On Topological Manifolds

William W. Flexner

The Annals of Mathematics, 2nd Ser., Vol. 32, No. 2 (Apr., 1931), 393-406.

Stable URL:
http://links.jstor.org/sici?sici=0003-486X%28193104%292%3A32%3A2%3C393%3AOTM%3E2.0.CO%3B2-P

The Annals of Mathematics is currently published by Annals of Mathematics.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://uk.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://uk.jstor.org/journals/annals.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.

http://uk.jstor.org/
Wed Nov 3 07:54:27 2004

ON TOPOLOGICAL MANIFOLDS.[1]

By William W. Flexner.

A topological manifold M_n is a compact separable space which has a complete set of neighborhoods each of which is a combinatorial n-cell. A combinatorial n-cell is a generalization due to Alexander of the ordinary n-simplex, and has the connectivity numbers and torsion coefficients of the n-simplex. Special cases of topological manifolds have been studied before in analysis situs. The manifolds (variétés) investigated by Poincaré in his first paper[2] are topological manifolds which have a certain restricted parametric representation. Wilson[3] and Hopf[4] have investigated the singular images upon each other of manifolds whose defining neighborhoods are n-cells, but have not considered the questions here dealt with. In this and a subsequent paper it is proposed to extend, using the methods of Veblen's *Colloquium Lectures on Analysis Situs* and Lefschetz's *Colloquium Lectures on Topology,* to topological manifolds the classical duality and homology theorems. This involves defining Betti numbers and torsion coefficients, proving their topological invariance, defining Kronecker Indices and proving the duality theorems.

Vietoris[5] has introduced the homology and group invariants of a general compact metric space. To define bounding and non-bounding cycles he uses infinite sequences of chains made up of ideal cells whose diameter decreases towards zero. Section 1 of this paper defines the homology characters in another way: in terms of a complex on M_n composed of singular chains which play the rôle of cells. By means of a deformation theorem modeled on that due to Alexander[6] it is proved that any singular chain on M_n can be deformed onto the singular complex. Therefore the incidence matrices of the singular complex give topologically invariant homology characters. This method shows that the homology theory of M_n can be derived from a finite singular complex, while for an arbitrary compact metric space the complex must be infinite.

---

[1] Received October 3, 1930. Presented to the American Mathematical Society, December 29, 1930.

[2] Poincaré, H., Analysis Situs. Journ. de l'Ec. Polyt. (2) 1 (1895), pp. 1–123.

[3] Wilson, W., Representations of Manifolds. Math. Ann., 100 (1928), pp. 552–578.

[4] Hopf, H., Zur Topologie der Abbildungen von Mannigfaltigkeiten. Math. Ann., 100 (1928), pp. 579–608, and 102 (1929), pp. 562–623.

[5] Vietoris, L. Math. Ann., 97 (1927), pp. 454–472, and 101 (1929), pp. 219–225.

[6] Alexander, J. W. Trans. Am. M. S., 16 (1915), pp. 148–154.

Section 2 begins by specifying when a manifold is orientable with respect to a particular set of defining neighborhoods E_n^i. Then the Kronecker Index $(\gamma_p \cdot \gamma_{n-p})$ of two singular cycles γ_p and γ_{n-p} on M_n is defined. The object of this and the next section is to serve as a basis for a proof of the Poincaré duality theorem for M_n. Since M_n cannot be cut up into cells the relations between p- and $(n-p)$-cycles cannot be obtained from a dual complex. The connectivity properties in the large are in this case brought in by considering M_n as immersed in a Euclidean space S_r of a sufficient number of dimensions,[7] r, and considering the intersection of M_n with chains C_{r-p}^i of S_r bounded by the cycles Γ_{n-p-1}^i in S_r which according to P. Alexandroff[8] link each non-bounding cycle γ_p^i on M_n. This intersection can be proved to be a cycle, γ_{n-p}^i, and to intersect γ_p^i with a Kronecker Index 1, so the duality theorem of Poincaré follows from section 2. The work just outlined has been completed by Lefschetz and Flexner[9] since this paper was first written.

Section 3 contains a proof, suggested to the writer by Professor Alexander, that M_n is homeomorphic to a subspace of Euclidean r-space.

It follows as a corollary of the duality theorem that the connectivity numbers of order higher than n are zero, a result proved in another way by Vietoris.

My thanks are due to Professor Alexander for suggesting the problem here treated to me and to both him and Professor Lefschetz for their very generous help during the course of this work.

1. Definitions; Invariance of the Betti numbers and torsion coefficients. Throughout this paper technical terms are used as in Lefschetz's *Topology*[10].

The manifold M_n here dealt with is a space satisfying two conditions.

1. M_n is a compact separable space. This condition implies that M_n is metric[11].

2. It is further required that there exist a complete set of neighborhoods $\{E_n^i\}$ for M_n each of which is a normal combinatorial n-cell. (L. T., p. 113, see also p. 106).

A normal combinatorial n-cell is an open simplicial complex whose boundary is a circuit with the Betti and torsion numbers of the $(n-1)$-sphere and which is itself the join (L. T., p. 111) of this boundary and

---

[7] Menger, K., Dimensionstheorie. Berlin (1929).

[8] Alexandroff, P. Annals of Math. (2) 30 (1928), pp. 101–187.

[9] Lefschetz, S., and Flexner, W. W. Proc. Nat. Acad. Sci., 16 (1930), pp. 530–533.

[10] Lefschetz, S., Topology. Am. Math. Soc., Colloquium Publications, Volume XII (1930): referred to in the sequel as "L. T.".

