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FREE DIFFERENTIAL CALCULUS III. SUBGROUPS

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1. Subgroups of free groups. Consider the free group X of rank $n(\leqq \infty)$ on a set of generators (x_j), a subgroup W of index $\upsilon(\leqq \infty)$, and its right cosets W_β (with $W_1 = W$). In each coset W_β select a representative element $| W_\beta |$, with $| W_1 | = 1$.

Denote by W^* the free group of rank $n\upsilon$ on a set of generators $(x_{j\beta})$; the generator $x_{j\beta}$ corresponds to the generator x_j and the coset W_β. Since W^* is a free group, a homomorpism τ of W^* into W is determined by defining

$$x_{j\beta}^{\tau} = | W_\beta | \cdot x_j \cdot | W_\beta x_j |^{-1}. \tag{1.1}$$

To each coset W_β there is a mapping, which will be denoted by the same symbol W_β, of X into W^* defined as follows:[1]

$$\begin{aligned} x_j^{W_\beta} &= x_{j\beta} \\ u^{W_\beta} &= \prod_{k=1}^{l} x_{j_k}^{\varepsilon_k W_\beta u(k)} \quad \text{if} \quad u = \prod_{k=1}^{l} x_{j_k}^{\varepsilon_k}. \end{aligned} \tag{1.2}$$

It is easily verified, from this definition, that

$$\begin{aligned} (uv)^{W_\beta} &= u^{W_\beta} v^{W_\beta u}, \\ 1^{W_\beta} &= 1, \\ (v^{-1})^{W_\beta} &= v^{-W_\beta v^{-1}}. \end{aligned} \tag{1.3}$$

From these formulae and the definition of τ, it follows, by induction on l, that[2]

$$(u_\beta)^{\tau} = | W_\beta | \cdot u \cdot | W_\beta u |^{-1}. \tag{1.4}$$

By (1.3), the restriction σ to W of the mapping W_1 is a homomorphism, and, by (1.4), the endomorphism $\tau\sigma$ of W is the identity. Hence σ is an isomorphism of W onto a subgroup W^σ of W^*, and τ is a homomorphism of W^* onto W, such that $\tau \mid W^\sigma = \sigma^{-1}$. Accordingly W will be identified with W^σ; the isomorphism σ thereby becomes the injection of W into W^*. Since $(x_{j\beta}^{\tau})$ generates W, the identification is consummated by setting $x_{j\beta}^{\tau}$ equal to $x_{j\beta}^{\sigma\tau}$, i.e. by the formula[2]

$$| W_\beta | \cdot x_j \cdot | W_\beta x_j |^{-1} = | W_\beta |_1 \cdot x_{j\beta} \cdot | W_\beta x_j |_1 . \tag{1.5}$$

Thus $X \cap W^* = W$, and τ is a retraction of W^* upon W. The mapping θ, defined by $a^\theta = a \cdot a^{-\tau}$, is a retraction of W^* upon the kernel of τ (since $\theta^2 = \theta$). Since $(ab)^\theta = ab^\theta a^{-1} \cdot a^\theta$, and $(a^{-1})^\theta = a^{-1} a^{-\theta} a$, it follows that

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[1] For the definition of the k^{th} initial section $u_{(k)}$ see formula (2.7) of FDCI.

[2] It is often convenient to write just u_β instead of u^{W_β}. To avoid possible confusion, I remark that, in this paper, the Greek letters α, β, γ are reserved for the designation of a variable coset of W. Of course the index 1, when applied as a subscript to a coset representative, refers to the coset mapping W_1 ; thus $| W_\beta |_1$ means $| W_\beta |^{W_1}$.

(1.6) *the kernel of* τ *is the consequence of the* $n\upsilon$ *elements* $x_{j\beta}^{\theta} = x_{j\beta}\overline{x_{j\beta}^{\tau}}$, *so that* $W = ((x_{j\beta}):(x_{j\beta} = x_{j\beta}^{\tau}))^{\tau}$.

It follows from (1.4) that $| W_\beta |_1$ belongs to the kernel of τ, and from (1.1) and (1.5) it follows that[2]

$$x_{j\beta}^{\theta} = (x_{j\beta} \cdot | W_\beta x_j |_1 \cdot x_{j\beta}^{-1}) \cdot | W_\beta |_1^{-1}.$$

Consequently

(1.7) *the kernel of* τ *is the consequence of the* $\upsilon - 1$ *elements* $| W_\beta |_1$, $\beta \neq 1$; *thus*[2]

$$W = ((x_{j\beta}):(| W_\beta |_1 = 1 \mid \beta \neq 1))^{\tau}.$$

Obviously the presentation (1.7) may be obtained from the presentation (1.6) by Tietze transformations (I) and $(I)^{-1}$.

The system Σ of coset representatives $| W_\beta |$ is called a *Schreier system* if the initial segments of any coset representative are also coset representatives. It is known that a Schreier system always exists. Schreier's proof [30] of this fact is quite straightforward and even proves a little more—that there always exists a minimal Schreier system, i.e. one for which the length $l(| W_\beta |)$ of any coset representative $| W_\beta |$ is not larger than the length $l(u)$ of any other element u of the coset W_β.

A generator $x_{j\beta}$ of W^* will be called a *trivial generator* if $x_{j\beta}^{\tau} = 1$, i.e. if

$$| W_\beta | \cdot x_j = | W_\beta x_j |.$$

Thus the occurrence of a trivial generator $x_{j\beta}$ corresponds to an occurrence of a coset representative v of length, say, l whose $(l - 1)^{\text{st}}$ initial segment u is also a coset representative. (Here $u \in W_\beta$ if $v = ux_j$, and $v \in W_\beta$ if $v = ux_j^{-1}$.) Thus there can be at most $\upsilon - 1$ trivial generators, and this maximum number obtains if and only if Σ is a Schreier system.

