi}\big\downarrow \\
Z_k & \longrightarrow & E(\mathbb{R}^k) & \longrightarrow & E(\mathbb{R}^{k+1}) \\
 & & {\scriptstyle i\cdot\text{forget}}\big\downarrow & & {\scriptstyle\text{forget}}\big\downarrow \\
 & & F^{\sharp}(\mathbb{R}^k) & \xrightarrow{\ j\ } & F(\mathbb{R}^{k+1})
\end{array}$$

where the upper and middle rows are fibration sequences up to homotopy, and F, $F^{\sharp}$, i and j are as in 6.1. The arrow from $E(\mathbb{R}^k)$ to $F^{\sharp}(\mathbb{R}^k)$ is the composition

$$H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times \mathbb{R}^k)_{\%}) \xrightarrow{\text{forget}} cA(M \times \mathbb{R}^k)_{\%} = F(\mathbb{R}^k) \xrightarrow{i} F^{\sharp}(\mathbb{R}^k)\,.$$

By 6.1 and 6.2, the lower square in the diagram is a homotopy pullback square. Therefore $\chi : Y_k \to Z_k$ is really the map between horizontal homotopy fibers in the commutative square

$$\begin{array}{ccc}
D(\mathbb{R}^k) & \longrightarrow & D(\mathbb{R}^{k+1}) \\
\big\downarrow & & \big\downarrow \\
F^{\sharp}(\mathbb{R}^k) & \longrightarrow & F(\mathbb{R}^{k+1})\,.
\end{array}$$

By 3.16 and sequel, χ is the Waldhausen map. $\square$

6.6. Lemma$^{\clubsuit}$**.** *The map*

$$\mathcal{S}(M) = D(0) \quad \longrightarrow \quad \operatorname{holim}\Big(E(0) \to {}^{pos}D(\mathbb{R}^\infty) \hookleftarrow {}^{pos}E(\mathbb{R}^\infty) \Big)$$

determined by (6.**i**) *induces a bijection on* π_0*.*

6.7. Lemma. *The restriction of the map in* 6.6 *to the component of some structure* $f : M' \to M$ *is* j*–connected if* j *is in the topological* h*–cobordism stable range for* M'*.*

Proof. We proved this in the case of the identity structure $\mathrm{id} : M \to M$, and we can deduce the general case by a translation argument. Replacing M by M' throughout, we obtain a commutative square

$$\text{(6.}\mathbf{v}\text{)} \qquad \begin{array}{ccc} D'(0) & \longrightarrow & {}^{pos}D'(\mathbb{R}^\infty) \\ {\scriptstyle \chi}\big\downarrow & & {\scriptstyle \chi}\big\downarrow \\ E'(0) & \longrightarrow & {}^{pos}E'(\mathbb{R}^\infty) \end{array}$$

where the primes $'$ have nothing to do with derivatives. For example, $D'(0) = \mathcal{S}(M')$, $E'(0) = H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M')_{\%})$, and so on. There is a a translation morphism from (6.**v**) to (6.**i**), as follows: Map $D'(0)$ to $D(0)$ by composing structures $g : M'' \to M'$ with the given $f : M' \to M$. Similarly, map ${}^{pos}D'(\mathbb{R}^\infty)$ to ${}^{pos}D(\mathbb{R}^\infty)$ by composing structures on $M' \times \mathbb{R}^i$ with $f \times \mathrm{id}$, for the appropriate i. Map $E'(0)$ to $E(0)$ by adding $\langle\!\langle M' \rangle\!\rangle - \langle\!\langle M \rangle\!\rangle$ (using the infinite loop space structure in $E(0)$. Map ${}^{pos}E'(\mathbb{R}^\infty)$ to ${}^{pos}E(\mathbb{R}^\infty)$ by adding $\langle\!\langle M' \times \mathbb{R}^i \rangle\!\rangle - \langle\!\langle M \times \mathbb{R}^i \rangle\!\rangle$. Note that these maps are homotopy equivalences, and the identity component of $D'(0)$ is mapped to the component of $f : M' \to M$. The lemma now follows from 6.5, applied with M' instead of M. $\square$

It remains to identify the right–hand vertical arrow in diagram (6.**i**), beginning of this section, with $\Xi_{\%}$ constructed in [WW2]. For this we need a precise formulation of a well–known principle. Briefly, it says that the image under $(1+T)$ of the surgery obstruction of a surgery problem is the difference of two invariants extracted from domain and codomain of the surgery problem, respectively. Put differently, a certain diagram is homotopy commutative. For the diagram, fix B, an ENR, and a sphere bundle γ on B with a distinguished section. Fix also $n \geq 0$. Let $\mathcal{C}$ be the category of retractive ENR's over B, equipped with the SW product which takes X_1, X_2 in $\mathcal{C}$ to $\Omega^n(X_1 \odot_\gamma X_2)$ (notation of 4.17). Let $\mathfrak{N}(B,\gamma,)$ be the bordism space of n–dimensional Poincaré spaces over $(B,\gamma,)$. "Points" in $\mathfrak{N}(B,\gamma)$ are n-dimensional Poincaré spaces with a map f to B and with Spivak normal bundle equal to $f^*\gamma$; "paths" in $\mathfrak{N}(B,\gamma)$ are bordisms between such objects, and so on. Details are given below. Let $\mathfrak{N}\mathfrak{N}(B,\gamma,-n)$ be the bordism space of n–dimensional Poincaré surgery problems over (B,γ) ; again, details below. The diagram is

$$\text{(6.}\mathbf{vi}\text{)} \qquad \begin{array}{ccc} \mathfrak{N}\mathfrak{N}(B,\gamma) & \xrightarrow{\text{difference}} & \mathfrak{N}(B,\gamma) \\ {\scriptstyle \mathrm{SWQR}}\big\downarrow & & {\scriptstyle \subset}\big\downarrow \\ \Omega^n L_n(\mathbb{Z}\pi_1(B)) \simeq L(\mathcal{C}) & \xrightarrow{(1+T)} & L^{\blacktriangledown}(\mathcal{C}) \end{array}$$

where SWQR is the Sullivan–Wall–Quinn–Ranicki map (a homotopy equivalence if $n \geq 5$). The arrow labelled "difference" is defined below.

6.8. Theorem$^\clubsuit$. *Diagram* (6.**vi**) *commutes up to a preferred homotopy.*

6.9. Details. $\mathfrak{N}\mathfrak{N}(B,\gamma)$ is the classifying space of a simplicial category whose objects in degree j are certain surgery problems

$$f : N \longrightarrow Y\,,$$

where N is a compact manifold modelled on $\mathbb{R}^n_\curlywedge \times \Delta^j$, and Y is a Poincaré space modelled on $\mathbb{R}^n_\curlywedge \times \Delta^j$ (which means: same corner structure as N). We assume that f restricts to a homotopy equivalence from ∂N to ∂Y, and that the Spivak normal bundles and reductions arise in the following way:

(1) Y comes with a map $g : Y \to B$.
(2) The bundle $f^*g^*(\gamma)$ with total space E, say, serves as normal bundle for N, so it comes with a reduction

$$\rho : (\mathbb{D}^p, \mathbb{S}^{p-1}) \wedge \Delta^j_+ \to (E/N, \partial E/\partial N)$$

(suitable p), respecting faces, which restricts to a *homeomorphism* from the complement of $\rho^{-1}(*)$ to $E \smallsetminus N$. We allow ρ to be "stable" (defined after many suspensions).
(3) The bundle $g^*(\gamma)$ serves as Spivak normal bundle for Y, with reduction $f_*(\rho)$.