[11] Urysohn, P., Math. Ann., 92 (1924), pp. 275–293.

a point. A property of the cells E_n^i often used is that every cycle on E_n^i bounds. In the sequel the normal combinatorial n-cell will be called simply the n-cell and the simplicial n-cell will be referred to with the prefix simplicial. The simplicial n-cell is a special case of the n-cell so that the class of manifolds M_n includes the type originally called "topological" made up of those compact separable spaces which can be covered by a finite number of overlapping simplicial n-cells.

A finite set of n-cell neighborhoods $\{E_n^i\}$ covering M_n is called the *covering set* $\{E_n^i\}$. Because of the Borel property, finite covering sets exist. All the n-cells of the same covering set which have a point in common with a given n-cell, E_n, of the set will be called a *nest of cells* with E_n as *center*. Each n-cell of the covering of M_n is itself an open simplicial manifold. This follows from the fact that each point of E_n^i has a neighborhood of arbitrarily small diameter which is an n-cell. Thus M_n can be covered by open sets which are simplicial manifolds. This is the essential property in the proof of the duality theorems.

THEOREM 1. *Given a manifold covered by a finite set U of overlapping n-cells, the manifold can be covered by another finite set U' of n-cells each so small that every nest of cells of U' is covered by a cell of U.*

Let $\{E_n^i\}$ be the complete set of neighborhoods of M_n. Each point is covered by an E_n^i of diameter less than ε. Therefore after removal of all neighborhoods of diameter greater than ε there remains a fundamental system of neighborhoods for M_n, of which, because M_n is closed and compact, a finite subset will cover M_n. So for every ε there is a finite covering $\{E'^i_n\}$ of cells whose diameter is less than ε. Given the finite covering $U = \{E_n^i\}$, the distance from any point of M_n to the boundary of some $E_n^i \supset P$ has a lower bound, $\eta > 0$. If then $\varepsilon < \frac{1}{3}\eta$, $\{E'^i_n\}$ is a covering of the sort required in the theorem where $U' = \{E'^i_n\}$. Remove from any covering U any n-cells entirely covered by other cells of U.

Now cover M_n with $n+2$ covering sets U^{-1}, U^0, U^1, $\cdots$, U^n such that:

i. U^{-1} is an original set covering M_n.

ii. The members of U^j are so small that theorem 1 applies and every nest of cells of U^j is completely covered by one n-cell of U^{j-1}.

Take arbitrarily a single point a_0^i in each n-cell E_n^{ni} of the covering U^n. Each such point is called an *elemental zero-cell* of M_n. The definition of an *elemental k-cell* is by induction. If E_n^{no}, E_n^{n1}, $\cdots$, E_n^{nk} are $k+1$ n-cells of the covering U^n all contained in the same n-cell $E_n^{n-k,\,i}$, of the covering U^{n-k} and such that any j of them ($j = 0, 1, \cdots, k$) are covered by an n-cell of the covering U^{n-j+1} the $k+1$ points a_0^i, where

$a_0^i \subset E_n^{ni}$, are the vertices of an oriented (L. T. p. 4) singular k-complex (L. T. p. 73), a_k, on M_n, with the properties:

1. If the n-cell E_n^{n-k+1, i_p} of the covering U^{n-k+1} covers $a_{k-1}^{i_p}$, then a_k is contained in an n-cell $E_n^{n-k, j}$ of the covering U^{n-k}. Such an $E_n^{n-k,j}$ exists because each E_n^{n-k+1, i_p} covers all but one of the cells $E_n^{n0}, \cdots, E_n^{ni_k}$ and hence these cells form a nest and are covered by a cell of U^{n-k}.

2. a_k is bounded by the elemental $(k-1)$-cells $a_{k-1}^{i_0}, \cdots, a_{k-1}^{i_k}$ determined by $E_n^{n0}, \cdots, E_n^{nk}$ taken k at a time. Because the definition is inductive, properties one and two are assumed for $a_{k-1}^{i_0}, \cdots, a_{k-1}^{i_k}$. Therefore the sum of these cells properly oriented is a cycle. Since the $(k-1)$st Betti number, $R_{k-1}(E_n^{n-k,j})$ of $E_n^{n-k,j}$ is zero, this cycle bounds a singular complex not necessarily an n-cell, on $E_n^{n-k,j}$ which can be taken to be a_k. This proves the existence of an a_k satisfying conditions one and two.

The *boundary* of an elemental k-cell is the sum of the oriented elemental $(k—1)$-cells that are obtained by omitting one at a time the $k+1$ vertices defining the elemental k-cell. In the sum each oriented elemental $(k—1)$-cell is affected with a sign as in L. T. (p. 14).

The definitions of *elemental chain, elemental cycle, elemental homology* are the same as those of chain, cycle and homology in L. T. (p. 16 et seq. and p. 21) with "elemental cell" written everywhere instead of "cell".

An elemental complex, $\mathfrak{K}$, on M_n is the set of all oriented elemental cells on M_n that can be constructed using as vertices a set of elemental zero-cells, one and only one in each n-cell E_n^{ni} of the covering U^n. The incidence relations (L. T. p. 16) of these cells with their oriented boundaries give a set of incidence matrices (L. T. p. 34) for M_n just as simplicial cells do for manifolds that can be cut up into simplicial cells. From these matrices Betti numbers and torsion coefficients (L. T. p. 34) can be calculated. In the sequel these will be referred to as the *elemental Betti and torsion numbers calculated from elemental homologies.* Because $\mathfrak{K}$ is made up of a finite number of elemental cells these numbers are finite.

A complex, chain, or cycle *on* M_n is the single valued continuous image on M_n of a simplicial complex chain or cycle. If a p-cell of a chain on M_n is mapped on an s-cell, $s < p$, of $\mathfrak{K}$, the p-cell will be given a zero coefficient in the chain, as in L. T. (p. 74). This is what Lefschetz calls a singular complex, chain or cycle. (L. T. p. 72). Here the word "singular" will be omitted.

