TWISTED ALEXANDER POLYNOMIALS OF HYPERBOLIC KNOTS

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Abstract. Given a hyperbolic knot K we use the torsion corresponding to the discrete and faithful $SL(2,\mathbb{C})$–representation to define a knot invariant $\mathcal{T}_K(t) \in \mathbb{C}[t^{\pm 1}]$ with no indeterminacy. We study the basic properties of the invariant $\mathcal{T}_K(t)$. In particular we show that it is always non-zero and we show that it gives lower bounds on the knot genus, fibering obstructions and amphichirality obstructions. We will furthermore show that $\mathcal{T}_K$ detects the knot genus, fiberedness and the chirality of all hyperbolic knots with up to twelve crossings.

1. Introduction

1.1. **Definition of hyperbolic torsion and basic properties.** Let $K \subset S^3$ be an oriented hyperbolic knot. The hyperbolic structure gives rise to a discrete and faithful representation $\pi_1(S^3 \setminus \nu K) \to PSL(2,\mathbb{C})$ which is unique up to an inner automorphism of $PSL(2,\mathbb{C})$ and up to complex conjugation, furthermore this representations lifts to a representation $\pi_K \to SL(2,\mathbb{C})$. (We refer to Section 3 for details). By fixing one type of lift and by normalizing the corresponding torsion invariant we obtain a well-defined invariant $\mathcal{T}_K(t)$ with no indeterminacy, which a priori lies in $\mathbb{C}(t^{\frac{1}{2}})$. We refer to $\mathcal{T}_K(t)$ as the *hyperbolic torsion of K*. The invariant $\mathcal{T}_K(t)$ has the following properties:

Theorem 1.1. *Let K be an oriented hyperbolic knot K. Then $\mathcal{T}_K(t)$ has the following properties:*

1. *$\mathcal{T}_K(t)$ lies in $\mathbb{C}[t^{\pm 1}]$,*
2. *$\mathcal{T}_K(\xi)$ is non-zero for any root of unity ξ, in particular $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are non-zero,*
3. *$\mathcal{T}_K(t) = \mathcal{T}_K(t^{-1})$,*
4. *$\mathcal{T}_K(t)$ is independent of the orientation of K,*
5. *if K^* denotes the mirror image of K, then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$,*
6. *if K is amphichiral, i.e. if $K = K^*$, then $\mathcal{T}_K(t)$ is a real polynomial, i.e. $\mathcal{T}_K(t) \in \mathbb{R}[t^{\pm\frac{1}{2}}]$.*

Remark. The hyperbolic torsion is surprisingly little studied. To our knowledge $\mathcal{T}_K(t)$ has so far only been studied for twist knots by Morifuji [Mo08], and in a slightly

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different setup, it has been studied for 2-bridge knots by Hirasawa and Murasugi [HM08] and Silver and Williams [SW09d]. A related invariant, namely the torsion of hyperbolic 3-manifolds corresponding to the adjoint representation, has been studied in detail by Dubois and Yamaguchi [DY09].

Remark. The torsion of hyperbolic manifolds corresponding to the discrete and faithful representation (i.e. the invariant $\mathcal{T}_K(1)$ in our notation) has been studied by Menal–Ferrer and Porti [MP09]. In particular Theorem 1.1 (2) follows in a straightforward way from their work. The invariant $\mathcal{T}_K(1)$ is an interesting invariant in its own right. It is conjectured that $\mathcal{T}_K(1)$, like the hyperbolic volume, is invariant under mutation.[1] Note though that the invariant $\mathcal{T}_K(1)$ is not related to the volume. For example using a variation on [Po97, Théorème 4.17] one can show ([Po09]) that there exists a sequence of knots K_n such that the volumes converge to a positive real number, whereas the numbers $\mathcal{T}_{K_n}(1)$ converge to zero.

