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FREE DIFFERENTIAL CALCULUS. II

The isomorphism problem of groups

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1. Group presentations

Consider the free group X generated by $\mathbf{x} = (x_1, x_2, \cdots)$, and let $\mathbf{r} = (r_1, r_2, \cdots)$ be any set of elements of X. The smallest normal subgroup of X that contains all of the elements $r_1, r_2, \cdots$ is the intersection R of all the normal subgroups of X that contain these elements; R consists precisely of those elements of X that are of the form $\prod^{l}_{k=1} u_k r_{i_k}^{\varepsilon_k} u_k^{-1}$, where $u_k \in X$ and $\varepsilon_k = \pm 1$. The elements of R are *consequences* of $\mathbf{r}$, and R itself is *the consequence* of $\mathbf{r}$.

The set $\mathbf{x}$ of generators and the set $\mathbf{r}$ of elements of X determine uniquely the quotient group $G = X/R$. The associated homomorphism ϕ of X on G carries $\mathbf{x}$ into a set $\mathbf{x}^{\phi} = (x_1^{\phi}, x_2^{\phi}, \cdots)$ of generators of G. Conventionally the set $\mathbf{x}$ itself may be called a set of generators for G. The set $\mathbf{r}$ is called a set of *relators*,[1] and the equations $r_i = 1$ are called *defining relations* for G. This somewhat awkward terminology has been inherited from former times when elements of G were not clearly distinguished from elements of X. To abbreviate the terminology the whole situation may be summarized symbolically

$$G = (\mathbf{x} : \mathbf{r}).$$

The symbol on the right I call a *presentation* of G; it consists of a set of generators $\mathbf{x}$ and a set of *relators* $\mathbf{r}$. The name 'presentation' was chosen because the situation is, in a sense, dual to that of group representation.

That every group has a presentation is simply a restatement of a well-known fact; however it may be very difficult to decide whether two given presentations define the same group or not. This is the *isomorphism problem.*

The cardinal number of the set $\mathbf{x}$ is the *rank of the presentation* $(\mathbf{x} : \mathbf{r})$. The *rank of a group* G is the minimum of the ranks of its presentations, i.e., the rank of G is the smallest n such that G may be generated by n of its elements. It is well-known that the free group X generated by $(x_1, \cdots, x_n)$ is of rank n. A group (or a presentation) is said to be *finitely generated* if it is of finite rank. In this paper the group G under consideration is assumed to be finitely generated, and only its finitely generated presentations will be considered. It will be sufficiently obvious where this restriction could be removed if desired; the theory to be developed is more or less ineffective on groups that are not finitely generated. Since a finitely generated group is necessarily countable the set of relators may be assumed enumerable.

A presentation $(\mathbf{x} : \mathbf{r})$ is *finite* if both of the sets $\mathbf{x}$ and $\mathbf{r}$ are finite; a group is

[1] Terminology suggested by H. Freudenthal.

finitely presented if it has a finite presentation. A finitely generated group need not be finitely presented [28].

It is convenient to generalize slightly the concept of a presentation of a group. Consider the free group X generated by $\mathbf{x} = (x_1, x_2, \cdots)$ and the free group $A = (a_1, a_2, \cdots)$. The free product $X * A$ is just the free group generated by $(\mathbf{x}; \mathbf{a}) = (x_1, x_2, \cdots; a_1, a_2, \cdots)$. Let $\mathbf{r} = (r_1, r_2, \cdots)$ be any set of elements of $X * A$ and R the consequence of $\mathbf{r}$ in $X * A$. In this situation there is uniquely determined not only the group $G = X * A/R$ but also the subgroup $F = RA/R \approx A/R \cap A$ of G. The associated homomorphism ϕ of $X * A$ on G maps A on F. The generators $a_1, a_2, \cdots$ are called *distinguished generators.* I write

$$(G, F) = (\mathbf{x}; \mathbf{a}: \mathbf{r})$$

and call the symbol on the right a *presentation of the pair* (G, F). A presentation of the pair $(G, 1)$ consisting of G and the trivial subgroup 1 is nothing else than a presentation of G. The isomorphism problem of groups may be extended to the *isomorphism problem of pairs*: given presentations $(\mathbf{x}; \mathbf{a}: \mathbf{r})$ and $(\mathbf{y}; \mathbf{b}: \mathbf{s})$, does there exist an isomorphism of $X * A/R$ on $Y * B/S$ that maps RA/R on SB/S?

The cardinal number of the set $\mathbf{x}$ of non-distinguished generators is the *rank of the presentation*. The *rank of a pair* (G, F) is the minimum of the ranks of its presentations, i.e., the rank of (G, F) is the smallest n such that G may be generated by F and n other elements of G. It is a trivial exercise to show that every pair has a presentation and that, moreover, a pair consisting of a finitely generated group and a finitely generated subgroup has a finitely generated presentation, i.e., one in which the sets $\mathbf{x}$ and $\mathbf{a}$ are finite. From now on we assume that F as well as G is finitely generated.

A presentation may be altered in several ways without changing the isomorphism type of group, or pair, presented. The basic alterations are the Tietze transformations (I), (II) of first and second kind [26, 27], whose definitions I generalize here to presentations of pairs. In a Tietze transformation of first kind $(\mathrm{I}) = (\mathrm{I})^{+1}$ one adjoins to the set of relators $\mathbf{r}$ of a presentation $(\mathbf{x}; \mathbf{a}: \mathbf{r})$ any set of consequences of $\mathbf{r}$. The inverse operation $(\mathrm{I})^{-1}$ deletes from the set of relators $\mathbf{r}$ any set of relators that are consequences of the remaining set. In a Tietze transformation of second kind one either (II′) adjoins to the set $\mathbf{a}$ of distinguished generators a new generator b and simultaneously adjoins to the set of relators $\mathbf{r}$ a new relator s of the form $s = b \cdot f^{-1}$ with $f \in A$, or (II) adjoins to the set of non-distinguished generators a new generator y and simultaneously adjoins to the set of relators a new relator of the form $s = y \cdot f^{-1}$ with $f \in X * A$.

It is easily verified that the pair presented is unaltered in isomorphism type by the Tietze operations (I), (II), (II′); of course the inverse Tietze operations $(\mathrm{I})^{-1}$, $(\mathrm{II})^{-1}$, $(\mathrm{II}')^{-1}$, when applicable, also do not change the isomorphism type of pair presented. The basic fact about groups and their presentations is the well-known Tietze theorem, which I generalize slightly to a theorem about pairs and their presentations.

(1.1) Tietze theorem [26, 27]. *If presentations*

$$(x_1, \cdots, x_m; \quad a_1, \cdots, a_n: \quad r_1, r_2, \cdots) \quad \textit{and}$$

$$(y_1, \cdots, y_p; \quad b_1, \cdots, b_q: \quad s_1, s_2, \cdots)$$

define isomorphic pairs then it is possible to pass from one to the other by a finite sequence of Tietze transformations.