THEOREM 2. *If γ_k is any k-chain on M_n it can be cut up into subdivisions so small that each can be covered by an n-cell of the covering U^n.*

This is apparent from the definition of γ_k.

THEOREM 3. *If γ_k is a k-cycle on M_n and γ'_k is a subdivision of γ_k and γ_k is the image of G_k, then $\gamma_k \sim \gamma'_k$ on M_n.* (Notation: L. T., p. 21).

A k-cycle is a special case of a k-chain. If G'_k is the subdivision of G_k producing γ'_k then $G_k - G'_k$ bounds a degenerate (L. T., p. 74) $(k+1)$-complex on G_k. Hence its image $\gamma_k - \gamma'_k$ bounds a $(k+1)$-chain on γ_k and hence on M_n.

THEOREM 4. *Fundamental Deformation Theorem: Given an r-cycle γ_r on M_n such that each cell of γ_r is in an n-cell of U^n, and an elemental complex $\mathfrak{K}$ on M_n then:*

Part 1. *Given any k-cell p^i_k of γ_k such that $p^i_k \to \sum_j t^i_j p^j_{k-1}$ (notation: L. T., p. 15) where t^i_j is an integer, there can be associated with p^i_k an elemental k-cell a^i_k of $\mathfrak{K}$ for which the following relations hold:*

i. *$a^i_k \to \sum_j t^i_j a^j_{k-1}$ where a^j_{k-1} is associated with p^j_{k-1}.*

ii. *There exists a $(k+1)$-chain q^i_{k+1} on M_n satisfying $q^i_{k+1} \to a^i_k - p^i_k - Q^i_k$ where $Q^i_k \to \sum_j t^i_j p^j_{k-1} - \sum_j t^i_j a^j_{k-1}$.*

Part 2. *If $p^1_k, p^2_k, \cdots, p^s_k$ are k-cells of γ_r and are all in E^{n0}_n then $a^1_k, a^2_k, \cdots, a^s_k$ lie in the same n-cell $E^{n-k-1,1}_n$ containing E^{n0}_n.*

Part 3. *If γ_k is a k-cycle on M_n such that $\gamma_k = \sum_i \mu_i p^i_k$, then $\sum_i \mu_i q^i_{k+1} \to \gamma_k - \Gamma_k$ where $\Gamma_k = \sum_i \mu_i a^i_k$ is a cycle.*

The proof is made by induction.

Step 0. Part 1. If a zero-cell p^i_0 is in the n-cell E^{ni}_n associate with it the elemental zero-cell a^i_0 of K in E^{ni}_n. If $E^{n-1,1}_n$ covers E^{ni}_n then $q^i_1 \to p^i_0 - a^i_0$ where q^i_1 is a segment in $E^{n-1,1}_n$ which, because the zeroth Betti number, $R_0(E^{n-1,1}_n)$, of $E^{n-1,1}_n$ is one, is bounded by p^i_0 and a^i_0.

Part 2. This follows immediately from the construction of a^i_0.

Part 3. Given that $\gamma_0 = \sum_i \mu_i p^i_0$ is a cycle, let $\Gamma_0 = \sum_i \mu_i a^i_0$. Since $q^i_1 \to p^i_0 - a^i_0$ it follows that $\sum_i \mu_i q^i_1 \to \gamma_0 - \Gamma_0$. Since the boundary of γ_0 vanishes and the boundary of Γ_0 corresponds to it cell for cell, the boundary of Γ_0 vanishes.

Step k. Imagine steps 0, 1, 2, $\cdots$, $k-1$ to have been taken for each of the $(0, 1, 2, \cdots, k-1)$-cells of γ_r.

Part 1. Given $p^i_k \to \sum_j t^i_j p^j_{k-1}$. Consider $\sum_j t^i_j a^j_{k-1}$ where a^j_{k-1} is associated by step $k-1$, part 1 with p^j_{k-1}. $\sum_j t^i_j a^j_{k-1}$ is a cycle by step $k-1$, part 3 and is in an n-cell $E^{n-k,i}_n$ containing E^{n0}_n by step $k-1$, part 2. So because $R_{k-1}(E^{n-k,i}_n) = 0$, $\sum_j t^i_j a^j_{k-1}$ bounds a^i_k in $E^{n-k,i}_n$. Associate a^i_k with p^i_k. By step $k-1$, part 3

$$\sum_j t_j^i q_k^j \to \sum_j t_j^i p_{k-1}^j - \sum_j t_j^i a_{k-1}^j .$$

Therefore $p_k^i - a_k^i - \sum_j t_j^i q_k^j$ is a cycle since its boundary vanishes. q_k^j is contained in an n-cell E_n^{n-k,a_j} covering p_{k-1}^j. Hence the set of cells $\{E_n^{n-k,a_j}\}$ $(j = 0, 1, \cdots, k)$ all have points in common with $E_n^{n-k,i}$ and so form a nest of cells. Therefore $p_k^i - a_k^i - \sum_j t_j^i q_k^j$ is in an n-cell $E_n^{n-k-1,1}$ of U^{n-k-1} and, being a cycle, bounds in that n-cell a $(k+1)$-complex q_{k+1}^i because $R_k(E_n^{n-k-1}) = 0$.

$$q_{k+1}^i \to p_k^i - a_k^i - \sum_j t_j^i q_k^j .$$

Let $\sum_j t_j^i q_k^j = Q_k^i$. This completes the proof of step k part 1.