1.2. **Topological information contained in $\mathcal{T}_K(t)$: Fiberedness and genus.** Let $K \subset S^3$ be a knot. We say that K is *fibered* if there exists a fibration $S^3 \setminus \nu K \to S^1$. The *genus* of the knot K is defined to be the minimal genus among all Seifert surfaces of K. The genus of K will be denoted by genus(K). We also write $x(K) = 2\text{genus}(K) - 1$ and refer to it as the *complexity* of K.

Given $p(t) = a_k t^k + \dots + a_l t^l \in \mathbb{C}[t^{\pm 1}]$ with $a_k, a_l \neq 0$ we say that $p(t)$ is *monic* if $a_k = 1$ and $a_l = 1$. Furthermore, the degree of $p(t)$ is defined to be $\deg(p(t)) = l - k$. We also define the degree of the zero polynomial to be zero. The following theorem is an immediate consequence of the work of Goda, Kitano and Morifuji [GKM05] (see also Cha [Ch03], Kitano and Morifuji [KM05], Pajitnov [Paj07], Kitayama [Kiy08], [FK06] and [FV09a, Theorem 6.2]).

Theorem 1.2. *Let $K \subset S^3$ be a hyperbolic knot. If K is fibered, then $\mathcal{T}_K(t)$ is monic and*

$$x(K) = \frac{1}{2} deg(\mathcal{T}_K(t))$$

holds.

The following is an immediate consequence of the definitions and of [FK06, Theorem 1.1].

Theorem 1.3. *Let $K \subset S^3$ be a hyperbolic knot. Then*

$$x(K) \geq \frac{1}{2} deg(\mathcal{T}_K(t))$$

[1][S] I am not quite sure what the status for this is. The invariants $\mathcal{T}_K(\pm 1)$ should be mutation invariant and Porti and Dubois should be able to prove it. But as of Dec 14th they have not proved it.

The calculations in [Ch03], [GKM05] and [FK06] gave evidence that twisted torsions corresponding to more general representations are very successful at detecting non–fibered manifolds and at detecting the genus of a knot. In fact in [FV08b] (see also [FV08a, FV07a, FV09a, FV09b]) it was shown that twisted torsion corresponding to regular representations coming from epimorphisms onto finite groups detect whether a knot is fibered or not. It is not known though whether twisted torsion always detects genus of a knot.

Instead of considering many different representations as in the earlier papers we now focus on one, canonical representation. We propose the following, possibly rather optimistic conjectures.

Conjecture 1.4. *Let $K \subset S^3$ be a hyperbolic knot. If $\mathcal{T}_K(t)$ is monic and if*

$$x(K) = \frac{1}{2}deg(\mathcal{T}_K(t))$$

holds, then K is fibered.

Conjecture 1.5. *Let $K \subset S^3$ be a hyperbolic knot. Then*

$$x(K) = \frac{1}{2}deg(\mathcal{T}_K(t)).$$

Note that Conjectures 1.4 and 1.5 combined give the conjecture that a knot K is fibered if and only if $\mathcal{T}_K(t)$ is monic. For a knot with at most twelve crossings we use *SnapPea* to determine the canonical representation $\alpha : \pi_1(X_K) \to SL(2,\mathbb{C})$ up to a small error. Using Fox calculus (see Section 4) we then compute $\mathcal{T}_K(t)$ up to about 10 digits for all knots in this range. Our calculations show that Conjectures 1.4 and 1.5 hold for all knots with up to twelve crossings. We refer to Section 6 for details.

In Section 5 we show that twisted torsion defines a $\mathbb{C}[t^{\pm 1}]$-valued function on the character variety of the knot exterior. We will show that the set of characters for which the corresponding torsion detects the knot genus is a Zariski open set and that the set of characters for which the corresponding torsion is monic is Zariski closed.

