ON RELATED PERIODIC MAPS.*

By E. E. Floyd.

1. Introduction. Consider a class of periodic maps defined on a topological space X. We are concerned with special cases of the following problem. Suppose the maps of the class are all related in some specified fashion. Are there, then, any implied relationships between the fixed point sets of the maps of the class?

A notable example of a problem of this sort has been solved recently by S. D. Liao [5]. If X is a finite dimensional compact Hausdorff space which has the homology groups of an n-sphere over the group I_p of integers mod p with p prime, and if T is periodic of period p on X, then, as P. A. Smith has proved ([8], p. 366), the fixed point set L has the homology groups of a r-sphere for some $-1 \leq r \leq n$. Liao settled a problem proposed by Smith by proving that if X also has finitely generated integral cohomology groups, then $n - r$ is even or odd according as T is orientation preserving or orientation reversing.

In section 1, we generalize Liao's result by proving that if X is a finite dimensional compact Hausdorff space with finitely generated integral cohomology groups, and if T is periodic of prime power period p^a on X, then the Lefschetz fixed point number of T is equal to the Euler characteristic of L (defined using I_p as coefficient group). We also extend a result of Smith ([9], p. 162) concerning the non-existence of certain types of periodic maps of arbitrarily large period on n-manifolds with negative Euler characteristic. The methods of this section depend heavily on recent results of Liao [5] and of the author [4] which in turn are based on the special homology groups of Smith [8].

In section 2, we consider a periodic map T of prime power period q^a and then consider the class of all periodic maps T_1 of the same period which are "sufficiently close" to T. Under these circumstances, we prove that the fixed point set L_1 of T_1 is close to L in the sense of Begle's metric [1] induced by the regular convergence introduced by Whyburn [11].

The author has read a pre-publication copy of Mr. Liao's paper [5], and wishes to thank Mr. Liao for that privilege.

* Received August 24, 1951; revised October 25, 1951.

2. The Lefschetz fixed point number of T. A periodic map on a space X generates a periodic linear isomorphism on the rational homology groups of X. We require later in the section an analysis of the latter. We dispose of this first, using a procedure similar to one used by Smith ([9], pp. 161-162) for a similar purpose.

Suppose V is a finite dimensional vector space over the rationals R. If W is a subspace of V, let dW denote the dimension of W. Let T be a linear transformation on V with $T^p =$ identity. There are associated with T the linear transformations $\sigma = 1 + T + \cdots + T^{p-1}$ and $\tau = 1 - T$. Clearly $\sigma\tau = \tau\sigma = 0$. We use the following preliminary remark (cf. [5], 4.11).

(2.1) *Image* $\sigma =$ *kernel* τ.

If m is a matrix presentation of T, then we call its characteristic equation $f(t)$ the characteristic equation of T. The characteristic roots of T are p-th roots of unity, for if $| m - \lambda I | = 0$, then $0 = | m^p - \lambda^p I | = (1 - \lambda^p)^{dV}$. Moreover, if no T^i, $0 < i < p$, has non-zero fixed points, then every characteristic root λ is a primitive p-th root of unity. For if $\lambda^l = 1$, then $| I - m^l | = | \lambda^l I - m^l | = 0$. Hence there exists $x \in V$, $x \neq 0$, with $T^l x = x$. But then $l = p$, so λ is a primitive p-th root.

Since $f(t)$ has rational coefficients and all its roots are p-th roots of unity, then $f(t) = f_{s_1}(t) \cdots f_{s_k}(t)$ where $f_{s_i}(t)$ is the cyclotomic equation of degree $\phi(s_i)$, and ϕ is Euler's ϕ-function, whose roots are the primitive s_i-th roots of unity. Moreover it may be seen that s_i divides p. In the following, we use $V(S)$ to represent the fixed point set of the linear transformation S.