The subgroup of W^* generated by the trivial generators $x_{j\beta}$ will be denoted by T, and the subgroup generated by the non-trivial generators $x_{j\beta}$ will be denoted by S. Thus $W^* = S * T$ and T is contained in the kernel of τ.

(1.8) *If* Σ *is a Schreier system, the elements* $| W_\beta |_1$ *all lie in* T. For then, if the reduced word representing $| W_\beta |$ is $\prod_{k=1}^{l} x_{j_k}^{\varepsilon_k}$, the image under W_1 of $| W_\beta |$ is $\prod_{k=1}^{l} x_{j_k\beta_k}^{\varepsilon_k}$, for certain indices $\beta_1, \cdots, \beta_l$, and it is easily verified that

$$x_{j_1\beta_1}, \cdots, x_{j_l\beta_l}$$

are, in fact, trivial generators.

It follows from (1.7) that, in this case, the kernel of τ is the consequence of T, and thus that $\tau \mid S$ maps S isomorphically upon W. Thus

(1.9) (Nielsen-Schreier) *The group* W *is isomorphic to the free group on the non-trivial generators* $x_{j\beta}$ *determined by the Schreier system* Σ. *Thus*

$$\begin{aligned} W &= ((x_{j\beta}):(x_{j\beta} = 1 \mid x_{j\beta} \in T))^{\tau} \\ &= ((x_{j\beta} \mid x_{j\beta} \in S))^{\tau}. \end{aligned}$$

It is easily verified that the rank N of W is equal to the first betti number of the coset diagram [30]. This is equal to $nv - v + 1$ if v is finite [30]; if v is infinite, N is known [30] to be infinite whenever W is normal, and may be finite when W is not normal.

It follows from (1.1), (1.5) and (1.8) that W^* is freely generated by the trivial generators $x_{j\beta}$ and the images under τ of the non-trivial generators $x_{j\beta}$. Thus

$$W^* = W * T. \tag{1.10}$$

2. Subgroups of arbitrary groups. Consider a subgroup F of index ν in a group $G = ((x_j):(r_i = 1))^\phi$. The free group X is mapped by ϕ homomorphically upon G, and the kernel R of ϕ is the consequence of the set of elements (r_i). The inverse image W of F is a subgroup of X that contains R, and its index v is the same as the index of F in G. The cosets of F are the images $F_\beta = W_\beta^\phi$ of the cosets of W. Since the normal subgroup R is the consequence in X of the elements r_i, it follows that R is the consequence in W of the elements

$$| W_\alpha | \cdot r_i \cdot | W_\alpha |^{-1} = | W_\alpha | \cdot r_i \cdot | W_\alpha r_i |^{-1} = r_{i\alpha}^\tau .$$

This proves the Reidemeister-Schreier theorem [26, 30, 31, 41]:

(2.1) THEOREM. *If Σ is a Schreier system then*

$$F = ((x_{j\beta}):(r_{i\alpha} = 1), (x_{j\beta} = 1 \mid x_{j\beta} \in T))^{\phi\tau}$$
$$= ((x_{j\beta} \mid x_{j\beta} \in S):(\hat{r}_{i\alpha} = 1))^{\phi\tau},$$

where $\hat{r}_{i\alpha}$ is obtained from $r_{i\alpha}$ by deleting the occurrences of trivial generators. The relations $r_{i\alpha} = 1$ and $x_{j\beta} = 1 \mid x_{j\beta} \in T$ are called in [26] the relations of the first and second kind respectively.

According to (2.1), the kernel of $\phi\tau$ is the consequence in W^* of $(r_{i\alpha})$ and T. Hence the kernel of ϕ is the consequence in W of $(r_{i\alpha}^\tau)$. The homomorphism ϕ of W upon F and the identity automorphism of T together determine a homomorphism ϕ^* of $W^* = W * T$ upon $F^* = F * T$, and the kernel of ϕ^* is then the consequence in W^* of $(r_{i\alpha}^\tau)$. But, by (1.4) and (1.5),

$$r_{i\alpha}^\tau = | W_\alpha |_1 \cdot r_{i\alpha} \cdot | W_\alpha |_1^{-1},$$

and hence the kernel of ϕ^* is the consequence in W^* of $(r_{i\alpha})$. Thus

$$F^* = ((x_{j\beta}):(r_{i\alpha} = 1))^{\phi^*}. \tag{2.2}$$

The presentation (2.2) has the notable feature that it does not depend on the choice of a system of coset representatives. In other words, without selecting any coset representatives at all, one can write down a presentation of the free product F^* of the given subgroup F and a free group T of rank $v - 1$. The reason for doing this is that, for some purposes, knowledge of $F * T$ is just as good as knowledge of F. An application is given in the next section.

It is well-known [32, Ch. II] that a subgroup F of G determines a representation $\rho = \rho_F$ of G upon a transitive group of permutations of the symbols

$$1, 2, \cdots, \upsilon.$$

This representation ρ, which is said to *belong* to F, is defined as follows:

$$g^\rho = \begin{pmatrix} 1 & 2 & \cdots & \upsilon \\ 1(g) & 2(g) & \cdots & \upsilon(g) \end{pmatrix}, \qquad g \in G, \tag{2.3}$$

where $F_{\beta(g)} = F_\beta g$. The coset F_β consists of those elements g of G for which $1(g) = \beta$; in particular $g \in F$ if and only if $1(g) = 1$. The correspondence between subgroups F of G and transitive representations ρ of G into the symmetric group of permutations of the symbols $1, 2, \cdots, \upsilon$ is one to one, provided that the symbol 1 is the index of the identity coset F in every case. It is convenient for some purposes to specify the subgroup F only indirectly by means of the representation ρ that belongs to it.

It may be noted that ρ^ϕ is the representation of X that belongs to W, since $F_\beta = W_\beta^\phi$, and that $\beta(u) = {}_\beta(g)$ if and only if g and u^ϕ lie in the same coset of F.

