Lie Groups in the Foundations of Geometry

HANS FREUDENTHAL
University of Utrecht, Utrecht, The Netherlands

1. Preliminary Remarks 145
2. The Helmholtz-Lie Space Problem 148
3. The Weyl-Cartan Space Problem 157
4. Geometries Connected with the Exceptional Simple Lie Groups 159
5. Tits Geometries 176
References 182
Appendix 189

1. Preliminary Remarks

1.1. The notion of *homogeneous space* will be used in this chapter. Though this notion is much older, the term has been coined by Elie Cartan [1].

A pair consisting of a space R and a transitive group F of topological mappings of R onto itself is called a *homogeneous space*. (Transitivity means: for every pair $x, y \in R$ there is an $f \in F$ such that $fx = y$.)

In a homogeneous space $[R, F]$ the set

$$J_{x_0} = \{f \in F \mid fx_0 = x_0\}$$

is a subgroup of F, the isotropy or *stability group* of x_0. If

$$\varphi x_0 = x_1, \qquad \varphi \in F, x_i \in R,$$

then

$$J_{x_1} = \varphi J_{x_0} \varphi^{-1}$$

so the stability groups of the points of R are *conjugate* in F. Often we will speak of *the* stability group of a homogeneous space without mentioning the point from which it was taken. If

$$gx_0 = x_1, \qquad g \in F, x_i \in R$$

then

$$gJ_{x_0} = \{f \in F \mid fx_0 = x_1\}$$

puts R in a (canonical) one-to-one relation with the set F/J_{x_0} of left-hand cosets of J_{x_0} in F.

In the sequel we restrict ourselves to "nice" cases without going into details about the nicety conditions which have to be fulfilled (see, e.g., Freudenthal [1]). Then we may state:

As a set of mappings of R onto itself, F is gifted with some topology in which it is a topological group represented in R, such that

$$\lceil x, f \rceil \to fx$$

is a continuous or even open mapping of $\lceil R, F \rceil$ onto R with

$$(fg)x = f(gx)$$

and J_x is a closed subgroup. Then, using the canonical identification of R with F/J, the homogeneous space $\lceil R, F \rceil$ may also be defined as the space of left-hand cosets of J in F with the usual topology and acted on by left-hand multiplications with elements of F.

Starting with a topological group F and a closed subgroup J, one can define a homogeneous space by putting $R = F/J$, providing R with the usual topology, and defining f_a for every $a \in F$, such that

$$f_a x = acJ \qquad \text{for all} \qquad x = cJ \in R.$$

This yields a representation of F in R. Its *kernel* consists of those a that fulfill $acJ = cJ$ identically in c; in other words, the kernel is the greatest normal subgroup N of F in J. The representation is faithful if, and only if, J does not contain a normal subgroup of F with more than one element. Otherwise, it leads to a homogeneous space $\lceil R, F/N \rceil$.

A homogeneous space is called *asystatic* (Lie) if the stability groups of all points are different, in other words, if J is not a normal subgroup of any larger subgroup of F.

1.2. We shall mainly be concerned with Lie groups. We admit "mixed" (not connected) Lie groups. The algebra of a Lie group is denoted by the corresponding boldface letter. Boldface type is also used for elements and subsets of Lie algebras.

The inner automorphisms of a Lie group G yield the adjoint representation $\tilde{G}$ of G. For $a \in G$ the inner automorphism taking $x \in G$ into axa^{-1} is called $\tilde{a}$. The adjoint group of G admits of a natural linear representation in G which is also called $\tilde{G}$. Likewise,

$$\tilde{a}\mathbf{x} = a\mathbf{x}a^{-1} \qquad \text{for} \quad a \in G, \quad \mathbf{x} \in \mathbf{G}.$$

The same notation is used in the adjoint algebra,

$$\tilde{\mathbf{a}}\mathbf{x} = [\mathbf{a}, \mathbf{x}].$$

A closed subgroup J of a Lie group is a Lie group too; the homogeneous space $R = F/J$ is an analytic variety acted on analytically by F.

Let $f_t \in F$ with $f_0 = 1$, $(df_t/dt)_{t=0} = \mathbf{f} \in \mathbf{F}$ define a differentiable curve $f_t J$ on R. Its tangential vector in $x_0 = J$ may be identified with $\mathbf{f} + \mathbf{J}$, so the tangential space T of R in x_0 may be identified with the linear space $\mathbf{F} \bmod \mathbf{J}$. *Then $j \in J$ takes $f_t J$ into $jf_t j^{-1} J$, hence $\mathbf{f} + \mathbf{J}$ into $\mathbf{j}\mathbf{f}\mathbf{j}^{-1} + \mathbf{J}$. J behaves in T as a linear group; every $j \in J$ acts upon T, such as $\tilde{j}$ does upon* $\mathbf{F} \bmod \mathbf{J}$.

However, this representation of J need not be faithful, in other words, generally J is not fully described by its linear behavior around x_0. This is a crucial point in solutions of the Helmholtz-Lie space problem. A counter example is the projective line with its projective group. If x_0 is the point at infinity, the translations leave x_0 and all vectors at x_0 invariant.

Generally, let K be the set of all $j \in J$ that leave T pointwise invariant, hence $k \in K$ if, and only if, $\tilde{\mathbf{k}}\mathbf{f} \equiv \mathbf{f}$ mod $\mathbf{J}$ for all $\mathbf{f} \in \mathbf{F}$. Then K is a normal subgroup of J. The behavior of J on T is described by J/K, the (first-order) *retrenchment* (Lie: Verkürzung) of J. (Higher order tangential spaces lead to higher order retrenchments with analogous properties. They shall not be considered in this chapter.)

The elements $\mathbf{k}$ of $\mathbf{K}$ are characterized by $\tilde{\mathbf{k}}\mathbf{f} \equiv 0 \bmod \mathbf{J}$. So $\mathbf{K}$ is the largest subset of $\mathbf{J}$ fulfilling $[\mathbf{K}, \mathbf{F}] \subset \mathbf{J}$. It is not known whether $\mathbf{K}$ is always solvable.

As a linear space $\mathbf{F}$ may be written as a direct sum $\mathbf{J} + \mathbf{F}_1$. By the local mapping

$$\mathbf{f}_1 \to (\exp \mathbf{f}_1)J \qquad (\mathbf{f}_1 \in \mathbf{F}_1),$$

$R = F/J$ may be locally identified with $\mathbf{F}_1$. Moreover, if $\mathbf{F}_1$ is invariant under all $\tilde{j}$ $(j \in J)$, J behaves locally in R, such as $\tilde{j}$ does in $\mathbf{F}_1$. Then a neighborhood of x_0 is pointwise invariant under K; if R is connected, this is true for all R, and K comes out to be trivial.

Generally, if K is trivial and hence J locally linear, the space is called *first-order homogeneous*. This happens to a connected space as soon as F splits linearly into $\mathbf{J} + \mathbf{F}_1$ with $\mathbf{F}_1$ invariant under $\tilde{j}$ $(j \in J)$, for instance, if J is compact or semisimple and connected.

2. The Helmholtz-Lie Space Problem

2.1. In the 19th century, until Hilbert [1] published his *Grundlagen der Geometrie* (1899), "foundations of geometry" meant a kind of research instigated in 1868 by Helmholtz' *Über die Tatsachen, die der Geometrie zum Grunde liegen* [2]; once, in 1902 Hilbert himself resumed that older terminology [2]. Its first occurrence, previous even to Helmholtz, is 1854 in the title of Riemann's "Habilitationsvortrag," *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, which was then published in 1867 [1].

2.2. Riemann [1] introduced what is now called Riemannian metrics. At the end of his lecture Riemann asked for those manifolds with Riemannian metric in which the shapes can be moved without deformation. They are characterized by constant curvature. The first proof for this assertion was given by Lipschitz [1]. It is evident that the change-free mobility has to be required for the (two-dimensional) surface elements only. According to Schur [1] from the third dimension onward it suffices to suppose that the surface elements can be freely turned; this capability implies that of being freely displaced.

2.3. Though the title of Helmholtz' paper aims at a philosophical reproof of Riemann's view (facts versus hypotheses), it is mathematically independent of Riemann's. Against Riemann, Helmholtz argued that metric presupposes congruence and, hence, the existence and free mobility of solid bodies. But as soon as this property is granted, one would be able to prove the ds^2 (square of the line element) to be a quadratic form on the tangential space, instead of postulating such a ds^2 as did Riemann.

This objection, repeated again and again in the 19th century (originally, even by Poincaré), is, of course, false. Physically, Riemannian metric does not depend on the existence of solid *bodies*; it only needs solid measuring *staffes* (one-dimensional bodies). The use of such measuring staffes, however, does not anticipate on sophisticated congruence axioms.

2.4. Helmholtz's approach is based on four axioms. The first postulates *that space is an n-dimensional manifold with differentiability properties.* The second axiom would read in a modern formulation: *There is a*

metric, and motions are the isometric transformations belonging to the component of identity. Notice, however, that Helmholtz does not suppose that distance is nonnegative or fulfills some other axioms of our metric spaces; distance is no more than a nice function. Helmholtz third axiom is *that of free mobility of solid bodies*. This means that *by motion every (finite) point set can be carried into any congruent one*. Helmholtz takes for granted that after fixing $n - 1$ general points the mobility is restricted to 1-parameter motions. His fourth axiom requires *that this motion should be periodic (and not spiraling)*. This is the so-called *monodromy* axiom.

It is Helmholtz' claim that his axioms characterize the class of *Euclidean* and *non-Euclidean* geometrics.

Helmholtz's axioms were meant as local assumptions, valid for general point sets, and nice functions. His formulations were surprisingly clear for that time. His axioms and methods anticipated geometric group theory and Lie groups, though possibly at that time Helmholtz had never heard of groups, and Lie groups had not yet come into existence. Helmholtz's exposition suffers from this lack of an explicit formulation of the underlying notions of group theory. Moreover, there is a serious gap in his proof, as pointed out by Lie [1, 2]. This criticism is not to belittle the merit of Helmholtz' work, which was the first serious approach to foundations of geometry after Steiner's failure and the first approach at all by group theory methods.

2.5. From about 1870 Lie studied continuous groups. In 1890 he applied his methods to the Helmholtz problem [1, 2]. When analyzing Helmholtz's axioms and proofs, he noticed a tacit substitution of *infinitesimal free mobility* for free mobility, such as formulated explicitly. By this substitution the problem had been simplified enormously. In this article we will refer to the problem formulated by Helmholtz as the *strong* Helmholtz-Lie space problem or *problem A*. The problem which has actually been dealt with by Helmholtz shall be called the *weak* Helmholtz-Lie space problem or *problem B*.

In our terminology of Section 1.2 we may say that Helmholtz tacitly assumed the group of motions to be locally linear (the space to be first-order homogeneous). He used the free mobility not for ordinary point systems, but for systems of infinitely adjacent points. Eventually, when doing so, one cannot benefit properly from the invariant distance function. On the other hand, one has to impose some upper boundary to the group, and this was done by a *mobility constraint*.

2.6. In modern terms Helmholtz' implicit conditions may be formulated as follows:

Problem B. $[R, F]$ *is a smooth first-order homogeneous space. The stability group* J *of the point* x_0 *induces a linear group* J_T *in the tangential space at* x_0 . J_T *is simply transitive over the sets of all total flags through* x_0 . (A total flag is an ascending sequence of i-dimensional linear subspaces T_i of T; $i = 0, 1, \ldots, n - 1$.)

Under these conditions $[R, F]$ is essentially a Euclidean or non-Euclidean geometry.

The condition to which the group J_T is subjected is commonly called free mobility in T. This is misleading. It is true that transitivity means a freedom of mobility. However, the adverb "simply" adds a constraint of mobility.

The kind of mobility that is postulated is exactly that of the group of rotations in n-space. A partial step in characterizing the class of Euclidean and non-Euclidean geometries by B is: B', the characterization of the rotation group by its simple transitivity over the flags.

If B has to be solved, B' is to be completed by B'': the inbedding of the stability groups characterized by B'.

Weyl [3] identified B' with the Helmholtz-Lie problem, and so did most of those who worked along Weyl's lines. However, Baer [1] stressed that B' is only a partial step. On the other hand, Birkhoff [3] called the other part B'', the Riemann-Helmholtz problem.

2.7. Lie's solution [2] of B is reasonably exact and simple. It proceeds by induction from $n = 3$. For Lie simple transitivity over the total flags is a local notion, which for $n = 2$ does not exclude groups with spiral orbits, though by global simple transitivity they are excluded. For $n = 2$, Lie has to rely on Helmholtz' monodromy axiom.

2.8. Weyl's proof [3] of B' is more involved, probably because Weyl did not use Lie group theory effectively. From B' Weyl stepped over into Riemannian geometry to prove B; in fact B' guarantees that the space R can be gifted with a Riemannian metric invariant under the group F.

2.9. A modern proof of B would be as follows:

1. J_T is irreducible.
2. For $n > 2$, J_T is even irreducible over the complex extension of T.

3. For $n > 2$, J_T is semisimple.

4. The only semisimple groups of n-space of the dimensionality $\frac{1}{2}n(n-1)$ required by the mobility condition are the real types of the orthogonal group.

