COMBINATORIAL NOVIKOV-MORSE THEORY

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0. Introduction

In classical Morse theory [23], one begins with a smooth manifold M and a smooth function on M. The main point is a very precise relationship between the critical points of the function and the topology of M. In [24, 25], Novikov introduced a generalization of this theory. Novikov's theory begins with a closed 1-form ω on M. Locally we can write $\omega = df$ for some function f which is well-defined up to the addition of a constant. In particular the critical points of f, and their indexes, are well-defined. Classical Morse theory appears as the case when ω is exact, that is when $\omega = df$ globally. In the generalized setting, Novikov found the appropriate topological data so that the corresponding Morse theorems remain true. We will say more about this later. The investigation of this theory has continued in a variety of directions (e.g [2] [5][6][18] [26][27][28][29][36][37][38]).

In [7, 8, 9] we presented a combinatorial Morse theory which can be applied to any to any CW complex, in particular the underlying topological space need not be a manifold. See [1], [3],[31], [11] for some applications of this theory. In this paper we extend the theory to include a combinatorial analog of Novikov's theory.

We begin in §1 by defining, and beginning the investigation of, the appropriate notion of a combinatorial differential form. In §2 we restrict attention to those combinatorial 1-forms which play a role in the combinatorial Novikov-Morse theory. We then show in §3 that the Novikov-Morse inequalities are true in this setting.

This work was partially supported by the National Science Foundation and the National Security Agency.

Another facet of Morse's theory is the Morse complex, a differential complex, which we denote by $\mathcal{M}$, which is constructed from the critical points and has the same homology as the underlying manifold. In the case of classical Morse theory, Milnor showed in [20] that

$$\text{Tor}\,(M) = \text{Tor}\,(\mathcal{M}) \tag{0.1}$$

where Tor (M) denotes the Reidemeister torsion of M, and Tor $(\mathcal{M})$ denotes the corresponding torsion of the Morse complex. Actually, both sides of 0.1 depend on the choice of a representation of $\pi_1(M)$, but we will be much more precise about all this shortly.

In [15] formula 0.1 was investigated for closed 1-forms with integral periods (or equivalently, for $\omega = df$ where f is a S^1-valued function). They found that in this case there is a correction term. Namely,

$$\text{Tor}\,(M) = \text{Tor}\,(\mathcal{M})\,\zeta(1,\omega) \tag{0.2}$$

where $\zeta(z,\omega)$ is a zeta-function built from the closed orbits of the flow along the vector field dual to ω. See [30] for more recent work in this direction.

It should be noted that formula 0.2 had previously been established for many vector fields with no zeros in [13, 14]. In the case of an S^1-valued function with no critical points, (0.2) is contained implicitly in [21] (as was pointed out by Fried). In fact, it seems likely that formula (0.2) holds for a generic vector field on M. Such a formula cannot hold for all vector fields since every manifold of dimension ≥ 3 with Euler characteristic 0 has a vector field with no zeroes and no closed orbits([17, 40]). The determination of the precise set of vector fields for which (0.2) holds remains an important question.

In §4 of this paper we present a proof of this formula in the combinatorial setting. Actually, we only outline the proof, as the major steps have already appeared in [9] and [10]. In [15], formula (0.2) is conjectured to be the value of a Seiberg-Witten invariant if $\dim(M) = 3$. The idea is that closed orbits of ω on a 3-manifold correspond to pseudo-holomorphic curves on a symplectic 4-manifold. Taubes has shown in

[35] that the Seiberg-Witten invariant counts such curves. We hope that this paper may ultimately play a role in a combinatorial understanding of the Seiberg-Witten invariants.

Now let us describe the contents of this paper in more detail. Let M be a finite regular CW complex. Let K denote the set of open cells of $M >$ We write $\alpha^{(p)}$ if α is a cell of dimension p, and $\alpha < \beta$ (or $\beta > \alpha$) if α lies in the boundary of β (and we say α is a face of β). In [7, 8, 9] we introduced the idea of a combinatorial Morse function. Let us quickly review this notion.

Let $\Omega^0(M)$ denote the set of functions

$$f : K \to \mathbb{R}.$$

We emphasize that $f \in \Omega^0(M)$ assigns a single real number to each cell of M. Say f is a *Morse function* if for each p-cell α

$$\#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} \leq 1,$$

and

$$\#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} \leq 1.$$

Say $\alpha^{(p)}$ is critical (of index p) if

$$\#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0,$$

and

$$\#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \leq f(\alpha)\} = 0.$$

Let $m_p(f)$ denote the number of critical cells of dimension p. Let $b_p = \dim H_p(M, \mathbb{R})$ denote the p^{th} Betti number of M.

Theorem 0.1. *(Corollaries 3.6 and 3.7 of* [8]*, Theorem 2.1 of* [9]*)*

(i) The Strong Morse Inequalities

For each $k = 0, 1, 2,$

$$m_k(f) - m_{k-1}(f) + \cdots \pm m_0(f) \geq b_k - b_{k-1} + \cdots \pm b_0.$$

(ii) The Weak Morse Inequalities

For all $k = 0, 1, 2$

$$m_k(f) \geq b_k.$$

Moreover,

$$m_0(f) - m_1(f) + \cdots + (-1)^n m_n(f) = b_0 - b_1 + \cdots + (-1)0^n b_n$$

where $n = \dim M$.

In fact the weak Morse Inequalities easily follow from the strong Morse inequalities. See [7, 8, 9] for a more complete development of this combinatorial Morse theory.

In order to present a combinatorial version of Novikov-Morse theory, we need the idea of a combinatorial 1-form. Let $\mathcal{F}$ denote the set of all pairs $\{\beta^{(p+1)} > \alpha^{(p)}\}$ as p ranges from 0 to $\dim M$. A *combinatorial 1-form* is a function

$$\omega : \mathcal{F} \to \mathbb{R}$$

Let $\Omega^1(M)$ denote the set of all combinatorial 1-forms on M. There is a natural differential

$$D : \Omega^0(M) \to \Omega^1(M).$$

Namely, if $f : K \to \mathbb{R}$ is a function, define $Df \in \Omega^1(M)$ by

$$Df(\beta^{(p+1)} > \alpha^{(p)}) = f(\beta) - f(\alpha).$$

Note that $Df = 0$ if and only if f is constant on each connected component of M.

Let ω be a combinatorial 1-form. Suppose we have a pair of cells $\alpha^{(p)} > \gamma^{(p-2)}$. Then there are exactly 2 $(p-1)$-cells β_1 and β_2 such that

$$\alpha > \beta_i > \gamma \quad i = 1, 2.$$

Say ω is *closed* if for all such $\alpha > \gamma$,

$$\omega(\alpha > \beta_1) + \omega(\beta_1 > \gamma) = \omega(\alpha > \beta_2) + \omega(\beta_2 > \gamma). \tag{0.3}$$

In §1 of this paper we show that there is a natural combinatorial deRham complex for which $\Omega^0(M)$ and $\Omega^1(M)$ are the $0-$ and $1-$cochains. That is, for each p we define a space $\Omega^p(M)$ of combinatorial p-forms, and a differential

$$D : \Omega^p(M) \to \Omega^{p+1}(M)$$

such that $D^2 = 0$. If ω is a combinatorial 1-form, then $D\omega = 0$ if and only if ω satisfies (0.3). Lastly, we show that the cohomology of this complex is isomorphic to the cohomology of M

$$H^*(\Omega^*, D) \cong H^*(M).$$

Let ω be a combinatorial 1-form. We say ω is a *Morse 1-form* if ω is closed and if for all p-cells α

$$\#\{\beta^{(p+1)} > \alpha \mid \omega(\beta > \alpha) \leq 0\} \leq 1,$$

and

$$\#\{\gamma^{(p-1)} < \alpha \mid \omega(\alpha > \gamma) \leq 0\} \leq 1.$$

We begin our investigation of such forms in §2. The main point is that there is a well-defined notion of a critical cell. Let ω be a Morse 1-form. Say $\alpha^{(p)}$ is critical (of index p) if

$$\#\{\beta^{(p+1)} > \alpha \mid \omega(\beta > \alpha) \leq 0\} = 0,$$

and

$$\#\{\gamma^{(p-1)} < \alpha \mid \omega(\gamma > \alpha) \leq 0\} = 0.$$

Let $m_p(\omega)$ denote the number of critical p-cells of ω. From our work in §1, we know that ω represents a class $[\omega] \in H^1(M, \mathbb{R})$. We show in §3 that one can define invariants $B_p([\omega])$, which depend on the pair $(M, [\omega])$ only up to homotopy, such that the Morse inequalities are true.

Theorem 0.2. *(i) The Strong Novikov-Morse Inequalities: For all* $k = 0, 1, 2, \ldots$

$$m_k(\omega) - m_{k-1}(\omega) + \cdots \pm m_0(\omega) \geq B_k([\omega]) - B_{k-1}([\omega]) + \cdots \pm B_0([\omega]).$$

(ii) The Weak Novikov-Morse Inequalities: For each $k = 0, 1, 2, \ldots$

$$m_k(\omega) \geq B_k([\omega]).$$

Moreover,

$$m_0(\omega) - m_1(\omega) + \cdots + (-1)^n m_n(\omega) = B_0([\omega]) - B_1([\omega]) + \cdots + (-1)^n B_n([\omega]).$$

If ω is exact, then $B_k([\omega]) = b_k$ as defined earlier, so these inequalities are, in fact, a generalization of the classical Morse inequalities. The topological invariants $B_k([\omega])$ are the same as these introduced in [24, 25] and [26]. They show, in some cases, how to define these invariants in terms of ingredients from classical algebraic topology.

Our presentation follows that of [26], modified for our setting, which in turn is based on the ideas of [40]. That is, we begin with the standard cellular chain complex of M

$$0 \to C_n(M,\mathbb{R}) \xrightarrow{\partial} C_{n-1}(M,\mathbb{R}) \xrightarrow{\partial} \cdots .$$

Given a combinatorial 1-form ω satisfying $D\omega = 0$, we define a 1-parameter family of differentials ∂_t be setting, for any oriented cells $\beta^{(p+1)} > \alpha^{(p)}$

$$\langle \partial_t \beta, \alpha \rangle = e^{t\omega(\beta<\alpha)} \langle \partial \beta, \alpha \rangle .$$

where $\langle , \rangle$ denotes the canonical inner product on $C_p(M,\mathbb{R})$ with respect to which the cells form an orthonormal basis. One can check directly that $\partial_t^2 = 0$ if and only if ω is closed in the sense of (0.3). In the case that $\omega = Df$ for some $f \in \Omega^0(M)$, this differential can equivalently be defined as

$$\partial_t = e^{-tf}\, \partial\, e^{tf}.$$

This is the form in which it appeared in [9].

The analysis of this differential is much easier if ω is *flat*, that is,

$$\omega(\beta > \alpha) \geq 0$$

for every $\beta < \alpha$. This idea was introduced in [9]. We also show in §2 that every Morse 1-form is equivalent to a flat Morse 1-form, in the sense that they have identical critical cells and gradient paths (we will define these shortly). Therefore, when proving the Novikov-Morse inequalities, it is sufficient to assume ω is flat.

We now consider the operator

$$\Delta_p(t) = \partial_t \partial_t^* \to \partial_t^* \partial_t : C_p(M,\mathbb{R}) \to C_p(M,\mathbb{R}).$$

We prove that as $t \to -\infty$ this operator has a limits $\Delta_p(-\infty)$ (the flatness of ω is required for this). Moreover,

$$\dim \ \mathrm{Ker}\ \Delta_p(-\infty) = m_p(\omega).$$

It is easy to see that dim Ker $\Delta_p(t)$ has a generic dimension for $t \in \mathbb{R}$. That is, there is a countable set $S \subseteq \mathbb{R}$ such that dim Ker $\Delta_p(t)$ is constant on $\mathbb{R} - S$. Standard theory implies that this generic dimension is a homotopy invariant of $(M, [\omega])$. We denote this invariant by $B_p([\omega])$. The Novikov-Morse inequalities easily follow.