1.3. **Topological information contained in $\mathcal{T}_K(t)$: Chirality and mutation.** In Theorem 1.1 we showed that if K is an amphichiral knot, then $\mathcal{T}_K(t)$ is a real polynomial. This turns out to be a rather good way to detect chirality. In fact we show, that if K is a hyperbolic knot with thirteen crossings or less, then K is chiral if and only if the imaginary part of $\mathcal{T}_K(t)$ is zero.

Hyperbolic invariants often do not detect mutation. For example Ruberman [Ru87] showed that the hyperbolic volume is unchanged under mutation. It is also known that the invariant trace field is unchanged under mutation ([MR03, Corollary 5.6.2]). Also note that the birationality type of the component of the character variety containing the discrete and faithful representation is unchanged under mutation (see [CL96], [Ti00, Ti04]). We conjecture that $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ also do not detect mutation. We again refer to Section 6 for details.

1.4. **Examples: The Conway knot and the Kinoshita-Terasaka knot.** Perhaps the most famous pair of knots are the Conway knot 11_{401} and the Kinoshita-Terasaka knot 11_{409}. These two knots are (positive) mutants with Alexander polynomial one. The genus of the Conway knot 11_{401} is equal to three and the genus of the Kinoshita-Terasaka knot 11_{409} is equal to two.

For the Conway knot we calculate that $\mathcal{T}_{K_{401}}(t)$ equals approximately

$$\begin{array}{lll} (4.8952-0.0992i)t^5 & +(-15.6857+0.2976i)t^4 & +(23.1036+0.0784i)t^3 \\ +(-26.9416-4.8451i)t^2 & +(38.3835+24.4943i)t & +(-43.3240-44.0806i) \\ +(38.3835+24.4943i)t^{-1} & +(-26.9416-4.8451i)t^{-2} & +(23.1036+0.0784i)t^{-3} \\ +(-15.6857+0.2976i)t^{-4} & +(4.8952-0.0992i)t^{-5}. & \end{array}$$

For the Kinoshita–Terasaka knot we calculate $\mathcal{T}_{K_{409}}(t)$ equals approximately

$$\begin{array}{lll} (4.4179+0.3760i)t^3 & +(-22.9416-4.8451i)t^2 & +(61.9644+24.0974i)t \\ +(-82.6954-43.4854i) & +(61.9644+24.0974i)t^{-1} & +(-22.9416-4.8451i)t^{-2} \\ +(4.4179+0.3760i)t^{-3}. & & \end{array}$$

Note that Conjectures 1.4 and 1.5 hold for both knots. Also note that the polynomials are not real, reflecting the fact that the knots are chiral. Finally note that the polynomials are different, which implies that $\mathcal{T}_K(t)$ is not invariant under mutation.

Remark. It is interesting to study the evaluations of $\mathcal{T}_K(t)$. We calculate

$$\begin{array}{lllll} \mathcal{T}_{K_{401}}(1) & \approx & \mathcal{T}_{K_{409}}(1) & \approx & 4.186003-4.228629i \\ \mathcal{T}_{K_{401}}(-1) & \approx & \mathcal{T}_{K_{409}}(-1) & \approx & 261.3432+102.1226i. \end{array}$$

This calculation reflects our conjecture that the invariants $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are unchanged under mutation. On the other hand we have

$$\begin{array}{lll} \mathcal{T}_{K_{401}}(i) & \approx & 33.7952-20.8122i \\ \mathcal{T}_{K_{401}}(i) & \approx & -33.7952+36.8122i. \end{array}$$

This suggests that the evaluation of $\mathcal{T}_K(\xi)$ is not a mutation invariant if $\xi \neq \pm 1$.

1.5. **Final remarks and open problems.**

Remark. (1) Let K be a hyperbolic knot and denote by $\alpha : \pi_1(X_K) \to PSL(2,\mathbb{C})$ the canonical representation. We can also consider the adjoint representation

$$\begin{array}{rcl} \alpha_{adj} : \pi_1(X_K) & \to & \operatorname{Aut}(\mathfrak{sl}(2,\mathbb{C})) \\ g & \mapsto & A \mapsto \alpha(g)A\alpha^{-1} \end{array}$$

associated to α. It is well-known that this representation is also faithful and irreducible. The corresponding twisted torsion was studied by Dubois and Yamaguchi [DY09], partly building on work of Porti [Po97]. We refer to Section 6.4 for a few calculations.