5. An indefinite quadratic is to be rejected.

6. $\dim F = n + \dim J_T = \frac{1}{2}n(n+1)$.

7. If F is semisimple, the dimension argument shows that it is a real type of the orthogonal group of $(n+1)$-space, so $\lceil F, R \rceil$ is a non-Euclidean geometry.

8. Let $\mathbf{N}$ be an abelian ideal of $\mathbf{F}$, then for $j \in J$, $\mathbf{N} + \mathbf{J}$ invariant under J; because of the irreducibility of J_T in T, $\mathbf{N} + \mathbf{J} = \mathbf{F}$; because of the semisimplicity of $\mathbf{J}$, $\mathbf{N} \cap \mathbf{J} = \mathbf{O}$. So $\exp \mathbf{N}$ is a normal subgroup of translations. $\lceil F, R \rceil$ is Euclidean geometry.

2.10. A simple proof for $n = 3$ has been given by Reidemeister [1].

2.11. B' has also been studied from an algebraic point of view. Iyanaga and Abe [1] still made additional assumptions; the same is true of Pickert's paper [1]. Baer [1] managed to remove the additional assumptions. Wilker's proof [1] is still more elementary than Baer's.

A noteworthy refinement has been introduced; transitivity is not postulated for total flags, but for total half-flags (built up from half-spaces). Half-flags up to the dimension n are required to characterize, besides the rotations group, the entire orthogonal group.

2.12. The fact that a bounded group of linear transformations which is transitive over the half-lines through the origin, possesses one, and essentially only one, quadratic invariant was advanced by Laugwitz [1] as a solution of the Helmholtz problem. The misinterpretation was possibly caused by Reidemeister's paper. Laugwitz's proposition is much weaker than B' which aims at a characterization of the rotations group.

2.13. In Section 2.2 Riemann's approach was mentioned. Birkhoff [3] undertook to characterize Euclidean and non-Euclidean geometries among n-dimensional Riemann metrics by group theory features, viz., by postulating the maximal free mobility, i.e., an $\frac{1}{2}n(n+1)$-dimensional group F of isometries. It is evident that this postulate is much too strong. It better suits the Helmholtz case, where the Riemannian character of the metric still has to be proved, than the Riemann case where this character is presupposed.

According to section 2.2 it follows that for $n \geqslant 3$, the stability group J of any point is supposed to be transitive over the set of bivectors of unit area. Then it is easily shown that J is transitive on the unit sphere, and even on the set of orthogonal pairs of points on the unit sphere. The orthogonal groups which are transitive on the unit sphere have been classified (Tits [6], Freudenthal [5]); one has still to single out those which are transitive on the set of orthogonal pairs. From the existence of such kind of isotropy groups in every point one can derive that of a transitive group F of isometries. Finally, one has to find all F in which J can be imbedded as an isotropy group (Freudenthal [5]).

2.14. In the realm of foundations of geometry Helmholtz's weak problem is unsatisfactory, because it cannot be formulated without differentiability assumptions.

Lie tackled the strong problem A, too, though only for $n = 3, 4$. Lie supposed an essentially positive distance function; but, on the other hand, he used free mobility for *triples* of points only (even for $n = 4$). By this means he succeeded to characterize the class of Euclidean and non-Euclidean geometries, but it is difficult to estimate to which degree his proofs can withstand modern demands of rigor. Of course, Lie adhered to the traditional differentiability assumptions.

2.15. The first purely topologic approach is credited to Hilbert. In Anhang IV (1902) of his *Grundlagen der Geometrie* [2] he formulated and solved the strong space problem for the plane in a topologic way. As an underlying topologic space he took the Euclidean plane, so he excluded spheric and elliptic geometry. He postulated the existence of a group F of orientation preserving homeomorphisms such that:

For every $x_0 \neq x_1$ *the orbit of* x_1 *under the stability subgroup of* x_0 *is infinite.*

For every sequence $f_n \in F$ *with* $\lim f_n(x_0) = x$, $\lim f_n(y_0) = y$, $\lim f_n(z_0) = z$, *there exists an* $f \in F$ *with* $f(x_0) = x$, $f(y_0) = y$, $f(z_0) = z$.

The first property is much weaker than Helmholtz' free mobility; the second property combines a kind of topologic completeness with a rigidity assumption (a substitute for Helmholtz's metric). Under the action of the group two different points cannot arbitrarily approach each other.

Hilbert proved that his postulates characterize Euclidean and hyperbolic plane geometry.

2.16. For many years Hilbert's result was the ultimate landmark in the space problem field, though attempts at partial solutions by Brouwer [1] in 1909-1910, Moore [1] in 1919, Süss [1] in 1925–1927, Cairns [1] in 1923, Lubben [1] in 1928, van Kérékjartó [1] in 1950, and Montgometry and Zippin [1] in 1940 were made.

2.17. In 1930 Kolmogorov [1] attempted to solve the general problem. His formulation is entirely topologic: *R should be a metrizable, locally compact, connected space. F should be a transitive group over R fulfilling a rigidity condition (called uniform continuity) and separation conditions at different levels.* On the first level the separation condition applies to $\lceil R, F \rceil$ itself. It means that, given two orbits of the stability group J of x, one of them separates R between x and the other. On the second level F is replaced by $F' =$ any J, and R by $R' =$ any orbit under this J; and the separation conditions are applied on $\lceil R', F' \rceil$, and so on. This process is supposed to stop on some finite level.

Kolmogorov asserted that his conditions characterize the classic solutions of the Helmholtz-Lie problem, but he did not publish the proof. It is still unknown whether this assertion is true without the additional assumption that F be complete.

2.18. Meanwhile, Birkhoff [1, 2] in 1941 and 1944 and Busemann [1] in 1941 and 1942 returned to the original, metric formulation of the Helmholtz problem. They considered a metric space and imposed conditions on its group of isometries: congruent sets (Birkhoff) of congruent 3-point subsets (Busemann) can be carried into each other by an isometry of the whole space. Their metrics, however, are not arbitrary; they have to fulfill rather strong conditions (a kind of convexity).

2.19. In 1951 and 1952 Wang [1, 2] eliminated such additional conditions. He classified all compact, connected metric spaces in which every pair of points can be carried in every congruent one by a space isometry. He tackled the same problem for locally compact spaces, too, though under additional assumptions, among which that of even dimensionality. The classification was made possible by the solution of Hilbert's fifth problem (on the analyticity of continuous groups).

2.20. The question was finally settled by Tits [1, 2], in 1952 who took the ideas of Kolmogorov. The first-level separation condition appeared to be strong enough for a successful classification. This is the

main step. At the second or third level (this depends on the exact formulation of the problem) the variety of solutions is restricted to the Euclidean and non-Euclidean geometries. F was not supposed to be connected; much attention was paid to the enumeration of all possible components of F. Tits published his proofs in his *Thèse d'Agrégé of* 1955 [6].

2.21. Freudenthal's investigations on this subject in 1954 [5] and 1956 [6], were instigated by Tits's first communication. His conditions are weaker, and his proof is quite different.

The space R is supposed to be locally compact and connected. The transitive group of homeomorphisms over R has to fulfill three conditions:

Rigidity. Given two disjoint closed sets A, B in R, one of which is compact, there is a nonvoid open set U such that for every $f \in F$ one of the sets $fU \cap A$ and $fU \cap B$ is void.

This is essentially the condition called uniform continuity by Kolmogorov. It causes a uniform structure in R invariant under F.

Completeness of F. Possibly this condition can be dismissed.

Separation. There exists an orbit of the stability group J of $x_0 \in R$ which dissects R.

This condition is weaker than Kolmogorov's which was used by Tits also.

Using topologic methods and especially the solution of the generalized fifth Hilbert problem (Yamabe [1]), one can prove that F is a transitive Lie group with a finite number of components and a compact stability group J, which in a suitable local coordinate system becomes a linear orthogonal group and transitive on the distance sphere. Using techniques of the representation theory of semisimple groups, one can draw up the following list* of possible $\mathbf{J}$ (with compact $\exp \mathbf{J}$):

$$\begin{array}{lll} \mathbf{O}, & & \\ \pi_2(\mathbf{B}_l) & (l \geqslant 2), & 2\pi_1(\mathbf{B}_1) \\ \pi_1(\mathbf{B}_3), & \pi_1(\mathbf{B}_4), & \\ \pi_3(\mathbf{D}_l) & (l \geqslant 3), & \\ \pi_1(\mathbf{G}_2), & & \end{array}$$

* We use Cartan's notation for simple Lie algebras and fundamental representations, adding $\mathbf{O}$ as a symbol for the null-algebra and $\mathbf{A}_0$ for the one-dimensional algebra. Also see the Appendix.

and the real representations (of the double dimension) belonging to

$\pi(\mathbf{A}_0)$,

$\pi_1(\mathbf{A}_l)$ $(l \geqslant 2)$ plus a facultative direct summand representing $\mathbf{A}_0$,

$\pi_1(\mathbf{C}_l)$ $(l \geqslant 1)$ plus a facultative direct summand representing $\mathbf{A}_0$ or $\mathbf{A}_1$.

As a next step, one has to determine the (possibly nonconnected) J belonging to these $\mathbf{J}$. Finally, one has to find the possible F in which the J can be embedded as stability groups. This problem can be dealt with by the method described in Section 2.9(7–8).

The definitive list of possible $\lceil F, R \rceil$ fulfilling the three conditions may be found in Tits [1, 2] or Freudenthal [5]. It consists of Euclidean, elliptic, spheric, and hyperbolic unitary geometries over the real and complex numbers, quaternions, octaves (the spheric geometry over the real numbers only, octave geometries for $n = 2$ only), and six other particular geometries.

When extending the separation condition to the second level, one is left with the real Euclidean, elliptic, spheric, and hyperbolic geometries, and four particular geometries; on the third level three of the particular geometries are eliminated, the last one disappearing on the fourth level.

The result applies immediately to the Birkhoff-Busemann-Wang formulation of the space problem. Indeed, if *F is transitive over the pair of points with the distance γ $(0 < \gamma <$ upper bound of all distances), the J_{x_0}-orbit consisting of the points in a distance γ from x_0 dissects R.*

Let F be the group of *all* isometries of the metric, connected, locally compact R. Let F be transitive over the pair of points with the distance γ, where γ is some positive number below the upper bound of the set of all distances in R. Then $\lceil R, F \rceil$ is a Euclidean, elliptic, spheric, or hyperbolic, unitary geometry over the real numbers, complex numbers, quaternions, or octaves (the spheric geometry over the reals only, octave geometries for $n = 2$ only). Let F be transitive over the triples with distances γ, γ, γ', where γ' is some positive number below the diameter of the distance sphere of radius γ. Then R, F is a Euclidean, elliptic, spheric, or hyperbolic geometry over the reals.

2.22. It should be observed that this solution of the strong Helmholtz-Lie problem, which looks so simple that it might be called final, excludes all kind of indefinite metric; though from the original Helmholtz point of view, with its local interpretation of transitivity, indefinite metrics should be admitted. It would be desirable to forge a frame in which

geometries based on an indefinite metric, like that of special relativity, can be fitted. Efforts to do this have not been as successful as in the definite case. Conditions which characterize Euclidean and non-Euclidean together with pseudo-Euclidean and pseudo-non-Euclidean geometries have been formulated, but they are more involved and less natural than in the former case (Freudenthal [12]).

First of all, though the usual metric spaces have appeared as a reasonable generalization of the metrics induced by positive quadratic forms, no analogue has been developed for indefinite quadratic forms. As a consequence, properties which should be derived from metrical assumptions have been enforced by group theory. Furthermore, the system of characterizing conditions is not free from inventions *ad hoc*.

The solution presented by Freudenthal [12] is as follows:

R is a locally compact, connected, separably metrizable space. ρ is a nontrivial, continuous quasi-metric on R, $\rho(x, x) = 0$. F is a transitive group of homeomorphisms of R with invariant ρ. F is topologized in the sense of continuous convergence, and in this topology F is locally compact and separably metrizable. F fulfills the conditions:

$\Theta_i(R, \rho, F)$: *If* $x_1, \ldots, x_i \in R$, $f_n \in F$, $\lim f_n x_j = x_j$, *then there are* $g_n \in F$, *such that* $\lim g_n = 1$ *and* $g_n x_j = f_n x_j$. (Actually the condition is only used for $i \leqslant 3$. For $i = 1$ it implies that the mapping of F/J_{x_0} onto R by means of $fJ \to fx_0$ is topologic. For $i > 1$ it implies the analogous property on the i-th level.)

For $x \in R$, L_x means the light cone of x, i.e., the set of y with $\rho(x, y) = 0$. $S_{x,\alpha}$ means the set of y with $\rho(x, y) = \alpha$.

$\Phi_1(R, \rho, F)$: *Let* $x_0 \neq x_1$. *Then for no open* W, $L_{x_0} \cap W = L_{x_1} \cap W$, *unless both sets are void.*

(This condition, which means that light cones cannot coincide locally, excludes two-dimensional relativity. This, however, is a minor defect. The condition can still considerably weakened:

Let $y \in L_{x_0}$, $y \neq x_0$. Let $M_{x_0 y}$ be the set of $x_1 \neq y$ with the property: there is a neighborhood W of y with $L_{x_0} \cap W = L_{x_1} \cap W$. Then $M_{x_0 y}$ is disconnected.)