These are the objects in degree j. The morphisms are isomorphisms, in the most rigorous sense.

The definition of $\mathfrak{N}(B,\gamma)$ is simpler: it is the classifying space of a simplicial category whose objects in degree j are certain Poincaré spaces Y modelled on $\mathbb{R}^n \times \Delta^j$. We assume that Y comes with a map $g : Y \to B$, and that $g^*\gamma$ serves as Spivak normal bundle for Y. Thus the reduction is a stable map

$$\rho : \mathbb{S}^p \wedge \Delta^j_+ \to E'/Y$$

respecting faces, where E' is the total space of $g^*\gamma$.

For the map labelled "difference", note that a surgery problem $f : N \to Y$ over B (details as above, modelled on $\mathbb{R}^n_\curlywedge \times \Delta^j$) determines a Poincaré space modelled on $\mathbb{R}^n \times \Delta^j$ as follows. Take

$$N \amalg Y/\sim$$

where $\sim$ identifies $x \in \partial N$ with $f(x) \in \partial Y$. Then $f \amalg g$ is still a map from $N \amalg Y/\sim$ to B, and the pullback of γ under $f \amalg g$ can serve as Spivak normal bundle: the appropriate reduction is $\rho \amalg (f_*(\rho))$, with domain

$$\mathbb{D}^p \amalg_{\mathbb{S}^{p-1}} \mathbb{D}^p,$$

which we have to identify with $\mathbb{S}^p$. Choose this identification in such a way that the first disk (corresponding to N) is mapped in an orientation preserving way ; then the other disk (corresponding to Y) will be mapped in an orientation–reversing way.

About the arrow SWQR: The traditional target for SWQR is the quadratic L–theory of the category of finitely generated free chain complexes of left modules over the group ring $\mathbb{Z}\pi_1(B)$. But the π–π theorem [WW2, §6] states that $L(\mathcal{C})$ has the same homotopy type.

The map labelled $\subset$ is in fact an inclusion, provided we interpret Poincaré spaces Y over B modelled on $\mathbb{R}^n \times \Delta^j$ as objects in $\mathcal{C}[j]$ (by taking disjoint union with B). Poincaré duality implies SW self duality in the usual way (end of §4).

A word about theorem 6.8: It requires proof. Our present construction of the arrow SWQR in the diagram, via Ranicki's quadratic construction and backwards through the π–π theorem, is too indirect to tell us much about $(1+T)\cdot SWQR$. The remedy is to lift Ranicki's quadratic construction from the chain complex level to the level of retractive spaces and spectra.

Let $\mathfrak{N}\mathfrak{M}^{\pitchfork}(B,\gamma)$ be the subspace of $\mathfrak{N}\mathfrak{M}(B,\gamma)$ consisting of the surgery problems $f : N \to Y$ over (B,γ) where Y is a manifold and $f_{|\partial N}$ is a homeomorphism to ∂Y. Like $\mathfrak{N}\mathfrak{M}(B,\gamma)$ this is an infinite loop space, but unlike $\mathfrak{N}\mathfrak{M}(B,\gamma)$ it behaves "excisively" as a functor of (B,γ). (That is, it takes homotopy pushout squares to homotopy pullback squares—thanks to transversality.) The difference construction takes $\mathfrak{N}\mathfrak{M}^{\pitchfork}(B,\gamma)$ to closed manifold bordism, which results in a commutative diagram

$$\begin{tikzcd}
\mathfrak{N}\mathfrak{M}^{\pitchfork}(B,\gamma) \arrow[r, "\subset"] \arrow[d] & \mathfrak{N}\mathfrak{M}(B,\gamma) \arrow[d, "\Xi"'] \\
\widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(B)^{\%}) \arrow[r] & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(B))\,.
\end{tikzcd}$$

Here Ξ is short for $\Xi\cdot(1+T)\cdot SWQR$, and by 6.8 this can be described as a difference construction followed by an Euler characteristic type map. Noting that closed manifolds have better Euler characteristics than Poincaré spaces, we have the left–hand vertical arrow and the commutativity. Since all this is natural in B, and since the lower left–hand corner of the diagram is the universal excisive approximation of the lower right–hand corner, we see that the left–hand vertical arrow is sufficiently determined and characterized by the rest of the diagram (including the excision property of the left–hand corner), and we *must* denote it by $\Xi^{\%}$. Furthermore, if $B = M$ and $\gamma = \nu^M$ and $n = \dim(M)$, then $\tilde{\mathcal{S}}(M)$ comes with a canonical map to the homotopy fiber of the upper row in the diagram, so the diagram extends:

$$\begin{tikzcd}
\tilde{\mathcal{S}}(M) \arrow[r] \arrow[d] & \mathfrak{N}\mathfrak{M}^{\pitchfork}(M,\gamma) \arrow[r, "\subset"] \arrow[d, "\Xi^{\%}"'] & \mathfrak{N}\mathfrak{M}(B,\gamma) \arrow[d, "\Xi"'] \\
\widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(B)_{\%}) \arrow[r] & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(B)^{\%}) \arrow[r] & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(B))\,.
\end{tikzcd}$$

Again, we *must* denote the left–hand arrow by $\Xi_{\%}$. To show that this $\Xi_{\%}$ is homotopic to χ (right–hand vertical in diagram (6.**i**)), we repeat the game of 1.7 and

1.8. This gives, in the notation of 1.8,

$$\begin{array}{ccc}
\tilde{\mathcal{S}}(M) = |\mathfrak{Y}| & \xrightarrow{\Xi_{\%}} & \big|k \mapsto H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}_{\lrcorner}(M \times \Delta^k)_{\%})\big| \\
\big\downarrow \subset & & \big\downarrow \subset \\
|\mathfrak{Z}| & \longrightarrow & \big|k \mapsto \bigcup_j H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}_{\lrcorner}(M \times \Delta^k \times \mathbb{R}^j)_{\%})\big| \\
\big\uparrow \subset & & \big\uparrow \subset \\
{}^{pos}c\mathcal{S}(M \times \mathbb{R}^\infty) \simeq \mathfrak{Z}[0] & \xrightarrow{\chi} & \bigcup_j H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times \mathbb{R}^j)_{\%})
\end{array}$$

where all the horizontal arrows are defined via Euler characteristics, the right–hand column consists of homotopy equivalences by inspection, and the left–hand column consists of homotopy equivalences by 1.8. $\square$