(2) If K is a knot in S^3 such that the complement is a graph manifold, then K is an iterated torus knot and it is well known that K is fibered. This is not true in more general set ups. For example there exists a knot K in an integral homology sphere Y such that $Y \setminus \nu K$ is a graph manifold with the following property: the ordinary Alexander polynomial is monic and its degree equals twice the genus, but the knot is not fibered.
(3) For simplicity we restrict ourselves in this paper to the study of knots. We expect that many of the results and conjectures are similarly valid for 3-manifolds in general. For 3-manifolds the appropriate question is whether twisted torsion detects the Thurston norm and fibered classes (cf. [FK06, FK08] and [FV08b] for more details).
(4) Let K be a knot and α a faithful representation. It is an open question whether the twisted invariant $\tau(K, \alpha)$ is necessarily non–trivial, i.e. not equal to ± 1 up to the indeterminacy. We refer to [Su04] and [SW09c] for more on twisted Alexander polynomials of groups in general and faithful representations.

We conclude this paper with a few questions and open problems:

(1) Does the invariant $\mathcal{T}_K(t)$ contain information about symmetries of the knot besides information on chirality?
(2) Does there exist a hyperbolic knot such that $\mathcal{T}_K(1) = 1$?
(3) Does there exist a hyperbolic knot such that $\mathcal{T}_K(t) = 1$?
(4) Show that the invariants $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are invariant under mutation.
(5) Do we always have $|\mathcal{T}_K(-1)| > |\mathcal{T}_K(1)|$? [2]
(6) Give examples of hyperbolic knots K_1 and K_2 such that $\mathcal{T}_{K_1}(t) = \mathcal{T}_{K_2}(t)$. Can such examples be found among fibered hyperbolic knots?
(7) Let K be an amphichiral hyperbolic knot. Then $\mathcal{T}_K(t)$ is a real polynomial. Is the top coefficient of $\mathcal{T}_K(t)$ always positive? Is it always at least one?
(8) If K is slice, does $\mathcal{T}_K(t)$ factor as $f(t) \cdot f(t^{-1})$? very unlikely!
(9) If K_1 and K_2 are two knots such that there exists an epimorphism $\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$. Does it follow that $\mathcal{T}_{K_1}(t)$ divides $\mathcal{T}_{K_2}(t)$? Also unlikely, but worth checking (cf. [KSW05] and [KS05]).

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2. Twisted invariants of 3–manifolds

2.1. **Twisted homology groups and twisted torsion.** Let X be a topological space. We write $\pi = \pi_1(X)$. Let $\alpha : \pi \to \mathrm{Aut}(V)$ be a representation where V is

[2] [S] That's a new question

a module over a ring R. We can thus view V as a left $\mathbb{Z}[\pi]$–module. We sometimes write V_α to indicate which action on V we consider.

Denote by $\widetilde{X}$ the universal cover of X. The chain complex $C_*(\widetilde{X})$ is a left $\mathbb{Z}[\pi]$–module via deck transformations. We can now consider the twisted cochain complex

$$C^*_\alpha(X;V) := C_*(\mathrm{Hom}_{\mathbb{Z}[\pi]}(\widetilde{X}), V),$$

we denote its homology groups by $H^*_\alpha(X;V)$. Using the natural involution $g \mapsto g^{-1}$ on the group ring $\mathbb{Z}[\pi]$, we can also view $C_*(\widetilde{X})$ as a right $\mathbb{Z}[\pi]$–module. We now obtain the twisted chain complex

$$C^\alpha_*(X;V) := C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V,$$

and we denote its homology groups by $H^\alpha_*(X;V)$. When α is understood we will drop it from the notation.