$\Phi_2(R, \rho, F)$: *Every* $x \in R$, $x \neq x_0$, *has a neighborhood W such that for any* $y \in W$ *with* $\rho(x_0, x) = \rho(x_0, y)$ *there exists an* $f \in J_{x_0}$ *with* $fx = y$.

(This condition, which means local transitivity of J on the distance spheres, would be entirely satisfactory if it were formulated with the provision $\rho(x_0, x) \neq 0$. However, we need the local transitivity on the light cones, too.)

$\Phi_3(R, F)$: *In some neighborhood of* x_0 *no* $S_{x_0,\alpha}$ $(\alpha \neq 0)$ *is* 0*-dimensional.*

(This condition excludes all one-dimensional geometries, but this is a minor defect. It is a condition *ad hoc*, which can not be dismissed.)

Under these conditions F appears to be a Lie group, and $\mathbf{J}$ a maximal Lie subalgebra of $\mathbf{F}$.

In the case of a nonsimple $\mathbf{F}$ the further analysis, which looks much like that of section 2.9(7–8), is not too laborious. For simple $\mathbf{F}$, however, the danger that $[R, F]$ might fail to be first-order homogeneous (see Section 1.2) is a source of difficulties. Possibly they are not essential; meanwhile second-level conditions are invoked to conquer them.

Let R_x be the x-component of the orbit J_{x_0} x, and F_x the group induced by J_{x_0} in R_x. Let the restriction of ρ to R_x still be called ρ.

The second level conditions are: $\Phi_i(R_x, F_x, \rho)$ *for all* x *in some neighborhood of* x_0 *with* $\rho(x_0, x) \neq 0$.

Assuming these conditions, one is able to reduce the classification of the admissible J to an algebraic problem:

J is a Lie group, linearly and irreducibly represented in a real finite-dimensional linear space T. J is semisimple up to a facultative one-dimensional compact direct factor. ρ is a nontrivial continuous real function on T, $\rho(0) = 0$. ρ is invariant under J. In some neighborhood of 0, $x_0 \neq 0$ can be carried into any neighboring x with $\rho(x) = \rho(x_0)$ by a $j \in J$ not far from 1.

One has to find all J fulfilling this condition.

Again, the local transitivity is needed on the light cones also. The list of admissible $\mathbf{J}$ is rather long; the tedious analysis depends on the theory of real simple Lie groups and real representations.

To embed an admissible J as a stability group into a suitable F, one has to work along the same lines as in the definite case.

Adding third-level conditions Φ one can finally reduce the variety of geometries to Euclidean, non-Euclidean, pseudo-Euclidean, and pseudo-non-Euclidean geometries.

3. The Weyl-Cartan Space Problem

3.1. In a smooth manifold R an affine connection is given when to every line element $[x, dx]$ of R corresponds a linear mapping $A(x, dx)$ of the tangential space T_x in x into T_{x+dx} in $x + dx$, such that A depends smoothly on x, linearly on dx, and $A(x, dx)\, \delta x = A(x, \delta x)\, dx$.

Let R be a generalized metric manifold, i.e., gifted with metrics

in the T_x depending smoothly on x. An affine connection is called compatible with the generalized metric, if for any line element $[x, dx]$ of R, $A(x, dx)$ is an (infinitesimal) isometric mapping of T_x into T_{x+dx}.

In the sequel it is supposed that the metric is essentially the same in every T_x, i.e., that the metric in T_x arises from that in x_0 by a linear mapping B_x (or rather by a linear mapping preserving elementary volumes defined smoothly in the various T_x).

3.2. In a note to his 1919 edition of Riemann's *Habilitationsvortrag* Weyl [1] raised the problem to characterize Riemannian manifolds among those generalized metric manifolds, and he formulated the conjecture that this could be done by the postulate:

Given the metric in T_{x_0}, there exists for any B an affine connection compatible with the generalized metric defined by B and the metric in T_{x_0}.

The problem solved by Weyl in 1922 [2, 3] is slightly different. (The difference was noticed by Laugwitz [2].) The generalized metric is replaced by a "group metric." This means that a linear Lie group is given in every T_x which again is related to that in T_{x_0}, called L, by a linear mapping B_x. The compatibility of an affine connection with such a group metric means that $A(x, dx)$ belongs to $B_{x+dx}LB_x^{-1}$. On the other hand, Weyl now postulated that the affine connection be uniquely determined by B.

Weyl's proof that this condition characterizes Riemannian metrics is extremely involved, mainly because no effective use was made of the techniques of the representation theory of semisimple Lie groups.

3.3. In 1923 É. Cartan [5] gave a much simpler proof, which applies to Weyl's first problem as well.* The variety of admissible groups L is sharply restricted by the assumption of the mere existence of the affine connection for any given B. Cartan obtained the groups of linear transformations of T leaving invariant

1. the volume
2. the volume and a given direction
3. a nondegenerate quadratic form
4. a nondegenerate skew bilinear form.

(Actually, the fourth group may still be excluded for $n \geqslant 4$; see Freudenthal [9].)

* The author could not find any justification for Scheibe's remark [1] (p. 198) that Cartan's proof should be "völlig unzureichend," and for Scheibe's much more involved own proof. The author did not find any gap in Weyl's proof either.

To come back to Weyl's problems, it suffices to add the postulate that the orbits are nowhere dense, in order to characterize Riemannian metrics. Of course, this postulate is much weaker than Weyl's uniqueness postulate.

3.4. Another proof of Cartan's problem was given by Freudenthal [9], who dropped the volume invariance condition. His list is somewhat longer; the fourth group is now admitted for $n = 4$.

3.5. The analogue of Weyl's problem for almost complex manifolds was dealt with by Klingenberg [1]. Though Klingenberg used Cartan's rather than Weyl's methods, he postulated both existence and uniqueness of the affine connection, as did Weyl. The result is a characterization of hermitean manifolds.

4. Geometries Connected with the Exceptional Simple Lie Groups

4.1. Though forshadowed by von Staudt's [1] and Wiener's [1] work,* Hilbert's discovery of the relation between geometric incidence theorems (lock theorems) and axioms for algebraic structures, is the most striking feature of his *Grundlagen der Geometrie* [1] of 1899. To formulate these relations, we choose projective geometry instead of Euclidean, as did Hilbert.

Adding *Desargues's* theorem as a "lock incidence theorem" to the "trivial" incidence axioms, one gets a class of geometries which can be described algebraically as that of projective geometries over a (non-necessarily commutative) field. Adding *Pappus-Pascal*'s theorem algebraically means postulating commutativity.

A momentous progress—the first one after Hilbert in this realm of ideas—was marked by Moufang's [1, 2] discovery and analysis of *harmonic* geometries. The harmonic lock incidence theorem says that a harmonic quadruple is uniquely determined by its first triple, so it does not depend on its particular construction. Moufang showed that affine coordinatization is still possible by means of the harmonic theorem, though the underlying algebraic structure may fail to be associative with respect to multiplication. Associativity was replaced by a weaker law, called *alternativity*, which means that the associator

$$\{a, b, c\} = (ab)c - a(bc)$$

* For the history of *Grundlagen der Geometrie*, see Freudenthal [7, 8].

is multiplied with -1 if a, b, c undergo an odd permutation. In rings, where $x = -x$ implies $x = 0$, this property is equivalent with

$$a(ab) = a^2b, \qquad (ab)a = a(ba), \qquad ab^2 = (ab)b.$$

An alternative ring with a one-element and inverses with the usual properties is called an alternative field. The class of harmonic geometrics can algebraically be described by the fact that they admit of affine coordinatizations over alternative fields.

An example of a harmonic non-Desarguean plane geometry was provided by the alternative field of the octaves. Higher-dimensional projective geometries are always Desarguean.

4.2. The octaves system* is one of the *Hurwitz algebras*, i.e., a finite-dimensional algebra with a one-element 1, an inner product with the usual properties, and a norm such that $|x|^2 = (x, x)$ and $|xy| = |x| \cdot |y|$. Over the reals (no other ground field will be admitted at the moment) there are four Hurwitz algebras, called $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_4$, $\mathscr{H}_8$ of dimensions 1, 2, 4, 8, the commutative fields of the reals, and of the complex numbers, the associative field of the quaternions, and the alternative field of the octaves.

A Hurwitz algebra possesses an involutory antiautomorphism $x \to \bar{x}$ with $\bar{x} = -x$ for $(x, 1) = 0$.

4.3. Another important tool is the so-called *Jordan algebras*, characterized by commutativity and the identity

$$a^2(ab) = a(a^2b).$$

The important Jordan algebras $J_n(\mathscr{H})$ are defined as follows. *They consist of the n-dimensional hermitean matrices over the Hurwitz algebra* $\mathscr{H}$, *the usual matrix product of a and b being replaced by the symmetrized product* $a \circ b = \frac{1}{2}(ab + ba)$. *If* $\mathscr{H}$ *is the algebra of octaves, we must restrict n to be* $\leqslant 3$.

4.4. Besides the large classes of simple and semisimple Lie algebras over the complex numbers, there are five exceptional ones,

$$\mathbf{G}_2, \quad \mathbf{F}_4, \quad \mathbf{E}_6, \quad \mathbf{E}_7, \quad \mathbf{E}_8$$

* A brief history of the octaves may be found in van der Blij's [1], an outline of their fundamental theory in van der Blij and Springer [1].

of dimensions

$$14, \quad 52, \quad 78, \quad 133, \quad 248.$$

For every one of these Lie algebras there exist real types other than the unitary one and the twin type ($\mathbf{L}_0$ and $\mathbf{L}_{**}$ if $\mathbf{L}$ is the complex type), viz.,†

$\mathbf{G}_{2.2}$ with maximal compact $\mathbf{A}_1 + \mathbf{A}_1$,

$\mathbf{F}_{4.1}$, $\mathbf{F}_{4.2}$ with maximal compact $\mathbf{B}_4$, $\mathbf{C}_3 + \mathbf{A}_1$,

$\mathbf{E}_{6.1}$, $\mathbf{E}_{6.2}$, $\mathbf{E}_{6.0.*}$ $\mathbf{E}_{6.2.*}$ with maximal compact $\mathbf{D}_5 + \mathbf{A}_0$, $\mathbf{A}_5 + \mathbf{A}_1$, $\mathbf{F}_4$, $\mathbf{C}_4$,

$\mathbf{E}_{7.1}$, $\mathbf{E}_{7.2}$, $\mathbf{E}_{7.3}$ with maximal compact $\mathbf{D}_6 + \mathbf{A}_1$, $\mathbf{E}_6 + \mathbf{A}_0$, $\mathbf{A}_7$,

$\mathbf{E}_{8.1}$, $\mathbf{E}_{8.2}$ with maximal compact $\mathbf{E}_7 + \mathbf{A}_1$, $\mathbf{D}_8$.

4.5. Whereas the groups of the large classes are given by simple definitions with intuitive geometric interpretations, the exceptional groups had originally been introduced in a purely formal way (É. Cartan [1, 2]).

However, in 1914 É. Cartan [4] mentioned without proof a representation of $G_{2,0}$ as the automorphisms group of the octaves (so is $G_{2,2}$ of the split octaves). A new progress was made by Chevalley and Schafer [1] in 1950. They discovered that $F_{4,0}$ appears as the automorphisms group of $J_3(\mathscr{H}_8)$ or, equivalently, as the invariance group of

$$\operatorname{tr}(x \circ x) \qquad \text{and} \qquad \operatorname{tr}(x \circ x \circ x)$$

in $J_3(\mathscr{H}_8)$. Actually, all that is needed for this discovery can be found in É. Cartan's paper [2]; however, in the case of the F_4 Cartan overlooked the cubic invariant though he acknowledged such an invariant in the case of E_6. Chevalley and Schafer also gave an interpretation of E_6 by means of $J_3(\mathscr{H}_8)$, but they did not mention the cubic invariant by which Cartan had introduced E_6.

4.6. An automorphism of the real projective plane is determined by the image of a general quadruple of points, so the group of automorphisms $(A_{3,2,*})$ depends on 8 parameters. For the complex projective plane the corresponding number is $2 \cdot 8$ $(A_{3,**})$; for the quaternion plane it is $4 \cdot 8 + 3$ $(A_{5,0,*})$, where the summand 3 accounts for the automorphisms of the quaternion field. For Moufang's octavian plane

† See Appendix. We retain Cartan's notations for simple algebras and Cartan's numberings of primitive root forms. There is no reason why every author should use his own system. For the real types we have proposed a more rational system of numbering.

analogous reasoning would give $8 \cdot 8 + 14 = 78$, which is the dimension of E_6. It seems that in the late 1940's several mathematicians guessed that some real form of the E_6 might be the automorphisms group of the octavian plane.

In 1950 Borel [1] noticed that Cartan's symmetric space $F_{4,0}/B_{4,0}$, which is 16-dimensional, is provided with a structure of a projective plane. This remark led him to calling this symmetric space the plane of octaves. Somewhat earlier, in 1949, Hirsch [1] had constructed the octavian plane by purely topologic means. At the same time Jordan [1] stated without proof a presentation of the octavian plane by the idempotents of $J_3(\mathscr{H}_8)$.