7. Localization

Here we are concerned with the right–hand side of the diagram in Theorem 0.7., that is, with

$$\iota : \mathcal{S}(\tau) \longrightarrow LA(M, \nu, -n)^{\%}.$$

The idea is to "localize" the construction(s) which led to the left–hand side. We localize near a closed subset $C \subset M$. Localization has a price: For instance, a homotopy equivalence $f : M' \to M$ (alias structure) restricts to a map $f^{-1}(U) \to U$ for any neighborhood U of C, but usually this restriction is not a homotopy equivalence. But it is still a degree one normal map. (In fact it is what we call an *unstable* degree one normal map: the unstabilized tangent bundle of $f^{-1}(U)$ pulls back from a bundle on U). We must try to make the best of that. In particular, one would hope that the expression

$$f_*\langle\!\langle f^{-1}(U)\rangle\!\rangle - \langle\!\langle U\rangle\!\rangle$$

can be given a meaning even though U is noncompact as a rule, and that it defines a homotopy fixed point of the appropriate involution on

$$\boldsymbol{A}(M@C)^{\%} := \boldsymbol{A}(M)^{\%}/\boldsymbol{A}(M \smallsetminus C)^{\%}\,.$$

If it can be done, in a way which is independent of the neighborhood U chosen, then surely the diagram

$$\text{(7.i)}\qquad \begin{array}{ccc}
\mathcal{S}(M) & \xrightarrow{\text{restriction}} & \left\{\begin{array}{c}\text{germs of unstable degree 1}\\ \text{normal maps about } C\end{array}\right\} \\
\chi\big\downarrow & & \chi\big\downarrow \\
H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) & \longrightarrow & H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M@C)^{\%})
\end{array}$$

commutes up to a preferred homotopy. The lower row is the composition

$$H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) \xrightarrow{\text{forget}} H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%}) \xrightarrow{\subset} H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M@C)^{\%})\,.$$

Our definition of the space of germs of unstable degree one normal maps about C is consistent with that of $\mathcal{S}(M)$. (It is not a "blockwise" definition.) This is an important point. Namely, suppose from now on that C does not contain any connected component of M. Then a miracle in the shape of Phillips–Gromov submersion theory will tell us that the upper right–hand corner in (7.**i)** is homotopy equivalent to $\mathcal{S}(\tau@C)$, the space of structures on $\tau = \tau^M$ defined in a neighborhood of C.

Suppose further that $C' \subset M$ is another closed set, not containing any connected component of M, and that $C \cup C' = M$. There are squares like (7.**i)** for C' and $C \cap C'$. Collecting all this in one square, we have

(7.**ii)**
$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\text{restriction}} & \Lambda_1 \\ \chi\big\downarrow & & \chi\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) & \longrightarrow & \Lambda_2 \end{array}$$

where Λ_1 and Λ_2 are the homotopy pullbacks of

$$\mathcal{S}(\tau@C) \longrightarrow \mathcal{S}(\tau@(C \cap C')) \longleftarrow \mathcal{S}(\tau@C')\ ,$$
$$H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M@C)^{\%}) \longrightarrow H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M@(C \cap C'))^{\%}) \longleftarrow H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M@C')^{\%})$$

respectively. Then clearly $\Lambda_1 \simeq \mathcal{S}(\tau)$ and $\Lambda_2 \simeq H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})$. Clearly the restriction map from $\mathcal{S}(M)$ to Λ_1 is ∇ (defined in the introduction). Taking all this into account, we may replace (7.**ii)** by a homotopy commutative diagram

(7.**iii)**
$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\ \nabla\ } & \mathcal{S}(\tau) \\ \chi\big\downarrow & & \chi\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)_{\%}) & \xrightarrow{\text{forget}} & H^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})\,. \end{array}$$

Moreover, being optimistic, we expect to see similar commutative diagrams

(7.**iv)**
$$\begin{array}{ccc} c\mathcal{S}(M \times V) & \xrightarrow{\ \nabla\ } & \mathcal{S}(\tau^{M \times V}) \\ \chi\big\downarrow & & \chi\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times V)_{\%}) & \xrightarrow{\text{forget}} & H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times V)^{\%}) \end{array}$$

for vector spaces V in $\mathcal{J}$. With these diagrams available, we are in a situation which is similar to that in the beginning §6, and we finish in exactly the same way (except that we need not go through the "pos" construction). That is, we let

$$D^{\tau}(V) := \mathcal{S}(\tau^{M \times V})\,, \qquad E^{\tau}(V) := H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M \times V)^{\%})$$

and we show that

(7.**v)**
$$\begin{array}{ccc} D^{\tau}(0) & \longrightarrow & D^{\tau}(\mathbb{R}^{\infty}) \\ \chi\big\downarrow & & \chi\big\downarrow \\ E^{\tau}(0) & \longrightarrow & E^{\tau}(\mathbb{R}^{\infty}) \end{array}$$

is a homotopy pullback square in a certain range. Then we verify

(1) $D^\tau(\mathbb{R}^\infty) \simeq \tilde{\mathcal{S}}(\tau)$,
(2) $E^\tau(\mathbb{R}^\infty) \simeq \widehat{H}^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}(M)^\%)$

and we identify the right–hand vertical arrow in (**7.v**) with the composition

$$\tilde{\mathcal{S}}(\tau) \longrightarrow L(M,\nu,-n)^\% \xrightarrow{\Xi^\%} H^\blacktriangledown(\mathbb{Z}_2; \boldsymbol{A}(M)^\%)\,.$$

Eventually, we shall have to discuss how much the whole construction depends on C and C' (not much).

7.1. Definitions—uncluttered. An *unstable degree one normal map* between oriented manifolds N_1, N_2 of the same dimension n consists of

a proper degree one map $f : N_1 \to N_2$,
an $\mathbb{R}^n$–bundle ζ on N_2,
and a (micro–)bundle isomorphism of $f^*\zeta$ with the tangent bundle of N_1.

Returning to M^n (closed and oriented for simplicity) and the closed subset $C \subset M$, we define $\mathcal{N}(M@C)$, the *space of germs of unstable degree one normal maps about* C. It is the classifying space of a topological category, say $\mathbf{nor}(M@C)$. An object in the category consists of

(1) an oriented N^n without boundary, and a map $f : N \to M$ which is proper and of degree one over C ;
(2) an $\mathbb{R}^n$–bundle ζ defined on a neighborhood of C in M ;
(3) a bundle isomorphism $\jmath : f^*\zeta \to \tau^N$, defined over a neighborhood of $f^{-1}(C)$.

Such an object has "subobjects", one for each open subset of N containing the compact set $f^{-1}(C)$. A *morphism* is an isomorphism of one object with a subobject of another. As regards the topology, we allow continuous variation of the f and $\jmath$ in an object $(N, f, \zeta, \jmath)$. We also allow continuous variation of the embedding $e : N \hookrightarrow N'$ and the bundle isomorphism $\theta : \zeta \to \zeta'$ in a morphism (e, θ) from $(N, f, \zeta, \jmath)$ to $(N', f', \zeta', \jmath')$.

There is a more discrete variant $\mathbf{nor}^\delta(M@C)$ where continuous variation of the embeddings e is no longer allowed (but f, $\jmath$ in objects and θ in morphisms can still vary continuously). The classifying space of this variant is $\mathcal{N}^\delta(M@C)$.

Again, it is a generally a good idea to replace topological categories by simplicial ones before taking realizations, but in the following informal discussion it is not a good idea—so please refrain.