Proof. By hypothesis there are given homomorphisms ϕ of $X * A$ upon G and ψ of $Y * B$ upon G such that R is the kernel of ϕ and S is the kernel of ψ and $\phi(A) = \psi(B) = F$. Let $\xi_1, \cdots, \xi_m$ be elements of $Y * B$ and $\eta_1, \cdots, \eta_p$ elements of $X * A$ such that $x_i^\phi = \xi_i^\psi$ and $\eta_j^\phi = y_j^\psi$. Similarly let $\alpha_1, \cdots, \alpha_n$ be elements of B and $\beta_1, \cdots, \beta_q$ elements of A such that $a_i^\phi = \alpha_i^\psi$ and $\beta_j^\phi = b_j^\psi$. Determine homomorphisms (actually retractions) ρ and σ of $X * A * Y * B$ upon $X * A$ and $B * A$ respectively by defining

$$\begin{array}{ll} x_i^\rho = x_i & x_i^\sigma = \xi_i \\ a_i^\rho = a_i & a_i^\sigma = \alpha_i \\ y_j^\rho = \eta_j & y_j^\sigma = y_j \\ b_j^\rho = \beta_j & b_j^\sigma = b_j . \end{array}$$

It is obvious that $\phi\rho = \psi\sigma$. The kernel of ρ is the consequence of $\mathbf{y}\eta^{-1}$ and $\mathbf{b}\beta^{-1}$; it follows that the kernel T of $\phi\rho$ is the consequence of $\mathbf{y}\eta^{-1}$, $\mathbf{b}\beta^{-1}$ and $\mathbf{r}$. Similarly T, as kernel of $\psi\sigma$, is found to be the consequence of $\mathbf{x}\xi^{-1}$, $\mathbf{a}\alpha^{-1}$ and $\mathbf{s}$. Thus

$$(G, F) = (\mathbf{x}, \mathbf{y}; \mathbf{a}, \mathbf{b}: \mathbf{x}\xi^{-1}, \mathbf{y}\eta^{-1}, \mathbf{a}\alpha^{-1}, \mathbf{b}\beta^{-1}, \mathbf{r}, \mathbf{s}).$$

But

$$\begin{array}{lr} (\mathbf{x}; \mathbf{a}: \mathbf{r}) \to (\mathbf{x}, \mathbf{y}; \mathbf{a}, \mathbf{b}: \mathbf{y}\eta^{-1}, \mathbf{b}\beta^{-1}, \mathbf{r}, \mathbf{s}) & \text{by } (\mathrm{II})^p(\mathrm{II}')^q, \\ \to (\mathbf{x}, \mathbf{y}; \mathbf{a}, \mathbf{b}: \mathbf{x}\xi^{-1}, \mathbf{y}\eta^{-1}, \mathbf{a}\alpha^{-1}, \mathbf{b}\beta^{-1}, \mathbf{r}, \mathbf{s}) & \text{by (I)}, \\ \to (\mathbf{x}, \mathbf{y}; \mathbf{a}, \mathbf{b}: \mathbf{x}\xi^{-1}, \mathbf{a}\alpha^{-1}, \mathbf{s}) & \text{by } (\mathrm{I})^{-1}, \\ \to (\mathbf{y}; \mathbf{b}: \mathbf{s}) & \text{by } (\mathrm{II})^{-m}(\mathrm{II}')^{-n}. \end{array}$$

The importance of this theorem is that it reduces the problem of showing that a given function of presentations is an invariant of the isomorphism type (of group or pair) presented to checking that it is unaltered by the Tietze transformations.

2. Jacobians

Let $(\mathbf{x}; \mathbf{a}: \mathbf{r})$ be a presentation of a pair (G, F) and let ϕ be the associated homomorphism of X upon G. For each non-distinguished generator x_j and relator r_i the ϕ-image $(\partial r_i/\partial x_j)^\phi$ of $(\partial r_i/\partial x_j)$ is an element of the group ring JG of G. The matrix $(\partial \mathbf{r}/\partial \mathbf{x})^\phi = (\partial(r_1, r_2, \cdots)/\partial(x_1, x_2, \cdots, x_m))^\phi =$

$\| (\partial r_i/\partial x_j)^\phi \|$ of elements of JG, whose rows $(\partial r_i/\partial \mathbf{x})^\phi$ correspond to the relators r_i, and whose columns $(\partial \mathbf{r}/\partial x_j)^\phi$ correspond to the non-distinguished generators x_j, is called *the Jacobian matrix of the presentation* [13].

The Jacobians of the various presentations of (G, F) are called *the Jacobians of* (G, F). In particular the Jacobians of $(G, 1)$ are called *the Jacobians of* G. Let us examine the effect of the Tietze transformations on the Jacobian of a presentation.

(I): If a new relator s is adjoined to the set of relators $\mathbf{r}$, the Jacobian matrix acquires a new row $(\partial s/\partial x_1, \cdots, \partial s/\partial x_n)^\phi$. *If s is a consequence of* $\mathbf{r}$ *this new row is a left-linear combination of the rows* $(\partial r_i/\partial x_1, \cdots, \partial r_i/\partial x_n)^\phi$.

PROOF. A consequence s of $\mathbf{r}$ is an element of X of the form $s = \prod_{k=1}^{l} u_k r_{i_k}^{\varepsilon_k} u_k^{-1}$, where $u_k \in X$ and $\varepsilon_k = \pm 1$. I claim that $(\partial s/\partial x_j)^\phi = \sum_{k=1}^{l} \varepsilon_k u_k^\phi (\partial r_{i_k}/\partial x_j)^\phi$; this calculation may be made directly, but it is enlightening to do it piecewise as follows.

(1) If $r_1, r_2 \in R$ then $(\partial(r_1 r_2)/(\partial x_j)^\phi = (\partial r_1/\partial x_j)^\phi + r_1^\phi(\partial r_2/\partial x_j)^\phi = (\partial r_1/\partial x_j)^\phi + (\partial r_2/\partial x_j)^\phi$;

(2) If $r_1 \in R$ then $(\partial r_1^{-1}/\partial x_j)^\phi = -r_1^{-\phi}(\partial r_1/\partial x_j)^\phi = -(\partial r_1/\partial x_j)^\phi$;

(3) If $r_1 \in R$ and $u \in X$ then $(\partial(u r_1 u^{-1})/\partial x_j)^\phi = (1 - u r_1 u^{-1})^\phi(\partial u/\partial x_j)^\phi + u^\phi(\partial r_1/\partial x_j)^\phi = u^\phi(\partial r_1/\partial x_j)^\phi$.

This shows, more explicitly, that

(1) *when two relators are multiplied together the corresponding rows are added together;*

(2) *when a relator is replaced by its inverse the corresponding row changes sign;*

(3) *when a relator is transformed the corresponding row is multiplied on the left by a group element.*

(II): *If a new generator y is adjoined to the non-distinguished generators $x_1, \cdots, x_m$ and a new relator $y \cdot f^{-1}$, where $f \in X * A$, is simultaneously adjoined to the relators $r_1, r_2, \cdots$ the Jacobian matrix acquires a new row and a new column. The entry in the intersection of the new row and column is* $(\partial(yf^{-1})/\partial y)^\phi = 1$, *the other elements of the new column are* $(\partial r_i/\partial y)^\phi = 0$. The other elements of the new row are $(\partial(yf^{-1})/\partial x_j)^\phi = -(\partial f/\partial x_j)^\phi$; the fact that these elements are not quite arbitrary might conceivably be useful but I know of no way to make use of it.

(II′): *If a new distinguished generator b and a new relator $b \cdot f^{-1}$, where $f \in A$, are simultaneously adjoined, the Jacobian matrix acquires a new row whose entries are* $(\partial(bf^{-1})/\partial x_j)^\phi = 0$.