4.7. Starting from the Chevalley and Schafer paper of 1950 though not acquainted with the investigations mentioned in Section 4.6 (except Hirsch's), Freudenthal in 1951 [2] looked for an algebraic presentation of the octavian plane and its automorphisms group.

The usual algebraization of projective n-space takes place in linear $(n+1)$-space by means of the lattice of its linear subspaces. Because of the lack of associativity in $\mathscr{H}_8$ this method would not work in the case of the octavian plane.

Another method of some importance in von Neumann's continuous geometry is to start with unitary $(n+1)$-space and the lattice of its orthogonal (hermitean) projections; the points of projective n-space would be described by the projections with a one-dimensional image. This algebraization of projective space takes place in $J_{n+1}(\mathscr{H})$ by means of the idempotents of this Jordan algebra; the points are described by the irreducible idempotents, and the inclusion relation $p \subset q$ is explained by $p \circ q = p$. The automorphisms are described by $p \to a^* p a$ with $\det a = 1$.

This method of algebraization was adopted by Freudenthal. It still works for the octavian plane which can be described by the lattice of idempotents of $J_3(\mathscr{H}_8)$. But while in the case of associative $\mathscr{H}$ any irreducible idempotent of $J_{n+1}(\mathscr{H})$ can be factorized as the matrix product xx^* of a vector x and its conjugate transposed x^*, this reduction to the first method of algebraization fails in the octavian case.

An important role is played in $J_3(\mathscr{H}_8)$ by a cubic form called det, defined by

$$\det\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ \bar{x}_1 & x_2 & \xi_3 \end{pmatrix} = \xi_1\xi_2\xi_3 - (\xi_1 \mid x_1 \mid^2 + \xi_2 \mid x \mid^2 + \xi_3 \mid x_3 \mid^2) + 2\mathrm{Re}(x_1x_2x_3).$$

det can be expressed by traces,

$$\det x = \tfrac{1}{3}\operatorname{tr} x^3 - \tfrac{1}{2}\operatorname{tr} x^2 \operatorname{tr} x + \tfrac{1}{6}(\operatorname{tr} x)^3.$$

A symmetric trilinear form $(, ,)$ is defined by

$$(x, x, x) = \det x.$$

An inner product is provided in $J_3(\mathscr{H}_8)$ by

$$(x, y) = \tfrac{1}{2}\operatorname{tr} xy.$$

By means of

$$(x \times y, z) = 3(x, y, z)$$

one can define a cross product of x and y.

The irreducible idempotents of $J_3(\mathscr{H}_8)$ are characterized by $x \times x = 0$, $\operatorname{tr} x = 1$. Defining P by

$$x \in P \leftrightarrow x \times x = 0$$

and taking the elements of P up to a scalar factor, we may consider P as the set of points of the octavian plane. If in the description of straight lines we replace the idempotents x with $\operatorname{tr} x = 2$ by $1 - x$, we can use the same set P as the set of straight lines. Then the incidence of a point x and a line u is described by

$$x \circ u = 0$$

or, equivalently, by

$$\operatorname{tr} xu = 0.$$

The straight lines are 8-dimensional spheres. The straight line through the different points x, y is given by

$$x \times y,$$

the intersection of two lines u, v by

$$u \times v.$$

The invariance group of det coincides on P with the group of automorphisms of the plane. It is the representation π_1(or π_3) of $E_{6,0,*}$. Its Lie algebra is the sum of two linear subspaces $\mathbf{K}$ and $\mathbf{L}$,

K consisting of the mappings $x \to a^*x + xa$, where a is a 3-matrix over $\mathscr{H}_8$ with $\operatorname{tr} a = 0$;

L consisting of the mappings induced by infinitesimal automorphisms of $\mathscr{H}_8$.

Let Φ and Φ' (infinitesimal elements of the invariance group of det) represent the same infinitesimal automorphisms of the plane, once with P interpreted as the point plane and afterwards as the lines plane. Then (up to a scalar factor) Φ and Φ' are related by $a \to -a^*$ in **K** and by the identity in **L**. If the point representation of E_6 is π_1, the line representation is π_3 (Freudenthal [3]).

The invariance group of det and the inner product together coincides with the group of automorphisms of $J_3(\mathscr{H}_8)$ (in the O-sense). Its Lie algebra consists of the $\Phi = \Phi'$. In the 26-dimensional subspace of $J_3(\mathscr{H}_8)$ of elements of trace 0, it is the representation π_1 of $F_{4,0}$. Geometrically, it can be interpreted as the transitive group of automorphisms of an *elliptic* geometry of the plane. By means of this group every element of $J_3(\mathscr{H}_8)$ can be put in a diagonal form, which is unique up to the order of the diagonal elements. (This theorem is the main tool by which the above properties were proved by Freudenthal [2, 3].)

4.8. In 1953 Tits [3] gave another proof (supplemented in [4]) for the identity of the group of automorphisms of the octavian plane with $E_{6,0,*}'$, using algebraic techniques of affine geometry. In the same paper he called attention to the polarities, especially the so-called hermitean polarities of the plane which on every straight line behave as inversions of the 8-sphere. The mapping carrying every point x into the line x is an elliptic polarity, defining the above elliptic geometry. Using a hyperbolic polarity, one gets a hyperbolic geometry of the plane with the group $F_{4,1}$ (Tits [4]).

The algebraic expression of the general hermitean polarity Φ is

$$\Phi a = 2(u \times u) \times a - (a, u)u;$$

the points a coinciding with its polar lines are given by $(a, u) = 0$ (Freudenthal [4, V]). Hermitean polarities and perspectivities have been extensively studied by Freudenthal [4, III; 4, V].

4.9. The algebra developed for the investigation of the plane over $\mathscr{H}_8$ works as well, if $\mathscr{H}_8$ is replaced by $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_4$, though some

traces appearing as numerical coefficients have to be changed. In the cases $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_4$, $\mathscr{H}_8$ the automorphisms groups yield:

elliptic:	$B_{1,0}$	$A_{2,0}$	$C_{3,0}$	$F_{4,0}$
hyperbolic:	$B_{1,1}$	$A_{2,1}$	$C_{3,1}$	$F_{4,1}$
projective:	$A_{2,0,*}$	$A_{2,**}$	$A_{5,0,*}$	$E_{6,0,*}$.

Of course, in the case of $\mathscr{H}_2$ one has to disregard noncontinuous automorphisms.

Complexification and quaternionization can, of course, be imposed on the four large classes of simple groups also. One has to start with their usual linear representations (respectively, π_1, π_2, π_1, π_3) and to interpret the invariant symmetric or skew form in the hermitean sense. One gets:

$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_4$
A_l	$A_l + A_l$	A_{2l+1}
B_l	A_{2l}	C_{2l+1}
C_l	A_{2l-1}	D_{2l}
D_l	A_{2l-1}	C_{2l}

It is easy to account for the real types too. Octavization, however, is possible for low l only.

4.10. Octavian planes have been investigated by purely algebraic methods too; the restriction to the real ground field was dropped, only the characteristic was supposed $\neq 2, 3$. Using as a tool the Peirce decomposition instead of transformation on the diagonal form, Springer [1] restated the essential geometric results. The group of automorphisms has been studied by Jacobson [1], Springer [4, 5], and Suh [1]. It becomes slightly more involved though it is still related to an algebraic group E_6. In the total group one has to distinguish the subgroup generated by the perspectivities and that generated by the perspectivities with united center and axis. The algebraic counterpart is the group of σ-semilinear f with

$$\det fx = \nu(\det x)^\sigma \qquad (\nu \neq 0)$$

and the subgroups defined by $\sigma = 1$, respectively, $\sigma = 1$, $\nu = 1$.

This equation also describes isomorphisms between different projective planes (over different octavian algebras). An extensive study of hermitian polarities, elliptic and hyperbolic plane geometries over octavian algebras with the same methods has been made by Springer and Veldkamp [1]. In the same paper a relation to the geometry of reflections has been established.

4.11. Though Freudenthal's investigations into the geometric nature of E_7 and E_8 started as early as 1953 [4, II; 4, III], they have in a latter stage gradually developed under the influence of parallel work by Tits. The influence was due to his paper [4], his manuscript [5], and mainly to personal communication. This makes it hard, if not impossible, to give a faithful account on details of the course of inventions. However, it should be noticed that from the beginning the framework of Tits' investigations was broader; in an early stage he obtained possession of a tool of huge heuristic importance which will be dealt with in Section V of this account.

Freudenthal [4, I] introduced linear spaces $\mathfrak{R}$ and $\mathfrak{L}$ of elements

$$P = \lceil x, u, \xi, \omega \rceil$$

respectively,

$$\Theta = \lceil \Phi, \rho, a, b \rceil$$

where x, u, a, $b \in J_3(\mathscr{H})$, Φ is an infinitesimal automorphism of $J_3(\mathscr{H})$ and ρ, ξ, ω are real numbers; furthermore, a commutative cross product on $\mathfrak{R}$ mapping $\lceil \mathfrak{R}, \mathfrak{R} \rceil$ on $\mathfrak{L}$, and a product ΘP mapping $\lceil L, K \rceil$ on $\mathfrak{R}$; finally, the varieties $\mathfrak{W}$ of P with $P \times P = 0$ and $\mathfrak{N}$ of Θ with $\Theta^2 = 0$ (i.e., $\Theta^2 \mathfrak{R} = 0$). For $\mathscr{H}_8$:

$$\dim \mathfrak{R} = 56, \quad \dim \mathfrak{L} = 133, \quad \dim \operatorname{proj} \mathfrak{M} = 27, \quad \dim \operatorname{proj} \mathfrak{N} = 33.$$

$\mathfrak{L}$ could be identified with $E_{7,2}$ represented in $\mathfrak{R}$ by left-hand multiplications according to π_1, leaving $\mathfrak{M}$ invariant, and characterized by this fact. $\mathfrak{N}$ appeared to be invariant under the adjoint transformations, both $\mathfrak{M}$ and $\mathfrak{N}$ being transitively transformed. Cartan's symplectic and fourth degree invariant of this representation could be identified as

$$\{P_1, P_2\} = (x_1, u_2) - (x_2, u_1) + \xi_1 \omega_2 - \xi_2 \omega_1,$$

and

$$\frac{1}{6}\{(P \times P)P, P\} = -\frac{1}{18}(P \times P, P \times P).$$

(These numerical coefficients are valid for $\mathscr{H}_8$.)

The main algebraic tool was an identity

$$(x \times x) \times (x \times x) = (\det x)x$$

for $x \in J_3(\mathscr{H})$, which has been proved by Springer [3] to be characteristic for det, and an identity for $\mathfrak{R}$,

$$(P \times P_1)P_2 - (P \times P_2)P_1 = \tfrac{1}{4}\{P_1, P_2\}P + \tfrac{1}{8}\{P, P_2\}P_1 + \tfrac{1}{8}\{P_1, P\}P_2.$$

This algebra led to a geometric interpretation: The elements of $\mathfrak{R}$ (up to scalar factors) are called points. $[\Theta, \Theta_1] = 0$ reads: the points are joined. Maximal sets of joined points are planes; intersections of different planes, if consisting of at least two points, are called straight lines.

The planes are described by the $P \in \mathfrak{M}$ in the sense that

$$\Theta P = 0$$

means incidence of the point Θ and the plane P. The planes then have the structure of projective planes over $\mathscr{H}$. For two different joined points there is one and only one line containing both of them. The intersection of different planes is void, a point, or a line. Given a plane P and a point Θ outside, there is one and only one plane passing through Θ and intersecting P in a line; algebraically, it is given by ΘP. Two planes P_1, P_2 intersect in a line, iff $P_1 \times P_2 = 0$, in a point iff $\{P_1, P_2\} = 0$, and $P_1 \times P_2 \neq 0$. Then $P_1 \times P_2$ is the intersection. Given a line and a point outside, there is either one plane containing both of them or one line meeting both of them, etc.

These properties are reminders of symplectic geometry of 5-dimensional projective space with its points, lines, planes, and conjugateness with respect to the fundamental skew bilinear form interpreted as jointness of points. When drawing this conclusion, Freudenthal was influenced by Tits [4] who discovered it in another context, to be displayed in Section V.

Indeed, if $\mathscr{H}_1$ is taken for $\mathscr{H}$, it *is* symplectic geometry of 5-dimensional space, though not presented in linear 6-space but in 14-space, where the symplectic group $C_{3,3}$ is represented by π_3 instead of π_1. For $\mathscr{H} = \mathscr{H}_2$ and $\mathscr{H}_4$ one gets symplectic geometry in complex and quaternion projective 5-space (with a hermitian skew form), and the groups $A_{5,1}$ represented by π_3, and $D_{6,1}$ represented by π_1.