The representation ρ is useful in writing down the presentation (2.2), because (1.2) can be written[1]

$$u_\beta = \prod_{k=1}^{l} x_{j_k\beta(u_{(k)})}^{\varepsilon_k}, \tag{2.4}$$

thus making it possible to write down the υ relations $r_{i\alpha} = 1$ $(\alpha = 1, 2, \cdots, \upsilon)$ in one operation, as will be illustrated below (§5).

3. Jacobians of subgroups. Let

$$\begin{aligned} \delta_{\alpha\beta}(u) &= 1 \quad \text{if} \quad W_\alpha u = W_\beta \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

for every $u \in X$. (Note that $\delta_{\alpha\beta}(1)$ is the ordinary Kronecker delta $\delta_{\alpha\beta}$.) For any group Γ, denote by $\mathfrak{M}(\Gamma)$ the ring of $\upsilon \times \upsilon$ matrices over the group ring $J\Gamma$. The formulae

$$\begin{aligned} u^\omega &= \| \delta_{\alpha\beta}(u) \cdot u^{W_\alpha} \| \text{ for } u \in X, \\ (f + g)^\omega &= f^\omega + g^\omega \qquad \text{for } f, g \in JX, \end{aligned} \tag{3.1}$$

determine a homomorphism ω of JX into $\mathfrak{M}(W^*)$; in fact

$$\begin{aligned} \| \delta_{\alpha\beta}(u)u_\alpha \| \cdot \| \delta_{\beta\gamma}(v)v_\beta \| &= \| \textstyle\sum_\beta \delta_{\alpha\beta}(u)\delta_{\beta\gamma}(v)u^{W_\alpha}v^{W_\beta} \| = \| \delta_{\alpha\gamma}(uv)u^{W_\alpha}v^{W_\alpha u} \| \\ &= \| \delta_{\alpha\gamma}(uv) \cdot (uv)_\alpha \|. \end{aligned}$$

The homomorphism $\tau\omega$ of JX into $\mathfrak{M}(W)$ is the monomial representation [32, Ch. V] of JX in JW.

The effect of ω on a derivative is especially interesting. Let $u = \prod_{k=1}^{l} x_{j_k}^{\varepsilon_k}$, so that[1] $\partial u/\partial x_j = \sum_{k=1}^{l} \varepsilon_k \delta_{j_k j} \cdot u_{(k)}$. Then

$$\left(\frac{\partial u}{\partial x_j}\right)^{\omega} = \sum_{k=1}^{l} \varepsilon_k \delta_{j_k j} \,\|\, \delta_{\alpha\beta}(u_{(k)}) \cdot u_{(k)}^{W_\alpha} \,\| = \|\, \sum_{k=1}^{l} \varepsilon_k \delta_{j_k j} \delta_{\alpha\beta}(u_{(k)} \cdot u_{(k)}^{W_\alpha} \,\|.$$

But $u_\alpha = \prod_{k=1}^{l} x_{j_k}^{\varepsilon_k W_\alpha u(k)}$, so that

$$\frac{\partial u_\alpha}{\partial x_{j\beta}} = \sum_{k=1}^{l} \varepsilon_k \, \delta_{j_k j} \, \delta_{\alpha\beta}(u_{(k)}) u_{(k)}^{W_\alpha}.$$

Thus

$$\left(\frac{\partial u}{\partial x_j}\right)^{\omega} = \left\| \frac{\partial u_\alpha}{\partial x_{j\beta}} \right\|. \tag{3.2}$$

If r_i is any relator of the presentation $(x:r)^\phi$ of G, then $\delta_{\alpha\beta}(r) = \delta_{\alpha\beta}$, and $(r_{i\alpha})^{\phi^*} = 1$, so that $r_i^{\phi^*\omega} = \| \delta_{\alpha\beta} \| = \mathbf{E} \in \mathfrak{M}(F^*)$. Thus $u^\phi = v^\phi$ implies that $u^{\phi^*\omega} = v^{\phi^*\omega}$, so that a homomorphism ω of JG into $\mathfrak{M}(F^*)$ is defined by the formula $u^{\omega\phi} = u^{\phi^*\omega}$, where ϕ^* denotes the homomorphism of $\mathfrak{M}(W^*)$ upon $\mathfrak{M}(F^*)$ that is induced by the homomorphism ϕ^* of W^* upon F^*. With this notation, there follows from (3.2) the following theorem.

(3.3) *Each Jacobian* $\| \partial r_i/\partial x_j \|^\phi$ *of* G *is mapped by* ω *into a Jacobian*

$$\| \partial r_i/\partial x_j \|^{\omega\phi} = \| \partial r_i/\partial x_j \|^{\phi^*\omega} = \| \partial r_{i\alpha}/\partial x_{j\beta} \|^{\phi^*}$$

of F^*. From (2.2) it therefore follows that[3]

(3.4) $\| \partial r_i/\partial x_j \|^{\phi^*\omega}$ *is equivalent to* $\| \mathbf{M} \; O \|$ *where* O *denotes the null matrix of* mv *rows and* $v - 1$ *columns, and* $\mathbf{M}$ *is a Jacobian of* F.

This algorithm is especially efficient when applied to the problem of determining the structure of $F/[F, F]$. Consider the homomorphism $\omega_0 = o\omega$ of JX into $\mathfrak{M}(1)$. Since $o\omega = o\phi^*\omega = o\omega\phi$, we have, by (3.3),

(3.5) *A relation matrix for* $F^*/[F^*, F^*]$ *is* $\| \partial r_i/\partial x_j \|^{\omega_0}$. *Consequently, if* G *is finitely generated and* $v < \infty$, *the torsion numbers of* $F/[F, F]$ *are the invariant factors of* $\| \partial r_i/\partial x_j \|^{\omega_0}$, *and the betti number of* $F/[F, F]$ *is equal to the nullity of* $\| \partial r_i/\partial x_j \|^{\omega_0}$ *decreased by* $v - 1$.