With the above calculations in mind we define two matrices over a ring to be *equivalent* if one can be obtained from the other by a finite number of *elementary transformations* (0), (I), (II), $(\mathrm{I})^{-1}$, $(\mathrm{II})^{-1}$, where these are defined as follows:

(0) Permute the rows in any way or permute the columns in any way;

(I) Adjoin to the matrix $\mathbf{A} = \| a_j^i \|$ any (countable) number of rows, each new row being a left-linear combination of the rows of $\mathbf{A}$;

(II) Adjoin to the matrix $\mathbf{A}$ a new row and a new column such that the entry in the intersection of the new row and column is 1 and the remaining entries in the new column are all 0; $\mathbf{A} \to \begin{Vmatrix} \mathbf{A} & 0 \\ * & 1 \end{Vmatrix}$.

Clearly the Tietze operations (I) and (II) induce in the Jacobian matrix the elementary transformations of the same designation. The Tietze operation (II′) induces a very special kind of elementary transformation (I), that is designated below by (I_0). Thus

(2.1) THEOREM. *The Jacobian matrices of the finitely generated presentations of a finitely generated group G all belong to a single equivalence class over JG. More generally, the Jacobian matrices of the finitely generated presentations of a pair (G, F) consisting of a finitely generated group G and a finitely generated subgroup F, all belong to a single equivalence class over JG.*

Among the Tietze transformations of first kind a special role is played by the transformation (I_0) that adjoins to the set of relations a number of *empty relations* $1 = 1$. The corresponding elementary transformation is (I_0): Adjoin to the matrix A a number of 0-rows $(0, \cdots, 0)$. [In order to avoid a certain type of mistake at a later stage of the development I have found it advisable to adjoin mentally a sufficient number of 0-rows to every Jacobian matrix.]

To adjoin a 0-column is *not* an elementary transformation; it would correspond to adjoining a new free generator (which would obviously change the isomorphism type presented). Thus the roles of row and column are not completely interchangeable; nevertheless, as will be shown below, they are *almost* interchangeable.

By compounding several Tietze transformations of the first kind one may obtain the transformation (III) that multiplies each relator of a given subset of $\mathbf{r}$ by an appropriate consequence of the relators in the complementary set. The corresponding elementary transformation is

(III) Add to each row of a given subset of the rows an appropriate left-linear combination of the rows in the complementary set; $\begin{Vmatrix} \mathbf{A} \\ \mathbf{B} \end{Vmatrix} \to \begin{Vmatrix} \mathbf{A} \\ \mathbf{B} + \mathbf{PA} \end{Vmatrix}$, where $\mathbf{P}$ is an arbitrary matrix of the proper size.

Surprisingly, the "analogous" transformation of columns is also an elementary transformation.

(III*) Add to a column a right-linear combination of other columns. This is done as follows:

$$\begin{Vmatrix} a_1^1 a_2^1 \cdots a_n^1 \\ a_1^2 a_2^2 \cdots a_n^2 \\ \cdot \;\; \cdot \;\; \cdots \;\; \cdot \end{Vmatrix} \quad \to \quad \left\Vert \begin{array}{cccc|ccc} a_1^1 & a_2^1 & \cdots & a_n^1 & 0 & \cdots & 0 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \hline c_2 & -1 & & 0 & 1 & & 0 \\ \vdots & & \ddots & & & \ddots & \\ c_n & 0 & & -1 & 0 & & 1 \end{array} \right\Vert \qquad \text{by (II)}$$

$$\to \left\|\begin{array}{cccc|ccc} a_1^1+\sum_{j=2}^{n} a_j^1 c_j & 0 & \cdots & 0 & a_2^1 & \cdots & a_n^1 \\ a_1^2+\sum_{j=2}^{n} a_j^2 c_j & 0 & \cdots & 0 & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \hline c_2 & -1 & & 0 & 1 & & 0 \\ \vdots & & \ddots & & & \ddots & \\ c_n & 0 & & -1 & 0 & & 1 \end{array}\right\| \text{ by (III)}$$

$$\to \left\|\begin{array}{cccc} a_1^1+\sum_{j=2}^{n} a_j^1 c_j & a_2^1 & \cdots & a_n^1 \\ a_1^2+\sum_{j=2}^{n} a_j^2 c_j & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \end{array}\right\| \text{ by (0) and (II).}$$

[(III*) need not correspond to any Tietze transformation of group presentations because the elementary transformations (II) used might not correspond to Tietze transformations (II).]

In the presence of (III*) we see that (II) can be replaced by the special case

$$(\mathrm{II}_0) \qquad \mathbf{A} \to \left\|\begin{array}{cc} \mathbf{A} & 0 \\ 0 & 1 \end{array}\right\|.$$

Thus the definition of equivalence may be given by the more nearly "symmetric" set of elementary transformations (0), (I_0), (II_0), (III), (III*) in place of the "unsymmetric" set (0), (I), (II).

In practice a useful elementary transformation is

(IV) Left-multiply a row by a unit e of the ring. [If $e = -1$ this corresponds to replacing a relator by its inverse; if e is a group element it corresponds to replacing a relator by one of its conjugates.]

In general it may be done as follows:

$$\left\|\begin{array}{ccc} \cdot & \cdots & \cdot \\ a_1^i & \cdots & a_n^i \end{array}\right\| \to \left\|\begin{array}{ccc} \cdot & \cdots & \cdot \\ a_1^i & \cdots & a_n^i \\ ea_1^i & \cdots & ea_n^i \end{array}\right\| \text{ by (I)}$$

$$\to \left\|\begin{array}{ccc} \cdot & \cdots & \cdot \\ 0 & \cdots & 0 \\ ea_1^i & \cdots & ea_n^i \end{array}\right\| \text{ by (III)}$$

$$\to \left\|\begin{array}{ccc} \cdot & \cdots & \cdot \\ ea_1^i & \cdots & ea_n^i \end{array}\right\| \text{ by (I)}^{-1}.$$

The "analogous" transformation of columns is also an elementary transformation [although it is harder to prove because there need not be any convenient 0-columns.]

(IV*) Right-multiply any column by a unit e of the ring

$$
\begin{aligned}
\left\| \begin{array}{cccc} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \end{array} \right\|
&\to \left\| \begin{array}{cccc|c} a_1^1 & a_2^1 & \cdots & a_n^1 & 0 \\ a_1^2 & a_2^2 & \cdots & a_n^2 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right\| \text{ by } (\mathrm{II}_0) \\
&\to \left\| \begin{array}{cccc|c} a_1^1 & a_2^1 & \cdots & a_n^1 & a_1^1 e \\ a_1^2 & a_2^2 & \cdots & a_n^2 & a_1^2 e \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right\| \text{ by } (\mathrm{III}^*) \\
&\to \left\| \begin{array}{c|ccc|c} 0 & a_2^1 & \cdots & a_n^1 & a_1^1 e \\ 0 & a_2^2 & \cdots & a_n^2 & a_1^2 e \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \hline -e^{-1} & 0 & \cdots & 0 & 1 \end{array} \right\| \text{ by } (\mathrm{III}^*) \\
&\to \left\| \begin{array}{c|cccc} 0 & a_2^1 & \cdots & a_n^1 & a_1^1 e \\ 0 & a_2^2 & \cdots & a_n^2 & a_1^2 e \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \hline 1 & 0 & \cdots & 0 & -e \end{array} \right\| \text{ by } (\mathrm{IV}) \\
&\to \left\| \begin{array}{cccc} a_1^1 e & a_2^1 & \cdots & a_n^1 \\ a_1^2 e & a_2^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \end{array} \right\| \text{ by } (\mathrm{II})^{-1} \text{ and } (0).
\end{aligned}
$$