7.2. Informal discussion. Define another category as follows: objects are objects $(N, f, \zeta, \jmath)$ in $\mathbf{nor}(M@C)$ together with a choice of point $z \in N$. Morphisms are morphisms in $\mathbf{nor}(M@C)$ sending the distinguished point to the distinguished point. The classifying space B of this category comes with a forgetful map to $\mathcal{N}(M@C)$. Each fiber of this map is a manifold N, equipped with an unstable degree one normal map to a neighborhood of C in M. It follows that $\mathcal{N}(M@C)$ parametrizes a giant family of unstable degree one normal maps to neighborhoods of C, and we would like to think of this family as a *universal* family. To be more explicit, suppose that X is a manifold with boundary and let $g : X \to \mathcal{N}(M@C)$ be a continuous map. The pullback g^*B of $B \to \mathcal{N}(M@C)$ is then a manifold with boundary, and the projection $g^*B \to X$ is a submersion with n–dimensional fibers. Each fiber is the domain of an unstable degree one normal map to a neighborhood of C in M. When $C = M$, the fibers are compact and the submersion $g^*B \to X$ is a fiber bundle. But in general, and especially when C does not contain any closed component, there is much less rigidity, and submersion theory will help us understand the homotopy type of $\mathcal{N}(M@C)$.

If g lifts to $\mathcal{N}^\delta(M@C)$, then g^*B acquires additional structure in the shape of a foliation by leaves of codimension n, transverse to the fibers of the submersion $E \to X$. (Equivalently, but more vaguely, g^*B comes with a flat connection.)

7.3. Theorem♣. *The inclusion $\mathcal{N}^\delta(M@C) \hookrightarrow \mathcal{N}(M@C)$ is a homology equivalence.*

Like 1.2, this is a Corollary to the McDuff–Segal–Mather theorems. It is valid without any assumptions on C. In the next theorem, we do assume that C does not contain a connected component of M.

7.4. Theorem. *A certain map from $\mathcal{N}(M@C)$ to $\mathcal{S}(\tau@C)$ is a homotopy equivalence.*

Definition and sketch proof. Defining the map amounts to showing that any unstable degree one normal map $f : N \to M$ about C, with bundle ζ, determines a stable fiber homotopy trivialization t of the Whitney sum $\zeta \oplus \nu^U$ over a small neighborhood U of C. In fact, for the purposes of this proof, $\mathcal{S}(\tau@C)$ is the space of such pairs (ζ, t).

To define this trivialization, informally, we can assume that $f : N \to U \subset M$ is proper of degree one, and that ζ lives on U. Choose an $\mathbb{R}^k$–bundle $\zeta^\perp$ on U and a trivialization of $\zeta \oplus \zeta^\perp$. This trivializes $f^*\zeta \oplus f^*(\zeta^\perp)$ and therefore $\tau^N \oplus f^*(\zeta^\perp$. Then $f^*(\zeta^\perp)$ is "the" normal bundle of N, which gives a canonical reduction $\rho : \mathbb{S}^{n+k} \to (T \cup \{\infty\})$, where T is the total space of $f^*(\zeta^\perp)$. Pushing this forward, we have a reduction $f_*\rho$ of $\zeta^\perp$ which, by the characterization of Spivak normal bundles (as in 5.2), determines a stable fiber homotopy equivalence of $\zeta^\perp$ with ν^U, hence a stable fiber homotopy trivialization of $\zeta \oplus \nu^U$.

For the proof of 7.4, we need the following facts from submersion theory.

(1) Suppose that $g : P \to Q$ is a map between manifolds without boundary, and that P has no compact component. Suppose there exists a locally trivial epimorphism $e : \tau^P \to g^*(\tau^Q)$ (details follow). Then the pair (g, e) is homotopic (through similar pairs) to a a pair (g_1, e_1) where g_1 is a submersion with derivative e_1.

(2) Suppose that g is already a submersion in a neighborhood of some closed subset $X \subset P$, and that the locally trivial epimorphism from τ^P to $g^*(\tau^Q)$ agrees near X with the derivative of g. Then the homotopy from (g, e) to (g_1, e_1) in (1) can be constructed rel X, provided no component of $P \smallsetminus X$ is relatively compact.

(3) Suppose that P and Q have boundaries after all, let $g : P \to Q$ take boundary to boundary, and assume that neither P nor ∂P has a compact component. Suppose there exists a locally trivial epimorphism $e : (\tau^P, \tau^{\partial P}) \to g^*(\tau^Q, \tau^{\partial Q})$. Then (g, e) is homotopic to (g_1, e_1), where g_1 is a submersion taking boundary to boundary, and e_1 is its derivative.

(4) Suppose further that g is already a submersion in a neighborhood of some closed subset $X \subset P$, etc. .
(5) The conclusions of (1), (2), (3) and (4) remain valid if locally trivial epimorphisms are replaced by locally trivial *stable* epimorphisms (details just below).

A map between microbundles on P is a *locally trivial epimorphism* if, in suitable local bundle charts, it looks like a projection

$$U \times \mathbb{R}^i \times \mathbb{R}^j \longrightarrow U \times \mathbb{R}^j \quad ; \quad (x, v, w) \mapsto (x, w)$$

(U open in P). A *stable* map between microbundles on P, say ξ_1 and ξ_2, is a map from $\xi_1 \oplus \varepsilon^j$ to $\xi_2 \oplus \varepsilon^j$, where ϵ^j is a trivial microbundle. For proofs and details, see [Gau]. The stable versions (5) follow from the unstable ones by obstruction theory.

We use all this to show that the relative homotopy groups of $\mathcal{N}(M@C) \to \mathcal{S}(\tau@C)$ are trivial, for any choice of base point in $\mathcal{N}(M@C)$. A typical representative for an element in one of these homotopy groups consists of the following:

(6) a neighborhood U of C in M ;
(7) a manifold P^{n+k-1} with a map $(g_1, g_2) : P \to \mathbb{S}^{k-1} \times U$ such that g_1 is a submersion, g_2 is proper, and (g_1, g_2) has degree one ;
(8) a microbundle ζ on $\mathbb{S}^{k-1} \times U$, and an isomorphism of $(g_1, g_2)^*(\zeta)$ with the vertical tangent bundle of P (where *vertical* is the direction of the fibers of g_1) ;
(9) an extension of the unstable normal invariant $\mathbb{S}^{k-1} \to \mathcal{S}(\tau@C)$ determined by (6), (7), (8) to a map from $\mathbb{D}^k$ to $\mathcal{S}(\tau@C)$.

Using transversality, "realize" the information in (9) (making U smaller if necessary) to obtain

(10) a manifold W^{n+k} with boundary $\partial W = P$, and a map $(\bar{g}_1, \bar{g}_2)$ from W to $\mathbb{D}^k \times U$, extending (g_1, g_2) in (7), such that $\bar{g}_2$ is proper and $(\bar{g}_1, \bar{g}_2)$ has degree one ;
(11) a microbundle $\bar{\zeta}$ with n–dimensional fibers on $\mathbb{D}^k \times U$, extending ζ, and a stable isomorphism of τ^W with $(\bar{g}_1, \bar{g}_2)^*(\bar{\zeta}) \oplus \varepsilon^k$ whose restriction over P is the stable isomorphism resulting from the submersion g_1 and the bundle isomorphism in (8).