If $F \supset [G, G]$ the structure of $F/[F, F]$ can be conveniently calculated in terms of the Alexander matrix $\| \partial r_i/\partial x_j \|^{\psi\phi}$. Here ψ is the abelianizing homomorphism that maps G upon $H = G/[G, G]$. Since $W \supset [X, X]$ we have $c^{\omega_0} = \| \delta_{\alpha\beta} \|$ for any $c \in [X, X]$ so that $u^{\psi\phi} = v^{\psi\phi}$ implies $u^{\omega_0} = v^{\omega_0}$, and a homomorphism ω_0 of JH into $\mathfrak{M}(1)$ is therefore defined by the formula $(u^{\psi\phi})^{\omega_0} = u^{\omega_0}$. Thus

(3.6) *If* $F \supset [G, G]$ *the matrix* $\| \partial r_i/\partial x_j \|^{\omega_0}$ *that appears in the statement* (3.5) *is equal to* $(\| \partial r_i/\partial x_j \|^{\psi\phi})^{\omega_0}$.

4. Covering spaces. If G is the fundamental group of a topological space $\mathfrak{Z}_0$, then to each subgroup F of G there belongs an unbranched covering space

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[3] This generalization of (3.5) is due to the late R. H. Kyle. The idea of replacing the representation ω_0 by permutation matrices by the more powerful "monomial representation" ω is also due to him.

$\mathfrak{X}$ and a base point $\mathfrak{p}$ of $\mathfrak{X}$ lying over the base point $\mathfrak{q}$ of $\mathfrak{Z}_0$; conversely, given $\mathfrak{X}$ and $\mathfrak{p}$, the subgroup F is uniquely determined. The fundamental group of $\mathfrak{X}$ is isomorphic to F. Accordingly, the results of §§2, 3 may be interpreted as algorithms for calculating the fundamental group of an unbranched covering space and its Jacobians, etc. In particular (3.5) and (3.6) describe the 1st homology group of an unbranched covering space. These results will now be extended to branched covering spaces.

Let $\mathfrak{Z}$ be a barycentrically subdivided, connected, locally finite complex and $\mathfrak{L}$ a subcomplex such that, for each vertex $\mathfrak{z}$ of $\mathfrak{L}$, the intersection $S(\mathfrak{z})$ of $\mathfrak{Z} - \mathfrak{L}$ with the open star $st\,\mathfrak{z}$ of $\mathfrak{z}$ is non-vacuous and connected. Let $\mathfrak{Y}$ be a finitely branched covering of $\mathfrak{Z}$ whose singular set $\mathfrak{Z}_s$ is a subcomplex of $\mathfrak{L}$. Let F be the subgroup of $G = \pi_1(\mathfrak{Z} - \mathfrak{L})$ to which the associated unbranched covering of $\mathfrak{Z} - \mathfrak{L}$ belongs. Then [33, §7] $\pi_1(\mathfrak{Y}) \approx F/N$, where N is the consequence of those elements a of F that are represented in at least one of the regions $S(\mathfrak{z})$. The relations $a = 1$ will be called *the branch relations*; the determination of $\pi_1(\mathfrak{Y})$ is seen to rest on calculation of a suitable "defining" set of branch relations. This is very simple to do if, for each vertex $\mathfrak{z}$, the elements of G that are represented in $S(\mathfrak{z})$ are all consequences of one such element. Such a situation arises [33, §6] if $\mathfrak{Z}$ is a connected, barycentrically subdivided, combinatorial q-dimensional manifold and $\mathfrak{L}$ is a combinatorial $(q - 2)$-dimensional manifold, polyhedrally imbedded in the interior of $\mathfrak{Z}$ in such a way that the star of each vertex of $\mathfrak{L}$ is flat in $\mathfrak{Z}$. (For then each $S(\mathfrak{z})$ belongs to the homotopy type of the circle.)

If b is an element of G represented in $S(\mathfrak{z})$ such that its consequence includes every element represented in $S(\mathfrak{z})$, then the branch relations at $\mathfrak{z}$ are just those elements $(gb^{\lambda}g^{-1})^{\pm 1}$, $g \in G$, $\lambda = 1, 2, \cdots$, that lie in F. Let u and v be elements of X such that $u^{\phi} = b$ and $v^{\phi} = g$. In order that $vu^{\lambda}v^{-1}$ lie in $W = \phi^{-1}(F)$, it is necessary and sufficient that the permutation $(vu^{\lambda}v^{-1})^{\rho\phi}$ leave fixed the symbol 1, i.e. that the permutation $(u^{\lambda})^{\rho\phi}$ leave fixed the symbol $\beta = 1(v^{\phi})$ into which the symbol 1 is sent by the permutation $v^{\rho\phi}$. Thus λ must be a multiple of the length of the cycle of $u^{\rho\phi}$ that contains β. If $u^{\rho\phi} = \cdots (\beta_1\, \beta_2 \cdots \beta_{\lambda}) \cdots$, where $\beta_1 = \beta$, say, then[2]

$$vu^{\lambda}v^{-1} = (vu^{\lambda}v^{-1})^{W_1} = v^{W_1}(u_{\beta_1}u_{\beta_2} \cdots u_{\beta_{\lambda}})v^{-W_1},$$

so that the corresponding branch relation is $u_{\beta_1}u_{\beta_2} \cdots u_{\beta_{\lambda}} = 1$. Thus

(4.1) *If $\mathfrak{Z}$ is a connected, barycentrically subdivided, combinatorial q-dimensional manifold and $\mathfrak{L}$ is a combinatorial $(q - 2)$-dimensional manifold, polyhedral and locally flat in the interior of $\mathfrak{Z}$, and if $\mathfrak{Y}$ is a finitely branched covering of $\mathfrak{Z}$ whose singular set is a subcomplex of $\mathfrak{L}$ and F is the subgroup of $G = \pi_1(\mathfrak{Z} - \mathfrak{L})$ to which the associated unbranched covering of $\mathfrak{Z} - \mathfrak{L}$ belongs, then a presentation of $\pi_1(\mathfrak{Y})$ is obtained from the presentation* (2.1) *of F by adjoining the branch relations*

$$u_{\beta_1}u_{\beta_2} \cdots u_{\beta_{\lambda}} = 1,$$

where $(\beta_1 \cdots \beta_{\lambda})$ ranges over the distinct cycles of $u^{\rho\phi}$, and the elements u are such that any element of G that is represented in some $S(\mathfrak{z})$ is a consequence of some u^{ϕ}.