(2.2) *Let T be a linear transformation on the finite-dimensional rational vector space V with $T^p =$ identity. Then*

(a) *if p is prime, there exists a non-negative integer k with $dV = dV(T) + k(p-1)$; moreover, trace $T = dV(T) - k$;*

(b) *if $p = q^a$ where q is prime and $a > 1$, then trace $T =$ trace $T \mid V(T^{q^{a-1}})$.*

Proof. To prove (a), decompose V into $V(T) \oplus V_1$, where $T(V_1) = V_1$ (cf. the proof of (2.1)). The characteristic equation of $T \mid V_1$ has as roots only primitive p-th roots of unity. Hence its characteristic equation is of the form $(f_p(t))^k$. Since the degree of $f_p(t)$ is $p - 1$, $dV = dV(T) + k(p-1)$. The trace of $T \mid V_1$ is then $k(\alpha_1 + \cdots + \alpha_{p-1})$, where the α_i's are the primitive p-th roots of unity. Hence the trace of $T \mid V_1$ is $-k$. So (a) follows.

To prove (b), decompose V into $V(T^{q^{a-1}}) \oplus V_1$, where $T(V_1) = V_1$. Then the characteristic equation of $T \mid V_1$ is of the form $(f_p(t))^k$, and the trace of $T \mid V_1$ is $k(\alpha_1 + \cdots + \alpha_{\phi(p)})$, where the α_i's are the primitive p-th roots of unity. It may then be seen that the trace of $T \mid V_1$ is 0. So (b) follows.

Suppose now that X is a compact Hausdorff space, and let T be a map of X into X. Let $H_n(X;F)$ denote the Čech homology group of X over the field F, and T_{*n} the induced linear transformation on $H_n(X;F)$. Define $\chi(X;F) = \Sigma(-1)^i dH_i(X;F)$, in case the right hand side is defined and finite, and call $\chi(X;F)$ the Euler characteristic of X over F. Also define $\alpha(T;F) = \Sigma(-1)^i$ trace T_{*i}, in case $\chi(X;F)$ exists, and call $\alpha(T;F)$ the Lefschetz fixed point number of T over F ([6], p. 319).

We suppose now that X is a finite dimensional compact Hausdorff space with finitely generated integral Čech cohomology groups. Let T denote a periodic map on X of prime period p. Let L denote the fixed point set of T, and Y the orbit decomposition space of T. We have occasion to use the following recent results. Of these, (2.3), (2.4), and (2.5) are due to Liao [5], and (2.6) to the author [4].

(2.3) (*Liao*). *Y has finitely generated cohomology groups.*

Liao ([5], Theorem 5.5) has given a proof for this in case X has the groups of an n-sphere over I_p. The proof used the extra assumption only to insure that L has finitely generated groups over I_p. Since this is true in the general case ([4], Theorem 4.2), the proof then holds.

(2.4) (*Liao*). $\chi(X;I_p) = \chi(X;R)$, $\chi(Y;I_p) = \chi(Y;R)$ ([5], Theorem 2.8).

(2.5) *If $\eta : X \to Y$ denotes the orbit decomposition map, then η_* maps $[x \mid x \in H_n(X;R), T_*x = x]$ isomorphically onto $H_n(Y;R)$.*

This result is more or less implicit in the work of Liao (cf. [5], 4.3, 4.11, 4.13). Because of its importance here, we outline, using the notation of [5; § 4], a direct argument. For each $b_{s\lambda} \in C_s(0(K_\lambda, T_\lambda);R)$, let $a_{s\lambda} \in C_s(K_\lambda, R)$ be such that $\eta_\lambda(a_{s\lambda}) = b_{s\lambda}$. Define $\xi_\lambda(b_{s\lambda}) = \sigma_\lambda a_{s\lambda}$. It may be verified that ξ_λ is uniquely defined, that $\partial\xi_\lambda = \xi_\lambda\partial$, and that $\pi_{\mu\lambda}\xi_\mu = \xi_\lambda\pi_{0\mu\lambda}$. Moreover, $\xi_\lambda\eta_\lambda = \sigma_\lambda$, and $\eta_\lambda\xi_\lambda(b_{s\lambda}) = pb_{s\lambda}$. Hence there is induced $\xi : H_s(0(X,T);R) \to H_s(X;R)$ with $\eta\xi(x) = px, x \in H_s(0(X,T);R)$, $\xi\eta(x) = \sigma(x), x \in H_s(X;R)$. Since $\eta\xi$ is an isomorphism onto, η maps image ξ isomorphically onto $H_s(0(X,T);R)$. Since η is onto and $\xi\eta = \sigma$, we have image ξ = image σ. But by (2.1) image σ = kernel τ. The assertion follows.