4.12. Hence, after the quadruple elliptic (hyperbolic) and projective plane geometries over the four Hurwitz algebras we found a quadruple

of symplectic geometries of 5-space with the groups (complex classification)

elliptic:	B_1	A_2	C_3	F_4
projective:	A_2	$A_2 + A_2$	A_5	E_6
symplectic:	C_3	A_5	D_6	E_7

which have the dimensions

3	8	21	52
8	16	35	78
21	35	66	133,

hence,

$$\begin{array}{rcccc} 5p - 2 + 0, & 0, & 3, & 14, \\ 8p \quad + 0, & 0, & 3, & 14, \\ 14p + 7 + 0, & 0, & 3, & 14, \end{array}$$

where $p = 1, 2, 4, 8$ and $0, 0, 3, 14$ are the dimensions of the continuous automorphisms groups of $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_4$, $\mathscr{H}_8$. (There are a few more arithmetic properties like this, for instance, the varieties of points, lines, and planes in symplectic geometry having the dimensions $1 + 4p$, $2 + 5p$, $3 + 3p$.)

The 3×4–rectangle of groups shows a symmetry, which suggests the completion by a fourth line

$$F_4 \quad E_6 \quad E_7 \quad *$$

with the dimensions

$$52 \quad 78 \quad 133 \quad *.$$

Extrapolating by

$$26p + 26 + 0, \quad 0, \quad 3, \quad 14,$$

one may guess that the dimension of the missing group must be 248, which is the dimension of E_8, and by such number mystical tricks one finds that the real types in the fourth line should be

$$F_{4,2} \quad E_{6,2} \quad E_{7,1} \quad E_{8,1}.$$

The first indication of the magic square arising from this addition is to be found in Tits's *Thèse d'agrégé* [6], though its arithmetical properties are not mentioned. It was probably independently found and used

as a heuristic tool by Freudenthal and Rozenfeld. The problem arose to fill the fourth line with a quadruple of geometries over the four Hurwitz algebras. In 1956 Rozenfeld [3] proposed a unified solution for the whole magic square, which will be considered in Section 4.16. In 1958 (or perhaps 1957) Tits and Freudenthal found independent explanations. The fourth geometry was called metasymplectic by Freudenthal [4, VIII], so the magic square now reads:

	$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_4$	$\mathscr{H}_8$
2-dim elliptic geometry:	B_1	A_2	C_3	F_4
2-dim projective geometry:	A_2	$A_2 + A_2$	A_5	E_6
5-dim symplectic geometry:	C_3	A_5	D_6	E_7
metasymplectic geometry:	F_4	E_6	E_7	E_8

Elliptic plane geometry has one kind of space element, points, (because straight lines connected to points by a fixed polarity are redundant). Projective plane geometry has two kinds, points and lines, 5-dim symplectic geometry has three, points, lines, and planes. Metasymplectic geometry may be expected to have a fourth kind, namely the symplectic geometries of the third line, called for short "symplecta" (plural of symplecton). The algebraic apparatus to deal with this geometry has been developed by Freudenthal in 1958, though partially it goes back as far as 1954 [4, II]. The knowledge of Tits' broader framework of geometries proved to be a useful clue in many details. The method applying to the real metasymplectic geometry was published in 1959 [4, VIII–IX], the unified method in 1963 [4, X–XI].

4.13. The exposition of the algebraic apparatus is postponed in order to start with a selection from the numerous geometric results.

In metasymplectic geometry there are three relations between points, in decreasing strength:

joined, i.e., contained in a line, which is unique if the points are different;

interwoven, i.e., contained in a symplecton, which is unique if the points are not joined;

hinged, i.e., joined to a third point, which is unique if the points are not interwoven.

There are dual relations between symplecta:

joined, i.e., intersecting along a plane, which is unique if the symplecta are different;

interwoven, i.e., intersecting, in a point if they are not joined;

hinged, i.e., joined to a third symplecton, which is unique if they are not interwoven.

If a point A and a symplecton Φ are given there is a point in Φ interwoven with A (and a symplecton through A interwoven with Φ). In the general case they are unique, otherwise A and Φ are called

half-incident, and then there is even a line in Φ joined to A, unique if A and Φ are not incident (and a pencil of symplecta through A joined to Φ, unique if A and Φ are not incident), and every point in Φ is hinged with A (every symplecton through A hinged with Φ).

If the points A, B are interwoven, and B, C joined, then A, C are hinged (and the analogue for symplecta).

In two given symplecta two joined points, one in each of both, may be found, iff they are hinged (and the dual).

A symplecton is a maximal set of pairwise interwoven points, and the set of symplecta through a given point is a maximal set of interwoven symplecta. (Notice, however, that the converses are not true.)

So far there was a duality—points $\leftrightarrow$ symplecta, lines $\leftrightarrow$ planes. This duality breaks down as soon as the algebraic structure of the symplecton (as a variety of points) is compared with that of the point (as a variety of the symplecta containing the point). The first one is just a symplecton, the other is a quadric of projective dimension 5, 6, 8, 12 and signature 1, 2, 4, 8.

The variety of the symplecta around a line has the structure of a real projective plane, whereas the variety of the points in a plane is just a plane over $\mathscr{H}$. The variety of the symplecta around a plane has the structure of a real projective line, whereas the variety of the points in a line is just a line over $\mathscr{H}$.

The automorphisms groups of the metasymplectic geometries are the groups of the fourth line of the magic square. They are transitive on the varieties of points, lines, planes, and symplecta.

4.14. The special algebraic apparatus for the real metasymplectic geometry (group $F_{4,2}$) is, actually, according to a remark of T. A. Springer, a Jordan algebra over split octaves. However, it was formulated in a nonoctavian way, using the well-known cubic form in the presentation

$$\det A + \det B + \det C - \operatorname{tr} ABC,$$

where A, B, C are 3-3-real matrices. This form is closely related to the 27 straight lines on a cubic surface.

The unified algebraic apparatus uses quite different tools. Let $\mathfrak{R}_4$ be a Lie algebra of the fourth line of the magic square. For $\Phi \in \mathfrak{R}_4$ a linear mapping $\langle \Phi, \Phi \rangle$ of $\mathfrak{R}_4$ into itself is defined by

$$\langle \Phi, \Phi \rangle \Phi^* = \tilde{\Phi}^2 \Phi^* + \epsilon_2^{-1}(\mathrm{tr}\ \tilde{\Phi}\Phi^*)\Phi - \epsilon_3^{-1}(\mathrm{tr}\tilde{\Phi}^2)\Phi^*$$

$$\epsilon_2 = 9, 12, 18, 30,$$

$$\epsilon_3 = \frac{9.52}{10}, 72, 126, 240.$$

$\tilde{\Phi}$ means the adjoint of Φ.

$\langle \Phi_1, \Phi_2 \rangle$ is the symmetric bilinear expression belonging to $\langle \Phi, \Phi \rangle$. $\mathfrak{R}_1$ is the linear space spanned by the $\langle \Phi_1, \Phi_2 \rangle$. The adjoint of $\mathfrak{R}_4$ is represented in $\mathfrak{R}_1$ in a natural way.

$\Phi \in \mathfrak{W}_4$ iff $\Phi \in \mathfrak{R}_4$ and $\langle \Phi, \Phi \rangle = 0$. The elements $\neq 0$ of $\mathfrak{W}_4$ up to scalar factors are the symplecta.

$A \in \mathfrak{W}_1$ iff A has the form $\langle \Phi_1, \Phi_2 \rangle$ with $[\Phi_1, \Phi_2] = 0$. The elements $\neq 0$ of $\mathfrak{W}_1$ up to a scalar factor are the points.

The symplecta Φ_1, Φ_2 are

joined iff $\langle \Phi_1, \Phi_2 \rangle = 0$;
interwoven iff $[\Phi_1, \Phi_2] = 0$;
hinged iff $tr\ \Phi_1\Phi_2 = 0$.

The points A, B are

joined iff $AB = 0$;
interwoven iff $[A, B] = 0$;
hinged iff $\mathrm{tr}\ AB = 0$.

The point A and the symplecton Φ are

incident iff there is a symplecton Φ^* with $\Phi = A\Phi^*$;
half incident iff $A\Phi = 0$.

4.15. Freudenthal's [4, V–VII] was dedicated to an *axiomatic* approach to octavian symplectic geometry. The axiomatic system is extremely simple. Its elements is a set $\mathfrak{P}$ of "points" and a binary reflexive symmetric relation, called jointness. The maximal sets of pairwise joined points are called planes; they form a set $\mathfrak{M}$. Intersections of two planes containing more than one point are called lines.

Axiom A_p: *Every plane with its points and lines is a plane over $\mathfrak{K}_p$.*

Axiom B: *If Θ is a point and P a plane, $\Theta \in P$, then the set of points in P joined to Θ is a line.*

Two disjoint planes P_1, P_2 are connected by a natural antiautomorphism $P_1 \to P_2$ carrying a point Θ of P into the line in P_2 which is pointwise joined to Θ.

Axiom C: $(P_3 \to P_1)(P_2 \to P_3)(P_1 \to P_2)$ *(for pairwise disjoint* P_i*) is a polarity of* P_1.

This axiom can also be formulated as a pure incidence theorem.

The axiomatic system characterizes the 5-dimensional symplectic geometries over the reals ($p = 1$) and over the octaves ($p = 8$). For $p = 2, 4$ it is possibly not sufficient.

A general axiomatic system for polar geometries comprising geometries on a quadric, unitary, and symplectic geometries over arbitrary fields has been developed by Veldkamp [1]. However, it does not cover the (non-Desarguean) symplectic geometry over $\mathscr{H}_8$.

Metasymplectic geometry has not yet been axiomatized.

4.16. The groups of the magic square possess symmetric spaces the dimension of which is a power of 2, viz., $2pq$ for the group in the p-th column and q-th row, where both columns and rows are numbered by $p = 1, 2, 4, 8$, $q = 1, 2, 4, 8$. These symmetric spaces belong to the real types

$$\begin{array}{cccc} B_{1.1} & A_{2.1} & C_{3.1} & F_{4.1} \\ & A_{2.*} & A_{5.2} & E_{6.1} \\ & & D_{6.4} & E_{7.1} \\ & & & E_{8.2}. \end{array}$$

Their stability groups are spin representations of certain orthogonal groups (see Cartan [6]).

This remark led Rozenfeld [3] to explain the compact types of the magic square groups as elliptic groups of planes over $H_p \otimes H_q$. The plane structure should be a weak one; there might be exceptional pairs of points with no unique joining line.

The origin of Rozenfeld's idea to admit tensor products of Hurwitz and related algebras is found in his book [2]. A projective geometry can be interpreted as an elliptic geometry by the following trick:

Consider pairs of points x, y *and pairs of hyperplanes* u, v *in a projective* n*-space.* Such quadruples possess a projective invariant

$$(x, v)(y, u)/(x, u)(y, v).$$

Combine x and u into one

$$A = xe_+ + ue_-$$

analogously,

$$B = ye_+ + ve_-$$

where

$$e_+ = \tfrac{1}{2}(1 + e), \qquad e_- = \tfrac{1}{2}(1 - e)$$

are "dual numbers,"

$$e^2 = 1.$$

A and B may be considered as points of a projective n-space over the algebra of dual numbers; indeed, multiplication of A with the dual number $\alpha e_+ + \beta e_-$ yields $\alpha x e_+ + \beta u e_-$. Hence, it means multiplication of x and u separately. Using the automorphism $\alpha \to \tilde{\alpha}$ of the algebra of dual numbers which is induced by $e \to -e$, one gets

$$A\bar{A} = (x, u), \qquad B\bar{B} = (y, v), \qquad A\bar{B} \cdot B\bar{A} = (x, v)(y, u).$$

So the above projective invariant of four elements may be written as an invariant of two points

$$A\bar{B} \cdot B\bar{A} / A\bar{A} \cdot B\bar{B}$$

which provides the projective space over the algebra of dual numbers with an elliptic structure.

Had we started with a projective geometry over the complex numbers or over the quaternions, the result would have been an elliptic geometry over an algebra of dual numbers with complex or quaternion coefficients, in other words, over the tensor product of the algebra of dual numbers with that of complex numbers or quaternions. (Here one has to be careful with the order of the factors in the invariant.)

Likewise, symplectic geometry can be interpreted as an elliptic geometry over split quaternions (basis 1, i, e, ei, multiplication rules as for quaternions except $e^2 = (ei)^2 = 1$). Lines are to be interpreted as points; the cross ratio of two lines with their polars in symplectic geometry becomes the elliptic invariant. Again one can start with symplectic geometry over complex numbers and quaternions to arrive at elliptic geometry over the tensor product of the algebra of split quaternions with that of complex numbers or quaternions.

This device has been used by Rozenfeld in a systematic way.

4.17. In a tensor product $\mathfrak{R}$ of Hurwitz algebras $\mathscr{H}_p$ or split Hurwitz algebras $\mathscr{H}_p'$ the conjugate of an element is defined by taking the conjugates in both factors. So if one factor is $\mathscr{H}_p$ or $\mathscr{H}_p'$ and the other $\mathscr{H}_q$ or $\mathscr{H}_q'$, the dimension of the subspace of real elements is

$$pq-p-q+2$$

of pure imaginary elements is

$$p+q-2.$$

Unitary groups of affine n-space over $\mathscr{K}$ are defined by imposing invariance of a hermitean form over $\mathscr{K}$. The dimension of such a group can be computed by counting the number of real parameters in the infinitesimal unitary skew hermitean matrices and adding the dimension of the automorphisms group of $\mathscr{K}$ which is the sum of those of the factor (0, 0, 3, 14). The matrix has n purely imaginary diagonal coefficients and $\binom{n}{2}$ general coefficients. So the dimension of the unitary group of affine n-space over $\mathscr{K}$ becomes

$$(p+q-2)n+pq\binom{n}{2}+(0,0,3,14)+(0,0,3,14)$$

according to $p = 1, 2, 4, 8$, $q = 1, 2, 4, 8$.