In §4 we investigate the Reidemeister torsion of M. Let

$$\Theta : \pi_1(M) \to O(k)$$

be an orthogonal representation. Let $\tilde{M}$ denote the universal cover of M, endowed with the cell structure induced from that of M, and let

$$C_p(M, \Theta) \subseteq C_p(\tilde{M}, \mathbb{R}^k)$$

denote those chains on $\tilde{M}$ which transform under $\pi_1(M)$ via Θ. (This is all done more explicitly in §4).

The differential ∂ preserves these spaces, and we let $H_*(M, \Theta)$ denote the homology of the resulting complex. We assume that Θ is acyclic, that is, $H_*(M, \Theta) = 0$. The spaces $C_p(M, \Theta)$ inherit natural inner products, so we can consider

$$\Delta_p(\Theta) = \partial\partial^* + \partial^*\partial : C_p(M, \Theta) \to C_p(M, \Theta).$$

If Θ is acyclic, then each $\Delta_p(\Theta)$ is invertible. Define the Reidemeister torsion of (M, Θ) by

$$\mathrm{Tor}\ (M, \Theta) = \Pi_{p=0}^n (\ \mathrm{Det}\ \Delta_p(\Theta))^{\frac{p+1}{2}(-1)^p}.$$

This is a topological invariant of the pair (M, Θ) [12]. This formula for torsion first appeared in [32]. The same formula can be used to define the torsion of any exact sequence endowed with an inner product.

Let

$$\mathcal{M}_p(\omega, \Theta) \subseteq C_p(M, \Theta) \leq C_p(\tilde{M}, \mathbb{R}^k)$$

denote those chains which transform via Θ and are supported on the lifts of the critical cells of ω. The next step is to define a differential

$$\tilde{\partial} : \mathcal{M}_p(\omega, \Theta) \to \mathcal{M}_{p-1}(\omega, \Theta).$$

This requires the notion of a *gradient path for* ω (or ω*-path*, for short).

Say a sequence of cells of $\tilde{M}$

$$c : \alpha_0^{(p)} = \beta_0^{(p)}, \gamma_0^{(p+1)}, \beta_1^{(p)}, \gamma_1^{(p+1)}, \cdots, \gamma_{k-1}^{(p+1)}, \beta_k^{(p)} = \alpha_1^{(p)}$$

is a ω-path from α_0 to α_1 (or index p and length k) if for each $i = 0, 1, \ldots, k-1$

1) $\beta_i < \gamma_i, \beta_{i+1} < \gamma_i$ and $\beta_i \neq \beta_{i+1}$

and

2) $\omega(\gamma_i > \beta_i) \leq 0$.

Note that it follows from the definitions that $\omega(\gamma_i > \beta_{i+1}) > 0$.

Similarly, define an ω-path on $\tilde{M}$ to be a sequence of cells of $\tilde{M}$ satisfying the same conditions, except that condition 2) should now read

2′) $\omega(\pi(\gamma_i) > \pi(\beta_i)) \leq 0$.

We now define an operator

$$\tilde{\partial} : C_{p+1}(\tilde{M}, \mathbb{R}^k) \to C_p(\tilde{M}, \mathbb{R}^k)$$

by setting, for any oriented cells $\beta^{(p+1)}$ and $\alpha^{(p)}$ of $\tilde{M}$,

$$\langle \tilde{\partial}\beta, \alpha \rangle = \sum_{\gamma^{(p)} < \beta} \langle \partial\beta, \gamma \rangle \sum_{c \in \Gamma_\omega(\gamma, \alpha)} \mu(c)$$

(where we have assigned orientations arbitrarily to the $\gamma < \beta$). In this formula $\Gamma_\omega(\gamma, \alpha)$ is the set of ω-paths from γ to α, and $\mu(c) = \pm 1$ is the algebraic multiplicity of c. The orientation on γ combined with an ω-path c from γ to α, induces an orientation on α. Set $\mu(c) = 1$ if this orientation agrees with the given orientation on α, and $\mu(c) = -1$ otherwise. Note that if we reverse the orientation on γ, the signs of $\langle \partial\beta, \gamma \rangle$ and $\mu(c)$ both switch, so the operator $\tilde{\partial}$ depends only on the orientations of α and β.

The operator $\tilde{\partial}$ maps $\mathcal{M}_p(\omega, \Theta)$ to $\mathcal{M}_{p-1}(\omega, \Theta)$, and, restricted to these Morse spaces, satisfies $\tilde{\partial}^2 = 0$. (This was established in [9]). Thus we can consider the

Morse complex

$$\mathcal{M}(\Theta) : 0 \to \mathcal{M}_n(\omega, \Theta) \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1}(\omega, \Theta) \xrightarrow{\tilde{\partial}} \cdots$$

For any representation Θ

$$H_*(\mathcal{M}(\Theta)) \cong H_*(M, \Theta)$$

(see [9]). Therefore, if Θ is acyclic, $\mathcal{M}(\Theta)$ is exact, and we can consider $\text{Tor}\ (\mathcal{M}(\Theta))$.

Theorem 0.3. *If there are no closed (= periodic) ω-paths on $\mathcal{M}$, then*

$$Tor\ (M, \Theta) = \ Tor\ (\mathcal{M}(\Theta)).$$

This is essentially Theorem 9.3 of [8] combined with Theorem 6.1 of [9].

We now define the correction factor if periodic ω-paths do exist. Define a zeta function

$$\zeta(z, \omega, \Theta) = \exp\left[\sum_{r=1}^{\infty} \frac{z^r}{r} \sum_{k=0}^{n} (-1)^k \sum_{c \in \mathcal{P}_k^r(\omega)} \mu(c)\ \text{tr}[\Theta(c)]\right]$$

where $\mathcal{P}_k^r(\omega)$ denotes the periodic ω-paths on M of length r and index k.

In [10] we studied the dynamics of such paths, and showed that one could recreate much of the theory of smooth dynamical systems. In particular, one can define the notion of a basic set. Suppose α is contained in a closed ω-path. Let Λ be the set of all cells β for which there is a closed ω-path continuing both α and β. We call Λ a non-trivial basic set. We proved in [10].

Theorem 0.4. *1) If, for each non-trivial basic set* $\Lambda, H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ *(where* $\dot{\Lambda} = \bar{\Lambda} - \Lambda$*), then* $\zeta(z, \omega, \Theta)$ *is regular at* $z = 1$.

2) If , for each non-trivial basic set Λ, $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$*, and* ω *has no critical cells, then* Θ *is acyclic and*

$$Tor\ (M, \Theta) = \zeta(1, \omega, \Theta).$$

It is somewhat to our embarrassment that we did not earlier put Theorem 0.3 and Theorem 0.4 together to get the general statement. This is the main result of §4.

Theorem 0.5. *If* $\Theta : \pi_1(M) \to O(k)$ *is acyclic and* $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ *for each non-trivial basic set* Λ, *then*

$$Tor\ (M, \Theta) = \ Tor\ (\mathcal{M}(\Theta))\ \zeta(1, \omega, \Theta).$$

The proof involves finding a Lyapunov function for the ω-paths, and using this function to deform the boundary operator ∂. We see that this separates the torsion into 2 terms, one involving the non-trivial basic sets, and the other involving the critical cells. The theorem then follows by applying the earlier theorems separately to the two terms.

It is interesting to note that in §4 we prove this formula for the flow along any combinatorial vector field (as defined in [10]), not just for the gradient paths of a Morse form.

The author would like to thank M. Hutchings for some very interesting converstaions on this these topics.

1. Combinatorial Differential Forms

In this section, we present a differential complex which seems to provide the appropriate context for our work on this subject. There is nothing really new here, but his complex does not seem to have been explicitly studied earlier.

Let M be a regular cell complex of dimension n, and

$$0 \to C_n(M) \xrightarrow{\partial} C_{n-1}(M) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M) \to 0$$

the real cellular chain complex of M, with respect to any coefficient field. Let

$$C_*(M) = \oplus_p C_p(M).$$

A linear map

$$\omega : C_*(M) \to C_*(M)$$

is said to be of *degree* d if for all $p = 1, 2, \dots, n$

$$\omega(C_p(M)) \leq C_{p+d}(M)$$

(if $p < 0$ or $p > n$ we interpret $C_p(M)$ to mean $\{0\}$).

We will be primarily interested in maps of nonpositive degree. Say a linear map ω of degree $-d \leq 0$ is *local* if, for each p and each oriented p-cell α, $\omega(\alpha)$ is a linear combination of oriented $(p-d)$-cells which lie in the closure of α.

We are now ready for the main definition for this section.

Definition 1. *For $d \geq 0$, we define the space of combinatorial differential d-forms $\Omega^d(M)$ by*

$$\Omega^d(M) = \{\textit{local linear maps } \omega : C_*(M) \to C_*(M) \textit{ of degree } -d\}.$$

We pause here to make a few remarks.

1) In the following sections in which we must work with real coefficients. However, the definitions and the main theorem of this section hold for any coefficient ring.
2) For each $d \geq 0$, $\Omega^d(M)$ is a module over the coefficient ring.

3) The definitions lead to the (perhaps unsettling) result that the boundary operator ∂ is a combinatorial differential 1-form (we will have more to say about this later).
4) Composition induces a map

$$\Omega^{d_1}(M) \times \Omega^{d_2}(M) \to \Omega^{d_1+d_2}(M).$$

The next step is to define a differential

$$D : \Omega^d(M) \to \Omega^{d+1}(M).$$

We do this via Leibniz's rule. That is, for any $\omega \in \Omega^d(M)$ and any p-chain c, define

$$(D\omega)(c) \in C_{p-(d+1)}(M)$$

by

$$(D\omega)(c) = \partial(\omega(c)) - (-1)^d \omega(\partial c)$$

[i.e., $D\omega = \partial \circ \omega - (-1)^d \omega \circ \partial$]. This formula is perhaps more suggestively written as

$$\partial(\omega(c)) = (D\omega)(c) + (-1)^d \omega(\partial c).$$

In Lemma 1.1, we summarize some relevant facts about D, which follow immediately from the definitions.

Lemma 1.1. *1) $D(\Omega^d(M)) \subseteq \Omega^{d+1}(M)$.*
2) $D^2 = 0$.

This leads us to consider the differential complex

$$\Omega^*(M) : 0 \to \Omega^0(M) \xrightarrow{D} \Omega^1(M) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n(M)) \to 0.$$

Theorem 1.2. *The cohomology of this complex is precisely the cohomology of M. That is,*

$$H^*(\Omega^*(M)) \cong H^*(M).$$

Proof. One way to prove this theorem is to observe that $\Omega^*(M)$ is the cellular cochain complex associated to a subdivision of M. Namely, place a vertex in each cell of M. Draw an edge between the vertices corresponding to each pair of cells of the form

$\alpha^{(p)} < \beta^{(p+1)}$. For any pair of cells $\alpha^{(p)} < \gamma^{(p+2)}$ there are precisely two $(p+1)$-cells β_1 and β_2 with $\alpha < \beta_i < \gamma$. Draw a 2-cell in γ bounded by the 4 edges corresponding to the pairs $\alpha < \beta_1, \alpha < \beta_2, \beta_1 < \gamma$ and $\beta_2 < \gamma$, etc. See Figure 1.1.