In the following we say that two representations $\alpha : \pi \to \mathrm{Aut}(V)$ and $\beta : \pi \to \mathrm{Aut}(W)$ are *equivalent* if there exists an isomorphism $\Psi : V \to W$ such that $\alpha(g) = \Psi^{-1} \circ \beta(g) \circ \Psi$ for any $g \in \pi$. It is well-known that the twisted cohomology and homology groups only depend on the equivalence class of α.

Now let X be a finite CW complex and let $\alpha : \pi_1(X) \to \mathrm{Aut}(V)$ be a representation where V is a vector space over a field $\mathbb{F}$. We denote by $\sigma_{i1}, \dots, \sigma_{ik_i}$ the set of i-cells (note that we picked a random ordering of the cells). For each cell σ_{ij} we then pick a lift $\tilde{\sigma}_{ij}$ to $\widetilde{X}$. This endows the free $\mathbb{Z}[\pi]$–module chain complex $C_*(\widetilde{X})$ with an ordered basis $\mathcal{B}$. We now consider the twisted cellular chain complex

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V.$$

Let $v_1, \dots, v_n$ be any basis of V. We endow $C_i(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V$ with the ordered basis

$$\{\tilde{\sigma}_{i1} \otimes v_1, \dots, \tilde{\sigma}_{ik_i} \otimes v_1, \dots, \tilde{\sigma}_{i1} \otimes v_n, \dots, \tilde{\sigma}_{ik_i} \otimes v_n\}.$$

If this complex is not acyclic, i.e. if $H^\alpha_*(X;\mathbb{F}^n) \neq 0$, then we define $\tau(X,\alpha,\mathcal{B}) := 0$. Otherwise we denote by $\tau(X,\alpha,\mathcal{B}) \in \mathbb{F}^\times := \mathbb{F} \setminus \{0\}$ the torsion of this based $\mathbb{F}$–complex. We will not recall the definition of torsion, we refer instead to the many excellent expositions, e.g. [Mi66], [Tu01, Tu02] and [Nic03]. Finally we define $\tau(X,\alpha) := \tau(X,\alpha,\mathcal{B})$. We summarize the key properties of $\tau(X,\alpha,\mathcal{B})$ and $\tau(X,\alpha)$ in the following lemma:

Lemma 2.1.
(1) *The invariant $\tau(X,\alpha,\mathcal{B}) \in \mathbb{F}$ is well-defined, i.e. independent of the choice of the basis for V.*
(2) *The invariant $\tau(X,\alpha) \in \mathbb{F}$ is well–defined up to multiplication by an element of the form ϵd where $d \in \det(\alpha(\pi))$ and $\epsilon \in \{-1,1\}$. If $\dim(V)$ is even, then $\epsilon = 1$.*
(3) *The invariants $\tau(X,\alpha,\mathcal{B})$ and $\tau(X,\alpha)$ depend only on the equivalence class of α.*

Proof. The first and the last statements can be easily verified. Now let $\mathcal{B}, \mathcal{B}'$ be bases of $C_*(\widetilde{X})$ corresponding to two different orderings and lifts of cells. If $\mathcal{B}'$ is obtained from $\mathcal{B}$ by switching the order of two cells, then it follows easily from the definitions that

$$\tau(X, \alpha, \mathcal{B}') = (-1)^{dim(V)} \tau(X, \alpha, \mathcal{B}).$$

Furthermore, if $\mathcal{B}'$ is obtained from $\mathcal{B}$ by acting by $g \in \pi_1(X)$ on a lift of a k-cell, then

$$\tau(X, \alpha, \mathcal{B}') = \det(\alpha(g))^{(-1)^k} \tau(X, \alpha, \mathcal{B}).$$

The second statement is now an immediate consequence of these observations, since any two $\mathcal{B}, \mathcal{B}'$ are related by a sequence of the above moves. □

Now suppose that M is a 3-manifold with empty or toroidal boundary and let $\alpha : \pi_1(M) \to \mathrm{Aut}(V)$ be a representation where V is a vector space over a field $\mathbb{F}$. We equip M with the structure of a finite CW complex X and we define $\tau(M, \alpha) = \tau(X, \alpha)$. Note that $\tau(M, \alpha)$ is independent of the choice of the underlying CW complex by Chapman's theorem [Chp74].