Thus

$$\pi_1(\mathcal{Y}) = ((x_{j\beta}):(r_{i\alpha} = 1),\ T = 1,\ (u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda} = 1))^{\Upsilon\phi\tau},$$

where Υ *denotes the homomorphism of* F *into* $\pi_1(\mathcal{Y})$ *induced by the injection.*

Since the kernel of $\Upsilon\phi\tau$ is the consequence in W^* of T, $(r_{i\alpha})$, and $(u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda})$, it follows that the consequence in W of $(r^\tau_{i\alpha})$ and $((u_{\beta_1}\cdot u_{\beta_2} \cdots u_{\beta_\lambda})^\tau)$ is the kernel of $\Upsilon\phi$. Denote by Υ^* the homomorphism of $F * T$ upon $\pi_1(\mathcal{Y}) * T$ determined by the homomorphism Υ of F upon $\pi_1(\mathcal{Y})$ and the identity automorphism of T. The kernel of $\Upsilon^*\phi^*$ is the consequence in W^* of

$$(r^\tau_{i\alpha}) = (|W_\alpha|_1\cdot r_{i\alpha}\cdot |W_\alpha|^{-1}) \quad \text{and} \quad ((u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda})^\tau)$$
$$= (|W_{\beta_1}|_1\cdot u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda}\cdot |W_{\beta_1}|^{-1}),$$

hence of $(r_{i\alpha})$ and $(u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda})$. Thus

$$(4.2) \qquad \pi_1(\mathcal{Y}) * T = ((x_{j\beta}):(r_{i\alpha} = 1),\ (u_{\beta_1}u_{\beta_2} \cdots u_{\beta_\lambda} = 1))^{\Upsilon^*\phi^*}.$$

Denoting by u_k, $k = 1, 2, \cdots$, the branch relators u^λ, we have

$$(4.3) \qquad \left\| \begin{matrix} \dfrac{\partial r_i}{\partial x_j} \\[2ex] \dfrac{\partial u_k}{\partial x_j} \end{matrix} \right\|^{\Upsilon^*\phi^*\omega}$$

is a Jacobian of $\pi(\mathcal{Y}) * T$, *and is therefore equivalent to* $\| \mathbf{N}\ O \|$, *where* O *denotes the null matrix of* $v - 1$ *columns, and* $\mathbf{N}$ *is a Jacobian of* $\pi_1(\mathcal{Y})$.

(4.4) *A relation matrix for the commutator quotient group of* $\pi_1(\mathcal{Y}) * T$ *is*

$$\left\| \begin{matrix} \dfrac{\partial r_i}{\partial x_j} \\[2ex] \dfrac{\partial u_k}{\partial x_j} \end{matrix} \right\|^{\omega_0}.$$

Hence, if G *is finitely generated and* $v < \infty$, *the torsion numbers of the* 1*-dimensional homology group* $H_1(\mathcal{Y}) = \pi_1(\mathcal{Y})/[\pi_1(\mathcal{Y}), \pi_1(\mathcal{Y})]$ *are the invariant factors of this matrix, and the betti number of* $H_1(\mathcal{Y})$ *is the nullity of this matrix decreased by* $v - 1$.

It follows, as at the end of the preceding paragraph, that

(4.5) *If* $F \supset [G, G]$ *the matrix*

$$\left\| \begin{matrix} \dfrac{\partial r_i}{\partial x_j} \\[2ex] \dfrac{\partial u_k}{\partial x_j} \end{matrix} \right\|^{\omega_0} \quad \textit{that appears in the statement (4,4) is equal to} \quad \left\| \begin{matrix} \dfrac{\partial r_i}{\partial x_j} \\[2ex] \dfrac{\partial u_k}{\partial x_j} \end{matrix} \right\|^{\omega_0\psi\phi}.$$

5. Some simply connected spaces. If, in (4.1), $\mathcal{Z}$ is the q-dimensional euclidean space (or the q-dimensional sphere) and $\mathcal{L}$ is a $(q - 2)$-dimensional manifold, polyhedral and locally flat in $\mathcal{Z}$, then there is a presentation

$$(x_1, \cdots, x_n : r_1, \cdots, r_m)^\phi$$

of G that has the following property: each x_j^ϕ is represented in some $S(\jmath)$, and every element of G that is represented in some $S(\jmath)$ is a transform of an appropriate power of one of the elements $x_1^\phi, \cdots, x_n^\phi$. (If $q = 3$ and $\mathcal{L}$ is a knot or link, such a presentation is, for example, any over presentation [29], a special case of which is the well-known Wirtinger presentation.)

Suppose that $x_j^{\rho\phi}$ is the product of c_j distinct cycles

$$x_j^{\rho\phi} = (\beta_1\beta_2 \cdots \beta_\lambda)(\cdots) \cdots (\cdots).$$

The corresponding branch relations

$$x_{j\beta_1}x_{j\beta_2} \cdots x_{j\beta_\lambda} = 1, \cdots$$

allow us to eliminate c_j of the generators $x_{j\beta_k}, \cdots$. None of the generators so eliminated need be trivial generators. For if $x_{j\beta_1}, x_{j\beta_2}, \cdots, x_{j\beta_\lambda}$ were all trivial generators then in X we would have $| W_{\beta_1} | x_j = | W_{\beta_2} |$, $| W_{\beta_2} | x_j = | W_{\beta_3} |, \cdots,$ and hence $| W_{\beta_1} | x_j^\lambda = | W_{\beta_1} |$, i.e. $x_j^\lambda = 1$, which is impossible, since $\lambda > 0$. Thus, by means of the branch relations and the "triviality" relations of the second kind we can eliminate $c_1 + c_2 + \cdots + c_n + v - 1$ of the generators. Hence

(5.1) *The rank of* $\pi_1(\mathcal{Y})$ *is at most* $nv - \sum_{j=1}^n c_j - v + 1$.