Finally note that the bundle isomorphism in (11) implies a stable locally trivial epimorphism from τ^W to ε^k. According to (1), (2), (3), (4) and (5), we can realize this by a submersion. In other words, we may *assume* that $\bar{g}_1$ is a submersion and that the stable isomorphism in (11) results from this and an isomorphism of the vertical tangent bundle of W with $(\bar{g}_1, \bar{g}_2)^*(\zeta)$ extending the isomorphism in (8). But then we are in the relative homotopy group of the identity map $\mathcal{N}(M@C) \to \mathcal{N}(M@C)$. $\square$

7.5. Definitions–cluttered. To be properly equipped for the construction of Euler characteristic type maps out of $\mathcal{N}(M@C)$ we have to add the following to (1), (2) and (3) in 7.1:

(4) a compact subset $Z \subset N$, call it *conductor*, containing $f^{-1}(C)$;

(5) a stable fiber homotopy equivalence of $f^*(\nu^M)$ with ν^N, defined near $f^{-1}(C)$, together with all the data needed to make this into a contractible choice (details follow).

Continuous variation of the conductor Z in (4), whatever that means, is not allowed. However, replacement of the conductor by a smaller conductor is allowed, and counts as a *morphism*. Thus any morphism in $\mathbf{nor}(M@C)$ (cluttered version) has the form

$$\text{isomorphism with a subobject \& conductor shrinkage}\,.$$

To understand (5), note that items (1), (2) and (3) in 7.1 do indeed determine, up to contractible choice, a stable fiber homotopy equivalence as in (5) (see first part of the proof of 7.4). It is left to you, gentle reader, to define the appropriate contractible space of choices. Continuous variation of all the material in (5) is allowed.

7.6. Definitions. Since we need Euler characteristics for the domains of *germs* of normal maps, we should search for descriptions of $\boldsymbol{A}^{\%}$ involving germs of retractive spaces. There are many definitions of this type in controlled A–theory. Suppose that $B \subset \bar{B}$ is a control space, and that $Z \subset (\bar{B} \smallsetminus B)$ is compact. We consider objects (U, X) where U is a neighborhood of Z in $\bar{B}$ and

$$X = \big(X \underset{s}{\overset{r}{\rightleftarrows}} U \cap B\big)$$

is a proper retractive ENR. Two such objects (U, X) and (U', X') are regarded as equal if there exists a smaller neighborhood U'' of Z such that the restrictions of X and X' to U'' are equal. We speak of a *germ near* Z of proper retractive ENR's over B. Define germs of controlled maps between such objects, germs of controlled homotopy equivalences, and germs of retractive maps. Let $\mathcal{C}$ be the category of such germs, where the morphisms are map germs over B and relative to B ; a morphism is a *cofibration* if it is the germ of an injection, and a weak equivalence if it it is the germ of a controlled homotopy equivalence (compare the definitions leading up to 3.6.) Write

$$\boldsymbol{A}(B@Z \subset \bar{B})$$

for the K–theory of $\mathcal{C}$. Note in passing that compactness of $\bar{B}$ was not used in any essential way, so we can allow $\bar{B}$ to be just locally compact. But compactness of Z is essential.

7.7. Proposition♣. *For Y, a compact ENR, the diagram*

$$\boldsymbol{A}(Y) \xrightarrow{\subset} \boldsymbol{A}\big(Y\times[0,\infty) \subset Y\times[0,\infty]\big) \longrightarrow \boldsymbol{A}\big(Y\times[0,\infty)@Y\times\{\infty\} \subset Y\times[0,\infty]\big)$$

(first arrow induced by inclusion of $Y \cong Y\times\{0\}$, second arrow by passage to germs, composite arrow equal to zero) is a (co–)fibration sequence up to homotopy. Consequently $\Omega\boldsymbol{A}\big(Y\times[0,\infty)@Y\times\{\infty\} \subset Y\times[0,\infty]\big)$ is a model for $\boldsymbol{A}(Y)^{\%}$ (by 3.6).

This is almost a corollary of the fibration theorem and the approximation theorem in [Wald3]. Suppose now that Z is a compact subset of Y, and that X is another compact ENR with a map $f : X \to Y \smallsetminus Z$. Then, using the models for $\boldsymbol{A}^{\%}$ proposed in 7.7, we have maps

$$\boldsymbol{A}(X)^{\%} \xrightarrow{f_*} \boldsymbol{A}(Y)^{\%} \xrightarrow{\text{forget}} \Omega\boldsymbol{A}\big(Y\times[0,\infty)@Z\times\{\infty\} \subset Y\times\{\infty\}\big)\,,$$

the second one by passage to smaller germs. The composition is zero by inspection. Keeping Y and Z fixed and allowing X and f to be truly arbitrary gives

$$\operatorname*{hocolim}_{f:X\to Y\smallsetminus Z} \boldsymbol{A}(X)^{\%} \longrightarrow \boldsymbol{A}(Y)^{\%} \longrightarrow \Omega\boldsymbol{A}\big(Y\times[0,\infty)@Z\times\{\infty\} \subset Y\times\{\infty\}\big).$$

and the composition is still zero. For the left–hand term we may write $\boldsymbol{A}(Y \smallsetminus Z)^{\%}$.

7.8. Proposition♣. *The diagram*

$$\boldsymbol{A}(Y \smallsetminus Z)^{\%} \longrightarrow \boldsymbol{A}(Y)^{\%} \longrightarrow \Omega\boldsymbol{A}\big(Y\times[0,\infty)@Z\times\{\infty\} \subset Y\times\{\infty\}\big)$$

is a (co–)fibration sequence up to homotopy. Hence $\Omega\boldsymbol{A}\big(Y\times[0,\infty)@Z\times\{\infty\} \subset Y\times\{\infty\}\big)$ *is a model for* $\boldsymbol{A}(Y)^{\%}/\boldsymbol{A}(Y\smallsetminus Z)^{\%} = \boldsymbol{A}(Y@Z)^{\%}$.

We use 7.8 to construct *local* microcharacteristics. The local microcharacteristic $\langle\!\langle Y@Z\rangle\!\rangle$ in $A(Y@Z)^{\%}$ is defined whenever Y is an ENR with a compact subset Z. It depends only on the germ of Y about Z. That is, for any neighborhood U of Z in Y, the equation $\langle\!\langle Y@Z\rangle\!\rangle = \langle\!\langle U@Z\rangle\!\rangle$ holds in $A(Y@Z)^{\%} = A(U@Z)^{\%}$. For this construction, we decree that $A(Y@Z)$ is the homotopy pullback of

$$* \longrightarrow \boldsymbol{A}\big(Y\times[0,\infty)@Z\times\{\infty\} \subset Y\times\{\infty\}\big) \hookleftarrow \boldsymbol{P}^{?}(Y,Z)$$

where $\boldsymbol{P}^{?}(Y,Z)$ is defined much like $\boldsymbol{P}(Y)$ in 3.7, except that *microequivalences* are replaced by *germs* of microequivalences near $Z\times\{\infty\}$. Then we proceed exactly as in 3.6 and sequel.

When $Y = Z$, we would like to say that $\langle\!\langle Y\rangle\!\rangle = \langle\!\langle Y@Z\rangle\!\rangle$. It is not quite true since these elements live in different spaces. However, the two spaces are related by a forgetful map (passage to germs) which is a homotopy equivalence and sends $\langle\!\langle Y\rangle\!\rangle$ to $\langle\!\langle Y@Z\rangle\!\rangle$.