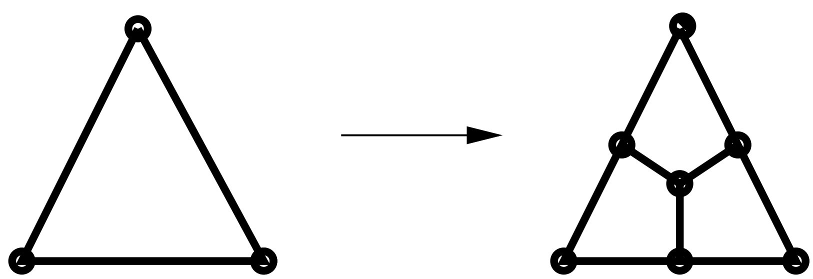

Figure 1

Instead, we will provide a direct algebraic proof which will also serve to illuminate some of the structure of the objects we have introduced. Let $\Omega^{a,b}(M)$ be the set of local linear maps from $C_a(M)$ to $C_b(M)$. Extending such a map to be 0 on $C_p(M),\ p \neq a$, we can think of $\Omega^{a,b}(M)$ as a subspace of $\Omega^{b-a}(M)$. In this way we have

$$\Omega^d = \bigoplus_{a=d}^{n} \Omega^{a,a-d}.$$

For any d-form ω, we can write

$$D(\omega) = D_1(\omega) + (-1)^{d+1} D_2(\omega)$$

where

$$\begin{aligned} D_1(\omega) &= \partial \circ \omega \\ D_2(\omega) &= \omega \circ \partial. \end{aligned}$$

It follows that

$$\begin{aligned} D_1(\Omega^{a,b}) &\leq \Omega^{a,b-1} \\ D_2(\Omega^{a,b}) &\leq \Omega^{a+1,b}. \end{aligned}$$

We can piece all of this together in a double complex.

$$\begin{tikzcd}
 & & & \Omega^{2,2} \arrow[r] \arrow[d, "D_1"] & \cdots \\
 & & \Omega^{1,1} \arrow[r, "D_2"] \arrow[d, "D_1"] & \Omega^{2,1} \arrow[r] \arrow[d, "D_1"] & \cdots \\
0 \arrow[r] & \Omega^{0,0} \arrow[r, "D_2"] & \Omega^{1,0} \arrow[r, "D_2"] \arrow[d] & \Omega^{2,0} \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 &
\end{tikzcd}$$

To calculate the cohomology of the total complex (Ω^*, D) we apply the general method of spectral sequences (see [19]). We first calculate the cohomology of the columns, that is, the D_1 cohomology. It follows immediately from the definitions that

$$Ker(D_1 : \Omega^{a,b} \to \Omega^{a,b-1})$$

is the space of linear maps

$$\omega : C_a \to C_b$$

which maps each a-cell α to a closed b-chain in $\bar{\alpha}$. Similarly,

$$Im(D_1 : \Omega^{a,b+1} \to \Omega^{a,b})$$

is the space of linear maps

$$\omega : C_a \to C_b$$

which maps each a-cell α to an exact b-chain in $\bar{\alpha}$. If we define

$$H^{a,b} \equiv \frac{\text{Ker } (D_1 : \Omega^{a,b} \to \Omega^{a,b-1})}{Im(D_1 : \Omega^{a,b+1} \to \Omega^{a,b})}$$

then we see that $H^{a,b}$ is isomorphic to the space of functions which assigns to each a-cell α an element of $H_b(\bar{\alpha})$. Since, for each α, $\bar{\alpha}$ is contractible, we see that

$$H^{a,b} \cong \begin{cases} 0 & \text{if } b \neq 0 \\ C^a(M) & \text{if } b = 0 \end{cases}$$

Therefore, the D-cohomology of the original complex is isomorphic to the cohomology of the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{0,0} & \xrightarrow{\bar{D}_2} & H^{1,0} & \xrightarrow{\bar{D}_2} & H^{2,0} & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ & & C^0(M) & & C^1(M) & & C^2(M) & & \end{array}$$

where $\bar{D}_2$ is the differential induced by the D_2 operator. To prove that the cohomology of this complex is isomorphic to the usual simplicial cohomology, we will show that, under the association of $H^{p,0}$ with $C^p(M,\mathbb{R})$, the operator $\bar{D}_2$ corresponds to the usual coboundary operator δ.

Let α be an oriented p-cell of M, and $\alpha^* \in C^p(M,\mathbb{R})$ the element which maps α to 1, and all other p-cells to 0. Choose a vertex $v \in \bar{\alpha}$. The p-form ω which maps α to v and all other cells to 0 is a representative of the element $[\omega] \in H^{p,0}$ which is identified with $\alpha^* \in C^p(M)$. For any $(p+1)$-cell β,

$$(D_2\omega)\beta = \langle \partial\beta, \alpha \rangle v.$$

We can see that $(D_2\omega)\beta \in H^{p+1,0}$ is identified with the element of $C^{p+1}(M,\mathbb{R})$ which maps each $(p+1)$-cell β to $\langle \partial\beta, \alpha \rangle$. This element is precisely $\delta\alpha^*$. $\square$

We end this section by presenting some examples of the ingredients we have just introduced.

1.1. **Combinatorial Differential 0-forms:** A combinatorial differential 0-form ω must map each oriented cell α to $c_\alpha \alpha$ for some constant c_α. In this way we can identify

ω with a function $F : K \to \mathbf{R}$, where $\mathbf{R}$ is the coefficient ring, by setting

$$F(\alpha) = c_\alpha.$$

Every function from K to $\mathbf{R}$ arises in this fashion. We will identify ω with the function F, and use the 2 notions interchangeably.

Given a 0-form ω corresponding to a function F, we can express its differential as follows. For any p-cell α

$$\begin{aligned}
(Dw)(\alpha) &= \partial(\omega(\alpha)) - \omega(\partial\alpha) \\
&= \partial(F(\alpha)(\alpha)) - \sum_{\gamma^{(p-1)}<\alpha} F(\gamma)\langle\partial\alpha,\gamma\rangle\gamma \\
&= \sum_{\gamma^{(p-1)}<\alpha} (F(\alpha) - F(\gamma))\langle\partial\alpha,\gamma\rangle\gamma.
\end{aligned}$$

From this we see

Theorem 1.3. *Let F be a combinatorial differential 0-form. Then $DF = 0$ if and only if for each pair of cells $\gamma^{(p-1)} < \alpha^{(p)}$*

$$F(\alpha) = F(\gamma).$$

It easily follows that $DF = 0$ if and only if F is constant on each connected component of M (that is, F takes on the same value on all cells in a connected component of M).

1.2. **Combinatorial Differential 1-forms:** Choose an orientation for each cell of M. A combinatorial 1-form ω must map each oriented p-cell $\alpha^{(p)}$ to a linear combination of the form $\sum_{\gamma^{(p-1)}<\alpha} c_{\alpha,\gamma}\gamma$. We can write this as

$$\omega(\alpha) = \sum_{\gamma^{(p-1)}<\alpha} c_{\alpha,\gamma}\langle\partial\alpha,\gamma\rangle(\langle\partial\alpha,\gamma\rangle\gamma).$$

We note that

$$c_{-\alpha,\gamma} = c_{\alpha,-\gamma} = -c_{\alpha,\gamma}$$

where $-\alpha$ and $-\gamma$ refer to the cells α and γ endowed with the opposite orientation. It follows that for each $\alpha^{(p)} > \gamma^{(p-1)}$, the quantity

$$c_{\alpha,\gamma}\langle\partial\alpha,\gamma\rangle$$

is independent of the chosen orientations on α and γ. Let $\mathcal{F}$ denote the set of pairs $\{\alpha^{(p)} > \gamma^{(p-1)}\}$ as p ranges from 0 to $\dim M$. We can identify ω canonically with a map

$$G : \mathcal{F} \to \mathbf{R}$$

where

$$G(\alpha^{(p)},\gamma^{(p-1)}) = c_{\alpha,\gamma}\langle\partial\alpha,\gamma\rangle$$

Every such function arises from a combinatorial 1-form. From now on, for each pair of cells $\alpha^{(p)} > \gamma^{(p-1)}$, we will write $\omega(\alpha > \gamma)$ for $G(\alpha,\gamma)$. We hope this will not cause undue confusion.

For example, from (1.1) it follows that for ant $F \in \Omega^0(M)$, and any $\beta^{(p+1)} > \alpha^p$,

$$DF(\beta^{(p+1)} > \alpha^p) = F(\beta) - F(\alpha).$$

Let ω be a combinatorial differential 1-form. We now derive an explicit representation of $D\omega$. For each p-cell α

$$\begin{aligned}
(D\omega)(\alpha) &= \partial(\omega(\alpha)) + \omega(\partial\alpha)\\
&= \partial(\sum_{\beta^{(p-1)}<\alpha} \omega(\alpha > \beta)\langle\partial\alpha,\beta\rangle\beta)\\
&+ \omega(\sum_{\beta^{(p-1)}<\gamma} \langle\partial\alpha,\beta\rangle\beta)\\
&= \sum_{\gamma^{(p-2)}<\beta<\alpha} \omega(\alpha > \beta)\langle\partial\alpha,\beta\rangle\langle\partial\beta,\gamma\rangle\gamma\\
&+ \sum_{\gamma^{(p-2)}<\beta<\alpha} \omega(\beta > \gamma)\langle\partial\alpha,\beta\rangle\langle\partial\beta,\gamma\rangle\gamma\\
&= \sum_{\gamma<\beta<\alpha} (\omega(\alpha > \beta) + \omega(\beta > \alpha))\langle\partial\alpha,\beta\rangle\langle\partial\beta,\gamma\rangle\gamma.
\end{aligned}$$

Fix a pair of cells $\gamma^{(p-2)} < \alpha^{(p)}$. Then there are precisely two $(p-1)$-cells $\beta_1^{(p-1)},\ \beta_2^{(p-1)}$ with

$$\gamma < \beta_1 < \alpha_1 \quad \gamma < \beta_2 < \alpha.$$

The coefficient of γ in $(D\omega)(\alpha)$ is

$$\pm\left[(\omega(\alpha > \beta_1) + \omega(\beta_1 > \gamma)) - (\omega(\alpha > \beta_2) + \omega(\beta_2 > \gamma))\right].$$

From this we see

Theorem 1.4. *Let ω be a combinatorial differential 1-form. Then $D\omega = 0$ if and only if for all cells*

$$\gamma^{(p-2)} < \beta_1^{(p-1)} < \alpha^{(p)}, \gamma < \beta_2^{p+1} < \alpha \quad \beta_1 \neq \beta_2$$

we have

$$\omega(\alpha > \beta_1) + \omega(\beta_1 > \gamma) = \omega(\alpha > \beta_2) + \omega(\beta_2 > \gamma). \tag{1.1}$$

Let ω be a closed 1-form, and F a 0-form. Then $\omega = DF$ if and only if for each pair of cells $\beta^{(p+1)} > \alpha^{(p)}$

$$\omega(\beta > \alpha) = F(\beta) - F(\alpha).$$

We can try to find F by "integrating" ω. This is, fix a cell $\alpha^{(p)}$ and assign $F(\alpha)$ arbitrarily. For any $\beta^{(p+1)} > \alpha$, define $F(\beta)$ by

$$F(\beta) = F(\alpha) + \omega(\beta > \alpha).$$

Similarly, if $\gamma^{(p11)} < \alpha$, define $F(\gamma)$ by

$$F(\gamma) = F(\alpha) - \omega(\alpha > \gamma).$$

continuing in this fashion, the identity (1.1) implies that the function F so constructed is well-defined on any simply-connected subcomplex of M which contains α.

Before leaving this section, we return to our earlier remark that for any regular cell complex M, the coboundary operator δ is a combinatorial differential 1-form. In fact, δ is closed. Therefore, from Theorem 1.2, δ represents a cohomology class of M.

This leads to the possibly surprising conclusion that for every regular cell complex M there is a canonical cohomology class

$$[\delta] = H^1(M, \mathbb{R}).$$

In fact, this cohomology class is trivial. Namely, define a 0-form F by setting, for any cell α

$$F(\alpha) = \text{ dimension } (\alpha).$$

Then $DF = \delta$.

2. Morse One-Forms

In this section we define, and investigate, the class of combinatorial one-forms to which our theory will apply. Let M be a compact regular CW complex, and $\omega \in \Omega^1(M, \mathbb{R})$ a 1-form satisfying $D\omega = 0$. Then, restricted to a contractible subcomplex N of M, we can write $\omega = Df$ for some $f \in \Omega^0(N, \mathbb{R})$. We say that ω is Morse, if f is a Morse function on N (as defined in [7, 8]) for all such N and f. We can define this notion in a more direct fashion as follows.