(2.6) $\chi(X; I_p) + (p-1)\chi(L; I_p) = p\chi(Y; I_p)$. [4].

We are now in a position to prove the main theorem of this section.

(2.7) THEOREM. *Let X be a finite dimensional compact Hausdorff space with finitely generated integral Čech cohomology groups. Let T be a periodic map on X of period q^a, q prime. Let L be the fixed point set of T. Then $\alpha(T; R) = \chi(L; I_q)$.*

Proof. We prove the theorem first for $a = 1$. Consider $T_{*n}: H_n(X; R) \to H_n(X; R)$. According to (2.5), the fixed point set of T_{*n} is isomorphic to $H_n(Y; R)$. Hence by (2.5),

$$dH_n(X; R) = dH_n(Y; R) + [dH_n(Y; R) - \text{trace } T_{*n}](p-1)$$

so that $dH_n(X; R) + (p-1) \text{ trace } T_{*n} = pdH_n(Y; R)$. Taking the alternating sum, we get $\chi(X; R) + (p-1)\alpha(T; R) = p\chi(Y; R)$. Using (2.4) and comparing with (2.6), we get $\alpha(T; R) = \chi(L; I_p)$.

Suppose $a > 1$ and suppose the theorem has been proven for $a - 1$. Consider $T_0 = T^{q^{a-1}}$. Let Y_1 denote the orbit space of the map T_0 on X, and $f: X \to Y_1$ the natural decomposition map. Define a map $S: Y_1 \to Y_1$ by $Sf = fT$. Then S is of period q^{a-1} on Y_1. Also, by (2.3), Y_1 has finitely generated integral cohomology groups. Hence, by the induction hypothesis, $\alpha(S; R) = \chi(L'; I_q)$, where L' is the fixed point set of S.

We point out that L and L' are homeomorphic. Clearly, $f(L) \subset L'$ and f is 1-1 on L. We prove that $f(L) = L'$. Let $y \in L'$, where $y = f(x)$, $x \in X$. Then $f(x) = Sf(x) = fT(x)$ so $T(x) = T_0{}^k(x)$ for some k. But then $kq^{a-1} - 1$ is a period for x, so $kq^{a-1} - 1$ divides q^a. Hence $k = 0$, so that $Tx = x$, and $x \in L$. So $\chi(L'; I_q) = \chi(L; I_q)$.

Finally, $\alpha(S; R) = \alpha(T; R)$. Let $F_n = [x; x \in H_n(X; R), T_{0*}x = x]$. Then, by (2.3), f_* maps F_n isomorphically onto $H_n(Y_1; R)$. Moreover, since $S_* f_* = f_* T_*$, we have trace $(S_{*n}; H_n(Y_1; R)) = \text{trace}(T_*, F_n)$. But by (2.2), $\text{trace}(T_*, F_n) = \text{trace}(T_*, H_n(X; R))$. It follows that $\alpha(S; R) = \alpha(T; R)$ and the theorem follows.

We now turn to some results concerned with properties of periodic maps of large period.

(2.8) (*Smith*). *Let V be a finite dimensional rational vector space. There exists a positive integer r associated with V so that if T is any linear transformation on V with $T^p = $ identity where $p > r$, then there exists $1 \leq j < p$ with $T^j = $ identity.*

Proof. We shall outline the proof ([9], pp. 161-162). Suppose $p = p_1^{a_1} p_2^{a_2} \cdots p_e^{a_e}$, where the p_i's are primes with $p_1 < p_2 < \cdots < p_e$. Define

$$\Phi(p) = \sum_1^e \phi(p_i^{a_i}) \text{ if } p_1 \neq 2 \text{ or } a_1 \neq 1; \ \Phi(p) = \sum_2^e \phi(p_i^{a_i}) \text{ otherwise.}$$

Then $\Phi(p) \to \infty$ as $p \to \infty$. We point out that if $\Phi(p) > dV$, then there exists $1 \leq j < r$ with $T^j =$ identity. For suppose this is not the case. Using the notation preceding (2.2), we have $f(t) = f_{s_1}(t) \cdots f_{s_k}(t)$, where $s_i \mid p$. Now each $p_i^{a_i}$ divides some s_j. For if not, each s_j divides $p/p_i = q$, so that $T^q =$ identity. But if each $p_i^{a_i}$ divides some s_j, it may be checked that $dV = \Sigma \phi(s_i) \geq \Phi(p)$. Hence $\Phi(p) \leq dV$, and the assertion follows.