2.2. **Twisted torsion of a knot.** Let $K \subset S^3$ be an oriented knot. Throughout the paper we write $X_K = S^3 \setminus \nu K$, $\pi_K = \pi_1(X_K)$ and we denote by $\phi_K : \pi_K \to \mathbb{Z}$ the homomorphism given by sending the oriented meridian to one. We will drop K from the notation if the knot is understood from the context.

Let $\alpha : \pi \to \mathrm{GL}(n, R)$ be a representation over a commutative domain R. We can now define a left $\mathbb{Z}[\pi]$–module structure on $R^n \otimes_R R[t^{\pm 1}] =: R[t^{\pm 1}]^n$ via $\alpha \otimes \phi$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)} p)$$

where $g \in \pi_1(X), v \otimes p \in R^n \otimes_R R[t^{\pm 1}] = R[t^{\pm 1}]^n$. Put differently, we get a representation $\alpha \otimes \phi : \pi \to \mathrm{GL}(n, R[t^{\pm 1}])$. We denote by $Q(t)$ the quotient field of $R[t^{\pm 1}]$. The representation $\alpha \otimes \phi$ allows us to view $R[t^{\pm 1}]^n$ and $Q(t)^n$ as left $\mathbb{Z}[\pi]$–modules. By Section 2.1 we can now consider the torsion invariant $\tau(X_K, \alpha \otimes \phi) \in Q(t)$. Throughout the paper we will often write $\tau(K, \alpha) = \tau(X_K, \alpha \otimes \phi)$.

Remark. The study of twisted polynomial invariants of knots was introduced by Lin [Li01]. It follows from the work of Kitano [Ki96] that our invariant $\tau(X_K, \alpha \otimes \phi)$ is equivalent to Wada's invariant (see [Wa94]) and closely related to Lin's invariant. We refer to [FV09a] for more on twisted invariants of knots and 3-manifolds.

Note that given $g \in \pi$ we have $\det((\alpha \otimes \phi)(g)) = t^{n\phi(g)} \det(\alpha(g))$. The following lemma is now a reformulation of Lemma 2.1 in our context.

Lemma 2.2. *Given an oriented knot $K \subset S^3$ and an n-dimensional representation α the invariant $\tau(K, \alpha) \in Q(t)$ is well–defined up to multiplication by an element of the form $\epsilon d t^{kn}$ where $d \in \det(\alpha(\pi))$ and $k \in \mathbb{Z}$. If n is even, then $\epsilon = 1$. Finally $\tau(K, \alpha)$ depends only on the equivalence class of α.*

2.3. **Twisted 0–th Alexander polynomials and non–trivial representations.** The twisted torsion of a knot is by definition a rational function. It is well–known though that the twisted torsion of a knot is in fact often a polynomial, provided the representation α is 'sufficiently non-trivial'. We refer to [Wa94, Proposition 8] or [KM05, Theorem 1.1] for examples of such results. The following theorem seems to be one of the most general results of this type:

Theorem 2.3. *Let K be a non-trivial oriented knot. Let $\alpha : \pi_K \to GL(n,\mathbb{F})$ be an irreducible representation over a field $\mathbb{F}$ which is non–trivial when restricted to Ker(ϕ_K). Then $\tau(K,\alpha) \in \mathbb{F}(t)$ is a polynomial, i.e. it lies in $\mathbb{F}[t^{\pm 1}]$.*