Definition 2. *Say a closed 1-form ω on M is a Morse form if for all p, and all p-cells α of M*

(1) $\#\{\beta^{(p+1)} > \alpha \mid \omega(\beta > \alpha) \leq 0\} \leq 1$

and

2) $\#\{\gamma^{(p-1)} < \alpha \mid \omega(\alpha > \gamma) \leq 0\} \leq 1.$

This concept can also be defined via "multi-valued functions." Let $\tilde{M} \xrightarrow{\pi} M$ be any cover of M satisfying $H^1(\tilde{M}, \mathbb{R}) = 0$. Let ω be a closed 1-form on M. Then $\tilde{\omega} = \pi^*\omega$ is an exact 1-form on $\tilde{M}$ so $\tilde{\omega} = D\tilde{f}$ for some $\tilde{f} \in \Omega^0(\tilde{M}, \mathbb{R})$. Then ω is a Morse form on M if and only if $\tilde{f}$ is a Morse function on $\tilde{M}$ (as defined in [7, 8]). We can think of $\tilde{f}$ as a multi-valued function on M.

Definition 3. *Let ω be a Morse form on M. Say $\alpha^{(p)}$ is a critical cell if*

(1) $\#\{\beta^{(p+1)} > \alpha \mid \omega(\beta > \alpha) \leq 0\} = 0$

and

(2) $\#\{\gamma^{(p-1)} < \alpha \mid \omega(\alpha > \gamma) \leq 0\} = 0.$

To be consistent with the standard terminology, if α is critical, we define the index of α to be the dimension of α.

In fact, Morse 1-forms have a bit more structure than may at first be apparent.

Theorem 2.1. *Let ω be a Morse 1-form on M, and $\alpha^{(p)}$ a p-cell, then either*

$$\#\{\beta^{(p+1)} > \alpha \mid \omega(\beta > \alpha) \leq 0\} = 0$$

or

$$\#\{\gamma^{(p-1)} < \alpha \mid \omega(\alpha > \gamma) \leq 0\} = 0$$

This is Theorem 1.1 of [7].

Corollary 2.2. *The cells of M can be partitioned into the critical cells, and pairs $\{\beta^{(p+1)} > \alpha^p\}$ where $\omega(\beta > \alpha) \leq 0$.*

We pause here to note the simple fact that every cohomology class can be represented by a Morse form.

Theorem 2.3. *Let $h \in H^1(M, \mathbb{R})$. Then there is a Morse 1-form ω with*

$$[\omega] = h.$$

Proof. Let ω_1 be any closed 1-form with $[\omega_1] = h$. Choose $c \geq 0$ so that for all p and all $\beta^{(p+1)} > \alpha^{(p)}$

$$\omega(\beta > \alpha) > -c.$$

Now define a function $F \in \omega^0(M, \mathbb{R})$ by setting for each cell $\alpha^{(p)}$

$$F(\alpha) = pc.$$

Let $\omega = \omega_1 + DF$. Then clearly ω is closed and

$$[\omega] = [\omega_1] = h.$$

For any $\beta^{(p+1)} > \alpha^{(p)}$

$$\begin{aligned}
\omega(\beta > \alpha) &= \omega_1(\beta > \alpha) + DF(\beta > \alpha) \\
&= \omega_1(\beta > \alpha) + F(\beta) - F(\alpha) \\
&= \omega_1(\beta > \alpha) + (p+1)c - pc \\
&> 0
\end{aligned}$$

which implies that ω is a Morse form. $\square$

Observe that the form ω constructed in this proof is trivial, in the sense that every cell is critical.

In our proofs, it will be convenient to restrict attention to Morse forms which satisfy an additional hypothesis.

Definition 4. *Let $\omega \in \Omega^1(M, \mathbb{R})$ be a Morse 1-form. Say ω is flat if for all p and all pairs $\beta^{(p+1)} > \alpha^{(p)}$*

$$\omega(\beta > \alpha) \geq 0.$$

The main goal of this section is to prove that nothing is lost by restricting attention to flat Morse 1-forms, as any Morse 1-form is equivalent, in a precise sense, to a flat Morse form.

Definition 5. *Say two closed 1-forms ω_1 and ω_2 are equivalent if*

1) $[\omega_1] = [\omega_2]$

where $[\omega_i] \in H^1(M, \mathbb{R})$ is the cohomology class represented by ω_i.

2) For all p, and all pairs $\beta^{(p+1)} > \alpha^{(p)}$

$$\omega_1(\beta > \alpha) > 0 \Leftrightarrow \omega_2(\beta > \alpha) > 0$$

Note that if ω_1 and ω_2 are equivalent, then

$$\omega_1 \text{ is a Morse form} \Leftrightarrow \omega_2 \text{ is a Morse form.}$$

We are now ready to state and prove the main theorem of this section.

Theorem 2.4. *Every Morse 1-form is equivalent to a flat Morse 1-form.*

Proof. Let ω be a Morse 1-form. Choose $c \geq 0$ so that for every p and every $\beta^{(p+1)} > \alpha^{(p)}$

$$\omega(\beta > \alpha) \geq -c.$$

Recall that the cells of M can be partitioned into those that are critical, and the disjoint pairs $\{\beta^{(p+1)}, \alpha^{(p)}\}$ where

$$\beta > \alpha$$

and

$$\omega(\beta > \alpha) \leq 0.$$

Define a function $F \in \Omega^0(M)$ as follows. For each critical cell $\alpha^{(p)}$ set

$$F(\alpha) = pc.$$

For each non-critical pair $\{\beta^{(p+1)}, \alpha^{(p)}\}$ set

$$\begin{aligned} F(\alpha) &= pc \\ F(\beta) &= pc - \omega(\beta > \alpha). \end{aligned}$$

Note that for any $\alpha^{(p)}$

$$(p-1)c \leq F(\alpha) \leq pc$$

which implies, in particular, that for any $\beta^{(p+1)}$ and $\alpha^{(p)}$

$$F(\beta) - F(\alpha) \geq 0.$$

Let

$$\omega^* = \omega + DF.$$

We will see that ω^* is flat and equivalent to ω. We first observe that ω^* is clearly closed, and

$$[\omega^*] = [\omega] \in H^1(M, \mathbb{R}).$$

For each $\beta^{(p+1)} > \alpha^{(p)}$ we have

$$\begin{aligned} \omega^*(\beta > \alpha) &= \omega(\beta > \alpha) + DF(\beta > \alpha) \\ &= \omega(\beta > \alpha) + F(\beta) - F(\alpha) \\ &\geq \omega(\beta > \alpha). \end{aligned}$$

Therefore,

$$\omega(\beta > \alpha) > 0 \Rightarrow \omega^*(\beta > \alpha) > 0.$$

Conversely, suppose $\beta^{(p+1)} > \alpha^{(p)}$ satisfy

$$\omega(\beta > \alpha) \leq 0.$$

Then

$$\begin{aligned}\omega^*(\beta > \alpha) &= \omega(\beta > \alpha) + F(\beta) - F(\alpha) \\ &= \omega(\beta > \alpha) + [pc - \omega(\beta > \alpha)] - [pc] \\ &= 0.\end{aligned}$$

It follows that for all $\beta^{(p+1)} > \alpha^{(p)}$

$$\omega^*(\beta > \alpha) \geq 0$$

and

$$\omega(\beta > \alpha) > 0 \Leftrightarrow \omega^*(\beta > \alpha) > 0$$

as desired. $\square$

3. The Morse Inequalities

Let ω be a Morse 1-form, and

$$h = [\omega] \in H^1(M, \mathbb{R}).$$

Let $m_p(\omega)$ denote the number of critical cells of ω of dimension p. In [7, 8, 9] we considered the case $h = 0$, the setting for classical Morse theory. We proved the Strong Morse Inequalities

$$m_p(\omega) - m_{p-1}(\omega) + \cdots \pm m_0(\omega) \geq b_p - b_{p-1} + \cdots \pm b_0, \quad \forall p = 0, 1, 2, \ldots$$

where

$$b_p = \dim H_p(M, \mathbb{R}).$$

These inequalities imply the Weak Morse Inequalities

$$m_p(\omega) \geq b_p \quad \forall p = 0, 1, 2, \ldots$$

$$m_0(\omega) - m_1(\omega) + \cdots + (-1)^n m_n(\omega) = b_0 - b_1 + \cdots + (-1)^n b_n.$$

Our goal in this section is to find analogous inequalities when $h \neq 0$. More precisely, for $h \in H^1(M, \mathbb{R})$ we wish to find topological invariants $B_p(h)$ so that if $[\omega] = h$ the corresponding Morse inequalities are true. In this section we will essentially be following the ideas of [26], see also [9] and [40].

We begin with a little of the standard theory of covering spaces, expressed in the language of this paper. Let

$$\pi : \tilde{M} \to M$$

denote the universal cover of M, with $\tilde{M}$ given the cell structure induced from M. Let $\tilde{\omega} = \pi^* \omega$. Then $\tilde{\omega}$ is exact, so we can find a function $\tilde{F} \in \Omega^0(\tilde{M}, \mathbb{R})$ satisfying $D\tilde{F} = \tilde{\omega}$. ($\tilde{F}$ is defined up to an additive constant).

The first step is to observe that for any covering transformation

$$g : \tilde{M} \to \tilde{M}$$

(i.e., so that $\pi = \pi \circ g$) the function $\tilde{F} \circ g - \tilde{F}$ is constant, That is,

$$\tilde{F}(g(\alpha)) - \tilde{F}(\alpha)$$

is independent of the cell α. This can be seen as follows

$$\begin{aligned} D(\tilde{F} \circ g - \tilde{F}) &= D(g^* \tilde{F}) - D\tilde{F} \\ &= g^* D\tilde{F} - D\tilde{F} \\ &= g^* \tilde{\omega} - \tilde{\omega} \\ &= g^* \pi^* \omega - \pi^* \omega \\ &= (\pi \circ g)^* \omega - \pi^* \omega \\ &= \pi^* \omega - \pi^* \omega \\ &= 0 \end{aligned}$$

Let $\rho_\omega(g) \in \mathbb{R}$ denote this constant. If g_1 and g_2 are any 2 covering transformations

$$\begin{aligned} \rho_\omega(g_1\, g_2) &= (g_1\, g_2)^* \tilde{F} - \tilde{F} \\ &= g_2^* g_1^* \tilde{F} - \tilde{F} \\ &= g_2^*(g_1^* \tilde{F} - \tilde{F}) + (g_2^* \tilde{F} - \tilde{F}) \end{aligned}$$

Since $g_1^* \tilde{F} - \tilde{F}$ is constant, it is left invariant by g_2^*. Therefore,

$$\begin{aligned} \rho_\omega(g_1 \circ g_2) &= (g_1^* \tilde{F} - \tilde{F}) + (g_2^* \tilde{F} - \tilde{F}) \\ &= \rho_\omega(g_1) + \rho_\omega(g_2). \end{aligned}$$

Therefore, in this way ω defines a representation

$$\rho_\omega : \pi_1(M) \to (\mathbb{R}, +).$$

In fact, it can easily be seen that ρ_ω depends only on $h = [\omega]$. Namely, if ω_1 is another 1-form with $[\omega_1] = h$ then

$$\omega_1 = \omega + Df$$

for some $f \in \Omega^0(M, \mathbb{R})$. Thus

$$\begin{aligned}\tilde{\omega}_1 = \pi^*\omega_1 &= \pi^*\omega + \pi^* Df \\ &= \tilde{\omega} + D\pi^* f \\ &= D\tilde{F} + D\pi^* f \\ &= D(\tilde{F} + \pi^* f)\end{aligned}$$

and for $g \in \pi_1(M)$

$$\rho_{\omega_1}(g) = g^*(\tilde{F} + \pi^* f) - (\tilde{F} + \pi^* f).$$

Since

$$g^*\pi^* f = \pi^* f$$

we have

$$\rho_{\omega_1}(g) = g^*\tilde{F} - \tilde{F} = \rho_\omega(g).$$

We now denote the representation ρ_ω by ρ_h. For every $t \in \mathbb{R}$ we can define a multiplicative representation

$$\eta_t : \pi_1(M) \to \mathbb{R}^*$$

by

$$\eta_t(g) = e^{t\rho_h(g)}.$$

For any representation

$$\eta : \pi_1(M) \to \mathbb{R}^*$$

we denote by

$$C_p(M, \eta) \subset C_p(\tilde{M}, \mathbb{R})$$

the p-chains of $\tilde{M}$ which transforms via η. That is,

$$C_p(M, \eta) = \{c \in C_p(\tilde{M}, \mathbb{R}) \mid g_* c = \eta(g)c \ \forall g \in \pi_1(M)\}.$$

We note that for each $g \in \pi_1(M)$, the boundary operator ∂ commutes with g_* and multiplication by $\eta(g)$. This implies that

$$\partial(C_p(M, \eta)) \subseteq C_{p-1}(M, \eta).$$

We can now consider the differential complex

$$C_*(M,\eta) : 0 \to C_n(M,\eta) \xrightarrow{\partial} C_{n-1}(M,\eta) \xrightarrow{\partial} \cdots \xrightarrow{\partial} (C_0(M,\eta) \to 0.$$

Let $H_p(M,\eta)$ denote the homology of this complex. Observe that if $\mathbf{1}$ denotes the trivial representation (so that $\mathbf{1}(g) = 1$ for all $g \in \pi_1(M)$) then we have a canonical isomorphism

$$C_p(M,\mathbf{1}) \cong C_p(M,\mathbb{R})$$

which is consistent with the boundary operators, so

$$H_p(M,\mathbf{1}) \cong H_p(M,\mathbb{R}).$$

Let

$$b_p(\eta) = \dim H_p(M,\eta).$$

We will later prove

Theorem 3.1. *There is a set $S \subseteq \mathbb{R}$, at most countably infinite, such that $b_p(\eta_t)$ is constant on $\mathbb{R} - S$. Denote this constant by $B_p(h)$. For $t \in S$*

$$b_p(\eta_t) \geq B_p(h).$$

In the statement of Theorem 3.1 we have defined the desired topological invariants $B_p(h)$. We are now ready to state the main theorem of this section.