(2.9) *As a consequence of* (2.8), *let X be a compact Hausdorff space with each $H_n(X;R)$ of finite dimension and $= 0$ for all but a finite number of n's. There exists a positive integer r so that if T is any periodic map on X, then $T_*^j : H_n(X;R) \to H_n(X;R)$ is, for some $1 \leq j \leq r$, the identity for all n.*

We denote the least such r by $r(X)$.

(2.10) THEOREM. *Let X be a finite dimensional compact Hausdorff space with finitely generated integral cohomology groups. Let T be a periodic map on X of period $p > r(X)$. There exists $1 \leq i < p$ such that $p/i = q$ is a prime, and such that if L_i denotes the fixed point set of T^i, then $\chi(X;R) = \chi(L_i, I_q)$.*

Proof. There exists, by (2.9), $1 \leq j \leq r$ with $T_{*n}^j =$ identity for all n. Suppose $p = j \cdot k \cdot q$, where k and q are positive integers with q prime. Let $i = j \cdot k$. Then $T_{*n}^i =$ identity for all n. Hence by (2.7), $\alpha(T^i;R) = \chi(X;R) = \chi(L_i, I_q)$.

The following is an extension of a result of Smith [9; 162]. It also generalizes the well-known theorem [11] that the periodic maps on a compact 2-manifold with negative Euler characteristic have uniformly bounded periods. It does not, however, provide the upper bound known for that case.

(2.11) THEOREM. *Let X be a compact manifold with $\chi(X;R) < 0$. Suppose T is a periodic map on X of period p, and such that if $1 \leq j < p$, then the dimension of the fixed point set of T^j is ≤ 1. Then $p \leq r(X)$.*

Proof. Suppose $p > r(X)$. Let i be the number given by (2.10). Then $\chi(X;R) = \chi(L_i;I_q) < 0$. But $\dim L_i \leq 1$, so that by a result of

Smith ([10], p. 704), L_i is the union of a disjoint collection of points and simple closed curves. Hence $\chi(L_i; I_q) \geq 0$, which is a contradiction.

(2.12) *The above theorem is not true if the restriction on the dimension of the fixed point set of T^j is removed.*

As an example, let X be a 2-sphere, and let Y be a 2-manifold with $\chi(Y) < 0$. Then $\chi(X \times Y) = \chi(X)\chi(Y) < 0$. But since X admits transformations of arbitrary period, so does $X \times Y$.

3. Convergence properties. We begin section 3 by stating an important result due to Smith [7] which is the basis for the work of this section. The result is stated and proved in the proof of Theorems I, II in [7].

(3.1) (*Smith*). *Let X be a locally compact n-dimensional Hausdorff space, $n < \infty$, and let T be a periodic map on X of prime period p. Denote by L the fixed point set of T. Suppose $0 \neq A_0 \subset A_1 \subset \cdots \subset A_m$, $m = pn + p$, is a sequence of compact subsets of X, with $T(A_i) = A_i$, and with every Čech cycle in A_i over I_p bounding in A_{i+1}. Then $L \cap A_m \neq 0$ and every cycle in $L \cap A_0$ over I_p bounds in $L \cap A_m$.*

We use also the concept of regular convergence introduced by Whyburn [12]. We shall phrase the definition in terms of Čech theory instead of Vietoris theory; these are interchangeable, as follows from the full equivalence of the two theories ([6], p. 277). Let X be a locally compact metric space, and let G be an abelian group. Let $[A_i]$ be a sequence of closed subsets of X, with A_i converging to a closed subset A of X. If n is a non-negative integer, then A_i converges n-regularly to A over G if and only if given $x \varepsilon A$ and a compact neighborhood U of x in X, there exists a closed neighborhood V of x (in X) with $V \subset U$, and a positive integer I, so that every Čech cycle in $V \cap A_i$ over G of dimension $\leq n$ bounds in $U \cap A_i$ for $i > I$. It may be seen that X is lc^n (i. e., homologically locally connected over G in the dimensions from 0 to n), if and only if the sequence $X, X, \cdots$ converges n-regularly to X.