Part 2. If $p_k^1, p_k^2, \cdots, p_k^s$ are in E_n^{no} then a_k^i associated with p_k^i lies in $E_n^{n-k,i}$ which contains E_n^{no} by step k, part 1. Hence the set $\{E_n^{n-k,i}\}$ $(i = 1, 2, \cdots, s)$, since each of its members contains E_n^{no}, forms a nest of cells and is covered by $E_n^{n-k-1,1}$ of the covering U^{n-k-1} and containing E_n^{no}.

Part 3. $\gamma_k = \sum_i \mu_i p_k^i$ is a cycle. Let $\Gamma_k = \sum_i \mu_i a_k^i$. Γ_k is a cycle because its boundary cells correspond to those of γ_k which vanish. From the result of step k, part 1 follows

$$\sum_i \mu_i q_{k+1}^i \to \sum_i \mu_i p_k^i - \sum_i \mu_i a_k^i - \sum_{ij} \mu_i t_j^i q_k^j .$$

But since γ_k is a cycle $\sum_i \mu_i t_j^i = 0$ for every j. Hence $q_{k+1}^i \to \sum_i \mu_i p_k^i - \sum_i \mu_i a_k^i$ and therefore $\sum_i \mu_i q_{k+1}^i \to \gamma_k - \Gamma_k$. This completes the induction and the proof of Theorem 4.

THEOREM 5. *To every k-cycle γ_k on M_n there is an elemental k-cycle Γ_k on M_n such that $\gamma_k - \Gamma_k \sim 0$.*

By Theorems 2 and 3 $\gamma_k - \gamma_k' \sim 0$ where γ_k' is a subdivision of γ_k satisfying the conditions put on γ_k in the statement of Theorem 4. By Theorem 4 $\gamma_k' - \Gamma_k \sim 0$ and therefore $\gamma_k - \Gamma_k \sim 0$.

THEOREM 6. *If γ_k is any k-cycle on M_n and Γ_k is the elemental k-cycle associated with it and $\gamma_k \sim 0$, then $\Gamma_k \sim 0$.*

By Theorem 5 $\gamma_k - \Gamma_k \sim 0$, so $\gamma_k \sim 0$ implies $\Gamma_k \sim 0$.

THEOREM 7. *If an elemental k-cycle Γ_k bounds a $(k+1)$-chain g_{k+1} on M_n, then Γ_k bounds an elemental $(k+1)$-chain G_{k+1}* on M_n.

Cut g_{k+1} up into cells so small that each of them is contained in an n-cell of U^n. Then apply the process of Theorem 4 to the i-cells of g_{k+1} $(i = 0, 1, 2, \cdots, k+1)$ associating with each of them an elemental i-cell.

For the elemental i-cells associated with the i-elements of the boundary use the i-elements of Γ_k themselves. All the k-cells of the boundaries vanish except those making up the boundary Γ_k. Hence Γ_k bounds the chain G_{k+1} of elemental i-cells associated with the cells of g_{k+1}. Theorems 5 and 7 are analogous to theorems proved by Veblen in his *Colloquium Lectures* Chapters 3 and 4 to obtain the topological invariance of the homology characters of a simplicial complex.

Instead of using the elemental cycles and homologies to calculate the Betti and torsion numbers of M_n it is possible to use the set of all cycles on M_n and their homologies (L. T. p. 75). These cycles and homologies will be called "topological" and the numbers topological Betti and torsion numbers. It will now be proved that the topological numbers are the same as the elemental numbers which were shown on page 396 to be finite. This implies that the topological numbers are finite. By Theorem 5 every cycle of M_n is homologous to an elemental cycle, so the elemental cycles form a basis for the cycles of M_n. Every relation of bounding among elemental cycles automatically implies a topological relation of bounding. Moreover any topological bounding relation among the elemental cycles, by Theorem 7, implies an elemental bounding relation. Therefore the homology group for any cycles on M_n is isomorphic to the homology group of the elemental complex which proves Theorem 8.

THEOREM 8. *The Betti numbers and torsion coefficients calculated from elemental homologies are finite and topologically invariant.*

Corollary. The connectivity and torsion numbers modulo m (L. T. p. 18 and p. 35) as obtained from the incidence relations of the elemental cells are topologically invariant. The proof is the same as that of Theorem 8 except that in adding the boundaries of the chains the coefficients of the individual cells of the boundary are reduced modulo m. This changes none of the details of the proof.

2. Orientation; Kronecker Index. It is now necessary to draw the distinction between manifolds orientable and non-orientable with respect to a covering set $\{E_n^i\}$ of n-cells. Suppose two n-cells of the set, E_n^i and E_n^j, overlap in a number of open sets of which R is one. Orient each n-cell by means of a complex on it. Cover each region R with an infinite complex, K^i, by the method described in Lefschetz's "Topology" (L.T., p. 311) using the complex on E_n^i to give K^i. K^i is connected and only two n-cells abut on each $(n-1)$-cell of K^i so K^i is an orientable simple circuit modulo an ideal element, Δ, its boundary. (L. T., p. 47 and p. 295). Hence it has a basic oriented n-cycle Γ_n^i mod Δ (L. T., p. 46 and p. 300). Because the Betti numbers of K^i are topologically invariant, Γ_n^j, the basic oriented n-cycle mod Δ obtained using, to cover R, an infine complex K^j derived from a complex on E_n^j, satisfies the relation $\Gamma_n^i \sim \varepsilon^j \Gamma_n^j \bmod \Delta$, where $\varepsilon^j = \pm 1$. If $p+1$ n-cells of $\{E_n^i\}$ cover R we have relations

$$\begin{array}{ccc} \Gamma_n^i \sim \varepsilon^{j_1} \Gamma_n^{j_1} & \bmod \Delta, \\ \vdots & \vdots & \vdots \\ \Gamma_n^i \sim \varepsilon^{j_p} \Gamma_n^{j_p} & \bmod \Delta. \end{array}$$

If the orientation of the cells E_n^i can be so chosen that for all i and j and all regions R, all the ε's corresponding to a given R are of the same sign, M_n is *orientable with respect to the covering* $\{E_n^i\}$, otherwise not.