Theorem 3.2. *Let ω be a Morse 1-form with $h = [\omega] \in H^1(M,\mathbb{R})$. Then with all notation as above we have*

(i) Strong Morse Inequalities:

$$m_p(\omega) - m_{p-1}(\omega) + \cdots \pm m_0(\omega) \geq B_p(h) - B_{p-1}(h) + \cdots \pm B_0(h) \quad for\, all\, p = 0,1,2,\ldots$$

(ii) Weak More Inequalities:

$$m_p(\omega) \geq B_p(h) \quad for\, all\, p = 0,1,2,\ldots$$

$$m_0(\omega) - m_1(\omega) + \cdots + (-1)^n m_n(\omega) = B_0(h) - B_1(h) + \cdots + (-1)^n B_n(h).$$

The rest of this section is devoted to proving Theorem 3.2. Along the way we will prove Theorem 3.1.

Out goal is to investigate the homology of the sequences

$$C_*(M,\eta_t) : 0 \to C_n(M,\eta_t) \xrightarrow{\partial} C_{n-1}(M,\eta_t) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,\eta_t) \to 0.$$

Rather than considering these varying spaces of chains, it is convenient to change variables so that the spaces remain constant, but the differential varies. Recall that $C_p(M,\mathbf{1})$ is canonically isomorphic to $C_p(M,\mathbb{R})$. Namely, if α is an oriented p-cell in M, choose a lift $\tilde{\alpha}$ of α (that is, a p-cell $\tilde{\alpha}$ of $\tilde{M}$ with $\pi(\tilde{\alpha}) = \alpha$). Define a p-chain $c_\alpha \in C_p(\tilde{M},\mathbb{R})$ by

$$c_\alpha = \sum_{g \in \pi_1(M)} g(\tilde{\alpha}).$$

It is easy to see that c_α is independent of the choice of $\tilde{\alpha}$, and that $c_\alpha \in C_p(M,\mathbf{1})$. The map $\alpha \mapsto c_\alpha$ extends linearly to an isomorphism

$$i : C_p(M,\mathbb{R}) \to C_p(M,\mathbf{1}),$$

and the $\{c_\alpha\}$ form a convenient basis for $C_p(M,\mathbf{1})$.

For each $t \in \mathbb{R}$, define a function

$$e^{t\tilde{F}} \in \Omega^0(\tilde{M},\mathbb{R})$$

by setting, for each cell α of $\tilde{M}$

$$e^{t\tilde{F}}(\alpha) = e^{t\tilde{F}(\alpha)}\alpha.$$

The key point is that $e^{t\tilde{F}}$ maps $C_p(M,\mathbf{1})$ isomorphically onto $C_p(M,\eta_t)$. For example, for each $c_\alpha \in C_p(M,\mathbf{1})$ and $g \in \pi_1(M)$

$$\begin{aligned}
g^*(e^{t\tilde{F}}c_\alpha) &= g^*(e^{t\tilde{F}})g^*c_\alpha \\
&= e^{tg^*(\tilde{F})}c_\alpha \\
&= e^{t(\rho_h(g)+\tilde{F})}c_\alpha \\
&= e^{t\rho_h(g)}(e^{t\tilde{F}}c_\alpha) \\
&= \eta_t(g)(e^{t\tilde{F}}c_\alpha)
\end{aligned}$$

so that

$$e^{t\tilde{F}} c_\alpha \in C_p(M, \eta_t).$$

Now consider the complexes

$$0 \to C_n(M, \mathbb{R}) \xrightarrow{\partial_t} C_{n-1}(M, \mathbb{R}) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} C_0(M, \mathbb{R}) \to 0$$

where

$$\partial_t = i^{-1}\, e^{-t\tilde{F}}\, \partial\, e^{t\tilde{F}}\, i.$$

For each $t \in \mathbb{R}$ the homology of this complex is isomorphic to $H_*(M, \eta_t)$. To proceed further, we will find a more explicit representation of the operator ∂_t.

Lemma 3.3. *For any p-cell α of M*

$$\partial_t \alpha = \sum_{\beta^{(p-1)} < \alpha} e^{t\omega(\alpha > \beta)} \langle \partial\alpha, \beta \rangle \beta$$

(where $<,>$ denotes the canonical inner product with respect to which the cells are orthonormal).

Proof. Let α be a p-cell of M and choose a lift $\tilde{\alpha}^{(p)}$ in $\tilde{M}$. For each $\beta^{(p-1)} < \alpha$, choose the lift $\tilde{\beta}$ such that $\tilde{\beta} < \tilde{\alpha}$. Then

$$\begin{aligned}
\partial_t \alpha &= (i^{-1}\, e^{-t\tilde{F}}\, \partial\, e^{t\tilde{F}}\, i)\alpha \\
&= (i^{-1}\, e^{-t\tilde{F}}\, \partial\, e^{t\tilde{F}}) \sum_{g \in \pi_1(M)} g(\tilde{\alpha}) \\
&= (i^{-1}\, e^{-t\tilde{F}}\, \partial) \sum_{g \in \pi_1(M)} e^{t\tilde{F}(g(\tilde{\alpha}))} g(\tilde{\alpha}) \\
&= (i^{-1}\, e^{-t\tilde{F}}) \sum_{g \in \pi_1(M)} e^{t\tilde{F}(g(\tilde{\alpha}))} \partial(g(\tilde{\alpha})) \\
&= (i^{-1}\, e^{-t\tilde{F}}) \sum_{g \in \pi_1(M)} e^{t\tilde{F}(g(\tilde{\alpha}))} g(\partial\tilde{\alpha}) \\
&= (i^{-1}\, e^{-t\tilde{F}}) \sum_{g \in \pi_1(M)} \sum_{\tilde{\beta}^{(p-1)} < \alpha} e^{t\tilde{F}(g(\tilde{\alpha}))} \langle \partial\tilde{\alpha}, \tilde{\beta} \rangle g(\tilde{\beta}) \\
&= i^{-1} \sum_{g \in \pi_1(M)} \sum_{\tilde{\beta} < \tilde{\alpha}} e^{t(\tilde{F}(g(\tilde{\alpha})) - \tilde{F}(g(\tilde{\beta})))} \langle \partial\tilde{\alpha}, \tilde{\beta} \rangle g(\tilde{\beta}) \qquad (3.1)
\end{aligned}$$

We now observe that

$$\langle \partial\tilde{\alpha}, \tilde{\beta}\rangle = \langle \partial\alpha, \beta\rangle$$

and

$$\begin{aligned}
\tilde{F}(g(\tilde{\alpha})) - \tilde{F}(g(\tilde{\beta})) &= \tilde{F}(\tilde{\alpha}) - \tilde{F}(\tilde{\beta}) \\
&= D\tilde{F}(\tilde{\alpha} > \tilde{\beta}) \\
&= \tilde{\omega}(\tilde{\alpha} > \tilde{\beta}) \\
&= \pi^*\omega(\tilde{\alpha} > \tilde{\beta}) \\
&= \omega(\alpha > \beta).
\end{aligned}$$

Substituting, we see that (3.1) is equal to

$$\begin{aligned}
& i^{-1} \sum_{\beta<\alpha} e^{t\omega(\alpha>\beta)} \langle \partial\alpha, \beta\rangle \sum_{g\in\pi_1(M)} g(\tilde{\beta}) \\
= \; & i^{-1} \sum_{\beta<\alpha} e^{t\omega(\alpha>\beta)} \langle \partial\alpha, \beta\rangle c_\beta \\
= \; & \sum_{\beta<\alpha} e^{t\omega(\alpha>\beta)} \langle \partial\alpha, \beta\rangle \beta
\end{aligned}$$

□

Using the canonical inner product $<,>$, we can define

$$\partial_t^* : C_p(M, \mathbb{R}) \to C_{p+1}(M, \mathbb{R})$$

to be the adjoint of the operator ∂_t, so that

$$\partial_t^* \alpha^{(p)} = \sum_{\beta^{(p+1)}>\alpha} e^{t\omega(\beta>\alpha)} \langle \partial\beta, \alpha\rangle \beta.$$

Now define the Laplace operator

$$\Delta_p^{(t)} = \partial_t \partial_t^* + \partial_t^* \partial_t : C_p(M, \mathbb{R}) \to C_p(M, \mathbb{R}).$$

More explicitly, for each p-cell α_1

$$\begin{aligned}\Delta_p(t)\alpha_1 &= \sum_{\alpha_2^{(p)}}\Bigg(\Bigg[\sum_{\beta^{(p+1)}>\alpha_1,\beta>\alpha_2} e^{t(\omega(\beta>\alpha_1)+\omega(\beta>\alpha_2))}\langle\partial\beta,\alpha_1\rangle\langle\partial\beta,\alpha_2\rangle\Bigg]\\ &+ \Bigg[\sum_{\gamma^{(p-1)}<\alpha_1,\gamma<\alpha_2} e^{t(\omega(\alpha_1>\gamma)+\omega(\alpha_2>\gamma))}\langle\partial\alpha_1,\gamma\rangle\langle\partial\alpha_2,\gamma\rangle\Bigg]\Bigg)\,\alpha_2 \qquad (3.2)\end{aligned}$$

The key point is that simple linear algebra implies that for each $t \in \mathbb{R}$

$$\text{Ker } \Delta_p(t) \cong H_p(M, \eta_t).$$

We are now ready to prove the main results of this section.

Proof of Theorem 3.1 As the above formula makes clear, the self-adjoint operators

$$\Delta_p(t) : C_p(M, \mathbb{R}) \to C_p(M, \mathbb{R})$$

depend analytically on t. It follows from Rellich's Theorem [33] that the eigenvalues of $\Delta_p(t)$ depend analytically on t. More precisely, for each $t^* \in \mathbb{R}$ there is an interval I containing t^*, and d_p analytic functions

$$\lambda_i : I \to \mathbb{R} \quad i = 1, 2, \ldots, d_p$$

(where d_p is the number of p-cells in M) such that for each $t \in I$

$$\text{spectrum } (\Delta_p(t)) = \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_{d_p}(t)\}.$$

Let $B_p(h)$ denote the number of λ_i's which are equal to 0 on all of I. Each of these eigenvalues must be 0 on all of $\mathbb{R}$. Each of the other λ_i's is 0 for at most finitely many values of $t \in I$. This implies the theorem.

Proof of Theorem 3.2. Let ω be a Morse 1-form. By Theorem 2.3 ω is equivalent to a flat Morse 1-form, so we will now assume that ω is flat. Our main idea is to investigate the behavior of $\Delta_p(t)$ as $t \to -\infty$. We think of $\Delta_p(t)$ as a matrix with respect to the basis of $C_p(M, \mathbb{R})$ consisting of the p-cells, so that the matrix elements are given by (3.2).