The proof of this theorem will require the remainder of this section. We find it convenient to rephrase the theorem in the language of twisted Alexander polynomials, the definition of which we now recall. Let X be a topological space, $\phi \in H^1(X;\mathbb{Z}) = \mathrm{Hom}(\pi_1(X),\mathbb{Z})$ non-trivial and $\alpha : \pi_1(X) \to \mathrm{GL}(n,R)$ a representation over a Noetherian UFD R. Recall that we can now define a tensor representation $\alpha\otimes\phi : \pi_1(X) \to \mathrm{GL}(n,R[t^{\pm 1}])$. We obtain a twisted module $H_i^{\alpha\otimes\phi}(X;R[t^{\pm 1}]^n)$ over the ring $R[t^{\pm 1}]$. Note that $R[t^{\pm 1}]$ is a Noetherian UFD and that $H_i^{\alpha}(X;R[t^{\pm 1}]^n)$ is therefore a finitely generated module over $R[t^{\pm 1}]$. We now denote by $\Delta^{\alpha}_{X,\phi,i} \in R[t^{\pm 1}]$ the order of $H_i^{\alpha\otimes\phi}(X;R[t^{\pm 1}]^n)$ and refer to it as the *i–th twisted Alexander polynomial of* (X,ϕ,α). We refer to [Tu01] or [FV09a, Section 2] for the precise definitions. Note that the twisted Alexander polynomials are well–defined up to multiplication by a unit in the ring $R[t^{\pm 1}]$.

We adopt the following naming conventions. If π is finitely presented group, then we define $\Delta^{\alpha}_{\pi,\phi,i} = \Delta^{\alpha}_{K(\pi,1),\phi,i}$. If K is an oriented knot in S^3, then we write $\Delta^{\alpha}_{K,i} = \Delta^{\alpha}_{X_K,\phi_K,i}$.

Twisted torsion and twisted Alexander polynomials are closely related invariants as the following well–known proposition shows:

Proposition 2.4. [KL99] [Tu01, Theorem 4.7] *Let $K \subset S^3$ be an oriented knot and let $\alpha : \pi_K \to GL(n,R)$ be a representation where R is a Noetherian UFD. Then $\Delta^{\alpha}_{K,\phi,0} \neq 0$ and*

$$\tau(X_K,\alpha\otimes\phi) = \frac{\Delta^{\alpha}_{K,\phi,1}}{\Delta^{\alpha}_{K,\phi,0}} \in R[t^{\pm 1}]$$

up to multiplication by a unit in $R[t^{\pm 1}]$.

Our main technical result of this section is now the following lemma, which we phrased in a slightly more general language than strictly necessary, hoping that the lemma will be of independent interest.

Lemma 2.5. *Let X be a topological space. We write $\pi = \pi_1(X)$. Suppose $\phi : \pi \to \mathbb{Z}$ is a non-trivial homomorphism such that Ker(ϕ) is non–trivial and let $\alpha : \pi \to GL(n,\mathbb{F})$ be an irreducible representation over a field $\mathbb{F}$ which is non–trivial when* *restricted to Ker(ϕ). Then*

$$\Delta^{\alpha}_{X,\phi,0} = 1 \in \mathbb{F}[t^{\pm 1}].$$

Note that Theorem 2.3 is now an immediate consequence of Proposition 2.4 and Lemma 2.5.

Proof of Lemma 2.5. Suppose that $\alpha : \pi \to \mathrm{GL}(n, \mathbb{F})$ is a representation over a field $\mathbb{F}$ which is non–trivial when restricted to $\mathrm{Ker}(\phi)$ and such that $\Delta^{\alpha}_{X,\phi,0} \neq 1 \in \mathbb{F}[t^{\pm 1}]$. We will show that α is not irreducible. We write $\Gamma = \mathrm{Ker}(\phi)$ and we pick $\mu \in \pi$ with $\phi(\mu) = 1$. We denote by $\alpha \otimes \phi : \pi \to \mathrm{Aut}(\mathbb{F}^n[t^{\pm 1}])$ the tensor representation. First recall that $\Delta^{\alpha}_{K,0} = 1$ if and only if $H^{\alpha\otimes\phi}_0(\pi; \mathbb{F}^n[t^{\pm 1}]) = 0$ (cf. e.g. [FK06, Lemma 2.2]). Also recall (cf. [HS97, Section VI]) that