It is now possible to define the Kronecker Index of two chains which do not intersect one another's boundaries and are on an orientable M_n. If M_n is not orientable the chains can be taken modulo 2 and the Kronecker Index computed modulo 2. Before defining the Index it is necessary to prove some theorems.

THEOREM 9. *Two cycles Γ_p and Γ_{n-p} are ε-deformable where $\varepsilon > 0$ is arbitrarily small, into Γ_p^s and Γ_{n-p}^s which intersect in a finite number of isolated points.*

Throughout the proof $F(E)$ means the boundary of E and $\bar{E}$ means $E + F(E)$, the closure of E. Suppose that $\{E_n^i\}$ is a finite set of cells covering M_n. On E_n^i take a complex K^i of mesh so small that if J^i is the sum of the closed cells of K^i with no points on $F(E_n^i)$, J^i covers M_n. Let 2τ be the maximum mesh of the complexes K^i.

Now make a subdivision of Γ_p and Γ_{n-p} into cells of mesh $\tau/4$ and call these subdivisions Γ_p and Γ_{n-p}. If the following assumptions about the reduction of the parts of Γ_p and Γ_{n-p} in $(i-1)$ of the domains $\{J^i\}$ be made, it can be proved that the reduction can be extended to another domain J and so, by induction, to all these domains. It is assumed that if

$$G^{i-1} = J^1 + J^2 + \cdots + J^{i-1},$$

then Γ_p and Γ_{n-p} are $\tau/4$-deformable into cycles Γ_p^{i-1} and Γ_{n-p}^{i-1} with mesh $\tau/4$ and such that the following three conditions are satisfied:

1. $$\Gamma_p^{i-1} = \gamma_p^{i-1} + \delta_p^{i-1}, \quad \Gamma_{n-p}^{i-1} = \gamma_{n-p}^{i-1} + \delta_{n-p}^{i-1},$$

where γ_p^{i-1} and γ_{n-p}^{i-1} are the sets of all closed cells of Γ_p^{i-1} and Γ_{n-p}^{i-1} respectively that have points in common with G^{i-1}.

2. γ_p^{i-1} intersects γ_{n-p}^{i-1} in a finite number of isolated points.

3. There exist neighborhoods N^{i-1} and N^{*i-1} such that N^{i-1} contains the sum of closed cells of γ_p^{i-1} that meet $F(G^{i-1})$, and N^{*i-1} contains the similar sum for γ_{n-p}^{i-1}, and such that

$$\Gamma_p^{i-1} \cdot N^{*i-1} = 0, \quad \Gamma_{n-p}^{i-1} \cdot N^{i-1} = 0, \quad N^{i-1} \cdot N^{*i-1} = 0.$$

In view of (3) the intersections of the γ's are interior to G^{i-1}, that is not on its boundary, and γ_p^{i-1} does not intersect δ_{n-p}^{i-1} nor does γ_p^{i-1} intersect δ_p^{i-1}. Furthermore

$$F(\delta_p^{i-1}) \subset N^{i-1}, \qquad F(\delta_{n-p}^{i-1}) \subset N^{*i-1}.$$

Let $G^i = J^1 + J^2 + \cdots + J^i$. It will now be proved that conditions one, two and three can be satisfied for the index i.

A. Consider the following quantities:

1. The distance from $F(\delta_p^{i-1})$ (which is on N^{i-1}) to $F(N^{i-1})$ and that from $F(\delta_{n-p}^{i-1})$ to $F(N^{*i-1})$.
2. The distances from δ_p^{i-1} and δ_{n-p}^{i-1} to G^{i-1}.
3. The distance from Γ_p^{i-1} to N^{*i-1} and from Γ_{n-p}^{i-1} to N^{i-1}.
4. The distance from $F(E_n^i)$ to $F(J^i)$.

The distances just mentioned are all positive and so is their lower bound. Choose a ζ less than a quarter of that lower bound and also less than $\tau/4$.

B. Subdivide Γ_p^{i-1} and Γ_{n-p}^{i-1} into subchains whose mesh ξ is to be at all events less than $\zeta/2$. Denote by γ_p and γ_{n-p} the set of all closed cells of δ_p^{i-1} and δ_{n-p}^{i-1} on E_n^i.

C. Let K be a complex on E_n^i which has J^i as a subcomplex and K^* be its dual such that the mesh of both complexes is ξ. The mesh of the subdivisions of Γ_p^{i-1} and Γ_{n-p}^{i-1} is small enough to assure that $F(\gamma_p)$ and $F(\gamma_{n-p})$ lie entirely in $E_n^i - J^i + N^{i-1}$ and $E_n^i - J^i + N^{*i-1}$ respectively.