Elliptic geometry of projective n-space over $\mathscr{K}$ should admit this group as its stability group. The group of motions should have pqn additional dimensions. Comparing its dimension with those of semi-simple groups, we find as elliptic groups of projective n-space over $\mathscr{K}$:

p, q	1	2	4	8
1	$D_{\frac{1}{2}(n+1)}$ or $B_{\frac{1}{2}n}$	A_n	C_{n+1}	F_4
2		$A_n \times A_n$	A_{2n+1}	E_6
4			D_{2n+2}	E_7
8				E_8

In the last column it is assumed that $n = 2$. The real types of these groups will depend on the signature of the underlying form and the division or split character of the factor algebras of $\mathscr{K}$.

Of course, this is only heuristic arguments. They are even incorrect in the cases of the last column, because then the lack of associativity in $\mathscr{K}$ forbids to define projective points as classes of scalar multiples of a vector. Outside the last column the argument can be validated by classic

methods. For $p = 8$, $q = 1$, Freudenthal's method of Jordan algebras can be used. Rozenfeld [3] claimed that it works in all cases $p = 8$, but this claim can be refuted for $q = 4, 8$. For $p = 8$, $q = 2$, Springer has proved [unpublished observations] that the method still works. Elliptic geometry over octaves with complex coefficients yields the compact, symmetric space belonging to $E_{6,1}$.

To justify Rozenfeld's assertions, which are undoubtedly true, one should prove that in any symmetric space of the magic square there exists:

1. an invariant set of pq-dimensional manifolds (straight lines) such that two general points are contained in one and only one;
2. an invariant polarity notion which imposes a structure of elliptic plane;
3. a way to describe the structure of the space by means of the algebras $\mathscr{H}$ (see Freudenthal [13]).

Rozenfeld found a few more arithmetical relations in the magic square, which still are waiting for an explanation.

4.18. Tits [8] discovered a unified presentation of all algebras of the magic square with all their real types:

K : a field of characteristic $\neq 2,3$;

$A = K$ or an alternative algebra of degree 2;

$C = K$ or a Jordan algebra of degree 3;

A_0, C_0 the kernels of the generic trace of A, C;

B, D the algebras of derivations of A, C.

For $a_1, a_2 \in A$

$$\langle a_1, a_2 \rangle b = \tfrac{1}{4}[[a, a']b] - \tfrac{3}{4}\{a, a', b\}$$

defines an element $\langle a_1, a_2 \rangle$ of B; for $c_1, c_2 \in C$

$$\langle c_1, c_2 \rangle d = c_1(c_2 d) - c_2(c_1 d)$$

defines an element of D.

In

$$L = B + A_0 \otimes C_0 + D$$

a Lie algebra structure is defined.

In B and D the commutator is the usual one,

$$[B, D] = (0),$$
$$[b, a \otimes c] = ba \otimes c \quad \text{for} \quad b \in B, \quad a \in A_0, \quad c \in C_0,$$
$$[d, a \otimes c] = a \otimes dc \quad \text{for} \quad d \in D, \quad a \in A_0, \quad c \in C_0,$$
$$[a_1 \otimes c_1, a_2 \otimes c_2] = (c_1, c_2)\langle a_1, a_2\rangle + (a_1a_2 - (a_1, a_2)) \otimes (c_1c_2 - (c_1, c_2)) + (a_1, a_2)\langle c_1, c_2\rangle \text{ for } a_i \in A_0, c_i \in C_0.$$

Taking for A and C the usual algebras, one gets all Lie algebras of the magic square.

This unified presentation is likely to play a major role in the future.

5. Tits Geometries

5.1. If (complex) A_l is presented as the group of linear mappings of $(l + 1)$-space R with determinant 1, the fundamental representation π_i of A_1 is just the representation induced in the space $R^{(i)}$ of the i-vectors (skew i-tensors) of R. The highest weight-vectors of π_i with respect to any (ordered) trunk (Cartan subgroup) are the pure i-vectors (exterior products of i vectors), which in the projective view represent the projective $(i - 1)$-subspaces of R in the Plücker coordinatization of the i-Grassmann variety Γ_i.

If the weights of π_1 are $\omega_1, \ldots, \omega_{l+1}$ $(\Sigma\omega_j = 0)$, the root forms are $\omega_i - \omega_j$, among which we can take $\rho_i = \omega_i - \omega_{i+1}$ in this order as primitive ones. Every positive root form is a sum $\rho_i + \rho_{i+1} + \ldots + \rho_j$ of primitive ones. The highest weight of π_i is $\omega_1 + \ldots + \omega_i$; it is called π_i too.

Γ_i acted on by $\pi_i(A_l)$ is a homogeneous space. To find its stability group, we look for the elements of $\pi_i(\mathbf{A}_l)$ which conserve the highest weight-vector up to a scalar factor. They are spanned by the whole trunk, by all branches $\mathbf{e}_\alpha$ belonging to positive root forms α, and by all branches $\mathbf{e}_{-\alpha}$, where α as a sum of primitive root forms is free from summands ρ_i.

Reversing this argument one can define the Grassmann varieties by group theory as homogeneous spaces of A_l, the stability groups of which are determined in a simple manner by the graph of $\mathbf{A}_l$.

5.2. This method can be generalized to arbitrary (complex) semi-simple groups.

Let **G** be a complex semisimple Lie algebra, **H** an ordered trunk, **B** the Borel subalgebra (hence, maximal solvable) spanned by **H** and the branches $\mathbf{e}_\alpha$ of positive root forms α. Let $\rho_1, \ldots, \rho_l$ be the primitive root forms, and $\mathbf{G}^{(i)}$ the subalgebra spanned by **B** and the branches $\mathbf{e}_{-\alpha}$, where α as a sum of primitive root form is free from ρ_i.

(According to Morozov [1] and Karpelevič [1] the $G^{(i)}$ are essentially all nonsemisimple maximal subalgebras of **G**.)

In the case $\mathbf{A}_l$, $G^{(i)}$ is the stability group of Γ_i, B the stability group of the homogeneous space of total flags. Γ_i can be also considered as the variety of conjugates of $G^{(i)}$, the variety of total flags as the variety of conjugates of B. Elements of different Γ_i are considered as incident (i.e., one of them contains the other) iff the intersection of their stability groups contains a conjugate of B.

Again these properties may be used as definitions in the general case.

In the case A_l the elements of all Γ_j incident with an element c of Γ_i and different from c fall into two subsets, the projective geometry within c with the group A_{i-1} and the projective geometry around c with the group A_{l-i}. Every element of the one is incident with every element of the other.

5.3. Such considerations led Tits [5–7] to the notion of a category of incidence geometries $J(W)$ belonging to graphs W of semisimple groups.

Every $J(W)$ consists of a graph W, a family of sets E_i corresponding to the nodes of W, and a binary symmetric reflexive incidence relation I on $\bigcup E_i$ which on every single E_i coincides with the identity relation.

The residual geometry of a $J(W)$ with respect to an $a \in E_i$ consists of the graph W' arising from W by removal of the node i and its bonds with other nodes, the family of sets E_j' of $c \in E_j$ incident with a $(j \neq i)$, and the restriction of I to $\bigcup E_j'$.

Axiom I: *Let W be the disjoint union of W' and W''. Then the sets of $J(W)$ are those of $J(W')$ and $J(W'')$, and the incidence relation of $J(W)$ coincides with those of $J(W')$ and $J(W'')$ in their definition domain and with the all-relation outside.*

Hence, if i and k are separated in W by j, then elements $a \in E_i$ and $c \in E_k$ are incident with each other as soon as both of them are incident with some element b of E_j.

Axiom II: *If $J(W)$ is given, let for any $x \in \bigcup E_i$, and some j, $\Phi(x)$ be the set of all elements of E_j incident with x. Then, if a and b are non-*

incident and if $\Phi(a) \cap \Phi(b)$ *is nonvoid, there exists some* c, *such that* $\Phi(c) \supset \Phi(a) \cap \Phi(b)$.

For instance, in the case A_l, the intersection of two nonvoid projective subspaces, if containing a point, is again a nonvoid projective subspace.

Axiom III: *For any* $J(W)$, $\bigcup E_i$ *does not split into two totally non-incident subsets except if* W *consists of one single node.*

Hence, two elements of $\bigcup E_i$ can always be joined by a finite chain of elements each of which is incident with the next one. One may even choose the links of the chain alternatively from two given sets E_i.

These axioms are fulfilled in the example with which we started, i.e., the geometries belonging to the complex semisimple Lie groups G, where $\bigcup E_i$ is the set of maximal subgroups of G containing a Borel subgroup (or equivalently, the set of conjugates of $G^{(i)}$), and $a \in E_i$, $b \in E_j$ are called incident if $a \cap b$ contains a Borel subgroup (conjugate of B). The main tool to prove this and to deal with these geometrics is the lemma of Bruhat which for the present use can be stated in the form:

Two Borel subgroups have a trunk in common.

By this fact the study of the relations between elements of E is enormously simplified. One can replace E_i by the set E_i^0 of maximal subgroups of G containing a given trunk H (or equivalently by the subgroups equivalent to $G^{(i)}$ under the kaleidoscope group), and E by $E^0 = \bigcup E_i^0$.

As $G^{(i)}$ is generated by H and the $\mathbf{e}_\alpha$ with $(\pi_i, \alpha) \geqslant 0$ ($\pi_i = i$-th fundamental weight), E_i^0 may be identified with the set Π_i of weights equivalent to π_i under the kaleidoscope group. Incidence of two elements of E^0 is then translated by the relation $(\lambda, \alpha)(\mu, \alpha) \geqslant 0$ for all root forms α between the corresponding $\lambda, \mu \in \Pi = \bigcup \Pi_i$.

This yields a simple algebraic tool to deal with the Tits geometries belonging to complex semisimple groups.

5.4. In a category subjected to these axioms the geometries $J(W)$ can be recursively studied on the basis of the knowledge of the geometries of rank 2, the W of which is the graph of $\mathbf{A}_2$, $\mathbf{B}_2$, or $\mathbf{G}_2$.

The Axioms I–III are a marvelous tool to derive incidence relations. Let us write a chain such as mentioned after Axiom III by a string of numbers i, each representing an element of E_i, but not necessarily the same one if the number appears several times. Then one can formulate incidence theorems of the kind:

For any pair of elements of E_i, E_j *there exists a joining chain iabcj.*

If for two given elements of E_i, E_j *there exists a joining chain iabcj, then for the same pair there exists a chain iuvj too.*

If for two given elements of E_i, E_j there exists a joining chain iabcj, it is either unique, or there exists a joining chain iuvj too.

In the case of $\mathbf{A}_l$ incidence theorems of this kind may read as follows:

Given a point and a line, there is a chain 1212 *joining them.*

If, given a i-space and a j-space, there is a k-space such that ikj, then there is for $k' = i + j - k - 1$ and $0 \leqslant k' < n$, a k'-space such that ik'j.

If, given a i-space and a j-space, there is a k-space $(k < i)$ such that ikj, then this k-space is either unique, or there is a $(k + 1)$-space such that $i(k + 1)j$.

n is called the length of the chain $a_0a_1 \ldots a_n$. A chain is called irreducible if all its elements are different. A geometry of rank $l = 2$ is called a n-gonic structure if for any two elements of $E_1 \cup E_2$ there exists a joining chain of length $\leqslant n$ and, at most, one joining chain of minimal length $< n$.

The characteristic incidence properties of this kind are for the geometries of rank 2: $\mathbf{A}_2$: a 3-gonic structure, $\mathbf{B}_2$: a 4-gonic structure, $\mathbf{G}_2$: a 6-gonic structure.

5.5. To show how the recursive method works, we derive in the geometry of $\mathbf{A}_l$ the existence of a line joining two given points:

In any case there is a joining chain 121212 ... 1. Deleting the node 1 we get a geometry $\mathbf{A}_{l-1}$ in which 2 plays the role of point. By induction we may suppose the existence of a 3 incident with the two first 2 of the above chain, hence 123212 ... 1. Here, by the consequence of Axiom I we may omit a 2 between 1 and 3, as to get a chain 1312 ... 1. Now, deleting the node 3 we get in the left-hand part of the graph a two-dimensional projective geometry which contains the first two 1's of the chain. These are points which can be joined by a line so we get a new chain 1212 ... 1 which is by two links shorter than the one we started with. This process can be repeated with the final result of the existence of 121.

To show the use of Axiom II, we prove the uniqueness of this line: If a pair of points and a pair of lines are such that every point is incident with every line, then according to Axiom II there is an element i incident with all of them. If the points and the lines are different, then $i > 2$. By induction with respect to l as in the former proof we can go back to $l = 2$ where the uniqueness is assumed as a base of the induction.

