Orthogonal Maslov Index

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UDC 517.9

To the memory of Sasha Sidorov

0. Let us start with a few words about the ordinary Maslov index [1]. Given an R-vector space V with a symplectic form $C(v_1, v_2)$ and a triple of Lagrangian subspaces $L_i^{\perp} = L_i \subset V$, i = 1, 2, 3, we define an integer $\mu(L_1, L_2, L_3)$ as follows. Let us consider another R-vector space $L_1 \oplus L_2 \oplus L_3$ with the quadratic form

$$B(l_1, l_2, l_3) := C(l_1, l_2) + C(l_2, l_3) + C(l_3, l_1)$$

The Maslov index $\mu(L_1, L_2, L_3)$ is defined as sign(B).

The Maslov index implies an explicit formula for a discrete cocycle of the symplectic group $\mathbf{Sp}(V, C)$ as follows. Fix a Lagrangian subspace $L_0 \subset V$. For $g_1, g_2 \in \mathbf{Sp}(V, C)$ we put

$$\mu(g_1,g_2):=\mu(L_0,g_1L_0,g_2g_1L_0).$$

The cohomology class of μ lies in $H^2_{\text{discr}}(\mathbf{Sp}(V, C), \mathbb{Z})$, that is in the second cohomology of the group $\mathbf{Sp}(V, C)$ considered as a discrete group.

Let us recall a more customary topological expression for this class. The cohomology of a discrete group coincides with the cohomology of its classifying space. There is a canonical map from this classifying space into the classifying space of the same group considered as a topological group. Let us denote this map by $\xi : \mathbf{BSp}_{discr}(V, C) \to \mathbf{BSp}(V, C)$. A maximal compact subgroup of $\mathbf{Sp}(V, C)$ is isomorphic to $\mathbf{U}(n)$, where $n = \frac{1}{2} \dim V$. Hence, the classifying space $\mathbf{BSp}(V, C)$ is homotopy-equivalent to $\mathbf{BU}(n)$. Denote by E the universal bundle over $\mathbf{BSp}(V, C)$ and denote by $c_1(E)$ its first Chern class. Then $\mu = 4\xi^*c_1(E)$ (see, e.g., [6]).

1. Now let W be a vector space over \mathbb{R} with a symmetric form $B\langle v_1, v_2 \rangle$ of zero signature. A subspace $V \subset W$ is said to be *self-orthogonal* if $V^{\perp} = V$. Roughly speaking, the *orthogonal Maslov index* is an integer-valued function of a set of five self-orthogonal subspaces V_1, \ldots, V_5 (actually it depends on some additional parameters).

As above, this index leads to an explicit formula for a cocycle $\nu \in H^4_{\text{discr}}(\mathbf{O}(W, B), \mathbb{Z})$. To describe the corresponding class in topological terms, we denote by ξ the map from the classifying space $\mathbf{BO}_{\text{discr}}(W, B)$ of the orthogonal group with the discrete topology into the classifying space $\mathbf{BO}(W, B)$ of orthogonal group as a topological group.

Since the signature of the form B is zero, the maximal compact subgroup of O(W, B) is isomorphic to $O(n) \times O(n)$, where $n = \frac{1}{2} \dim W$. The classifying space can also be factorized: BO(W, B) is homotopyequivalent to $BO(n) \times BO(n)$. Let E_1 and E_2 be the corresponding universal bundles over BO(W, B). Then $\nu = \xi^*(p_1(E_1) - p_1(E_2))$, where p_1 stands for the first Pontryagin class.

2. This note has originated in attempts to understand the simplest cohomological invariants of manifolds arising from the simplest topological field theories. These invariants measure the obstructions to the additivity of the signature. The first obstruction has been calculated by Wall [2]. We just tried to model his calculations in the second simplest case. In [3] Atiyah computed these obstructions in a different way. The comparison yields the formula for ν given at the end of the previous section.

The idea to investigate topological field theories related to the signature, and thus the whole idea of this note, belongs to D. Kazhdan. I am deeply grateful to him. I am also obliged to P. Bressler, A. Goncharov, and D. Kaledin for their tolerance during numerous conversations. The results of this note were obtained at the beginning of 1992, while the author was studying at the Harvard University. It is a pleasure to acknowledge the hospitality and support of this institution. Finally, thanks are due to the referee for valuable comments and to B. Feigin for invaluable explanations.