D. By ξ-deformations γ_p and γ_{n-p} can be reduced to subchains of K and K^* respectively. Add the deformation chains of $F(\gamma_p)$ and $F(\gamma_{n-p})$ and call the new chains γ_p and γ_{n-p} and the new cycles Γ_p^i and Γ_{n-p}^i. In addition if γ_p $(p < n)$ has p-cells on $F(J^i)$, γ_p can be modified by another ξ-deformation as in the similar case, L. T., p. 153, so as to remove them from $F(J^i)$. The process is as follows. Because of D, J^i is a subchain of K. If $p < n$ it can be arranged that γ_p has no p-cell on $F(J^i)$ for if the cell σ_p of γ_p is on $F(J^i)$, σ_p can be replaced by the p other p-faces not on $F(J^i)$ of a $(p+1)$-simplex of which σ_p is one edge. If $p = n$ no n-cell can be on $F(J^i)$ because $F(J^i)$ is $(n-1)$-dimensional. Now call γ'_p the sum of the cells of γ_p whose closure meets J^i. Also γ'_{n-p} is the similar sum of cells of γ_{n-p}.

E. Set

$$\gamma_p^i = \gamma_p^{i-1} + \gamma'_p, \qquad \delta_p^i = \Gamma_p^i - \gamma_p^i,$$

and similarly for $n-p$.

It is now necessary to construct non-overlapping neighborhoods N^i and N^{*i} such that the sum of the closed p-cells of a suitable γ_p^i that meet $F(G^i)$ is contained in N^i and such that no points of Γ_{n-p}^i meet N^i. A similar neighborhood N^{*i} must be constructed for the $(n-p)$-chain. The first step in this construction is to create neighborhoods M^i and M^{*i} of the intersection of γ'_p and γ'_{n-p} with $F(J^i)$ such that

$$\begin{aligned} &\text{(a)}\quad M^i \cdot N^{*i-1} = 0, \qquad M^{*i} \cdot N^{i-1} = 0; \\ &\text{(b)}\quad M^i \cdot M^{*i} \;\;= 0; \\ &\text{(c)}\quad M^i \cdot \Gamma^i_{n-p} = 0, \qquad M^{*i} \cdot \Gamma^i_p = 0. \end{aligned}$$

(a) It follows from A 1 and B that γ'_p does not enter N^{*i-1} and similarly for γ'_{n-p} and N^{i-1}, so (a) can be satisfied.

(b) γ'_p was so constructed (D) that it does not meet γ'_{n-p} on $F(J^i)$. Therefore (b) can be satisfied.

(c) To prove that (c) can be fulfilled it suffices to show that

$$[\gamma'_p \cdot F(J^i)] \cdot \Gamma^i_{n-p} = 0,$$

and this will follow if

$$\begin{aligned} &\mathrm{C^I}.\quad [\gamma'_p \cdot F(J^i)] \cdot \gamma^{i-1}_{n-p} = 0, \\ &\mathrm{C^{II}}.\quad [\gamma'_p \cdot F(J^i)] \cdot \gamma'_{n-p} = 0, \\ &\mathrm{C^{III}}.\quad [\gamma'_p \cdot F(J^i)] \cdot \delta^i_{n-p} = 0, \end{aligned}$$

because

$$\Gamma^i_{n-p} = \gamma^{i-1}_{n-p} + \gamma'_{n-p} + \delta^i_{n-p}.$$

$\mathrm{C^I}$. γ'_p is outside G^{i-1} or in N^{i-1}. γ^{i-1}_{n-p} has no points in either of these sets so $\mathrm{C^I}$ holds.

$\mathrm{C^{II}}$. For the same reason as in case (b) equation $\mathrm{C^{II}}$ is true.

$\mathrm{C^{III}}$. Any closed cell of δ^i_{n-p} is either in N^{*i-1} where there are no points of γ'_p or else does not meet J^i which verifies $\mathrm{C^{III}}$.

It has been shown that a sufficiently small neighborhood M^i of $\gamma'_p \cdot F(J^i)$ will have the properties (a), (b), (c). Exactly the same proof holds for M^{*i} and the $(n-p)$-chains.

Suppose now γ_p and γ_{n-p} to be subjected to a sufficiently small subdivision and γ'_p and γ'_{n-p} to be defined as the sums of the closed cells of this new sub-division meeting J^i. Then because the old γ'_p and γ'_{n-p} did not intersect on $F(J^i)$, the sums of the closed cells of the new ones meeting $F(J^i)$ do not intersect one another. New γ^i_p, δ^i_p, γ^i_{n-p} and δ^i_{n-p} are now of course constructed from the new γ'_p and γ'_{n-p} as before from the old.

If the subdivisions have been chosen sufficiently small the M's will contain the sums of the closed cells of the respective chains γ'_p and γ'_{n-p} which meet $F(J^i)$.

F. Now let $M^i + N^{i-1} = N^i$, $M^{*i} + N^{*i-1} = N^{*i}$, and verify that condition three holds for N^i and N^{*i}.

But

$$F(G^i) = F(G^{i-1} + J^i) = \overline{F(G^{i-1}) \cdot (M_n - J^i)} + F(J^i) \cdot (M_n - G^{i-1}).$$

Cells of γ_p^{i-1} meeting $F(G^i)$ all meet $F(G^{i-1})$ and then they are in $N^{i-1} \subset N^i$. Also the closed cells of γ'_p not on N^{i-1} are exterior to G^{i-1} by A 1 and A 4, hence if they meet $F(G^i)$ they meet it in $F(J^i)$ and so lie in M^i. It follows since $\gamma_p^i = \gamma_p^{i-1} + \gamma'_p$ that the sum of the closed cells of γ_p^i meeting $F(G^i)$ is on N^i and likewise for γ_{n-p}^i and N^{*i}.

It remains to prove that $N^i \cdot N^{*i} = 0$ and that $\Gamma_{n-p}^i \cdot N^i = \Gamma_p^i \cdot N^{*i} = 0$. The first statement follows from E(a) and E(b). Due to A 3 and A 4, Γ_{n-p}^i does not meet N^{i-1} and by condition (c) it does not meet M^i. Therefore $\Gamma_{n-p}^i \cdot N^i = 0$. Similarly $\Gamma_p^i \cdot N^{*i} = 0$.