The local microcharacteristic is lax natural for homeomorphism germs in the following sense: given (Y,Z) and (Y',Z') as above, and a map of pairs $f : Y \to Y'$ such that $f^{-1}(Z') \subset Z$ there is an induced map $A(Y@Z) \to A(Y'@Z')$. If f restricts to a homeomorphism from a neighborhood of $f^{-1}(Z')$ to a neighborhood of Z', then f determines a path in $A(Y'@Z')$ from $f_*\langle\!\langle Y@Z\rangle\!\rangle$ to $\langle\!\langle Y'@Z'\rangle\!\rangle$ … and so on as in 3.1 and sequel.

Next we use local microcharacteristics to construct a map from $\mathcal{N}(M@C)$ to $A(M@C)^{\%}$. We proceed exactly as in 3.8 and sequel, noting first of all that it is sufficient to construct a map from $\mathcal{N}^{\delta}(M@C)$ to $A(M@C)^{\%}$. Then we think of $\mathcal{N}^{\delta}(M@C)$ as the classifying space of a simplicial category (formerly topological category) and replace it by a homotopy equivalent classifying space of an ordinary category. What does this ordinary category look like ? At this stage we have to pay attention to items (1), (2), (3) in 7.1 and particularly to item (4) in 7.5, but not to (5) in 7.5. So our ordinary category has objects $(N^n, f, \zeta, \jmath, Z, k)$ where

(1) $f : \Delta^k \times N \to \Delta^k \times M$ is a map over Δ^k, proper and of degree one over a neighborhood of C in M ;
(2) an $\mathbb{R}^n$–bundle ζ defined on a neighborhood of C ;
(3) a bundle isomorphism $\jmath$ of $f^*\zeta$ with the vertical tangent bundle of $\Delta^k \times N$;
(4) a compact $Z \subset N$ such that $\Delta^k \times Z$ contains $f^{-1}(\Delta^k \times C)$.

From such an object we extract the local microcharacteristic $\langle\!\langle N@Z \rangle\!\rangle$ in $A(N@Z)^{\%}$, and the diagram

(7.vi) $$A(N@Z)^{\%} \xleftarrow{\simeq} A(\Delta^k \times N@\Delta^k \times Z)^{\%} \xrightarrow{p_* f_*} A(M@C)^{\%}$$

where $p : \Delta^k \times M \to M$ is the projection. Then we play the quasifibration–quasisection game. The functor F sending our object $(N^n, f, \zeta, \jmath, Z, k)$ to the homotopy pushout of (7.**vi**) determines a quasifibration on the classifying space of the category. The composition

$$A(M@C)^{\%} \xrightarrow{+\langle\!\langle M@C \rangle\!\rangle} A(M@C)^{\%} \xrightarrow{\subset} F(N^n, f, \zeta, \jmath, Z, k)$$

(where $+\langle\!\langle M@C \rangle\!\rangle$ is a translation map) is a homotopy equivalence, giving a (quasi) trivialization of the quasifibration ; finally the local microcharacteristic in

$$A(N@Z)^{\%} \subset F(N^n, f, \zeta, \jmath, Z, k)$$

determines a quasisection, which in view of the trivialization amounts to a map from base to fiber (i.e. from $\mathcal{N}^\delta(M@C)$ to $A(M@C)^{\%}$). By 7.3, this determines another map from $\mathcal{N}(M@C)$ to $A(M@C)^{\%}$ (same discussion as in sequel of 3.8). As usual, we leave it to the reader to make everything as functorial as possible.

There is a forgetful map from $\mathcal{S}(M)$ to $\mathcal{N}(M@C)$, given by passage to germs—strictly speaking, we should redefine $\mathcal{S}(M)$ as a subspace of $\mathcal{N}(M@M)$ to have such a map. If we do, then we have a commutative square

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\text{forget}} & \mathcal{N}(M@C) \\ {\scriptstyle 3.8 \text{ and sequel}}\downarrow & & \downarrow{\scriptstyle \text{just constructed}} \\ A(M)_{\%} & \longrightarrow & A(M@C)^{\%} \end{array}$$

The lower horizontal arrow is composition of the forgetful map $A(M)_{\%} \to A(M)^{\%}$ with the localization $A(M)^{\%} \to A(M@C)^{\%}$.

Practically no extra work is required to show that the map from $\mathcal{N}(M@C)$ to $A(M@C)^{\%}$ just constructed lifts to the homotopy fixed point space of a suitable SW duality involution. This gives the right–hand column in a commutative square predicted earlier in this section (diagram (7.**i**)). For this refinement, it is of course important to remember and use item (5) in 7.5. Also, we must "declare" the appropriate SW duality involution. Return therefore to the situation of 7.6, where B is an ENR, open dense in a locally compact $\bar{B}$, and $Z \subset \bar{B} \smallsetminus B$ is closed. Let γ be a spherical fibration with section, defined on $U \cap B$, where U is a neighborhood of Z in $\bar{B}$. For X and Y in $\mathcal{C}$ (notation of 7.6), define $X \odot_\gamma Y$ by the formula of 4.18, replacing proper retractive ENR's over B by germs of such near Z throughout.

7.9. Proposition♣. *The rule $(X,Y)\mapsto X\odot_\gamma Y$ is an SW product on $\mathcal{C}$.*

In practice this needs to be "normalized". For example, if B happens to be an n–manifold, and γ is its normal bundle, then the appropriate SW product tends to be $(X,Y)\mapsto \Omega^n(X\odot_\gamma Y)$.

As explained earlier in this section, diagram (7.**i)** leads to diagram (7.**ii)** and diagram (7.**iii)**, and we leave it there while studying $M\times V$, for a vector space V. This is a manifold and, with the compactification $(M\times V\subset M*S(V))$, a Poincaré control space (5.4). Therefore it has a Spivak normal fibration (5.2), and therefore

$$\nabla : c\mathcal{S}(M\times V)\longrightarrow \mathcal{S}(\tau^{M\times V})$$

could be defined exactly as in §0, with M replaced by $M\times V$. Alternatively, we could define it using 7.4, which is what we really are going to do.