It may be convenient, in practice, to deal directly with the group ring JG. Instead of group-homomorphism ϕ of X on G with kernel R associated with a presentation $(\mathbf{x}\colon \mathbf{r})$ we consider the ring-homomorphism of JX on JG. Its kernel is the ideal $\mathfrak{R}$ generated by the elements $r_i - 1$. Thus we are led to consider a presentation $(\mathbf{x}\colon '\mathbf{r})$ of a ring JG where $'r_i = r_i - 1$ (or more generally $(\mathbf{x}\colon \mathbf{q})$ where q_i is an element of the fundamental ideal $\mathfrak{X}$) and $\mathfrak{R}$ is the *consequence* of $'\mathbf{r}$, i.e. the ideal generated by the elements $'r_i$. It is easily verified that Tietze's theorem holds for ring-presentations with the following modifications: *the consequences* of $'\mathbf{r}$, i.e. the elements of $\mathfrak{R}$, are the elements of the form $\sum_{k=1}^{l} a_k \, 'r_{i_k} b_k$, where a_k, $b_k \in JX$; the new ring-relator in Tietze (II) is $y - f$ where y is the new generator and $f \in X$ (or more generally $f \in JX$ such that

$f^{\circ} = 1$). Since $'r_i^{\phi} = 0$, it is especially easy to verify that Tietze (I) in its ring form adjoins new rows that are left-linear combinations of the old rows. For $(\partial \sum a\,'rb/\partial x_j)^{\phi} = \sum\{(\partial a/\partial x_j)^{\phi}\,'r^0 b^0 + a^{\phi}(\partial\,'r/\partial x_j)^{\phi} b^0 + a^{\phi}\,'r^{\phi}(\partial b/\partial x_j)^{\phi}\} = \sum a^{\phi}(\partial' r/\partial x_j)^{\phi} b^0$.

3. Homomorphs of the Jacobians

A homomorphism[2] ψ of the ring JG maps the Jacobians $(\partial \mathbf{r}/\partial \mathbf{x})^{\phi}$ of (G, F) into matrices $(\partial \mathbf{r}/\partial \mathbf{x})^{\psi\phi}$ whose entries belong to the ring $(JG)^{\psi}$. I call $(\partial \mathbf{r}/\partial \mathbf{x})^{\psi\phi}$ *the Jacobian of* $(\mathbf{x}; \mathbf{a}: \mathbf{r})$ *at* ψ, *or a Jacobian of* (G, F) *at* ψ. Clearly

(3.1) *The Jacobians of* (G, F) *at* ψ *belong to a single equivalence class over the ring* $(JG)^{\psi}$.

Suppose that for every group G of a certain type there is assigned a homomorphism $\psi = \psi_G$ of the group ring JG; the group rings JG of these groups are then said to have a *generic homomorphism* ψ. Let $G^{(1)} = (\mathbf{x}: \mathbf{r})$ and $G^{(2)} = (\mathbf{y}: \mathbf{s})$ be groups of the type considered. Then

(3.2) *In order that* $G^{(1)} \approx G^{(2)}$ *it is necessary that* $(JG^{(1)})^{\psi_1} \approx (JG^{(2)})^{\psi_2}$. *If this condition is satisfied it is then further necessary that the matrices* $(\partial \mathbf{r}/\partial \mathbf{x})^{\theta\psi_1\phi_1}$ *and* $(\partial s/\partial y)^{\psi_2\phi_2}$ *should be equivalent over* $(JG^{(2)})^{\psi_2}$ *for some isomorphism* θ *of* $(JG^{(1)})^{\psi_1}$ *upon* $(JG^{(2)})^{\psi_2}$.

In applications to knot theory the following sharper statement is required:

(3.3) *Suppose that the group rings* JG *of the group* G *of a certain type have a generic homomorphism* ψ *into a given ring* $(JG)^{\psi}$. *Then two groups,* $G^{(1)} = (\mathbf{x}: \mathbf{r})$ *and* $G^{(2)} = (\mathbf{y}: \mathbf{s})$, *of this type can be isomorphic only if the matrices* $(\partial \mathbf{r}/\partial \mathbf{x})^{\psi_1\phi_1}$ *and* $(\partial \mathbf{s}/\partial \mathbf{y})^{\psi_2\phi_2}$ *are equivalent over* $(JG)^{\psi}$.

A simple example of a generic homomorphism is the endomorphism o: $JG \to J$. Another generic homomorphism is the abelianizing homomorphism ψ: $JG \to JH$, where H denotes the commutator quotient group G/G_2. These and various intermediate possibilities have the practical advantage that the image rings are commutative. In a later part of this paper representations of JG by matrices over a ring will be considered; generally speaking, these will not be generic.

Of all the choices for ψ certainly the least prepossessing is the endomorphism o. It is therefore very auspicious that

(3.4) *The commutator quotient group* $H = G/G_2$ *is determined by the Jacobian class of* G *at* o.

This follows from the noteworthy fact that

(3.5) *The Jacobians of* G *at* o *are relation matrices for* H, *which follows immediately from the observation* [FDCI §2] *that* $(\partial r_i/\partial x_j)^{\circ}$ *is the exponent sum of* x_j *in* r_i. *Thus*

(3.6) *The torsion numbers of* H *are the invariant factors of* $(\partial \mathbf{r}/\partial \mathbf{x})^{\circ}$ *and the betti number of* H *is the nullity* (the number of columns minus the rank) *of* $(\partial \mathbf{r}/\partial \mathbf{x})^{\circ}$.

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[2] Usually ψ will be the extension to JG of a group-homomorphism ψ of G (so that $(JG)^{\psi}$ would be the group ring of G^{ψ}). However this need not be the case. For instance a homomorphism of JG into the ring of integers of an algebraic number field may be useful.

Similarly it may be shown that the Jacobians of (G, F) at o are relation matrices for G divided by its smallest normal subgroup that contains G_2 and F. (Notice that $(\partial \mathbf{r}/\partial \mathbf{x})^o$ is unaltered if we set equal to 1 the distinguished generators $a_1, a_2, \cdots$).

The above indicates that the Jacobian class of G at a homomorphism ψ contains information about the structure of G that is destroyed by ψ itself. Roughly speaking, the Jacobian class of G at ψ determines the structure of G modulo the commutator subgroup of the kernel of ψ; the exact statement may be found in [10].

4. The Alexander matrices

Consider a matrix $\mathbf{A}$ over an arbitrary commutative ring and an arbitrary non-negative integer d. The ideal generated by the minor determinants of $\mathbf{A}$ of order $n - d$, where n is the number of columns, is called the d^{th} *elementary ideal* $\mathfrak{E}_d(\mathbf{A})$ of $\mathbf{A}$. It is to be understood that $\mathfrak{E}_d(\mathbf{A}) = (1)$ for $d \geqq n$, and that $\mathfrak{E}_d(\mathbf{A}) = (0)$ if $\mathbf{A}$ has fewer than $n - d$ rows. Clearly $\mathfrak{E}_d(\mathbf{A}) \subset \mathfrak{E}_{d+1}(\mathbf{A})$; thus to each matrix $\mathbf{A}$ there is associated its *chain of elementary ideals*

$$\mathfrak{E}_0(\mathbf{A}) \subset \mathfrak{E}_1(\mathbf{A}) \subset \cdots .$$

By the *length* of this chain is meant the smallest integer d for which $\mathfrak{E}_d(\mathbf{A}) = (1)$. The smallest integer d for which $\mathfrak{E}_d(\mathbf{A}) = (0)$ is the *nullity* of $\mathbf{A}$.