Now it has been shown that the conditions assumed for $i - 1$ can all be realized for i, so the induction is complete.

The first step of the induction is possible since G^0 can be taken to be zero and $\gamma_p^0 = \gamma_{n-p}^0 = 0$, $\delta_p^0 = \Gamma_p$ and $\delta_{n-p}^0 = \Gamma_{n-p}$. Then the induction is started. It also comes to an end. Because $\{J^i\}$ covers M_n, a finite number s of steps will reduce δ_p and δ_{n-p} to zero and Γ_p^s and Γ_{n-p}^s will result which intersect in a finite number of points, each point interior to an n-cell of $\{E_n^i\}$. The deformations applied to Γ_p and Γ_{n-p} in order to produce Γ_p^s and Γ_{n-p}^s are finite in number so the total deformation is arbitrarily small.

THEOREM 10. *If Γ_p and Γ_{n-p} are two chains on M_n which do not intersect each other's boundaries, then Theorem 9 applies to them.*

The proof is the same as that of Theorem 9 provided that $F(\gamma)$ and $F(\delta)$ be everywhere replaced by the part of the boundaries of the partial chains γ and δ which are not on $F(\Gamma)$.

When chains have the following properties: their boundaries do not intersect one another, the chains intersect in a finite number of isolated points, about each intersection their cells are on a complex and its dual respectively; they are said to constitute a *regular pair*.

The Kronecker Index, $(C_p \cdot C_{n-p})$, for a regular pair of chains C_p and C_{n-p} is defined as follows. If $A^1, A^2, \cdots, A^r$ are the isolated intersections then there exist for A^i subchains C_p^i and C_{n-p}^i of C_p and C_{n-p} which include all their cells through A^i, do not intersect elsewhere and are on an n-cell E_n^i of M_n. Then $(C_p^i \cdot C_{n-p}^i)$ is defined as in L. T., p. 194. $(C_p \cdot C_{n-p})$ is defined as $\sum_i (C_p^i \cdot C_{n-p}^i)$. It follows immediately that for regular pairs the Kronecker Index is additive and obeys the same permutation laws as for simplicial manifolds.

THEOREM 11. *Let C_p and C_{n-p} be a regular pair and let there exist $C_{p+1} \rightarrow C_p$ such that C_{p+1} does not meet $F(C_{n-p})$. Then $(C_p \cdot C_{n-p}) = 0$.*

a. If C_{p+1} is on an n-cell E_n of M_n then the theorem follows from L. T., p. 170.

b. C_{p+1} can be written $C_{p+1} = \sum_{i=1}^{k} C_{p+1}^i$ where all the cells C_{p+1}^i that have a point in common are on a single n-cell, E_n^i, of the covering and no $(p-1)$-cell of C_{p+1}^i contains a point of $C_p \cdot C_{n-p}$. It will be shown that without changing $(C_p \cdot C_{n-p})$ the situation can be so modified as to replace k by $r < k$.

Let A be one of the intersections of C_p and C_{n-p} and call D the sum of the chains $F(C_{p+1}^i)$ meeting A. $D = D' + D''$ where D' is on C_p and D'' has no p-cells on C_p. $F(D'')$ does not meet $C_p \cdot C_{n-p}$. Deform D'' and C_{n-p} into a regular pair according to Theorem 10 by a deformation acting on a subchain of C_{n-p} not containing A and so small that no new intersections with C_p are brought about. Add the deformation chain of the p-cells of the old D'' to the $(p+1)$-cells to whose boundary it belongs if the $(p+1)$-cell is one of the sum whose boundary is D. If the $(p+1)$-cell is not in D subtract the deformation chain. Now replace C_p by $C_p - D$. Since $D' + D''$ bounds a sub-chain of C_{p+1} in an n-cell of M_n, and that subchain does not meet $F(C_{n-p})$, and D and C_{n-p} are a regular pair, it follows from part a. that $(C_p \cdot C_{n-p}) = ((C_p - D) \cdot C_{n-p})$. Because D'' and C_{n-p} are a regular pair, $D'' \cdot C_{n-p}$ is on a p-cell of D'' so no $(p-1)$-cell of the new $\sum_{i=1}^{r} C_{p+1}^i$ $r < k$ meets $D'' \cdot C_{n-p}$. Thus the situation is as before with k replaced by r. Repeating this process will ultimately reduce $\sum_i C_{p+1}^i$ to a sum of cells on a single n-cell of M_n which is case a.

Kronecker Index of arbitrary chains. If C_p and C_{n-p} are arbitrary chains neither of which meets the boundary of the other, then by Theorem 10 they can be deformed by a deformation T onto a regular pair, C_p' and C_{n-p}'. $(C_p \cdot C_{n-p})$ is then defined as $(C_p' \cdot C_{n-p}')$.

THEOREM 12. *The definition of* $(C_p \cdot C_{n-p})$ *just given is unique provided the deformation* T *is small enough.*

Let C_p^1, C_{n-p}^1 and C_p^2, C_{n-p}^2 be two regular pairs approximating C_p and C_{n-p}. Let A be a generic point of the intersection of C_p^1 and C_{n-p}^1 and B the same for C_p^2 and C_{n-p}^2. There are three possibilities.

a. There is no A point on C_{n-p}^2 and no B point on C_p^1.