The graph of $\mathbf{D}_l$ yields the geometry on a nondegenerate quadric in projective $(2l - 1)$-space. E_3 is the set of points on the quadric, the geometry around a point has the graph of D_{n-1}, which means that it is the geometry on a quadric in $(2l - 3)$-space. For $3 < i < l$, we get the $(i - 3)$-projective geometry within an element of E_i, and the geometric on a quadric in $(2l - 2i + 3)$-space around that element; E_i is the set

of projective $(i - 3)$-spaces on the quadric in $(2l - 1)$-space. By such reasons an element of E_l is a projective $(l - 3)$-space on the quadric, and the axis of two pencils of $(l - 1)$-spaces, which are elements of E_1 and E_2. Incidence is defined by means of the inclusion relation as in the case of $\mathbf{A}_l$, except for $a_i \in E_i$ $(i = 1, 2)$, where it means intersecting in a $(l - 2)$-space.

In the same way the graphs of the other simple groups may be used to study the nature of the geometric elements. A large number of incidence theorems such as mentioned above, are found in Tits' paper [7].

5.6. The case F_4 is particularly interesting. Its graph is

$$\dot{a} - \dot{b} \Rightarrow \dot{c} - \dot{d}$$

It shows that the geometry within an element of E_d is the symplectic one of projective 5-space; hence, an element of E_d is a 5-symplecton, the points, lines, planes of which are elements of E_a, E_b, E_c. The geometry around a point (element of E_a), however, is that of a quadric, the points, lines, planes of which are the symplecta, planes, lines in the earlier sense. The symplecta around a plane form a pencil (projective line), whereas the geometry within a plane is just plane geometry etc.

These are well-known properties of metasymplectic geometry. In Tits' paper [12] a few incidence properties of the F_4-geometry are found. Between two points one gets the chains *aa* (identical), *aba* (joined), *ada* (interwoven), *ababa* (hinged), and *abababa* (generic). Two lines can be in 17 different relative positions, which in the general case cannot be described by single chains (e.g., the condition on two lines of lying in one symplecton and intersecting in a point).

5.7. The axioms of Section 5.3 do not define uniquely a geometry belonging to a given graph. The incidence properties mentioned thus far are "trivial" ones which have to be supplemented by nontrivial, lock incidence theorems of the kind of Desargues and Pappus-Pascal in usual projective geometry.

In this respect $\mathbf{G}_2$-geometry is particularly novel. The algebraic tools to deal with it are split octaves. The set of points a is defined by $a^2 = 0$ (equivalently $(a, a) = 0$ and $\mathrm{Re}\, a = 0$); for two points a, b the relation of being joined is $ab = 0$; lines are maximal sets of joined points. Points and lines form a 6-gonic structure. Schellekens [1] has studied lock incidence theorems in ths actual $\mathbf{G}_2$-geometry, by which this geometry is characterized axiomatically among 6-gonic structures. This work was done in a broader context, which will be sketched in Section 5.9.

The axiomatic approach to polar geometries by Veldkamp has been mentioned in Section 4.15.

5.8. Thus far we dealt with the notion of geometries $J(W)$ subjected to the Axioms I–III with a view to geometries of complex semisimple Lie groups. The notion has proved still more useful in the investigations on geometries of real types of semisimple Lie groups.

On a quadric in real projective $(2l-1)$-space characterized by a number of p positive and $q = 2l - p$ negative squares in its quadratic form ($p \leqslant q$), there are projective subspaces only up to the dimension $p - 1$. So there are no more than p kinds E_i of geometrical objects, where p is the real rank of the group G of the quadric (a real form of D_l). The incidence structure in $\bigcup E_i$ looks much like that of B_p or C_p.

In a group theory interpretation the loss of geometrical objects means that maximal subgroups might be lost as soon as the complex group is restricted to a real type. Maximal subgroups of G might cease to be maximal in the complex extension, they might be the intersection of complex conjugate maximal subgroups of complex G.

Tits [12] fitted geometries of real semisimple groups into the frame of geometries $J(W)$:

$\bigcup E_i$ consists of the maximal proper subgroups of G containing a Borel subgroup over the complex field, the E_i being classes of conjugacy; two elements of $\bigcup E_i$ are incident, if their intersection still contains a Borel subgroup over the complex field. The graph belonging to this system of E_i is defined *a posteriori*: To every E_i belongs a node of W. To know the numbers of bonds by which two nodes i, j are to be joined, we form the residual geometry with respect to a flag which has one element common with each E_k ($k \neq i, j$). If this residual geometry is a m-gone, i and j are joined by a $(m-2)$-fold bond. (So we have to use a 4-fold, instead of the traditional 3-fold, bond for $\mathbf{G}_2$.)

It is not difficult to establish the list of graphs of real types of semisimple groups (see Tits [12]). No new kind of graphs appear.

It is a new striking feature of the magic square that the real groups of every line show the same graph, viz., the 1-point-graph for the hyperbolic plane geometries, the $\mathbf{A}_2$-graph for projective plane geometries, the $\mathbf{C}_3$-graph for 5-dimensional symplectic geometries, and the $\mathbf{F}_4$-graph for metasymplectic geometries. This means that the trivial incidence properties are the same for geometries of the same line of the magic square.

5.9. Tits' formulations are still more general. They apply to algebraic groups also. A special case [13] dealt with is that of B_2 over a perfect

field of characteristic 2. There the permutation of the two nodes of B_2 can be extended to an outer automorphism. Geometrically, this means a duality between E_1 and E_2 (points and lines). Polarities and their sets of selfconjugated points (ovoids) have been studied and their groups have been related to the Suzuki groups.

Another special case is related to the trialities on quadrics in projective 7-space classified by Tits [11]. These are automorphisms of the D_4-geometry mapping $E_1 \to E_2 \to E_3 \to E_1$ with the period 3. According to T. A. Springer [unpublished observations] one defines a $*$-product for split octaves over K by

$$a * b = \tau^2 \bar{a} \cdot \tau \bar{b},$$

where τ is an automorphism of K of period 3. Generalizing the method in Section 5.7 one defines points by $a * a = 0$, the relation of being joined by $a * b = 0$, and lines as maximal sets of joined points (see Schellekens [1]). The group of this geometry is a G_2, if $\tau = 1$. If, however, $\tau \neq 1$, the group may be an exceptional D_4 over the subfield of τ-invariants. In Schellekens' earlier mentioned axiomatic approach to $\mathbf{G}_2$-geometry the existence of this kind of hexagonic structure is used to proving the independence of certain lock incidence axioms.

References

R. Baer

1. Free mobility and orthogonality, *Trans. Am. Math. Soc.* **68** (1950), 439-460.

G. Birkhoff

1. Metric foundations of geometry, *Proc. Natl. Acad. Sci. U.S.* **27** (1941), 402-406.
2. Metric foundations of geometry I, *Trans. Am. Math. Soc.* **55** (1944), 465-492.
3. Extensions of Lie groups, *Math. Z.* **53** (1950), 226-235.

F. van der Blij

1. History of the octaves, *Simon Stevin* **34** (1961), 106-125.

F. van der Blij and T. A. Springer

1. Octaves and triality, *Nieuw Archief v. Wiskunde* **8** (1960), 158-169.

A. Borel

1. Le plan projectif des octaves et les sphères commes espaces homogènes, *Compt. Rend. Acad. Sci.* **230** (1950), 1378-1380.

L. E. J. Brouwer

1. Die Theorie der endlichen kontinuierlichen Gruppen, unabhängig von den Axiomen von Lie, *Math. Ann.* **67** (1909), 246-267; **69** (1910), 181-203.

H. Busemann

1. Metric methods in Finsler spaces, *Ann. Math. Studies* No. 8 (1942).

St. S. Cairns

1. An axiomatic basis for plane geometry, *Trans. Am. Math. Soc.* **35** (1923), 235-244.

É. Cartan

1. Sur la structure des groupes de transformations finis et continus. Ph. D. Thesis, Paris, 1894; Œuvres.
2. Sur la réduction à sa forme canonique de la structure d'un groupe de transformations fini et continu, *Am. J. Math.* **18** (1896), 1-61; Œuvres, I," pp. 137-292.
3. Les groupes projectifs continus réels qui ne laissent invariante aucune multiplicité plane, *Bull. Soc. Math. France* **41** (1913), 53-96; "Œuvres, I," pp. 493-530.
4. Les groupes réels simples finis et continus, *Ann. Sci. École Norm. Sup.* **31** (1914), 263-355; "Œuvres, I," pp. 399-491.
5. Sur un théorème fondamental de M. H. Weyl, *J. Math. Pures Appl.* **2** (1923), 167-192.
6. Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, *Ann. Sci. École Norm. Sup.* **44** (1927), 345-467; "Œuvres, I," pp. 867-989.
7. Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces, *Rend. Circ. Mat. Palermo* **53** (1929), 217-252.

C. Chevalley and R. D. Schafer

1. The exceptional simple Lie algebras F_4 and E_6, *Proc. Natl. Acad. Sci. U.S.* **36** (1950), 137-141.

H. Freudenthal

1. Einige Sätze über topologische Gruppen, *Ann. Math.* **37** (1936), 46-56.
2. "Oktaven, Ausnahmegruppen und Oktavengeometrie." Mathematisch Instituut der Rijksuniversiteit te Utrecht, 1951 (mimeographed); new revised edition, 1960.
3. Zur ebenen Oktavengeometrie, *Proc. Koninkl. Ned. Akad. Wetenschap.* **A56** (1953), 195-200.
4. Beziehungen der E_7 und E_8 zur Oktavenebene, I-XI,
   I-II: *Proc. Koninkl. Ned. Akad. Wetenschap.* **A57** (1954), 218-230, 363, 368.
   III-IV: *Proc. Koninkl. Ned. Akad. Wetenschap.* **A58** (1955), 151-157, 277-285.
   V-VII: *Proc. Koninkl. Ned. Akad. Wetenschap.* **A62** (1959), 165-201.
   VIII-IX: *Proc. Koninkl. Ned. Akad. Wetenschap.* **A62** (1959), 447-474.
   X-XI: *Proc. Koninkl. Ned. Akad. Wetenschap.* **A66** (1963), 457-487.
5. La topologie dans les fondements de la géométrie, *Internat. Congr. Math.* (Amsterdam, 1954) **III**, 178-184.
6. Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems, *Math. Z.* **63** (1956), 374-405.
7. Zur Geschichte der Grundlagen der Geometrie, *Nieuw Archief v. Wiskunde* **5** (1957), 105-142.
8. Die Grundlagen der Geometrie um die Wende des 19. Jahrhunderts, *Math. Phys. Semesterber.* **7** (1960), 1-25.

9. Zu den Weyl-Cartanschen Raumproblemen, *Arch. Math.* 11 (1960), 107-115.
10. Im Umkreis der sogenannten Raumprobleme. *In* "Essays on the Foundations of Mathematics," dedicated to A. H. Fraenkel, Jerusalem, 1961, pp. 322-327.
11. Das Helmholtz-Liesche Raumproblem bei indefiniter Metrik, *Math. Ann.* (1964), in press.
12. Symplektische und metasymplektische Geometrien. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959*, pp. 31-33 (1962).
13. Bericht über die Theorie der Rosenfeldschen elliptischen Ebenen. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959*, pp. 35-37 (1962).

H. Helmholtz

1. Über die tatsächlichen Grundlagen der Geometrie (Vortrag vom 22. Mai 1866?), *Verhandl. Naturhist. med. Verein Heidelberg* **4** (1868), 197-202; *Wiss. Abhandl.* **II** (1883), 610-617.
2. Über die Tatsachen, die der Geometrie zum Grunde liegen. *Nachr. Ges. Wiss. Göttingen* (1868), pp. 193-221; *Wiss. Abhandl.* **II** (1883), 618-639.

D. Hilbert

1. Grundlagen der Geometrie. Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Leipzig, 1899.
2. "Grundlagen der Geometrie," 2nd ed., 1902, Suppl. IV; 9th ed. 1962, Suppl. IV.

G. Hirsch

1. La géométrie projective et la topologie des espaces fibrés, *Colloq. Intern. Centre Nat. Rech. Sci.* No. 12 (1949), 35-42.

S. Iyanaga and M. Abe

1. Über das Helmholtzsche Raumproblem. *Proc. Imp. Acad. Tokyo* **19** (1943), 174-180, 540-543.

N. Jacobson

1. Some groups of transformations defined by Jordan algebras III, *J. Reine Angew. Math.* **207** (1961), 61-85.

P. Jordan

1. Über eine nicht-desarguessche ebene projektive Geometrie, *Abhandl. Math. Sem. Univ. Hamburg* **16** (1949), 74-76.

F. Karpelevič

1. On the (non-semi-simple) maximal subalgebras of semi-simple Lie algebras (Russian), *Dokl. Akad. Nauk SSSR* **76** (1951), 775-778.

B. de Kérékjarto

1. Sur le groupe des homographies et des antihomographies d'une variable complexe, *Comm. Math. Helv.* **13** (1950), 68-82.