With $C\subset M$ as before, we write $\mathcal{N}(M\times V@C)$ and the like to mean $\mathcal{N}(M\times V@C\times\{0\})$ and the like. Treating $M\times V$ as a manifold like any other manifold, we have

$$\chi : \mathcal{N}(M\times V@C)\to H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M\times V@C)^{\%})$$

which is the right–hand column of a commutative square

$$\text{(7.}\mathbf{vii}\text{)}\qquad \begin{array}{ccc} c\mathcal{S}(M\times V) & \xrightarrow{\text{forget}} & \mathcal{N}(M\times V@C) \\ \chi\big\downarrow & & \chi\big\downarrow \\ H^{\blacktriangledown}(\mathbb{Z}_2;c\boldsymbol{A}(M\times V)_{\%}) & \longrightarrow & H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M\times V@C)^{\%})\,. \end{array}$$

The lower horizontal arrow is induced by the composition of the forgetful map from $cA(M\times V)_{\%}$ to $cA(M\times V)^{\%}$ with the localization $cA(M\times V)^{\%}\to A(M\times V@C)^{\%}$. This *localization* can be understood in two ways: first of all we can describe it as passage to germs, which gets us from the homotopy fiber of

$$\boldsymbol{A}(M\times V\subset M*S(V))\hookrightarrow \boldsymbol{A}\big(M\times V\times[0,\infty)\subset M*S(V)\times[0,\infty]\big)$$

(our model for $c\boldsymbol{A}(M\times V)^{\%}$) to

$$\Omega\boldsymbol{A}\big((M\times V\times[0,\infty))@(C\times\{0\}\times\{\infty\})\subset(M\times V\times[0,\infty)\big)$$

(our model for $\boldsymbol{A}(M\times V@C)^{\%}$, allowing for technical modifications). Using 2.8 and 3.12, however, we can describe it as the forgetful map

$$\text{(7.}\mathbf{viii}\text{)}\qquad \operatorname*{holim}_{\text{compact } Z\subset M\times V}\boldsymbol{A}(M\times V@Z)^{\%}\quad\longrightarrow\quad \boldsymbol{A}(M\times V@C)^{\%}\,.$$

Supposing now that $M=C\cup C'$, where both C and C' are closed and neither contains a component of M, we write Λ_1^V and Λ_2^V for the homotopy limits of

$$\mathcal{N}(M\times V@C)\to\mathcal{N}(M\times V@(C\cap C'))\leftarrow\mathcal{N}(M\times V@C')\,,$$
$$H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M\times V@C)^{\%})\to H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M\times V@(C\cap C'))^{\%})\leftarrow H^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{A}(M\times V@C')^{\%})$$

respectively. We see from 7.4 and from (7.**viii**) just above that

$$\Lambda_1^V \simeq \mathcal{S}(\tau^{M\times V})\,, \qquad \Lambda_2^V \simeq H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M\times V)^{\%})\,.$$

Therefore, combining (7.**vii**) and the corresponding diagrams for C' and $C\cap C'$, we obtain the commutative diagram (7.**iv**) predicted earlier in this section.

Now let $D^\tau(V) := \mathcal{S}(\tau^{M\times V})$ and $E^\tau(V) := H^{\blacktriangledown}(\mathbb{Z}_2; c\boldsymbol{A}(M\times V)^{\%})$ as planned, and regard $\chi : D^\tau(V) \longrightarrow E^\tau(V)$ as a natural transformation between *continuous* functors in the variable V, using 3.14. Clearly $D^\tau(\mathbb{R}^\infty)$ is homotopy equivalent to $\tilde{\mathcal{S}}(\tau)$, the block version of $\mathcal{S}(\tau)$, and $E^\tau(\mathbb{R}^\infty)$ is a model for $\widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})$ by resolution arguments as in §6. One of our tasks is to identify χ from $D^\tau(\mathbb{R}^\infty)$ to $E^\tau(\mathbb{R}^\infty)$ with the composition

$$\tilde{\mathcal{S}}(\tau) \longrightarrow L(M,\nu,-n)^{\%} \xrightarrow{\Xi^{\%}} \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})\,.$$

This is similar to what we saw in §6. Here is a sketch: First observe that $\tilde{\mathcal{S}}(\tau) \simeq \tilde{\mathcal{N}}(M)$, and that $\tilde{\mathcal{N}}(M)$ behaves well (in contrast to $\mathcal{N}(M)$) in the sense that the square of restriction maps

$$\begin{array}{ccc}
\tilde{\mathcal{N}}(M) & \longrightarrow & \tilde{\mathcal{N}}(M@C) \\
\Big\downarrow & & \Big\downarrow \\
\tilde{\mathcal{N}}(M@C') & \longrightarrow & \tilde{\mathcal{N}}(M@(C\cap C'))
\end{array}$$

is a homotopy pullback square. Using this and bisimplicial arguments (as in end of §6), identify χ from $D^\tau(\mathbb{R}^\infty)$ to $E^\tau(\mathbb{R}^\infty)$ as the composition

$$\tilde{\mathcal{S}}(\tau) \xrightarrow{\simeq} \tilde{\mathcal{N}}(M) \to \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})$$

where the second arrow is defined directly by taking the difference of the microcharacteristics of domain and codomain in a normal map. Then use 6.8 to identify this second arrow with

$$\tilde{\mathcal{N}}(M) \simeq L(M,\nu,-n)^{\%} \xrightarrow{\Xi^{\%}} \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2; \boldsymbol{A}(M)^{\%})\,.$$

Finally we must show that the commutative diagram

$$\begin{array}{ccc}
D^\tau(0) & \longrightarrow & D^\tau(\mathbb{R}^\infty) \\
{\scriptstyle\chi}\Big\downarrow & & {\scriptstyle\chi}\Big\downarrow \\
E^\tau(0) & \longrightarrow & E^\tau(\mathbb{R}^\infty)
\end{array}$$

(labelled (7.**v**) earlier in this section) is highly connected. The proof of 6.5 can serve as a model: there the essence was to show that a certain map between spectra is a homotopy equivalence, or induces a homotopy equivalence of the (-1)–connected covers. We did this mostly by referring to Waldhausen's work. Here we also have a map between two spectra, which we must show is a homotopy equivalence. The two spectra are made from the continuous functors D^τ and E^τ on $\mathcal{J}$, and their n–th terms are equal to

$$\begin{aligned} &\mathrm{hofiber}\left(D^\tau(\mathbb{R}^n) \to D^\tau(\mathbb{R}^{n+1})\right), \\ &\mathrm{hofiber}\left(E^\tau(\mathbb{R}^n) \to E^\tau(\mathbb{R}^{n+1})\right) \end{aligned}$$

respectively. To understand the second of these better, we introduce another functor F^τ on $\mathcal{J}$ with $F^\tau(V)$ equal to $cA(M \times V)^{\%}$ (compare 2.8). Then we analyse F^τ as in 6.1, 6.2 and 6.3, and conclude that the spectrum associated with E^τ is homotopy equivalent to $\boldsymbol{A}(M)^{\%}$. Write $\Omega\,\mathbf{Wh}(\tau^M)$ for the spectrum associated with D^τ. We are trying to show that

$$\chi : \Omega\,\mathbf{Wh}(\tau^M) \longrightarrow \boldsymbol{A}(M)^{\%}$$

is a homotopy equivalence. Rather than blame it directly on Waldhausen, let's try to simplify the task by exploiting excision properties.

7.10. Abstraction. Let $\mathcal{CO}_n$ be the following category. Objects are pairs (M, C) where M is an n–manifold without boundary and $C \subset M$ is a compact subset not containing any connected component of M. A morphism from (M, C) to (M', C') is an embedding $f : M \to M'$ such that $f(C) \supset C'$. We call such a morphism an pseudoisomorphism if $f(C) = C'$. Let Ψ be a covariant functor from $\mathcal{CO}_n$ to spaces which satisfies the following.

(1) Ψ takes pseudoisomorphisms to homotopy equivalences.
(2) For an object (M, C) where C is a union of two compact subsets C_1 and C_2, the commutative square

$$\begin{array}{ccc} \Psi(M,C) & \longrightarrow & \Psi(M,C_1) \\ \downarrow & & \downarrow \\ \Psi(M,C_2) & \longrightarrow & \Psi(M,C_1 \cap C_2) \end{array}$$

(arrows induced by obvious morphisms) is a homotopy pullback square.