Independent Moscow University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 29, No. 1, pp. 92–95, January-March, 1995. Original article submitted October 13, 1994.

3. The main construction. For the definition and properties of the ordinary (symplectic) Maslov index see [1].

(a) Let W be a vector space over \mathbb{R} with a nondegenerate symmetric bilinear form \langle , \rangle of signature 0. A subspace L of a space V with a bilinear form B is said to be *self-orthogonal* if $L^{\perp} = L + \text{Ker}(B)$.

(b) The second part of our data is a set of five self-orthogonal subspaces V_1, \ldots, V_5 of W.

(c) For any triple $I = \{i_1, i_2, i_3\} \subset \{1, \ldots, 5\}$ we define an auxiliary space

 $V_{I} := \operatorname{Ker}(s_{I} : V_{i_{1}} \oplus V_{i_{2}} \oplus V_{i_{3}} \to W, s_{I}(v_{1}, v_{2}, v_{3}) = v_{1} + v_{2} + v_{3})$

with the bilinear form $B_I((v_1, v_2, v_3), (u_1, u_2, u_3)) = \langle v_1, u_2 \rangle$. It is easy to verify that the form B_I is skew-symmetric.

(d) The third part of our data is formed by self-orthogonal subspaces $L_I \subset V_I$ for any triple I.

(e) For any quadruple $J = \{j_1, j_2, j_3, j_4\} \subset \{1, \ldots, 5\}$ we define a subspace $V_J := \bigoplus_{I \subset J} V_I$ with a skew-symmetric form $B_J := \bigoplus_{I \subset J} (-1)^I B_I$, where for $I = J - j_s$ we set $(-1)^I := (-1)^s$.

(f) We define a subspace L_J in V_J for any J:

$$L_J := \operatorname{Ker}(s_J \colon V_J \to V_{j_1} \oplus V_{j_2} \oplus V_{j_3} \oplus V_{j_4}),$$

where s_J denotes the component-wise summation.

Lemma 1. The subspace L_J in V_J is self-orthogonal with respect to the form B_J .

Unfortunately, the only proof I know requires case by case inspection of the possible configurations of the quadruple of subspaces V_{j_1}, \ldots, V_{j_4} listed in [4].

(g) We define a space $V := \bigoplus_J V_J$ with a skew-symmetric form $B := \bigoplus_J (-1)^J B_J$.

(h) Note that $(V, B) = \bigoplus_I (V_I \oplus V_I, B_I \oplus B_I)$. Finally, we define a self-orthogonal subspace $L \subset V$ as $\bigoplus_I (\Delta \oplus (\operatorname{Ker} B_I, 0))$. Here Δ is the diagonal in $V_I \oplus V_I$. We also note that $\operatorname{Ker} B_I$ is equal to the image of the operator $s^I \colon V^I \to V_I$, where $V^I = V_{i_1} \cap V_{i_2} \oplus V_{i_2} \cap V_{i_3} \oplus V_{i_3} \cap V_{i_1}$, such that $s^I(x, y, z) := (x - z, y - x, z - y)$.

(i) Now we have three self-orthogonal subspaces in V:

$$L, \quad \bigoplus_J L_J, \quad \bigoplus_{I \subset J} L_I.$$

We define the orthogonal Maslov index

$$\mu(V_1,\ldots,V_5;L_{123},\ldots,L_{345}):=\mu\bigg(L;\bigoplus_J L_J;\bigoplus_{I\subset J}L_I\bigg).$$

The orthogonal Maslov index leads to the 4-cochain $\nu(g_1, \ldots, g_4)$ of the orthogonal group $\mathbf{O}(W)$ in the usual way (cf. Sec. 0). Namely, let us choose a self-orthogonal subspace $L_{123} \subset V_{123}$ for every triple of self-orthogonal subspaces $V_1, V_2, V_3 \subset W$ in an arbitrary way (in the notation of (c) above). Select a self-orthogonal subspace $V_0 \subset W$ and put $V_1 := V_0, V_2 := g_1V_0, V_3 := g_2g_1V_0, V_4 := g_3g_2g_1V_0$, and $V_5 := g_4g_3g_2g_1V_0$ for any quadruple g_1, g_2, g_3, g_4 of orthogonal operators. Finally, we set

$$\nu(g_1, g_2, g_3, g_4) := \mu(V_1, \ldots, V_5; L_{123}, \ldots, L_{345})$$

Lemma 2. (a) ν is a cocycle;

(b) the class of ν does not depend on the choice of L_I .