In particular $\mathcal{Y}$ is simply connected if $\sum_{j=1}^n c_j = nv - v + 1$. This occurs when the coset diagram is, aside from rëentrant edges, a tree. In the case of a knot $\mathcal{L}$ in 3-space $\mathcal{Z}$, the generators $x_1^\phi, \cdots, x_n^\phi$ are conjugate in G, so that $c_1 = \cdots = c_n$, and thus the condition is satisfied if and only if

$$c = v - \frac{v-1}{n}.$$

The simplest instance of this is $n = 2$, $v = 3$, and $c = 2$, i.e.

$$x_1 \rightarrow (1 \quad 2)$$

$$x_2 \rightarrow (2 \quad 3).$$

Such a representation can be found for a large class of knot groups (those for which $\Delta(-1) \equiv 0 \pmod 3$) [34]). The simplest such knot is the overhand knot 3_1. Its group is

$$G = (x_1, x_2 : x_1x_2x_1 = x_2x_1x_2).$$

The irregular 3-sheeted covering $\mathcal{Y}$ of this knot is therefore simply connected. The same holds for the knots 6_1, 7_4, 7_7, etc. Of course these covering spaces are probably 3-spheres; in the case of the covering of 3_1, I am almost certain that $\mathcal{Y}$ is a 3-sphere. Assuming that it is, it is interesting to examine the situation of the branch curves in $\mathcal{Y}$, and this affords an illustration of the use of the algorithms developed in the preceding sections.

Since $c = 2$ the knot 3_1 is covered by a link of multiplicity 2 in the presumed 3-sphere $\mathfrak{Y}$; the index of branching of these two curves is 1 and 2. According to §2 the fundamental group of the unbranched covering $\mathfrak{X}$ of $\mathfrak{Z} - \mathfrak{L}$ is

$$\begin{aligned}(x_{11}, x_{12}, x_{13} &: x_{11}x_{22}x_{13} = x_{21}x_{11}x_{22}, \\ x_{21}, x_{22}, x_{23} & x_{12}x_{21}x_{11} = x_{22}x_{13}x_{23}, \\ & x_{13}x_{23}x_{12} = x_{23}x_{12}x_{21}, \\ & x_{11} = 1, \\ & x_{22} = 1) \\ = (x_{12}, x_{21} &: (x_{21}x_{12})^2 = (x_{12}x_{21})^2),\end{aligned}$$

which is the group of the link made up of two circles with linking number 2. It would therefore appear that the overhand knot is covered by two circles, of branching index 1 and 2, respectively, which are doubly linked. (Cf. [35].) In the same way it would appear that the knot 6_1 is covered by a circle (with branching index 2) and an overhand knot (with branching index 1) doubly linked.

6. The cyclic coverings of a knot. Consider now a knot $\mathfrak{L}$ in the 3-sphere $\mathfrak{Z}$, and an over-presentation $(x_0, \cdots, x_n : r_1, \cdots, r_n)^\phi$ of G. The commutator quotient group $G/[G, G]$ is infinite cyclic, and the generators $x_0^\phi, \cdots, x_n^\phi$ all belong to the coset of $[G, G]$ that generates $G/[G, G]$. Thus, for each positive integer v, there is a representation

$$\rho : x_j^\phi \to (1 \quad 2 \cdots v)$$

of G upon a cyclic group of permutations. The associated covering spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are called the v^{th} cyclic coverings of $\mathfrak{L}$ (unbranched and branched respectively).

To discuss the 1^{st} homology group of $\mathfrak{X}$ and $\mathfrak{Y}$ it is most convenient to pass from the given over presentation to a presentation

$$G = (x, a_1, \cdots, a_n : s_1, \cdots, s_n)^\phi$$

which is such that the elements a_j^ϕ all belong to the commutator subgroup $[G, G]$. This is easily done by introducing the new generators $a_1, \cdots, a_n$ by the formulae $a_j = x_j x_0^{-1}$, writing x for x_0 and $s_i(x, a_1, \cdots, a_n)$ for

$$r_i(x_0, a_1x_0, \cdots, a_nx_0),$$

and writing again ϕ for the new canonical homomorphism. The abelianizing homomorphism ψ maps G upon an infinite group H generated by an element which will be denoted by t. Then $x^{\psi\phi} = t$ and $a_j^{\psi\phi} = 1$. By the fundamental formula [FDC I (2.3)] we have $(t - 1)(\partial s_i/\partial x)^{\psi\phi} = 0$, hence $(\partial s_i/\partial x)^{\psi\phi} = 0$. Hence the Alexander matrix [FDC II p. 204] is

$$\mathbf{A}(t) = \| O \ \mathbf{F}(t) \|$$

where O denotes the null column vector, $\mathbf{F}(t) = \| f_{ij}(t) \| i, j = 1, \cdots, n$ and $f_{ij}(t) = (\partial s_i/\partial a_j)^{\psi\phi}$.