Suppose $\alpha_1 \neq \alpha_2$ are two p-cells of m, and $\beta^{(p+1)}$ satisfies $\beta > \alpha_1$ and $\beta > \alpha_2$. Since ω is a flat Morse form, $\omega(\beta > \alpha_1)$ and $\omega(\beta > \alpha_2)$ are both non-negative, and at least one is positive. Thus

$$\omega(\beta > \alpha_1) + \omega(\beta > \alpha_2) > 0.$$

Similarly, if $\gamma^{(p-1)}$ satisfies $\gamma < \alpha_1$ and $\gamma < \alpha_2$ then

$$\omega(\alpha_1 > \gamma) + \omega(\alpha_2 > \gamma) > 0.$$

This implies that the off-diagonal terms of $\Delta_p(t)$ vanish exponentially fast as $t \to -\infty$.

Now we consider the diagonal entries

$$\langle \Delta_p(t)\alpha, \alpha\rangle = \sum_{\beta^{(p+1)} > \alpha} e^{2t\omega(\beta > \alpha)} + \sum_{\gamma^{(p-1)} < \alpha} e^{2t\omega(\alpha > \gamma)}.$$

If α is critical, then for each $\beta^{(p+1)} > \alpha$, $\omega(\beta > \alpha) > 0$, and for each $\gamma^{(p-1)} < \alpha$, $\omega(\alpha > \gamma) > 0$, so the corresponding diagonal entry vanishes exponentially fast as $t \to -\infty$. If α is not critical then either there is precisely one $\beta^{(p+1)} > \alpha$ with $\omega(\beta > \alpha) = 0$ or there is precisely one $\gamma^{(p-1)} > \alpha$ with $\omega(\alpha > \gamma) = 0$, but not both, and all other exponents are positive. This implies that the corresponding diagonal entry approaches 1 exponentially fast as $t \to -\infty$. Summarizing

$$\Delta_p(t) \overset{t \to -\infty}{\sim} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix} + 0(e^{\epsilon t})$$

where the $2x2$ black matrix is with respect to the slitting

$$C_p(M, R) = \mathcal{M}_p \oplus \mathcal{M}_p^{\perp},$$

Where $\mathcal{M}_p$ denotes the span of the critical p-cells.

This implies that as $t \to -\infty$, $m_p(\omega)$ eigenvalues of $\Delta_p(t)$ approach 0, and all of the remaining eigenvalues approach 1. Choose a $T > 0$ so that for all p and all $t < -T$, $\frac{1}{2}$ is not an eigenvalue of $\Delta_p(t)$. For $t < -T$ let $\mathcal{W}_p(t) \subseteq C_p(M, \mathbb{R})$ denote the span of the eigenvectors corresponding to eigenvalues less that $\frac{1}{2}$. Then

$$\partial_t(\mathcal{W}_p(t)) \leq \mathcal{W}_{p-1}(t)$$

so we can form the Witten complex

$$\mathcal{W}(t) : 0 \to \mathcal{W}_n(t) \xrightarrow{\partial_t} \mathcal{W}_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} \mathcal{W}_0(t) \to 0.$$

It is easy to see that

$$H_*(\mathcal{W}(t)) \cong H_*(M, \eta_t).$$

Choose a $t > T$ so that

$$\dim H_p(M, \eta_t) = B_p(h).$$

Consider the complex $\mathcal{W}(t)$ for this t. The equalities

$$\begin{aligned} \dim \mathcal{W}_p(t) &= m_p(\omega) \\ \dim H_p(\mathcal{W}(t)) &= B_p(h) \end{aligned}$$

along with some simple linear algebra, imply the desired Morse inequalities.

4. Reidemeister Torsion

In this section we use the techniques of [9] and [10] to prove a formula for Reidemeister torsion in terms of the dynamics of the gradient vector field of a Morse 1-form. In the case of the differential of a smooth S^1-valued function on a smooth manifold, this formula recently appeared in [15]. Earlier, special cases appeared in [20] and [13, 14].

To explain the desired formula, we begin with a review of Reidemeister torsion. (See [12], [32], [9] for more details). Let

$$\Theta : \pi_1(M) \to O(k)$$

be a representation (where $O(k)$ denotes the space of $n \times n$ real orthogonal matrices). Consider the complex of $\mathbb{R}^k$-valued chains on $\tilde{M}$.

$$0 \to C_n(\tilde{M}, \mathbb{R}^k) \xrightarrow{\partial} C_{n-1}(\tilde{M}, \mathbb{R}^k) \xrightarrow{\partial} \cdots .$$

Define

$$C_p(M, \Theta) \subseteq C_p(\tilde{M}, \mathbb{R}^k)$$

to be the chains which transform via Θ. More precisely, choose an orientation for each cell of $\tilde{M}$ such that for each cell α of $\tilde{M}$ and each $g \in \pi_1(\tilde{\alpha})$, the map $g : \alpha \to g(\alpha)$ is orientation preserving. Then every $\mathbb{R}^k$-valued p-chain c of $\tilde{M}$ can be expressed as $c = \sum_{\alpha^{(p)}} c_\alpha \alpha$ for some $c_\alpha \in \mathbb{R}^k$. Then $c \in C_p(M, \Theta)$ if for each $g \in \pi_1(M)$

$$g(c) = \Theta(g)c.$$

That is, if

$$\sum_{\alpha^{(p)}} c_\alpha g(\alpha) = \sum_{\alpha^{(p)}} \{[\Theta(g)](c_\alpha)\}\alpha,$$

or, equivalently, if for each p-cell α of $\tilde{M}$ and each $g \in \pi_1(M)$

$$c_{g^{-1}(\alpha)} = [\Theta(g)](c_\alpha).$$

Note that

$$\dim C_p(M, \Theta) = k \cdot \#\{p\text{-cells of } M\}.$$

The boundary operator ∂ maps $C_p(M,\Theta)$ to $C_{p-1}(M,\Theta)$, so we can consider the complex

$$0 \to C_n(M,\Theta) \xrightarrow{\partial} C_{n-1}(M,\Theta) \xrightarrow{\partial} \cdots .$$

Denote the homology of this complex by $H_*(M,\Theta)$. We assume that the representation is acyclic, that is, $H_*(M,\Theta) = 0$.

We now observe that there is a standard L^2 inner product on $C_p(M,\Theta)$. Namely, for each cell α of M choose a lift $\tilde{\alpha}$. Then, for every $c_1, c_2 \in C_p(M,\Theta)$, set

$$\langle c_1, c_2 \rangle = \sum_{\alpha \in C_p(M)} (c_1(\tilde{\alpha}), c_2(\tilde{\alpha})),$$

where $(\ ,\ )$ denotes the standard inner product on $\mathbb{R}^k$. Because Θ is an orthogonal representation, the inner product $\langle\ ,\ \rangle$ is independent of the chosen lifts.

Let $\partial^* : C_p(M,\Theta) \to C_{p+1}(M,\Theta)$ denote the adjoint of ∂ with respect to this inner product, and

$$\Delta_p(\Theta) = \partial\partial^* + \partial^*\partial : C_p(M,\Theta) \to C_p(M,\Theta)$$

the corresponding Laplace operator. For every representation Θ

$$\text{Ker } \Delta_p(\Theta) \cong H_p(M,\Theta).$$

In particular, if Θ is acyclic then for each p, $\Delta_p(\Theta)$ is a strictly positive operator. With these definitions in hand, we are now ready to define the Reidemeister Torsion of M with respect to the representation Θ, $T(M,\Theta)$, by

$$Tor(M,\Theta) = \prod_{p=1}^{\dim M} (\text{ Det } \Delta_p(\Theta))^{\frac{p+1}{2}(-1)^p}.$$

This formula for $Tor(M,\Theta)$ first appeared in [32].

Let ω be a Morse 1-form. The goal of this section is to prove a formula for $T(M,\Theta)$ in terms of the dynamics of ω. This formula will have two types of contributions. One from the Morse complex of ω which is built from the critical points. The other is a zeta function built from the periodic orbits. We now describe these ingredients more precisely. Our presentation will be very brief. For a complete treatment of the combinatorial Morse complex see [9]. For a complete treatment of the combinatorial zeta functions see [10].

Let $\tilde{\omega}$ denote the lift of ω to $\tilde{M}$, and $\tilde{f}$ a function $\tilde{M}$ such that $D\tilde{f} = \tilde{\omega}$. For each p, let

$$\mathcal{M}_p(\omega, \Theta) \subseteq C_p(M, \Theta)$$

denote those chains which are supported on the critical simplices of $\tilde{\omega}$, and transform via Θ. We will now define a differential

$$\tilde{\partial} : \mathcal{M}_p(\omega, \Theta) \to \mathcal{M}_{p-1}(\omega, \Theta).$$

This requires the introduction of the notion of a gradient path of $\tilde{f}$. For p-cells α_0, α_1 of $\tilde{M}$, a *gradient path of $\tilde{f}$ (of dimension p and length k)* from α_0 to α_1 , or equivalently an $\tilde{\omega}$-path from α_0 to α_1, is a sequence of cells of $\tilde{M}$

$$c : \alpha_0 = \beta_0^{(p)}, \gamma_0^{(p+1)}, \beta_1^{(p)}, \gamma_1^{(p+1)}, \cdots, \gamma_{k-1}^{(p+1)}, \beta_k^{(p)} = \alpha_1$$

such that for every $i = 0, 1, \ldots, k-1$

1) $\beta_i < \gamma_i$ and $\beta_{i+1} < \gamma_i$

2) $\tilde{f}(\beta_i) \geq \tilde{f}(\gamma_i) > \tilde{f}(\beta_{i+1})$ [$\Leftrightarrow \tilde{\omega}(\gamma_i > \beta_i) \leq 0$ and $\tilde{\omega}(\gamma_i > \beta_{i+1}) > 0$]

Let c be such a gradient path, and suppose α_0 and α_1 have been endowed with an orientation. Choose an orientation for each of the other β_i's and γ_i's. We define the (algebraic) multiplicity of c by

$$\mu(c) = \prod_{i=1}^{k-1} -\langle \partial\gamma_i, \beta_1 \rangle \langle \partial\gamma_i, \beta_{i+1} \rangle.$$

Note that $\mu(c)$ is independent of the chosen orientations on the γ_i's, and the β_i's other than $\beta_0 = \alpha_0$ and $\beta_k = \alpha_1$,.

For any oriented critical cells $\beta^{(p)}$ and $\alpha^{(p-1)}$ of $\tilde{M}$, set

$$\langle \tilde{\partial}\beta, \alpha \rangle = \sum_{\alpha_1^{(p-1)} < \beta} \langle \partial\beta, \alpha_1 \rangle \sum_{c \in \Gamma(\alpha_1, \alpha)} \mu(c)$$

where $\Gamma(\alpha_1, \alpha)$ denotes the set of all gradient paths from α_1 to α. This operator $\tilde{\partial}$ extends linearly to a map

$$\tilde{\partial} : \mathcal{M}_p(\tilde{M}, \mathbb{R}^k) \to \mathcal{M}_{p-1}(\tilde{M}, \mathbb{R}^k),$$

and, in fact, preserves those chains which transform via Θ. Thus we can consider

$$\mathcal{M}(\omega, \Theta) : 0 \to \mathcal{M}_n(\omega, \Theta) \stackrel{\tilde{\partial}}{\to} M_{n-1}(\omega, \Theta) \stackrel{\tilde{\partial}}{\to} \cdots .$$

It is not at all obvious, but true nonetheless, that $\tilde{\partial}^2 = 0$, so $\mathcal{M}(\omega, \Theta)$ forms a differential complex which we call the Morse complex (see [9] for the basic properties of the operator $\tilde{\partial}$). The crucial fact is that

$$H_*(\mathcal{M}(\omega, \Theta)) \cong H_*(M, \Theta).$$

We have assumed that Θ is acyclic so $\mathcal{M}(\omega, \Theta)$ is exact. Note that each $\mathcal{M}_p(\omega, \Theta)$ inherits an inner product from $C_p(M, \Theta)$. We can now define the adjoint operator $\tilde{\partial}^*$, the corresponding Laplace operators $\tilde{\Delta}$, and thus, using the formula (4), the torsion $T(\mathcal{M}(\omega, \Theta))$ of this complex.