(4.1) *Equivalent matrices have the same chain of elementary ideals.*

This is most easily checked by showing that $\mathfrak{E}_d$ is unaltered by the elementary transformations (I_0), (III), (III*) and (II_0). For (I_0), (III) and (III*) this is immediate. (Note that the fact that $\mathfrak{E}_d = (0)$ if $\mathbf{A}$ has fewer than $n - d$ rows enters into the consideration of (I_0).) The elementary transformation (II_0) replaces $\mathbf{A}$ by $\begin{Vmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{Vmatrix}$. Clearly every minor determinant of order $n - d$ of the matrix $\mathbf{A}$ appears as a minor determinant of order $n + 1 - d$ of the matrix $\begin{Vmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{Vmatrix}$, and conversely every minor determinant of $\begin{Vmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{Vmatrix}$ of order $n + 1 - d$ is a linear combination of minor determinants of $\mathbf{A}$ of order $n - d$. Thus $\mathfrak{E}_d\left(\begin{Vmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{Vmatrix}\right) = \mathfrak{E}_d(\mathbf{A})$ for every $d < n$. Furthermore

$$\mathfrak{E}_n\left(\begin{Vmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{Vmatrix}\right) = (\mathfrak{E}_{n-1}(\mathbf{A}), 1) = (1) = \mathfrak{E}_n(\mathbf{A}).$$

The theory of the Jacobians at a homomorphism into a commutative ring is dominated by the theory of the Jacobians at the abelianizing homomorphism $\psi: JG \to JH$. I call a Jacobian matrix at ψ an *Alexander matrix*.[3] By the d^{th} *elementary ideal* of (G, F) will be meant the d^{th} elementary ideal of an Alexander matrix of (G, F). It follows from (3.1) and (4.1) that this ideal of the ring JH

[3] Such matrices generalize the matrices introduced by Alexander in [1].

does not depend on the presentation of (G, F) used; it will be denoted by $\mathfrak{E}_d(G, F)$. By the d^{th} *elementary ideal* $\mathfrak{E}_d(G)$ *of* G will be meant $\mathfrak{E}_d(G, 1)$. The elementary ideals of (G, F) form *the chain of elementary ideals of* (G, F)

$$\mathfrak{E}_0(G, F) \subset \mathfrak{E}_1(G, F) \subset \cdots .$$

From (3.2) it follows that

(4.2) $G^{(1)} \approx G^{(2)}$ *only if* $H^{(1)} \approx H^{(2)}$ *and some isomorphism of* $H^{(1)}$ *upon* $H^{(2)}$ *transforms the chain of elementary ideals of* $G^{(1)}$ *into the chain of elementary ideals of* $G^{(2)}$.

Let us consider now two subgroups F and E of G, where $E \subset F$. Let the rank of (F, E) be c and consider a presentation $(x_1, \cdots, x_n; a_1, \cdots, a_m : \mathbf{r})$ of (G, E) which is such that $(x_1, \cdots, x_{n-c}; x_{n-c+1}, \cdots, x_n, a_1, \cdots, a_m : \mathbf{r})$ is a presentation of (G, F). (It is easily seen that such a one always exists.) Consider a minor determinant of order $n - d$ of the Alexander matrix $(\partial \mathbf{r}/\partial(x_1, \cdots, x_n))^{\psi\phi}$ of the given presentation of (G, E). Its Laplace expansion according to those of the columns $(\partial \mathbf{r}/\partial x_{n-c+1})^{\psi\phi}, \cdots, (\partial \mathbf{r}/\partial x_n)^{\psi\phi}$ that are present shows that it belongs to one of the ideals $\mathfrak{E}_{d-e}(G, F)$ for some $e = 0, 1, \cdots, c$, hence to the ideal $\mathfrak{E}_d(G, F)$. Thus $\mathfrak{E}_d(G, E) \subset \mathfrak{E}_d(G, F)$. On the other hand a minor determinant of order $(n - c) - d$ of the Alexander matrix $(\partial \mathbf{r}/\partial(x_1, \cdots, x_{n-c}))^{\psi\phi}$ of the given presentation of (G, F) is a minor determinant of order $n - (c + d)$ of the Alexander matrix $(\partial \mathbf{r}/\partial(x_1, \cdots, x_n))^{\psi\phi}$ of (G, E). Thus $\mathfrak{E}_d(G, F) \subset \mathfrak{E}_{d+c}(G, E)$. Summarizing:

(4.3) *If* $E \subset F \subset G$ *then* $\mathfrak{E}_d(G, E) \subset \mathfrak{E}_d(G, F) \subset \mathfrak{E}_{d+c}(G, E)$, *where* c *is the rank of* (F, E).

In order to compare the elementary ideals of G and a homomorph G/N of G it must be assumed that $(G/N)/(G/N)_2 \approx G/G_2$, i.e. that $N \supset G_2$. It is also only reasonable to assume $F \supset N$. If $(G, F) = (\mathbf{x}; \mathbf{a}: \mathbf{r})$ then $(G/N, F/F \frown N)$ has a presentation $(\mathbf{x}; \mathbf{a}: \mathbf{r}, \mathbf{s})$, where each s_i belongs to A.

Then $\partial(\mathbf{r}, \mathbf{s})/\partial \mathbf{x} = \left\| \begin{matrix} \partial \mathbf{r}/\partial \mathbf{x} \\ 0 \end{matrix} \right\|$, so that, for each d, $\mathfrak{E}_d(G/N, F/F \frown N) = \mathfrak{E}_d(G, F)$. Thus we get

(4.4) $\mathfrak{E}_d(G/N, F/F \frown N) = \mathfrak{E}_d(G, F)$, *whenever* N *is a normal subgroup of* G *such that* $G_2 \subset N \subset F$. *In particular,* $\mathfrak{E}_d(G/N) = \mathfrak{E}_d(G, N)$ *if* $G_2 \subset N$.

If, in an Alexander matrix of (G, E), c columns are deleted, the result is an Alexander matrix of a pair (G, F) where $F \supset E$ and is such that the rank of (F, E) is $\leqq c$. Thus $\mathfrak{E}_{d+c}(G, E) = \sum_F \mathfrak{E}_d(G, F)$, where the summation is extended over certain subgroups F for which the rank of (F, E) is $\leqq c$. On the other hand if F is any such subgroup there can be found a presentation of (G, E) such that an Alexander matrix for (G, F) is obtained from the Alexander matrix of this presentation of (G, E) by deleting c properly chosen columns. Hence

(4.5) $\mathfrak{E}_{d+c}(G, E) = \sum \mathfrak{E}_d(G, F)$, *summed over those subgroups* F *that are of rank* c *over* E. *In particular* $\mathfrak{E}_d(G) = \sum \mathfrak{E}_0(G, F)$, *summed over the subgroups* F *of rank* d.