W. Klingenberg

1. Eine Kennzeichnung der Riemannschen sowie der Hermiteschen Mannigfaltigkeiten, *Math. Z.* **70** (1959), 300-309.

A. Kolmogoroff

1. Zur topologisch-gruppentheoretischen Begründung der Geometrie. *Nachr. Ges. Wiss. Göttingen* (1930) pp. 208-210.

N. H. Kuiper

1. A real analytic non-desarguian plane. *Nieuw Archief v. Wiskunde* **5** (1957), 19-24.

D. Laugwitz

1. Über die Invarianz quadratischer Formen bei linearen Transformationen und das Raumproblem. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. IIa* **2** (1956), 21-25.
2. Über eine Vermutung von Hermann Weyl zum Raumproblem, *Arch. Math.* **9** (1958), 128-133.

S. Lie

1. Über die Grundlagen der Geometrie. *Ber. Ges. Wiss. Leipzig* **42** (1890), 284-321, 355-418; *Ges. Abhandl.* **II**, 380-468.
2. Theorie der Transformationsgruppen III, Abt. V." Leipzig, 1893.

R. Lipschitz

1. Fortgesetzte Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen, *J. Reine Angew. Math.* **72** (1870), 1-56.

R. G. Lubben

1. The double elliptic case of the Lie-Riemann-Helmholtz problem of the foundation of geometry, *Fund. Math.* **11** (1928), 34-95.

N. S. Mendelson

1. Non-desarguesian projective plane geometries which satisfy the harmonic point axiom, *Canad. J. Math.* **8** (1956), 522-562.

D. Montgomery and L. Zippin

1. Topological group foundations of rigid space geometry, *Trans. Am. Math. Soc.* **48** (1940), 21-49.

R. L. Moore

1. On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry, *Am. J. Math.* **41** (1919), 299-319.

V. V. Morozov

1. The non-semi-simple maximal subgroups of simple groups (Russian). Ph.D. Thesis, Kazan, 1943.

R. Moufang

1. Alternativkörper und der Satz vom vollständigen Vierseit (D_9), *Abhandl. Math. Sem. Univ. Hamburg* **9** (1932), 207-222.
2. Zur Struktur von Alternativkörpern, *Math. Ann.* **110** (1934), 207-222.

G. Pickert

1. Elementare Behandlung des Helmholtzschen Raumproblems, *Math. Ann.* **120** (1947, 1949), 492-501.
2. Bemerkungen über die projektive Gruppe einer Moufang-Ebene, *Illinois J. Math.* **3** (1959), 169-173.

K. Reidemeister

1. Das Lie-Helmholtzsche Raumproblem und ein Satz von Maschke, *Abhandl. Math. Sem. Univ. Hamburg* **4** (1926), 172-173.

B. Riemann

1. Über die Hypothesen, welche der Geometrie zugrunde liegen, 1854 gehaltener Habilitationsvortrag (1867), *Abhandl. Ges. Wiss. Göttingen* **15** (1876), 254-269.

B. A. Rozenfeld

1. The compact simple group E_6 as the group of motions of the compact non-euclidean plane of octaves (Russian), *Dokl. Akad. Nauk Azerb. SSR* **10** (1954), 829-833.
2. "Non-euclidean Geometries" (Russian). Moscow, 1955.
3. Geometrical interpretation of the compact simple Lie groups of the class E (Russian), *Dokl. Akad. Nauk SSSR* **106** (1956), 600-603.
4. Simple Lie groups and non-euclidean geometry (Russian), *Uch. Zap. Kolomensk. Ped. Inst.* **1** (1956), 11-26.
5. Geometrical interpretation of the symmetric spaces with simple fundamental groups (Russian), *Dokl. Akad. Nauk SSSR* **110** (1956), 23-26.
6. Interpretation of symmetric spaces with simple fundamental groups as manifolds with symmetry features (Russian.), *Uch. Zap. Kolomensk. Ped. Inst.* **2** (1958), 19-37.
7. Einfache Lie-Gruppen und nichteuklidische Geometrien. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959* pp. 135-155 (1962).

H. Salzmann

1. Kompakte zweidimensionale projektive Ebenen, *Math. Ann.* **145** (1962), 401-428.
2. Topologische projektive Ebenen. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959* pp. 157-163 (1962).

E. Scheibe

1. Über das Weylsche Raumproblem, *J. Reine Angew. Math.* **197** (1957), 162-207.

G. J. Schellekens

1. On a hexagonic structure, I-II, *Proc. Koninkl. Akad. Wetenschap.* **A65** (1962), 201-234.

F. Schur

1. Über den Zusammenhang der Räume constanten Riemannschen Krümmungsmasses mit den projektiven Räumen, *Math. Ann.* **27** (1886), 537-567.

T. A. Springer

1. The projective octave plane I-II, *Proc. Koninkl. Akad. Wetenschap.* **A63** (1960), 74-101.
2. The classification of reduced exceptional simple Jordan algebras, *Proc. Koninkl. Akad. Wetenschap.* **A63** (1960), 414-422.
3. Characterization of a class of cubic forms, *Proc. Koninkl. Akad. Wetenschap.* **A65** (1962), 259-265.
4. The projective octave plane. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959* pp. 173-174 (1962).
5. On the geometric algebra of the octave planes, *Proc. Koninkl. Akad. Wetenschap.* **A65**, 451-468 (1962).

T. A. Springer and F. D. Veldkamp

1. Elliptic and hyperbolic octave planes, I-III, *Proc. Koninkl. Akad. Wetenschap.* **A66** (1963), 413-451.

G. K. Ch. von Staudt

1. "Geometrie der Lage." Nürnberg, 1847.

Tae-il Suh

1. On isomorphisms of little projective groups of Cayley planes, I-II, *Proc. Koninkl. Akad. Wetenschap.* **A65** (1962), 320-339.

W. Süss

1. Beiträge zur gruppentheoretischen Begründung der Geometrie, *Tohoku Math. J.* **26** (1926), 365-385; **27** (1926), 214-242; **28** (1927), 228-241; *Japan. J. Math.* **4** (1927), 97-119.

J. Tits

1. Étude de certains espaces métriques, *Bull. Soc. Math. Belg.* **1952** (1953), 44-52.
2. Sur un article précédent: Études de certains espaces métriques, *Bull. Soc. Math. Belg.* **1953** (1953), 124-125.
3. Le plan projectif des octaves et les groupes de Lie exceptionnels, *Bull. Acad. Roy. Belg. Sci.* **39** (1953), 309-329.
4. Le plan projectif des octaves et les groupes exceptionnels E_6 et E_7, *Bull. Acad. Roy. Belg. Sci.* **40** (1954), 29-40.
5. Étude géométrique d'une classe d'espaces homogènes, *Compt. Rend. Acad. Sci.* **239** (1954), 466-468.

6. Sur certaines classes d'espaces homogènes de groupes de Lie, *Mém. Acad. Roy. Belg. Sci.* **29** (1955), fasc. 3.
7. Les groupes de Lie exceptionnels et leur interprétation géométrique. *Bull. Soc. Math. Belg.* **8** (1956), 48-81.
8. "Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles." Manuscript.
9. Sur la géométrie des R-espaces, *J. Math. Pures Appl.* **36** (1957), 17-38.
10. Sur les analogues algébriques des groupes semi-simples complexes. *Colloque d'Algèbre supérieure, 1956* pp. 261-289.
11. Sur la trialité et certains groupes qui s'en déduisent, *Publ. Math. Inst. Hautes Études Sci.* **2** (1959), 13-60.
12. Groupes algébriques semi-simples et géométries associées. Algebraical and topological foundations of geometry, *Proc. Colloq. Utrecht, 1959* pp. 175-192 (1962).
13. Ovoïdes et groupes de Suzuki, *Arch. Math.* **13** (1962), 187-198.

F. D. Veldkamp

1. Polar geometry I-V, *Proc. Koninkl. Akad. Wetenschap.* **A62** (1959), 512-551; **A63** (1960), 207-212.

H. C. Wang

1. Two-point homogeneous spaces, *Ann. Math.* **55** (1952), 177-191.
2. Two theorems on metric spaces, *Pacific J. Math.* **1** (1951), 473-480.

H. Weyl

1. Note to lecture by B. Riemann, "Über die Hypothesen..." p. 27, Berlin, 1919.
2. Die Einzigartigkeit der pythagoraeischen Massbestimmung, *Math. Z.* **12** (1922), 114-116.
3. "Mathematische Analyse der Raumproblems." Springer, Berlin, 1923.

H. Wiener

1. Weiteres über die Grundlagen und Aufbau der Geometrie, *Jahresber. Deut. Mat.-Ver.* **3** (1894), 70-80.

P. Wilker

1. Algebraische Lösung des Helmholtzschen Raumproblems für reelle Vektorräume, *Math. Ann.* **139** (1960), 433-445 .

H. Yamabe

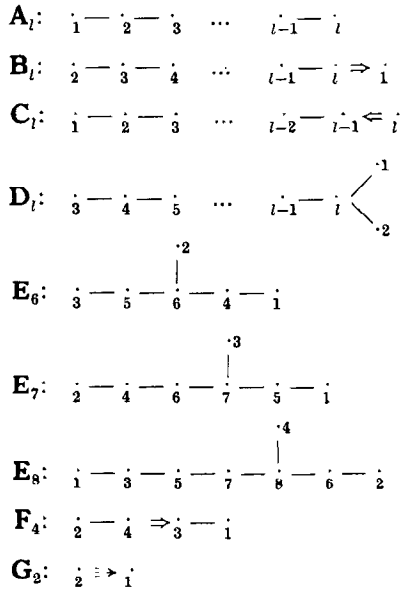
1. A generalization of a theorem of Gleason. *Ann. Math.* **58** (1953), 351-365.

I. Yokota

1. Embeddings of projective spaces into elliptic projective Lie groups. *Proc. Japan Acad.* **35** (1959), 281-283.

Appendix

1. Numbering of primitive root forms and fundamental weights according to É. Cartan:



If $\rho_1, \ldots, \rho_l$ are the primitive root forms, the fundamental weight π_i is defined by

$$2\frac{(\pi_i, \rho_j)}{(\rho_j, \rho_j)} = \begin{matrix} 0 & (i \neq j) \\ 1 & (i = j). \end{matrix}$$

A_0 means the one-dimensional Lie algebra.

2. Let $\mathbf{M}$ be a real Lie algebra and $\mathbf{M} \otimes \mathrm{co}$ its complex extension. Let C the involutory semimorphism of $\mathbf{M} \otimes \mathrm{co}$ which leaves invariant just the elements of $\mathbf{M}$. Let C_0 be an involutory automorphism which defines a unitary restriction of $\mathbf{M} \otimes \mathrm{co}$. If $\mathbf{M}$ is simple, three things may happen:

1. $\mathbf{L} = \mathbf{M} \times \mathrm{co}$ is simple; CC_0 is an inner automorphism; $\mathbf{M}$ is an *inner* real type of $\mathbf{L}$.

2. $\mathbf{L} = \mathbf{M} \otimes \mathrm{co}$ is simple; CC_0 is an outer automorphism; $\mathbf{M}$ is an *outer* real type of $\mathbf{L}$.

3. $\mathbf{M} \times \mathrm{co}$ splits as $\mathbf{L} + CC_0\mathbf{L}$; $\mathbf{M}$ is a *twin* real type of $\mathbf{L}$.

CC_0 may be supposed to conserve a given trunk (Cartan subalgebra), in case

1. elementwise,

2. by a fixed nontrivial automorphism A of the graph, $A\mathbf{e}_\rho = \mathbf{e}_{A\rho}$ for primitive root forms ρ.

In (1) and (2) $T = CC_0$ respectively, CC_0A may be characterized by an integer $j > 0$ indicating that

$$T\mathbf{e}_\alpha = -\mathbf{e}_\alpha \text{ or } \mathbf{e}_\alpha$$

iff the primitive rootform ρ_j occurs with an odd or even coefficient in α; or by $j = 0$ iff $T\mathbf{e}_\alpha = \mathbf{e}_\alpha$ for all rootforms α.

The corresponding real types are indicated by

(1) $\mathbf{L}_j$, (2) $\mathbf{L}_{j,*}$ (3) $\mathbf{L}_{**}$.

Using this notation one can easily read the maximal compact subalgebra from the graph.

Notice that j need not be unique.

3. With respect to an irreducible representation f of complex semi-simple $\mathbf{L}$ with highest weight λ, and a real type $\mathbf{L}_{re}$ of $\mathbf{L}$ with the semi-morphism C, three cases can occur:

(1) λ and its conjugate $C\lambda$ are not equivalent,

(2) λ and $C\lambda$ are equivalent, $\epsilon = -1$,

(3) λ and $C\lambda$ are equivalent, $\epsilon = 1$,

where ϵ is defined by

$$C\lambda - \lambda = \sum q_\nu \rho_\nu,$$

$$\epsilon_\nu = -1 \quad \text{for} \quad \nu \neq j,$$

$$\epsilon_\nu = 1 \quad \text{for} \quad \nu = j,$$

$$\epsilon = \prod \epsilon^{q_\nu}.$$

In (1) and (2) the representation f of L_{re} is essentially complex; to get a real representation one has to double the number of dimensions. In (3) it is essentially real. In (2) f can be considered as a representation in quaternion space of half-dimensionality.

Reference

H. Freudenthal
"Lie Groups." Mimeographed lectures, Yale University, 1961.