A Schreier system of representatives of the cosets of $F = \pi_1(\mathfrak{X})$ is

$$(| W_\beta | = x^{\beta-1}, \beta = 1, 2, \cdots, v).$$

The τ-images of the corresponding generators of W^* are $x_\beta^\tau = x^{\beta-1}\cdot x\cdot x^{-\beta}$, $\beta = 1, \cdots, v-1$; $x_v^\tau = x^{v-1}\cdot x\cdot 1^{-1}$; $a_{j\beta}^\tau = x^{\beta-1}a_jx^{-(\beta-1)}$, $\beta = 1, \cdots, v$, hence $x_1, \cdots, x_{v-1}$ are the trivial generators. Thus, by (2.1),

$$\pi_1(\mathfrak{X}) = (x_v, (a_{j\beta}):(s_{i\alpha} = 1))^{\phi\tau}.$$

The corresponding relation matrix is obtained from the relation matrix for $F^*/[F^*, F^*]$, i.e. the matrix $\mathbf{A}(\mathbf{T})$, where $\mathbf{T}$ denotes the $v \times v$ matrix

$$\left\| \begin{matrix} & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \\ 1 & & & & & \end{matrix} \right\| = t^{\omega_0},$$

by deleting the first $v - 1$ columns. Thus a relation matrix for $H_1(\mathfrak{X})$ is the $nv \times (nv + 1)$ matrix $\| O\ \mathbf{F}(\mathbf{T}) \|$. Since the branch relation is $x_1x_2 \cdots x_v = 1$, i.e. $x_v = 1$, the $nv \times nv$ matrix $\mathbf{F}(\mathbf{T})$ is a relation matrix for $H_1(\mathfrak{Y})$. Thus emerges the following well-known fact:

(6.1) *The group $H_1(\mathfrak{X})$ is the direct sum of the group $H_1(\mathfrak{Y})$ and the infinite cyclic group generated by $x_v^{\psi\phi\tau}$.*

The invariants of the matrix $\mathbf{F}(\mathbf{T})$ have been investigated by knot-theoretical methods [36, 18, 37]. On the other hand the determinant and the nullity of $\mathbf{F}(\mathbf{T})$ can be calculated by elementary algebra [38, 39, 40, 16]. To do this the integral group ring JH of H is replaced by its group ring CH over the field C of complex numbers; although the elementary ideals of a matrix are, generally speaking, destroyed by equivalence over CH, its determinant and nullity are obviously preserved.

It is well known that one can find $n \times n$ matrices $\mathbf{P}, \mathbf{Q}$ over the field R of rational numbers, such that $\mathbf{P}\cdot\mathbf{F}(t)\cdot\mathbf{Q}$ is a diagonal matrix

$$\textstyle\sum_{j=1}^{\prime n} \Delta_j(t) = \left\| \begin{matrix} \Delta_1(t) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \Delta_n(t) \end{matrix} \right\|$$

and the product $\Delta_1(t) \cdots \Delta_{n-d+1}(t)$ of the first $n - d + 1$ diagonal elements is the g.c.d. in RH of the elementary ideal $\mathfrak{E}_d$, $d = 1, 2, \cdots, n$, i.e. a generator of the smallest principal ideal in RH that contains $\mathfrak{E}_d$. It is easy to see that

$$\mathbf{P}(\mathbf{E})\cdot\mathbf{F}(\mathbf{T})\cdot\mathbf{Q}(\mathbf{E}) = \textstyle\sum_{j=1}^{\prime n} \Delta_j(\mathbf{T}),$$

where $\sum_{j=1}^{\prime n} \Delta_j(\mathbf{T})$ denotes the direct sum of the $\upsilon \times \upsilon$ matrices $\Delta_j(\mathbf{T})$, and $\mathbf{P}(\mathbf{E})$, $\mathbf{Q}(\mathbf{E})$ are the $n\upsilon \times n\upsilon$ matrices obtained from $\mathbf{P}$ and $\mathbf{Q}$ by replacing the entries p_{ij} and q_{ij} by the $\upsilon \times \upsilon$ matrices $p_{ij}\mathbf{E}$, $q_{ij}\mathbf{E}$. Thus *the nullity of* $\mathbf{F}(\mathbf{T})$ *is the sum of the nullities of* $\Delta_j(\mathbf{T})$, $j = 1, \cdots, n$, *and the determinant of* $\mathbf{F}(\mathbf{T})$ *is the product of the determinants of* $\Delta_j(\mathbf{T})$, $j = 1, \cdots, n$.

Let $\mathbf{W}_0 = \| \zeta^{\alpha\beta} \|_{\alpha,\beta=1,2,\cdots,\upsilon}$, where ζ is a primitive υ^{th} root of unity; its inverse is $\mathbf{W}_0^{-1} = 1/\upsilon \| \zeta^{-\alpha\beta} \|_{\alpha,\beta=1,2,\cdots,\upsilon}$. Furthermore $\mathbf{W}_0^{-1}\mathbf{T}\mathbf{W}_0 = \mathbf{U} = \sum_{\beta=1}^{\prime \upsilon} \zeta^{\beta}$. Thus $\mathbf{W}_0^{-1} \cdot \Delta_j(\mathbf{T}) \cdot \mathbf{W}_0 = \Delta_j(\mathbf{U}) = \sum_{\beta=1}^{\prime \upsilon} \Delta_j(\zeta^{\beta})$. Hence *the nullity of* $\Delta_j(\mathbf{T})$ *is equal to the number of distinct* υ^{th} *roots of unity that are roots of the equation* $\Delta_j(t) = 0$; *and the determinant of* $\Delta_j(\mathbf{T})$ *is equal to* $\prod_{\beta=1}^{\upsilon} \Delta_j(\zeta^{\beta})$. From this the following conclusions may immediately be drawn (cf. [38, 39, 40, 16]):

(6.2) *The* 1*-dimensional betti number of* $\mathfrak{Y}$ *is equal to* $\sum_{j=1}^{n} b_j$, *where* b_j *denotes the number of distinct* υ^{th} *roots of unity that are zeros of the* j^{th} *elementary divisor* $\Delta_j(t)$ of $\mathbf{F}(t)$.

(6.3) *The order of the* 1*-dimensional homology group of* $\mathfrak{Y}$ *is equal to the resultant of the polynomial* $t^{\upsilon} - 1$ *and the Alexander polynomial* $\Delta(t)$ $(= \det \mathbf{F}(t))$ *of the knot* $\mathfrak{L}$.