Now let us consider the free product $G = G^{(1)} * G^{(2)}$ of groups $G^{(1)}$ and $G^{(2)}$. The commutator quotient group of $G^{(1)} * G^{(2)}$ is the direct product $H = H^{(1)} \cdot H^{(2)}$ of the commutator quotient groups $H^{(1)}$ and $H^{(2)}$ of $G^{(1)}$ and $G^{(2)}$. Thus the rings $JH^{(1)}$ and $JH^{(2)}$ are imbedded in JH in a natural way, so that Alexander matrices of $G^{(1)}$, $G^{(2)}$ and G can be compared. If $G^{(1)} = (\mathbf{x}; \mathbf{r})$ and $G^{(2)} = (\mathbf{y}: \mathbf{s})$ then $G = (\mathbf{x}, \mathbf{y}: \mathbf{r}, \mathbf{s})$; hence an Alexander matrix of G is $\left\| \begin{matrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{matrix} \right\|$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are Alexander matrices of $G^{(1)}$ and $G^{(2)}$. Hence

$$(4.6) \qquad \mathfrak{E}_d(G^{(1)} * G^{(2)}) = \textstyle\sum_{d_1+d_2=d} \mathfrak{E}_{d_1}(G^{(1)}) \cdot \mathfrak{E}_{d_2}(G^{(2)}).$$

Since the free group of rank n is the free product of n infinite cyclic groups it follows that

(4.7) *The d^{th} elementary ideal of the free group of rank n is*

$$\begin{aligned} \mathfrak{E}_d &= (0) \textit{ if } d < n, \\ &= (1) \textit{ if } d \geqq n. \end{aligned}$$

Next we consider abelian groups.

(4.8) *The d^{th} elementary ideal of the free abelian group H of rank $n \geqq 1$ is*

$$\begin{aligned} \mathfrak{E}_d &= (0) \textit{ if } d = 0, \\ &= \mathfrak{H}^{n-d} \textit{ if } 1 \leqq d \leqq n-1, \\ &= (1) \textit{ if } d \geqq n, \end{aligned}$$

where $\mathfrak{H}$ denotes the fundamental ideal of JH.

PROOF. That $\mathfrak{E}_0 = (0)$ is proved in §5 below. That $\mathfrak{E}_n = (1)$ follows from (4.7), (4.4) and (4.3). This proves (4.8) for $n = 1$. If $n > 1$ the group $H = H^{(n)}$ has the presentation $(x_1, \cdots, x_n : [x_1, x_2], \cdots, [x_{n-2}, x_n], [x_{n-1}, x_n])$. Denote its Alexander matrix by $\mathbf{A}_n$. We have

$$\mathbf{A}_n = \left\| \begin{array}{ccc|c} & \mathbf{A}_{n-1} & & 0 \\ \hline 1 - x_n & & & x_1 - 1 \\ & \ddots & & \vdots \\ & & 1 - x_n & x_{n-1} - 1 \end{array} \right\|.$$

Since each entry of $\mathbf{A}_n$ is an element of $\mathfrak{H}$ we have $\mathfrak{E}_d(\mathbf{A}_n) \subset \mathfrak{H}^{n-d}$. On the other hand it may be seen that if $d \geqq 1$ any minor determinant D of $\mathbf{A}_{n-1}$ of order $(n-1) - d$ may be enlarged to a minor determinant of $\mathbf{A}_n$ of order $n - d$ whose value is $(x_j - 1)D$ where j is any of the indices $1, \cdots, n$. Thus $\mathfrak{E}_d(\mathbf{A}_n) \supset (x_1 - 1, \cdots, x_n - 1)\mathfrak{E}_d(\mathbf{A}_{n-1})$. Repetition of this argument shows that $\mathfrak{E}_d(\mathbf{A}_n) \supset (x_1 - 1, \cdots, x_n - 1) \cdot (x_1 - 1, \cdots, x_{n-1} - 1) \cdots (x_1 - 1, x_2 - 1, \cdots, x_{d+1} - 1)$. But the same formula must hold with the indices $(1, \cdots, n)$ permuted in any way. Hence $\mathfrak{E}_d(\mathbf{A}_n) \supset \mathfrak{H}^{n-d}$.

In the same way it may be shown that

(4.9) *The d^{th} elementary ideal of the direct product $H = H^{(1)} \times H^{(2)} \times \cdots \times H^{(n)}$ of cyclic groups of respective orders $p_1, p_2, \cdots, p_n$, where $p_k \geqq 0$, is*[4]

$$\begin{aligned} \mathfrak{E}_d &= (\sigma_1\sigma_2 \cdots \sigma_n) && \text{if } d = 0, \\ &= \textstyle\sum_{k_1<k_2<\cdots<k_{n-d}} \mathfrak{H}_{k_1}\cdot\mathfrak{H}_{k_2} \cdots \mathfrak{H}_{k_{n-d}} && \text{if } 1 \leqq d \leqq n-1, \\ &= (1) && \text{if } d \geqq n, \end{aligned}$$

where σ_k denotes the sum of the elements of the group $H^{(k)}$ if $p_k > 0$, and $\sigma_k = 0$ if $p_k = 0$, and $\mathfrak{H}_k$ denotes the ideal (H, σ_k) consisting of those elements u for which $u^0 \equiv 0 \pmod{p_k}$.

5. The order ideal

The 0^{th} elementary ideal is called the *order ideal.*

(5.1) *The order ideal of a group G depends only on its commutator quotient group H. In fact $\mathfrak{E}_0(G) = (\sigma)$, where*

$$\begin{aligned} \sigma &= \textstyle\sum_{h\in H} h \text{ if } H \text{ is finite} \\ &= 0 \text{ if } H \text{ is infinite.} \end{aligned}$$

Proof. Since H is a finitely generated abelian group, $H = H^{(1)} \times H^{(2)} \times \cdots \times H^{(\mu)}$ where $H^{(j)} = (t_j : t_j^{p_j})$ and $p_j \geqq 0$. Choose a presentation $(\mathbf{x}: \mathbf{r})$ of G such that $x_j^{\psi\phi} = t_j$ for $j = 1, \cdots, \mu$ and $x_j^{\psi\phi} = 1$ for $j = \mu + 1, \cdots, n$ (cf. [1]). Let

$$\begin{aligned} \sigma_j &= \frac{t_j^{p_j} - 1}{t_j - 1} = 1 + t_j + \cdots + t_j^{p_j - 1} && \text{if } p_j > 0, \\ &= 0 && \text{if } p_j = 0. \end{aligned}$$

It is easily verified by direct calculation that, $u(t_j - 1) = 0$ for every j if and only if $u \equiv 0 \pmod{\sigma_1 \cdots \sigma_n}$. By the fundamental formula [FDCI (2.3)], $(\partial\mathbf{r}/\partial x_j)^{\psi\phi}(x_j^{\psi\phi} - 1) = 0$. Hence, for any subset $\mathbf{r}' = (r_{i_1}, \cdots, r_n)$ of the relators, and any index $k = 1, \cdots, n$, we find

$$\begin{aligned} \det\left(\frac{\partial\mathbf{r}'}{\partial\mathbf{x}}\right)^{\psi\phi}\cdot(x_k^{\psi\phi} - 1) &= \det\left(\frac{\partial\mathbf{r}'}{\partial x_1}, \cdots, \frac{\partial\mathbf{r}'}{\partial x_k}(x_k - 1), \cdots, \frac{\partial\mathbf{r}'}{\partial x_n}\right)^{\psi\phi} \\ &= -\sum_{j\neq k} \det\left(\frac{\partial\mathbf{r}'}{\partial x_1}, \cdots, \frac{\partial\mathbf{r}'}{\partial x_j}(x_j - 1), \cdots, \frac{\partial\mathbf{r}'}{\partial x_n}\right)^{\psi\phi} \\ &= 0. \end{aligned}$$

Consequently

$$\det\left(\frac{\partial\mathbf{r}'}{\partial x}\right)^{\psi\phi} \equiv 0 \pmod{\sigma}.$$

[4] The case $d = 0$ is proved in the next section.