4. Orthogonal Maslov index as an obstruction to the additivity of the signature. Now let us apply the above construction to the calculation of the signature defect, in the spirit of [2]. So let Mbe a 2⁴-manifold (see [5]), or in other words a manifold modeled on the boundary of a 5-simplex, or yet in other words a manifold M glued up of five manifolds with boundary M_J , $J \subset \{1, \ldots, 5\}$, #J = 4, so that the boundary ∂M_J is equal to the union $\bigcup_{I \subset J} M_I$ and the boundary of each M_I is contained in the union of smaller manifolds, and so on. Suppose that the smallest manifold M_{\emptyset} (without boundary) is of dimension 4k, so that dim M = 4k + 4. Let $W = H_{2k}(M_{\emptyset}, \mathbb{R})$ with the intersection form \langle , \rangle . It is a nondegenerate symmetric form of signature zero, since $M_{\emptyset} = \partial M_i$ for any $i \in \{1, \ldots, 5\}$.

Let $\{V_i := \operatorname{Ker}(H_{2k}(M_{\varnothing}) \to H_{2k}(M_i)) = \operatorname{Im}(\partial_i : H_{2k+1}(M_i, M_{\varnothing}) \to H_{2k}(M_{\varnothing})), i = 1, \ldots, s\}$ be a set of five self-orthogonal subspaces in W.

Recall the notation V_I introduced in Sec. 3(c). It is easy to see that for any triple $I = \{i_1, i_2, i_3\}$ the preimage $(\partial_{i_1} \oplus \partial_{i_2} \oplus \partial_{i_3})^{-1}(V_I)$ in $H_{2k+1}(M_{i_1}, M_{\varnothing}) \oplus H_{2k+1}(M_{i_2}, M_{\varnothing}) \oplus H_{2k+1}(M_{i_3}, M_{\varnothing})$ actually lies in $H_{2k+1}(M_{i_1} \cup M_{i_2} \cup M_{i_3})$. Denote by S_I its intersection with the kernel of the map induced by the inclusion of manifolds: $H_{2k+1}(M_{i_1} \cup M_{i_2} \cup M_{i_3}) \to H_{2k+1}(M_I)$.

Finally, let $L_I := (\partial_{i_1} \oplus \partial_{i_2} \oplus \partial_{i_3})(S_I) \subset V_I$. One can verify that L_I is a self-orthogonal subspace of V_I with respect to the symplectic form introduced in Sec. 3(c). Thus, it makes sense to consider the orthogonal Maslov index $\mu(W; V_1, \ldots, V_5; L_{123}, \ldots, L_{345})$.

Theorem 1.
$$sign(M) = \sum_{\#J=4} sign(M_J) + \mu(W; V_1, \ldots, V_5; L_{123}, \ldots, L_{345}).$$

The proof follows the scheme of [2].

Before stating the next corollary, we recall the notation of Secs. 1 and 3. Namely, ν denotes the cohomology class of degree 4 of $\mathbf{BO}_{discr}(W, B)$ (or equivalently, the discrete cohomology class of the group $\mathbf{O}(W, B)$) defined before Lemma 2. We denote by ξ the canonical map from $\mathbf{BO}_{discr}(W, B)$ to $\mathbf{BO}(W, B)$ and denote by E_1 and E_2 the universal bundles over $\mathbf{BO}(W, B)$.

Corollary 1. $\nu = \xi^*(p_1(E_1) - p_1(E_2))$.

The proof follows from the comparison of our Theorem 1 and the General Remarks (Sec. 4) of [3].

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Translated by M. Finkelberg