It follows from (6.1 and (6.2) or (6.3) that the 1-dimensional betti group $B_1(\mathfrak{X})$ is infinite cyclic, generated by the coset $\tilde{x}_{\upsilon}$ containing $x_{\upsilon}^{\psi\phi\tau}$, whenever $\Delta(t) = 0$ and $t^{\upsilon} = 1$ have no common roots. A Jacobian of $\pi_1(\mathfrak{X}) * T$ is

$$\left\| \frac{\partial s_i}{\partial x} \frac{\partial s_i}{\partial a_1} \cdots \frac{\partial s_i}{\partial a_n} \right\|_{i=1,\cdots,n}^{\phi*\omega} \qquad \text{where } x^{\omega} = \begin{Vmatrix} & x_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & x_{\upsilon-1} \\ x_{\upsilon} & & & & & \end{Vmatrix}$$

and

$$a_j^{\omega} = \begin{Vmatrix} a_{j1} & & & & \\ & a_{j2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & a_{j\upsilon} \end{Vmatrix}.$$

The homomorphism $\psi_0 : \pi_1(\mathfrak{X}) * T \rightarrow B_1(\mathfrak{X}) * 1 = B_1(\mathfrak{X})$ maps $x_1^{\phi*}, \cdots, x_{\upsilon-1}^{\phi*}$ and all $a_{j\beta}^{\phi*}$ into 1, hence also $x^{\phi*\omega}$ into

$$\mathbf{X} = \begin{Vmatrix} & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \\ \tilde{x}_{\upsilon} & & & & & \end{Vmatrix} \qquad \text{and} \qquad a_j^{\phi*\omega} \text{ into } \mathbf{E} = \begin{Vmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{Vmatrix}.$$

Since kernel $\psi \subset$ kernel $\psi_0\omega$, a homomorphism ω of JH into $\mathbf{M}(B_1(\mathfrak{X}))$ is defined by the formula $(g^{\psi})^{\omega} = g^{\psi_0\omega}$, and $t^{\omega} = \mathbf{X}$, therefore, since $\mathbf{X}$ and $\mathbf{E}$ commute, the matrix $\mathbf{A}(\mathbf{X}) = \| O \; \mathbf{F}(\mathbf{X}) \|$, where O denotes the $nv \times v$ null matrix, is a Jacobian of $\pi_1(\mathfrak{X}) * T$ at ψ_0. Hence a Jacobian matrix of $\pi_1(\mathfrak{X})$ at $\psi_0 : \pi_1(\mathfrak{X}) \to B_1(\mathfrak{X})$ is $\| O \; \mathbf{F}(\mathbf{X}) \|$, where O denotes here the null column-vector of dimension nv. Thus the 1[st] elementary ideal $\mathfrak{E}_1$ of the Jacobian of $\pi_1(\mathfrak{X})$ at ψ_0 is a principal ideal whose generator $\tilde{\Delta}(\tilde{x}_v) = \det(\mathbf{F}(\mathbf{X}))$ is an L-polynomial in $\tilde{x}_v$. This polynomial, which is determined only up to a factor $\pm\tilde{x}_v^{\lambda}$, will be called tentatively, the *Alexander polynomial of* $\mathfrak{X}$ *over its betti group.*

As shown above, $\mathbf{F}(\mathbf{X})$ is equivalent, over the rationals, to the diagonal matrix $\sum_{j=1}^{\prime n} \Delta_j(\mathbf{X})$, so that $\tilde{\Delta}(\tilde{x}_v) = \prod_{j=1}^{n} \det \Delta_j(\mathbf{X})$. Let us now adjoin to the ring $CB_1(\mathfrak{X})$ a new indeterminate c whose v^{th} power is $\tilde{x}_v$, and define

$$\mathbf{W} = \| c^{\alpha}\zeta^{\alpha\beta} \|_{\alpha,\beta=1,2,\cdots,v},$$

where, as before, ζ is a primitive v^{th} root of unity. Then

$$\mathbf{W}^{-1} = 1/v \| c^{-\beta}\zeta^{-\alpha\beta} \|_{\alpha,\beta=1,2,\cdots,v}$$

and $\mathbf{W}^{-1}\mathbf{X}\mathbf{W} = c\mathbf{U}$, so that $\mathbf{W}^{-1}\Delta_j(\mathbf{X})\mathbf{W} = \Delta_j(c\mathbf{U})$. Therefore

$$\tilde{\Delta}(\tilde{x}_v) = \prod_{j=1}^{v} \Delta(c\zeta^{j}).$$

Hence

(6.4) *If no root of* $\Delta(t) = 0$ *is a* v^{th} *root of unity, the Alexander polynomial* $\tilde{\Delta}(\tilde{x}_v)$ *of the* v^{th} *cyclic unbranched covering* $\mathfrak{X}$ *over its betti group is equal to the resultant of the polynomials* $t^{v} - \tilde{x}_v$ *and* $\Delta(t)$.

For example if $\mathfrak{L}$ is the overhand knot 3_1 we find

$$\begin{aligned}\tilde{\Delta}(\tilde{x}_v) &= 1 - \tilde{x}_v + \tilde{x}_v^2 \text{ if } v \equiv \pm 1 \pmod 6\\ &= 1 + \tilde{x}_v + \tilde{x}_v^2 \text{ if } v \equiv \pm 2 \pmod 6\\ &= 1 + 2\tilde{x}_v + \tilde{x}_v^2 \text{ if } v \equiv 3 \pmod 6.\end{aligned}$$

(For $v \equiv \pmod 6$ the betti group is not cyclic.)

For $\mathfrak{L}$ the figure eight knot 4_1 we find

$$\tilde{\Delta}(\tilde{x}_2) = 1 - 7\tilde{x}_2 + \tilde{x}_2^2$$

$$\tilde{\Delta}(\tilde{x}_3) = 1 - 18\tilde{x}_3 + \tilde{x}_3^2$$

etc.

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