Thus $\mathfrak{E}_0(G) \subset (\sigma)$. Since $u\sigma = u^\circ\sigma$ for any $u \varepsilon JH$, the ideal (σ) consists of the integral multiples of σ. Hence there is a non-negative integer a such that $\mathfrak{E}_0(G) = (a\sigma)$. Hence $(\mathfrak{E}_0(G))^\circ = (ap)$, where $p = p_1p_2 \cdots p_\mu$. On the other hand $(\partial\mathbf{r}/\partial\mathbf{x})^\circ$ is a relation matrix for H, according to (3.5), so that $(\mathfrak{E}_0(G))^\circ$, the order ideal of $(\partial\mathbf{r}/\partial\mathbf{x})^\circ$, must be the ideal (p), since p is the order of H. Thus $a = 1$, and, consequently, $\mathfrak{E}_0(G) = (\sigma)$.

6. The 1st elementary ideal

Given any two elements g_1 and g_2 of G there may be found a presentation $(x_1, \cdots, x_n : \mathbf{r})$ of G such that $x_1^\phi = g_1$ and $x_2^\phi = g_2$. Then, by the fundamental formula [FDCI (2.3)], for any set $\mathbf{r}' = (r_{j_1}, \cdots, r_{i_{n-1}})$ of $n - 1$ relators, we find

$$\begin{aligned} \det\left(\frac{\partial\mathbf{r}'}{\partial(x_1, x_3, \cdots, x_n)}\right)^{\psi\phi}(x_1^{\psi\phi} - 1) &= \det\left(\frac{\partial\mathbf{r}'}{\partial x_1}(x_1 - 1), \frac{\partial\mathbf{r}'}{\partial x_3}, \cdots, \frac{\partial\mathbf{r}'}{\partial x_n}\right)^{\psi\phi} \\ &= \det\left(-\sum_{j=2}^{n}\frac{\partial\mathbf{r}'}{\partial x_j}(x_j - 1), \frac{\partial\mathbf{r}'}{\partial x_3}, \cdots, \frac{\partial\mathbf{r}'}{\partial x_n}\right)^{\psi\phi} \\ &= -\det\left(\frac{\partial\mathbf{r}'}{\partial x_2}(x_2 - 1), \frac{\partial\mathbf{r}'}{\partial x_3}, \cdots, \frac{\partial\mathbf{r}'}{\partial x_n}\right)^{\psi\phi} \\ &= -\det\left(\frac{\partial\mathbf{r}'}{\partial(x_2, \cdots, x_n)}\right)^{\psi\phi}\cdot(x_2^{\psi\phi} - 1). \end{aligned}$$

Thus, denoting by $F^{(1)}$ and $F^{(2)}$ the subgroups of G generated by g_1 and g_2 respectively,

$$E_0(G, F^{(2)})\cdot(g_1^\psi - 1) = E_0(G, F^{(1)})\cdot(g_2^\psi - 1). \tag{6.1}$$

Since $\sum_{g_2 \epsilon G}(g_2^\psi - 1)$ is the fundamental ideal $\mathfrak{H}$ of H, it follows from (4.5) that

$$\mathfrak{E}_1(G)\cdot(g_1^\psi - 1) = \mathfrak{E}_0(G, F^{(1)})\cdot\mathfrak{H}. \tag{6.2}$$

From this we derive

(6.3) *If H is the infinite cyclic group generated by t and if F is the subgroup of G generated by an element g for which $g^\psi = t^\lambda$, then $\mathfrak{E}_0(G, F) = \mathfrak{E}_1(G)\cdot(t^\lambda - 1)/(t - 1)$.*

(6.4) *If H is the free abelian group of rank $\mu \geqq 2$ then $\mathfrak{E}_1(G) = \mathfrak{D}\cdot\mathfrak{H}$ where $\mathfrak{D}$ is a certain ideal, and if F is the subgroup of G generated by an element g then $\mathfrak{E}_0(G, F) = \mathfrak{D}\cdot(g^\psi - 1)$.*

Proof. Let $(u_1, \cdots, u_\alpha)$ be a basis for the ideal $\mathfrak{E}_1(G)$ and let $(v_1, \cdots, v_\beta)$ be a basis for the ideal $\mathfrak{E}_0(G, F)$. A basis for the fundamental ideal $\mathfrak{H}$ of JH is $(t_1 - 1, \cdots, t_\mu - 1)$ where $t_1, \cdots, t_\mu$ is a basis for H. By (6.2) we must have

$$u_i(g^\psi - 1) = \textstyle\sum_{j,k} a_{ijk}v_j(t_k - 1) \qquad (i = 1, \cdots, \alpha) \tag{6.5}$$

$$\textstyle\sum_l b_{jkl}u_l(g^\psi - 1) = v_j(t_k - 1) \qquad (j = 1, \cdots, \beta; k = 1, \cdots, \mu). \tag{6.6}$$

Suppose first that H is infinite cyclic. Then $\mu = 1$ and $g^{\psi} = t^{\lambda}$ for some integer λ, so that (6.6) becomes

$$\sum_l b_{jl} u_l \frac{t^{\lambda} - 1}{t - 1} \cdot (t - 1) = v_j(t - 1).$$

It follows that $\sum_l b_{jl} u_l \cdot (t^{\lambda} - 1)/(t - 1) = v_j$, since $t - 1$ is not a divisor of zero in JH. Thus $\mathfrak{E}_0(G, F) \subset \mathfrak{E}_1(G) \cdot (t^{\lambda} - 1)/(t - 1)$. On the other hand, using again the fact that $t - 1$ is not a zero-divisor, we get from (6.5) that $u_i \cdot (t^{\lambda} - 1)/(t - 1) = \sum_{jk} a_{ijk} v_j$, so that $\mathfrak{E}_1(G) \cdot (t^{\lambda} - 1)/(t - 1) \subset \mathfrak{E}_0(G, F)$.

Suppose next that H is free abelian of rank $\mu \geqq 2$ and let $g^{\psi} = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_{\mu}^{\lambda_{\mu}}$. If $\lambda_1 \neq 0$, say, the polynomials $g^{\psi} - 1$ and $t_2 - 1$ are relatively prime. Since, by (6.6) with $k = 2$, $g^{\psi} - 1$ divides $v_j(t_2 - 1)$, it follows that

$$v_j = w_j(g^{\psi} - 1) \qquad j = 1, \cdots, \beta. \tag{6.7}$$

Denote by $\mathfrak{D}$ the ideal $(w_1, \cdots, w_{\beta})$, so that (6.7) says that $\mathfrak{E}_0(G, F) = \mathfrak{D} \cdot (g^{\psi} - 1)$. Substituting from (6.7) into (6.5) and (6.6) and dividing out by the non-zero-divisor $g^{\psi} - 1$ yields

$$u_i = \sum_{jk} a_{ijk} w_j(t_k - 1) \qquad (i = 1, \cdots, \alpha)$$

$$\sum_l b_{jkl} u_l = w_j(t_k - 1) \qquad (j = 1, \cdots, \beta; k = 1, \cdots, \mu)$$

which say that $\mathfrak{E}_1(G) \subset \mathfrak{D} \cdot \mathfrak{H}$ and $\mathfrak{E}_1(G) \supset \mathfrak{D} \cdot \mathfrak{H}$ respectively. If $g^{\psi} = 1$ it follows from (6.6) that $\mathfrak{E}_0(G, F) = (0)$.