In [9] we proved that if ω is exact, then

$$Tor(M, \Theta) = Tor(\mathcal{M}(\omega, \Theta)).$$

Our goal now is to find the appropriate correction factor if $[\omega] \in H^1(M, \mathbb{R})$ is not 0.

It is now better to work on M rather than $\tilde{M}$. Define a *ω-path of dimension p and length k* to be a sequence of cells of M

$$\gamma : \beta_0^{(p)}, \gamma_0^{(p+1)}, \beta_1^{(p)}, \gamma_1^{(p+1)}, \ldots , \gamma_{k-1}^{(p+1)}, \beta_k^{(p)}$$

such that for all $i = 0, 1, \ldots , k - 1$

1) $\beta_i < \gamma_i$ and $\beta_{i+1} < \gamma_i$

2) $\omega(\gamma_i > \beta_i) \leq 0$ and $\omega(\gamma_i > \beta_{i+1}) > 0$.

Say γ is *closed* (or *periodic*) if $\beta_0 = \beta_k$.

The dynamics of such paths was studied extensively in [10]. Define the *chain recurrent set* $\mathcal{R}$ of ω to be the set of cells which are either critical cells of ω, or which are contained in some non-trivial closed path. The chain recurrent set can be decomposed into a disjoint union of *basic sets*

$$\mathcal{R} = \bigcup_i \Lambda_i$$

where two cells α and β in $\mathcal{R}$ belong to the same basic set if and only if there is a closed ω-path which contains α and β.

The next step is to introduce a zeta function which keeps track of the closed ω-paths. First note that if $c = \beta_0^{(p)}, \gamma_0^{(p+1)}, \ldots, \beta_0^{(p)}$ is a closed ω-path then we can define the multiplicity of $c, \mu(c)$, as in (4.2). For a closed path, $\mu(c)$ is independent of all choices of orientations. Let $\tilde{\beta}_0$ be a p-cell of $\tilde{M}$ which is a lift β_0. Then we can lift c to an $\tilde{\omega}$-path $\tilde{c}$ of $\tilde{f}$ beginning with $\tilde{\beta}_0$. Then $\tilde{c}$ need not be closed, but must end at $g(\tilde{\beta}_0)$ for some $g \in \pi_1(M)$. The element g is not quite well-defined, because replacing $\tilde{\beta}_0$ by another lift of β_0 has the effect of conjugating g by some element of $\pi_1(M)$. Associating c with this conjugacy class in $\pi_1(M)$ we can define $trace[\Theta(c)]$, which we denote by $\mathrm{tr}\Theta(c)$. With these definitions in mind, we define the desired zeta function by

$$\zeta(z, \omega, \Theta) = exp\left[\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{p=0}^{n} (-1)^p \sum_{c \in \mathcal{P}_k^{(p)}} \mu(c)\ \mathrm{tr}\Theta(c)\right]$$

where $\mathcal{P}_k^{(p)}$ denotes the closed ω-paths of dimension p and length k. From Theorem 5.12 of [10] we learn the following theorem.

Theorem 4.1. *The power series $\zeta(z, \omega, \Theta)$ has a positive radius of convergence. Moreover, $\zeta(z, \omega, \Theta)$ can be analytically continued to a meromorphic function on the entire complex plane.*

Say a basic set Λ is *non-trivial* if Λ contains a non-trivial closed path (otherwise Λ consists of a critical cell of ω). The following theorem is proved in §6 of [10]. (See Theorem 6.1 of [10]. There it is required that ω has no critical points, but that hypothesis is unnecessary if all we care about is the following theorem.)

Theorem 4.2. *Suppose $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ for each non-trivial basic set Λ. Then the zeta function $\zeta(z, \omega, \Theta)$ is analytic at $z = 1$.*

We can now state the main theorem of this section.

Theorem 4.3. *Suppose $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ for each non-trivial basic set Λ. Then*

$$Tor(M, \Theta) = \; Tor\,(\mathcal{M}(\omega, \Theta))\,\zeta(1, \omega, \Theta).$$

Before, beginning the proof, we will rewrite this theorem in the language of combinatorial vector fields, as introduced in [8, 10] (see also [4][34]).

A *combinatorial vector field V on M* is simply a disjoint collection of pairs $\{\beta^{(p+1)} > \alpha^{(p)}\}$ of cells of M. This is equivalent to the definition which we used in [7] and [9], but is closer to the notion as it appeared earlier, in another language in [4] and [34].

Let V be a combinatorial vector field on M. A *rest point of V* is a cell which is not contained in any pair in V. A V-path of index p and length k is any sequence of cells

$$\gamma : \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \dots , \beta_{k-1}^{(p)}, \alpha_k^{(p)}$$

such that for each $i = 0, 1, \dots , k-1$

1) $\alpha_i < \beta_i$ and $\alpha_{i+1} < \beta_i$

2) $\alpha_i \neq \alpha_{i+1}$

3) $\{\beta_i > \alpha_i\} \in V$

A Morse 1-form ω gives rise to a vector field V, which we call the dual vector field, by declaring

$$\{\beta^{(p+1)} > \alpha^{(p)}\} \in V \Leftrightarrow \omega(\beta > \alpha) \leq 0.$$

Then V-paths, as defined above, are precisely the same as ω-paths as defined earlier. The rest points of V correspond to the critical points of ω. For any combinatorial vector field V, one can define the chain recurrent set, basic sets, and, given a representation Θ of $\pi_1(M)$, the corresponding Morse complex $\mathcal{M}(V, \Theta)$ and zeta function $\zeta(z, V, \Theta)$. Theorem 4.3 can now be stated in terms of combinatorial vector fields.

Theorem 4.4. *Let V be a combinatorial vector field on M, and $\Theta : \pi_1(M) \to O(k)$ a representation. Suppose $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ for each non-trivial basic set Λ. Then*

$$T(M, \Theta) = \text{ Tor } (\mathcal{M}(V, \Theta))\, \zeta(1, V, \Theta).$$

Theorem 4.4 clearly implies Theorem 4.3. It is not yet clear to the author, however, if Theorem 4.4 is more general than the previous theorem. That is, if V is a combinatorial vector field such that $H_*(\bar{\Lambda}, \dot{\Lambda}, \Theta) = 0$ for each non-trivial basic set Λ (so that, in particular, V has no homotopically trivial closed paths). Then is there a Morse 1-form ω such that V is dual to ω?

The remainder of this section is devoted to a proof of Theorem 4.4. Our presentation will be rather sketchy, as the main steps have already been proved in [9] and [10].

The first step is the introduction of a Lyapunov function for V. A Lyapunov function is function $g \in \Omega^0(M)$ such that for all $\beta^{(p+1)} > \alpha^{(p)}$

1) $g(\beta) \geq g(\alpha)$.
2) $g(\beta) = g(\alpha)$ if and only if $\{\beta > \alpha\} \in V$ or α and β are elements of the same basic set.

The existence of such a function is established in Theorem 2.4 of [10].

Note that for any representation Θ and any $g \in \Omega^0(M)$, there is a natural action of g on $C_*(M, \Theta)$. Namely, g acts on $C_*(\tilde{M}, \mathbb{R}^k)$ by multiplying any cell α of $\tilde{M}$ by $g(\pi(\alpha))$. It is easy to see that this preserves $C_*(M, \Theta)$.

We now fix an acyclic representation $\Theta : \pi_1(M) \to O(k)$, a combinatorial vector field V, and a Lyapunov function g for V. Consider the one-parameter family of differential complexes

$$0 \to C_n(M, \Theta) \xrightarrow{\partial_t} C_{n-1}(\mathcal{M}, \Theta) \xrightarrow{\partial_t} \cdots$$

where

$$\partial_t = e^{tg} \partial e^{-tg},$$

($e^{tg} \in \Omega^0(M)$ is the function which maps a cell α to $e^{tg(\alpha)}(\alpha)$). The homology of these complexes is constant in t. Let

$$\Delta_p(t,\Theta) = \partial_t^*\partial_t + \partial_t^*\partial_t : C_p(M,\Theta) \to C_p(M,\Theta)$$

denote the corresponding Laplace operator. For every $t \in \mathbb{R}$ we have

$$\text{Ker } \Delta_p(t,\Theta) \cong H_p(M,\Theta).$$

These operators were introduced (in the smooth setting) in [40], and previously studied in the combinatorial setting in [9] and [10].

The main idea of the proof is to consider the torsion of the ∂_t-complexes, and to let $t \to \infty$. That is, define

$$T(t) = \sum_{p=0}^{n} (\text{ Det } \Delta_p(t,\Theta))^{\frac{p+1}{2}(-1)^p}.$$

Then

$$T(0) = \text{ Tor } (M,\Theta).$$

We showed in Lemma 6.2 of [9] (see also (6.7) of [9]) that

$$\frac{d}{dt}\log T(t) = k\sum_{p=0}^{n}\sum_{\text{critical } \alpha^{(p)}} g(\alpha)$$

so that $e^{-\kappa t}T(t)$ is constant, where

$$\kappa = k\sum_{p=0}^{n}\sum_{\text{critical } \alpha^{(p)}} g(\alpha).$$

Now let us take a closer look at the operator ∂_t. Let $e_1,\dots,e_k$ denote the standard basis for $\mathbb{R}^k$. For each p-cell α of $\tilde{M}$, and $i \in \{1,\dots,k\}$ let

$$c_{\alpha,i} = \sum_{g\in\pi_1(M)} [\Theta(g)](e_i)g^{-1}(\alpha).$$

Then

$$c_{\alpha,i} \in C_p(M,\Theta).$$

For any cell α of $\tilde{M}$

$$\partial_t\alpha = \sum_{\beta^{(p-1)}<\alpha} e^{t(g(\beta)-g(\alpha))}\langle\partial\alpha,\beta\rangle\beta$$

where we have written $g(\alpha)$ for $g(\pi(\alpha))$. It follows that for each i

$$\partial_t c_{\alpha,i} = \sum_{\beta^{(p-1)}<\alpha} e^{t(g(\beta)-g(\alpha))}\langle\partial\alpha,\beta\rangle c_{\beta,i}.$$

Since $g(\beta)-g(\alpha)\leq 0$ for each $\beta^{(p-1)}<\alpha^p$, ∂_t converges exponentially fast to an operator ∂_∞. The corresponding Laplace operator Δ_t converges exponentially fast to $\Delta(\infty)$, where

$$\Delta(\infty)c_{\alpha,i} = \partial_\infty\partial_\infty^* + \partial_\infty^*\partial_\infty c_{\alpha,i}$$

$$= \sum_{\alpha_1^{(p)}}[\sum_{\beta^{(p+1)}\approx\alpha,\beta\approx\alpha_1}\langle\partial\beta,\alpha\rangle\langle\partial\beta,\alpha_1\rangle + \sum_{\gamma^{(p-1)}\approx\alpha,\gamma\approx\alpha_1}\langle\partial\alpha,\gamma\rangle\langle\partial\alpha,\gamma\rangle]c_{\alpha_1,i}$$

For each p-cell α of M, choose a lift α^* in $\tilde{M}$. Then $\{c_{\alpha^*,i}\}$, as α ranges over the p-cells of M and i ranges from 1 to k, forms a basis for $C_p(M,\Theta)$.

Suppose α is critical. Then for all $\beta^{(p+1)}>\alpha$, $g(\beta)>g(\alpha)$ so that $\alpha\not\approx\beta$, and for all $\gamma^{(p-1)}<\alpha$, $g(\gamma)<g(\alpha)$, so that $\alpha\not\approx\gamma$. This implies that for all i

$$[\Delta(\infty)]c_{\alpha^*,i}=0.$$

Suppose $\alpha^{(p)}$ is not in the chain recurrent set. Then either

i) There exists exactly one $(p+1)$-cell β with $\beta>\alpha$ and $\alpha\approx\beta$. Moreover, for all $\alpha_1^{(p)}\neq\alpha$ with $\beta>\alpha$ then $g(\beta)>g(\alpha_1)$, so that $\alpha_1\not\approx\beta$.

or,

ii) There exists exactly one $(p-1)$-cell γ with $\gamma<\alpha$ and $\alpha\approx\gamma$. Moreover, for all $\alpha_1^{(p)}\neq\alpha$ with $\alpha_1>\gamma$ then $g(\alpha)>g(\gamma)$, so that $\alpha_1\not\approx\gamma$

but not both. Therefore, if $\alpha\notin\mathcal{R}$ then for all i

$$[\Delta(\infty)]c_{\alpha^*,i}=c_{\alpha^*,i}$$

(see the proof of Lemma 6.3 in [10] for a more complete discussion).