In this case deform, using Theorem 10, a subchain of C_p^1 that is away from A and C_{n-p}^1, and deform a subchain of C_{n-p}^2 away from B and C_p^2, by a deformation T' so small that no intersections with the other pairs are changed and a regular pair, again called C_p^1 and C_{n-p}^2, results. If T and T' are small enough the deformation chain D, of dimensionality $p+1$, connecting C_p^1 and C_p^2, obtained in getting C_p^1 and C_p^2 from C_p, is very near C_p, so $F(C^1_{n-p})$ does not meet it. Its boundary is $C^1_p - C^2_p$ plus a deformation chain which does not meet the $(n-p)$-chains. Hence by Theorem 11,

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

If $p = 0$ (or $n - p = 0$), C^1_p, C^2_{n-p} and C^2_p, C^1_{n-p} are regular pairs so $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p})$ follows for all such cases from the argument made at the end of the last paragraph.

b. $p \neq 0$, no A point is a B point. This includes the three cases: 1) Points of type A are on C^2_{n-p} but none of type B are on C^1_p; 2) There is no A point on C^2_{n-p} but there are some B points on C^1_p; 3) A has points on C^2_{n-p} and B has points on C^1_p but no A point is a B point.

Because no points are of both type A and B, and because $n - p < n$, a small deformation applied to C^1_p away from C^1_{n-p} and to C^2_{n-p} away from C^2_p will reduce Case b to Case a by removing C^2_{n-p} from A, removing C^1_p from B, and not changing A or B.

c. A and B points coincide, $p \neq 0$.

In this case deform C^2_{n-p} in such a way that: 1) $C^2_p \cdot C^2_{n-p}$ no longer meets A points, 2) the set of points in which the new C^2_{n-p} and the old differ, can be covered by an n-cell E_n of M_n. This deformation is possible if it is taken small enough because $n - p < n$. But by Theorem 11, part a, the deformation leaves $(C^2_p \cdot C^2_{n-p})$ unchanged and reduces Case c to Case b.

So now it is proved that in all cases

$$(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^1_{n-p}).$$

Similarly it can be shown that $(C^2_p \cdot C^2_{n-p}) = (C^2_p \cdot C^1_{n-p})$. Therefore $(C^1_p \cdot C^1_{n-p}) = (C^2_p \cdot C^2_{n-p})$ which was to be proved.

THEOREM 13. *The Kronecker Index of two arbitrary chains on M_n which do not meet one another's boundaries has the permutation properties of the Index for chains on a simplicial manifold. If the chain γ_p is a cycle and $\gamma_p \sim 0$ then $(\gamma_p \cdot \gamma_{n-p}) = 0$ for every γ_{n-p}.*

The first part of the theorem follows from the definitions of the Kronecker Index. If $\gamma_p \sim 0$ and γ^0_p and γ^0_{n-p} are a regular pair constructed homologous to γ_p and γ_{n-p} according to Theorem 9, then $\gamma^0_p \sim 0$ and, by Theorem 11, $(\gamma^0_p \cdot \gamma^0_{n-p}) = 0$. Therefore $(\gamma_p \cdot \gamma_{n-p}) = 0$ which proves the second part.

3. Immersion of M_n in S_r.

THEOREM 14. *A topological manifold M_n can be homeomorphically mapped upon a subset of an Euclidean space, S_r.*

The idea of the proof is as follows. A topological manifold M_n may be covered by a finite number of simplicial n-complexes K^i such that each point of M_n is interior to at least one complex K^i. Corresponding to each complex K^i can be determined a finite set of bounded, continuous functions, $x^i_1, x^i_2, \cdots, x^i_{r^i}$, defined at all points of K^i and such that:

1) The functions x^i_s all vanish on the boundary of K^i.

2) At each interior point P of K^i at least one of the functions x^i_s does not vanish.

3) If P and Q are two distinct interior points of K^i at least one of the functions x^i_s has a value at Q different from its value at P.

The domain of definition of the functions x^i_s is extended over the entire manifold M_n by putting them equal to zero at all points of M_n not on K^i. The combined functions x^i_s corresponding to all values of i will have the property that at two arbitrarily chosen distinct points P and Q of the manifold at least one of them will have a value at Q different from its value at P. They may consequently be regarded as the coordinates of a point P of M_n in a Euclidean space of $r = \sum_i r^i$ dimensions. The problem then reduces to that of defining the functions just described over the complex K^i.

Let L^i be the boundary of K^i. It may always be assumed that no two simplexes of K^i have the same vertices and that each n-simplex of K^i has at least one vertex not on the boundary of L^i. For if K^i does not have this property initially it can be replaced by its first derived which is a complex which does. Suppose K^i has α vertices and L^i has β ($\beta < \alpha$) vertices. Then K^i is homeomorphic to a sub-complex τ of an $(\alpha-1)$-simplex σ in Euclidean $(\alpha-1)$-space $S_{\alpha-1}$ obtained by associating each vertex A^j of σ with one of K^i and drawing between the vertices A^j the simplexes corresponding to those determined by the associated vertices of K^i. Moreover, the boundary L^i of K^i corresponds to a sub-complex of a $\beta-1$ face δ of the simplex σ. Now, pass an $(\alpha-2)$-dimensional hyperplane $P_{\alpha-2}$ of $S_{\alpha-1}$ through the vertices of δ in such a manner that it does not pass through any vertex of σ other than those of δ. Consider $S_{\alpha-1}$ to be a projective space and $P_{\alpha-2}$ to be the plane at infinity. Then an inversion through a point O of $S_{\alpha-1}$ not on τ will transform the interior of τ into a homeomorphic image τ' and the boundary of τ into the point O. The point O may now be taken as the origin of $\alpha-1$ Cartesian coordinates, $x^i_1, x^i_2, \cdots, x^i_{\alpha-1}$. The values of these coordinates at a point of τ' will be by definition the values being sought of the functions x^i_s at the corresponding points of K^i. This completes the argument.