Theorems (6.3) and (6.4) are handy for calculating $\mathfrak{E}_1(G)$ when H is torsion-free. Theorem (6.3) goes back to Alexander [1].

Of special interest are those groups G that have presentations in which there are more generators than relations. For instance the group of a knot of multiplicity $\mu \geqq 1$ has this property. If $\mu = 1$ the ideal $\mathfrak{E}_1(G)$ is a principal ideal; the generator of $\mathfrak{E}_1(G)$ is the polynomial $\Delta(t)$, determined of course only up to a factor $\pm t^{\lambda}$, that was defined by Alexander [1] for the group of a single knot. If $\mu \geqq 2$, the ideal $\mathfrak{D}$ is a principal ideal; its generator is a polynomial $\Delta(t_1, \cdots, t_{\mu})$ that is determined only up to a factor $\pm t_1^{\lambda_1} \cdots t_{\mu}^{\lambda_{\mu}}$. For any value of $\mu = 1, 2, \cdots$ I have called $\Delta(t_1, \cdots, t_{\mu})$ the *Alexander polynomial* of G. Thus G has an Alexander polynomial $\Delta = \Delta_G$ whenever G/G_2 is torsion-free and G can be presented with more generators than relations. If this last condition is not fulfilled we may speak of the *Alexander ideal,*—$\mathfrak{E}_1(G)$ if $\mu = 1$, and $\mathfrak{D}$ if $\mu \geqq 2$. Recent investigations of the Alexander polynomial may be found in [12], [16], [18] and [29].

If H is not torsion-free the ring JH has divisors of zero and the situation becomes more complicated. This case may occur in important applications, but I am, at the moment, uncertain as to the proper way to treat it.

It may be observed from (4.3), (4.4) and (4.8) that, $\mathfrak{E}_d(G)$ is contained in $\mathfrak{H}^{\mu-d}$ whenever H is free abelian of rank μ and $1 \leqq d \leqq \mu - 1$. Hence, in particular, $\mathfrak{E}_1(G) \subset \mathfrak{H}^{\mu-1}$. Thus, if $\mu \geqq 3$, the Alexander polynomial $\Delta(t_1, \cdots, t_{\mu})$ must be of the form $\sum a_{k_1, \cdots, k_{\mu-2}} (t_{k_1} - 1) \cdots (t_{k_{\mu-2}} - 1)$. Furthermore if $\mu = 1$, the fact that $(\mathfrak{E}_1(G))^{\circ}$ is the 1^{st} elementary ideal of $(\partial \mathbf{r}/\partial \mathbf{x})^{\circ}$, which is a relation matrix for H, shows that $(\mathfrak{E}_1(G))^o = (1)$, hence that $\Delta(1) = \pm 1$ (cf. [1]).

7. The cohomology groups of a group

Recently Eilenberg and MacLane [9] have associated with an arbitrary group G and an arbitrary right JG-module $\mathfrak{A}$ groups $H_n(G, \mathfrak{A})$ and $H^n(G, \mathfrak{A})$, called, respectively, the homology and cohomology groups of G over $\mathfrak{A}$.

(7.1) *The groups $H_n(G, \mathfrak{A})$ and $H^n(G, \mathfrak{A})$ are determined, for every n and $\mathfrak{A}$, by an arbitrary Jacobian matrix $(\partial \mathbf{r}/\partial \mathbf{x})^\phi$ of G.*

To calculate these groups one first constructs a sequence of "incidence matrices" $\mathbf{M}_{n-1}$ $(n = 1, 2, \cdots)$ over JG as follows:

$$\mathbf{M}_0 = \left\| \begin{matrix} x_1 - 1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - 1 \end{matrix} \right\|^\phi \qquad \mathbf{M}_1 = \left\| \begin{matrix} \dfrac{\partial r_1}{\partial x_1} & \cdots & \dfrac{\partial r_1}{\partial x_n} \\ & \cdots & \\ \dfrac{\partial r_m}{\partial x_1} & \cdots & \dfrac{\partial r_m}{\partial x_n} \end{matrix} \right\|^\phi,$$

$\mathbf{M}_n$ is a matrix of α_{n+1} rows and α_n columns whose row space contains exactly those vectors $\mathbf{v} = (v_1, \cdots, v_{\alpha_n})$ such that $\mathbf{v} \cdot \mathbf{M}_{n-1} = 0$; $\alpha_0 = 1, \alpha_1 = n, \alpha_2 = m$. If $n > 2$, there is no guarantee that α_n is finite; in fact it is an interesting question as to when $\mathbf{M}_2$ can be chosen to be a finite matrix. We consider the additive group $Z_n(G, \mathfrak{A})$ whose elements are the vectors $\mathbf{v} = (v_1, \cdots, v_{\alpha_n})$ over $\mathfrak{A}$ which have the property $\mathbf{v} \cdot \mathbf{M}_{n-1} = 0$, and the additive group $Z^n(G, \mathfrak{A})$ whose elements are the vectors $\mathbf{v} = (v_1, \cdots, v_{\alpha_n})$ over $\mathfrak{A}$ which have the property[5] $\mathbf{v} \cdot \overline{\mathbf{M}}'_n = 0$. The vectors $\mathbf{w} \cdot \mathbf{M}_{n-1}$ form a subgroup $B_{n-1}(G, \mathfrak{A})$ of $Z_{n-1}(G, \mathfrak{A})$, and the vectors $\mathbf{w} \cdot \overline{\mathbf{M}}'_n$ form a subgroup $B^{n+1}(G, \mathfrak{A})$ of $Z^{n+1}(G, \mathfrak{A})$. The homology and cohomology groups are $H_n(G, \mathfrak{A}) = Z_n(G, \mathfrak{A})/B_n(G, \mathfrak{A})$ and $H^n(G, \mathfrak{A}) = Z^n(G, \mathfrak{A})/B^n(G, \mathfrak{A})$. For details and proof see [11].

In the so-called case of "simple operation" the multiplication of an element $\mathfrak{a}$ of $\mathfrak{A}$ by an element u of JG is defined to be $\mathfrak{a} \cdot u = \mathfrak{a} u^o$. In this case $\mathbf{v} \cdot \mathbf{M}_n = \mathbf{v} \cdot \mathbf{M}_n^o$, so that the groups $H_n(G, \mathfrak{A})$ and $H^n(G, \mathfrak{A})$ are determined by the sequence $\mathbf{M}_0^o, \mathbf{M}_1^o, \mathbf{M}_2^o, \cdots$ of integral matrices.

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[5] $\overline{\mathbf{M}}'$ denotes the conjugate transpose of $\mathbf{M}$, where "conjugation" is defined by $\overline{\Sigma_{g \in G} a_g g} = \Sigma_{g \in G} a_g g^{-1}$.