Suppose $\alpha\in\Lambda$ for some non-trivial basic set Λ. Then for any $(p+1)$ or $(p-1)$-cell β,

$$\alpha\approx\beta\Leftrightarrow\beta\in\Lambda.$$

This implies that for any p-cells α and α_1 of M, if the coefficient of $c_{\alpha_1^* i}$ in $[\Delta(\infty)]c_{\alpha^*,j}$ is non-zero, then $\alpha_1 \in \Lambda$. (see the paragraph following the proof of Lemma 6.3 in [10] for a more complete discussion.)

Let $\Lambda_1, \dots, \Lambda_\ell$ denote the non-trivial basic sets. Let $\mathcal{M}_p(\Theta) \subseteq C_p(M,\Theta)$ denote the span of the critical p-cells, $\mathcal{N}_p(\Theta) \subseteq C_p(M,\Theta)$ the span of the non-recurrent p-cells, and $C_p(\Lambda_i,\Theta) \subseteq C_p(M,\Theta)$ the span of the p-cells in Λ_i. Expressing $\Delta_p(\infty) : C_p(M,\Theta) \to C_p(M,\Theta)$ as a block matrix with respect to the decomposition

$$C_p(M,\Theta) = \mathcal{M}_p(\Theta) \oplus \mathcal{N}_p(\Theta) \oplus \oplus_{i=1}^{\ell} C_p(\Lambda_i,\Theta),$$

we see that it has the block diagonal form

$$\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \Delta_p^1 & & \\ & & & \ddots & \\ & & & & \Delta_p^\ell \end{pmatrix} \tag{4.1}$$

where

$$\Delta_p^i = \Delta_p(\infty) \mid_{C_p(\Lambda_i,\Theta)} .$$

We take a moment to consider the operator Δ_p^i. Extend the cell structure of Λ_i to a cell structure on $\bar{\Lambda}_i$ (this is not necessary if $\bar{\Lambda}_i$ is a subcomplex of M). Then there is a canonical isomorphism between $C_p(\bar{\Lambda}_i,\dot{\Lambda}_i,\Theta)$ and $C_p(\Lambda_i,\Theta)$, where $\dot{\Lambda}_i = \bar{\Lambda} - \Lambda$. Namely, if $c = \sum_{\alpha^p \subset \bar{\Lambda}_i} c_\alpha \alpha$ represents an element of $C_p(\bar{\Lambda}_i,\dot{\Lambda}_i,\Theta)$, then identify c with $\sum_{\alpha^p \subset \Lambda_i} c_\alpha \alpha$. With this identification the operator

$$\partial : C_p(\bar{\Lambda}_i,\dot{\Lambda},\Theta) \to C_{p-1}(\bar{\Lambda}_i,\dot{\Lambda},\Theta)$$

corresponds to the operator

$$\partial_\infty : C_p(\Lambda_i,\Theta) \to C_{p-1}(\Lambda_i,\Theta),$$

so that

$$\text{Ker } \Delta_p^i \cong H_p(\bar{\Lambda}_i \dot{\Lambda}_i,\Theta).$$

It follows that if, for each i, $H_*(\bar{\Lambda}_i,\dot{\Lambda}_i,\Theta) = 0$, then for each i and p

$$\det \Delta_p^i \neq 0.$$

Let

$$\zeta_{\Lambda_i}(z,V,\Theta) = \exp\left[\sum_{r=1}^{\infty}\frac{z^r}{r}\sum_{p=0}^{n}(-1)^p \sum_{\gamma\in\mathcal{P}_r^{(p)}(\Lambda_i)} m(\gamma)tr[\Theta(\alpha)]\right]$$

where $\mathcal{P}_r^{(k)}(\Lambda_i)$ denotes the closed paths of index p and length r consisting of cells in Λ_i. (Recall that by the definition of a basic set, each closed path is a subset of a single basic set.) Then $\zeta_{\Lambda_i}(z,V,\Theta)$ can be analytically continued to a meromorphic function on the entire complex plane. The following is Theorem 6.3 of [10].

Theorem 4.5. *Suppose $H_*(\bar{\Lambda}_i,\dot{\Lambda}_i,\Theta)=0$. Then $\zeta_{\Lambda_i}(z,\Theta)$ is analytic at $z=1$, and*

$$\begin{aligned} \mathit{Tor}\ (\bar{\Lambda}_i,\dot{\Lambda}_i,\Theta) &\equiv \prod_p(\ \mathit{Det}\ \Delta_p^i)^{\frac{p+1}{2}(-1)^p} \\ &= \zeta_{\Lambda_i}(1,V,\Theta). \end{aligned}$$

Let $\text{Det}\,'\Delta_p(\infty)$ denote the product of the non-zero eigenvalues of $\Delta_p(\infty)$. Then Theorem 4.5 has the following implication.

Corollary 4.6. *Suppose that $H_*(\bar{\Lambda},\dot{\Lambda},\Theta)=0$ for each non-trivial basic set Λ. Then $\zeta(z,V,\Theta)$ is analytic at $z=1$, and*

$$\begin{aligned} \prod_p(\ \mathit{Det}\,'\Delta_p(\infty))^{\frac{p+1}{2}(-1)^p} &= \prod_p[\prod_i\ \mathit{Det}\ \Delta_p^i]^{\frac{p+1}{2}(-1)^p} \\ &= \prod_i \zeta_{\Lambda_i}(1,V,\Theta) = \zeta(1,V,\Theta). \end{aligned}$$

It remains to understand the behavior of the eigenvalues of $\Delta_p(t)$ which tend to 0 as $t\to\infty$. From (4.1) we see that the number of such eigenvalues is

$$k\cdot\#\{\text{critical } p\text{-cells}\}.$$

Since the eigenvalues vary continuously with t, there are $T^*,\epsilon>0$ such that for all $t>T^*$

$$\#\{\text{eigenvalues of } \Delta_p(t)<\epsilon\} = k\cdot\#\{\text{critical } p\text{-cells}\}.$$

For $t>T^*$, define the Witten space

$$\mathcal{W}_p(t,V,\Theta)\subseteq C_p(M,\Theta)$$

to be the span of the eigenfunctions corresponding to eigenvalues less than ϵ.

The operator ∂_t commutes with $\Delta_p(t)$, so ∂_t preserves $\mathcal{W}_*(t, V, \Theta)$. Thus we can consider the complex

$$\mathcal{W}(t, V, \Theta) : 0 \to \mathcal{W}_n(t, V, \Theta) \xrightarrow{\partial_t} \mathcal{W}_{n-1}(t, V, \Theta) \xrightarrow{\partial_t} \cdots .$$

It follows that for all t

$$H_*(\mathcal{W}(t, V, \Theta)) \cong H_*(M, \Theta).$$

Since Θ is acyclic, $\mathcal{W}(t, V, \Theta)$ is exact, and we can consider, for each t,

$$\text{Tor } (\mathcal{W}(t, V, \Theta)) = \prod_p (\text{ Det } \Delta_p(t) \mid_{\mathcal{W}_p(t,V,\Theta)})^{\frac{p+1}{2}(-1)^p}.$$

Summarizing what we know so far, we see that for each t

$$\text{Tor } (M, \Theta) = e^{-\kappa t} T(t) = e^{-\kappa t} \text{ Tor } (\mathcal{W}(t, V, \Theta))\, \zeta(1, V, \Theta) + O(e^{-\epsilon t}). \tag{4.2}$$

From (4.1) we learn that

$$\lim_{t \to \infty} \mathcal{W}_p(t, V, \Theta) = \mathcal{M}_p(V, \Theta) \subseteq C_p(M, \Theta) \tag{4.3}$$

where $\mathcal{M}_p(V, \Theta)$ is the Morse space of p-chains supported on the rest points of V. Let

$$\pi(t) : C_p(M, \Theta) \to \mathcal{W}(t, V, \Theta)$$

be the orthogonal projection. For each critical p-cell α, and for each $i \in \{1, \ldots, k\}$, let $\omega_{\alpha,i}(t) = \pi(t) c_{\alpha,i}$. Then for t large enough $\{\omega_{\alpha,i}(t)\}$ forms a basis for $\mathcal{W}_p(t, V, \Theta)$.

Let $I(t)$ denote the square matrix with rows and columns indexed by pairs $\{\alpha, i\}$ with α a critical p-cell and $i \in \{1, \ldots, k\}$, and

$$\begin{aligned} I_{\{\alpha_1,i_1\}\{\alpha_2,i_2\}}(t) &= \langle \omega_{\alpha_1,i_1}(t), \omega_{\alpha_2,i_2(t)} \rangle \\ &\overset{t \to \infty}{\to} \delta_{\{\alpha_1,i_1\}\{\alpha_2,i_2\}} \end{aligned}$$

Let

$$\omega^*_{\alpha,i}(t) = I^{-\frac{1}{2}}(t) \omega_{\alpha,i}(t).$$

Then $\{\omega^*_{\alpha,i}(t)\}$ forms an orthonormal basis of $\mathcal{W}_p(t, V, \Theta)$.

The following is Theorem 4.1 of [9].

Theorem 4.7. *For any critical cells $\beta^{(p+1)}$ and $\alpha^{(p)}$, and any $i,j \in \{1,\dots,k\}$, as $t \to \infty$*

$$\langle \partial_t \omega^*_{\beta,i}, \omega^*_{\alpha,j}\rangle = e^{t(g(\alpha)-g(\beta))}(\langle \tilde{\partial} c_{\beta,i}, c_{\alpha,j}\rangle + 0(e^{-t\epsilon}))$$

for some $\epsilon > 0$.

Let

$$G : \mathcal{W}_p(t,V,\Theta) \to \mathcal{W}_p(t,V,\Theta)$$

denote the linear map which satisfies

$$G(\omega^*_{\alpha,i}) = g(\alpha)\omega^*_{\alpha,i}.$$

Let

$$\tilde{\partial}_t = e^{-tG}\partial_t e^{tG}$$

Then Theorem 4.7 implies that

$$\lim_{t\to\infty} \tilde{\partial}_t = \tilde{\partial} \tag{4.4}$$

Denote by $\tilde{\mathcal{W}}(t,V,\Theta)$ the complex

$$0 \to \mathcal{W}_n(t,V,\Theta) \stackrel{\tilde{\partial}}{\to}_t \mathcal{W}_{n-1}(t,V,\Theta) \stackrel{\tilde{\partial}}{\to}_t \cdots .$$

Of course,

$$H_*(\mathcal{W}(t,V,\Theta)) \cong H_*(\mathcal{W}(t,V,\Theta)) = 0$$

so we can consider the torsion of $\tilde{\mathcal{W}}$. From (4.4) and (4.3) we see that

$$\lim_{t\to\infty} \text{ Tor } (\tilde{\mathcal{W}}(t,V,\Theta)) = \text{ Tor } (\mathcal{M}(V,\Theta)).$$

It follows form Lemma 6.2 of [8] that

$$\text{Tor } (\tilde{\mathcal{W}}(t,V,\Theta)) = e^{\kappa t} \text{ Tor } (\mathcal{W}(t,V,\Theta)).$$

Substituting into (4.1) we learn

$$\text{Tor } (M,\Theta) = \text{ Tor } (\tilde{\mathcal{W}}(t,V,\Theta))\,\zeta(1,V,\Theta) + O(e^{-\epsilon t}).$$

Letting $t \to \infty$ yields

$$\text{Tor } (M,\Theta) = \text{ Tor } (\mathcal{M}(V,\Theta))\,\zeta(1,V,\Theta)$$

as desired.

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