Let X and Y be metric spaces. Let A_i be a sequence of closed subsets of X which converges to a subset A of X. Let $f_i: A_i \to Y$, $f: A \to Y$ be continuous. We shall say that f_i *converges continuously* to f if and only if whenever $x_i \to x$, $x_i \varepsilon A_i$, then $f_i(x_i) \to f(x)$. This specializes, in case $A_i = A$, to the notion of continuous convergence introduced by Carathéodory ([2], p. 58).

(3. 2) THEOREM. *Let X be a locally compact n-dimensional metric space. $n < \infty$. Suppose $[A_i]$ is a sequence of closed subsets of X converging n-regularly over I_p, p prime, to the subset A of X. Let T_i be a continuous periodic transformation of period p on A_i, such that $[T_i]$ converges continuously to the continuous function T on A. Then the fixed point set $[F_i]$ of T_i converges n-regularly over I_p to the fixed point set F of T.*

Proof. The reader may verify that if $x \varepsilon F$ and U is a neighborhood of x (in X), then there exists a neighborhood V of x and a positive integer I such that if $i > I$, then $\bigcup_j T_i^j (V \cap A_i) \subset U$.

Let $x \varepsilon F$ and let U be a compact neighborhood of x. There exists a sequence $U = U_{2m+1} \supset U_{2m} \supset \cdots \supset U_0$, $m = pn + p$, of compact neighborhoods of x (in X) and a positive integer I, such that $U_0 \cap A_i \neq 0$, for $i > I$, and (a) if $i > I$, then $\bigcup_j T_i^j (U_k \cap A_i) \subset U_{k+1}$ for $k = 0, \cdots, 2m$, and (b) for $i > I$ every cycle in $U_r \cap A_i$ over I_p bounds in $U_{r+1} \cap A_i$.

For each $0 \leq k \leq m$ and each $i > I$, define $V_{k,i} = \bigcup_j T_i^j (U_{2k} \cap A_i)$. Then $V_{k,i} \subset U_{2k+1} \cap A_i$, and $T_i(V_{k,i}) = V_{k,i}$. Moreover, since $V_{k+1,i} \supset U_{2k+2} \cap A_i$, every cycle in $V_{k,i}$ bounds in $V_{k+1,i}$. Hence we may apply (3. 1) to the sequence $V_{0,i} \subset V_{1,i} \subset \cdots \subset V_{m,i}$, and the transformation T_i. It follows that $V_{m,i} \cap F_i \neq 0$, and every cycle in $V_{0,i} \cap F_i$ bounds in $V_{m,i} \cap F_i$. Hence, for $i > I$ every cycle in $U_0 \cap F_i$ bounds in $U \cap F_i$, and $U \cap F_i \neq 0$.

To finish the proof, the reader has only to note that if $x_{m_i} \varepsilon F_{m_i}$, and $x_{m_i} \to x$, then $x \varepsilon F$. This follows easily from continuous convergence.

(3. 3) COROLLARY. *Let X be a locally compact n-dimensional metric space, $n < \infty$, which is lc^n over I_p, p prime. Let $[T_j]$ be a sequence of periodic maps on X of common period p^a, which converges continuously to the continuous map T. Then the fixed point set F_j of T_i converges n-regularly over I_p to the fixed point set F of T.*

Proof. The proof is a straight-forward combination of (3. 2) together with a procedure used often by Smith for extending proofs from period p to period p^a ([8], p. 367).

(3. 4) COROLLARY. *Let the hypotheses be those of* (3. 3) *and suppose in addition that X is compact. Then there exists I such that for $i > I$, we have $H_j(F_i; I_p) \approx H_j(F; I_p)$ for all j. In particular, suppose X an n-sphere. Then there exists an integer r so that F_i, $i > I$, and F are all homological r-spheres over I_p.*

Proof. This follows from a theorem of Begle [1].

(3.5) COROLLARY. *Let X be an n-dimensional compact metric space, $n < \infty$, which is lc^n over I_p, p prime. Let T be a periodic map on X of period p^a with fixed point set L. There is an $\epsilon > 0$ such that if T_1 is periodic on X of period p^a, $\rho(T(x), T_1(x)) < \epsilon$ for all $x \varepsilon X$, and L_1 denotes the fixed point set of T_1, then $H_j(L; I_p) \approx H_j(L_1; I_p)$ for all j.*

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