(The example to keep in mind is: $\Psi(M, C) = \mathcal{N}(M@C) \simeq \mathcal{S}(\tau^M@C)$ as in 7.4.) For a closed manifold M, we define

$$\Psi(M, \forall) := \operatorname*{holim}_C \Psi(M, C)$$

where the homotopy limit is taken over all closed subsets $C \subset M$ not containing any connected component of M. (Note that id_M is a morphism from (M, C) to (M, C') provided $C' \subset C$.) Exercise: If M itself is the union of two closed subsets C and C' neither of which contains a connected component, then the square

$$\begin{array}{ccc} \Psi^!(M,\forall) & \longrightarrow & \Psi(M,C) \\ \downarrow & & \downarrow \\ \Psi(M,C') & \longrightarrow & \Psi(M,C \cap C') \end{array}$$

is commutative up to a preferred homotopy, and as such it is a homotopy pullback square. (In our example, $\Psi(M,\forall)$ is homotopy equivalent to $\mathcal{S}(\tau^M)$, but usually *not* to $\mathcal{N}(M)$.)

Next, suppose that $\mathbf{\Psi}$ is a covariant functor from $\mathcal{CO}_n$ to spectra. Make the following assumptions: $\mathbf{\Psi}$ satisfies (1) and (2), and in addition

(1) 3 Isotopic morphisms in $\mathcal{CO}_n$, say from (M,C) to (M',C'), induce homotopic maps from $\mathbf{\Psi}(M,C)$ to $\mathbf{\Psi}(M',C')$.

(2) If $C=\cap_i C_i$ where $C_i\subset C_{i-1}\subset M$ for all $i>0$ and (M,C_0) is in $\mathcal{CO}$, then

$$\operatorname{hocolim}\mathbf{\Psi}(M,C_i) \quad \xrightarrow{\simeq} \quad \mathbf{\Psi}(M,C)\,.$$

Suppose that $\mathbf{\Theta}$ from $\mathcal{CO}_n$ to spectra also satisfies the four conditions, and let $e:\mathbf{\Psi}\to\mathbf{\Theta}$ be a natural transformation. Then, using excision and handlebody covers [KiSi] one can show: If $e:\mathbf{\Psi}(M,C)\to\mathbf{\Theta}(M,C)$ is a homotopy equivalence whenever C is a singleton, then e is a homotopy equivalence for all (M,C) in $\mathcal{CO}_n$. Consequently $e:\mathbf{\Psi}(M,\forall)\to\mathbf{\Theta}(M,\forall)$ is a homotopy equivalence for arbitrary closed M.

7.11. Example. Let $\mathbf{\Psi}(M,C) := \Omega\,\mathbf{Wh}(\tau^M@C)$,the spectrum made from the continuous functor on $\mathcal{J}$ given by

$$V\mapsto \mathcal{N}(M\times V@C)$$

(notation as in diagram (7.**vii**)). Let $\mathbf{\Theta}(M,C) := \boldsymbol{A}(M@C)^{\%}$, and think of it as the spectrum made from the continuous functor on $\mathcal{J}$ given by

$$V\mapsto H^{\blacktriangledown}(\mathbb{Z}_2;c\boldsymbol{A}(M\times V@C))\,.$$

Let $e:=\chi:\mathbf{\Psi}(M,C)\to\mathbf{\Theta}(M,C)$ be the map of spectra induced by the natural transformation of the same name χ (right–hand column of diagram (7.**vii**)). If we can show that e is a homotopy equivalence whenever C is a singleton, then according to 7.10 we have shown that

$$\chi:\mathbf{\Psi}(M,\forall)=\Omega\,\mathbf{Wh}(\tau^M) \quad\longrightarrow\quad \mathbf{\Theta}(M,\forall)=\boldsymbol{A}(M)^{\%}$$

is a homotopy equivalence for an arbitrary closed n–manifold M. After all, this is still what we want to show. Clearly

$$\chi:\Omega\,\mathbf{Wh}(\tau^M@\{z\})\to\boldsymbol{A}(M@\{z\})^{\%} \tag{7.ix}$$

is rather independent of M and z, up to isomorphism; only the dimension n seems to matter. But a change in dimension only results in a shift, upwards or downwards. Then it must be enough to show that (7.**ix**) is approximately $2n$–connected ($n=\dim(M)$). We can prove this using the commutative diagram

$$\begin{array}{ccc}
\Omega\,\mathbf{Wh}(M) & \xrightarrow{\lambda\nabla} & \Omega\,\mathbf{Wh}(\tau^M@\{z\}) \\
{\scriptstyle\chi}\big\downarrow & & {\scriptstyle\chi}\big\downarrow \\
\boldsymbol{A}(M)_{\%} & \xrightarrow{\lambda\cdot\text{forget}} & \boldsymbol{A}(M@\{z\})^{\%}
\end{array}$$

where $M = \mathbb{S}^n$ and z is the north pole. The upper horizontal arrow is the composition of $\nabla : \Omega\,\mathbf{Wh}(M) \longrightarrow \Omega\,\mathbf{Wh}(\tau^M)$ with localization near z, and the lower horizontal arrow is the composition of the forgetful map $\boldsymbol{A}(M)_{\%} \to \boldsymbol{A}(M)^{\%}$ with localization near z. It is a well known consequence of Morlet's disjunction lemma [Mor3], [BuLaRo] and the sliced smoothing theory of Morlet [Mor1], [Mor2], [BuLa2] that there exists a map g from the stable sphere $\boldsymbol{S}^n$ to $\Omega\,\mathbf{Wh}(\tau^M @\{z\})$ making

$$\lambda\nabla \vee g : \Omega\,\mathbf{Wh}(M) \vee \boldsymbol{S}^n \quad \longrightarrow \quad \Omega\,\mathbf{Wh}(\tau^M @\{z\}$$

approximately $2n$–connected (always assuming $M = \mathbb{S}^n$). See [Wald2] for explanations. Similar things can be said about the lower row of the diagram, either as a consequence of Waldhausen's theorem or as an application of Goodwillie calculus. Since we already know that the left–hand χ in the diagram is a homotopy equivalence, it is enough to prove that χ from $\Omega\,\mathbf{Wh}(\tau^M @\{z\})$ to $\boldsymbol{A}(M)^{\%}$ is an isomorphism on π_n. Equivalently, writing τ^0 for the tangent bundle of a point:

7.12. Lemma♣. *The map $\chi : \Omega\,\mathbf{Wh}(\tau^0) \to \boldsymbol{A}(*)$ induces an isomorphism on π_0.*

* * *

Since this paper is the last part in a series, this may be the place to correct errors in earlier parts.

(1) Proposition 2.2 in [WW1] is false. (The homotopy constructed in the so–called proof is not defined everywhere.) This has no serious consequences for [WW1], because the specific coordinate free spectra (e.g. [WW1, 1.11]) used to prove the main theorems are in fact sufficiently associative. Prove this by hand, or use a superior construction, given in [We], which produces strictly associative coordinate free spectra.

(2) The reference in the proof of [WW2, 2.10] should be to [42, Lemma 5.2], not [4,2 Lemma 5.2].

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