$$H^{\alpha\otimes\phi}_0(\pi; \mathbb{F}^n[t^{\pm 1}]) = \mathbb{F}^n[t^{\pm 1}]/((\alpha \otimes \phi)(g)(v) - v \mid v \in \mathbb{F}^n[t^{\pm 1}], g \in \pi).$$

By our assumption we have $H^{\alpha\otimes\phi}_0(\pi; \mathbb{F}^n[t^{\pm 1}]) \neq 0$. It is straightforward to see that this implies that $H^{\alpha}_0(\Gamma; \mathbb{F}^n) = \mathbb{F}^n/(\alpha(g)(v) - v \mid v \in \mathbb{F}^n, g \in \Gamma)$ is also non–trivial. Now pick a non–singular form $\langle\, ,\, \rangle$ on $\mathbb{F}^n$ and denote by $\alpha^* : \pi \to \mathrm{GL}(n, \mathbb{F})$ the unique representation which satisfies $\langle \alpha^*(g)v, \alpha(g)w\rangle = \langle v, w\rangle$ for all $g \in \pi, v, w \in \mathbb{F}^n$. We now let $Y = K(\Gamma, 1)$ and we denote by $\widetilde{Y}$ the universal cover of Y. We write $(\mathbb{F}^n)_\alpha$ to denote $\mathbb{F}^n$ with the Γ–action given by α, and similarly we write $(\mathbb{F}^n)_{\alpha^*}$. Using the inner product we then get an isomorphism of $\mathbb{F}$–module chain complexes:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\widetilde{Y}), (\mathbb{F}^n)_{\alpha^*}) & \to & \mathrm{Hom}_{\mathbb{F}}\big(C_*(\widetilde{Y}; (\mathbb{F}^n)_\alpha), \mathbb{F}\big) = \mathrm{Hom}_{\mathbb{F}}\big(C_*(\widetilde{Y} \otimes_{\mathbb{Z}[\Gamma]} (\mathbb{F}^n)_\alpha, \mathbb{F}\big) \\ f & \mapsto & ((c \otimes w) \mapsto \langle f(c), w\rangle). \end{array}$$

Note that this map is well–defined since $\langle \beta(g^{-1})v, w\rangle = \langle v, \overline{\beta}(g)w\rangle$. It is now easy to see that this defines in fact an isomorphism of $\mathbb{F}$–module chain complexes. It now follows that

$$\begin{array}{rcl} H^i_{\alpha^*}(\Gamma; \mathbb{F}^n) & = & H_i(\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\widetilde{Y}), (\mathbb{F}^n)_{\alpha^*})) \\ & \cong & H_i(\mathrm{Hom}_{\mathbb{F}}\big(C_*(\widetilde{Y}; (\mathbb{F}^n)_\alpha), \mathbb{F}\big)) \\ & \cong & H_i(C_*(\widetilde{Y}; (\mathbb{F}^n)_\alpha)) \\ & = & H^{\alpha}_i(\Gamma; \mathbb{F}^n). \end{array}$$

Note that the second to last isomorphism is given by the universal coefficient theorem. Recall (cf. again [HS97, Section VI]) that

$$H^0_{\alpha^*}(\Gamma; \mathbb{F}^n) = \{v \in \mathbb{F}^n \mid \alpha^*(g)(v) = v \text{ for all } g \in \Gamma, v \in \mathbb{F}^n\}.$$

We now write $V := \{v \in \mathbb{F}^n \mid \alpha^*(g)(v) = v \text{ for all } g \in \Gamma, v \in \mathbb{F}^n\}$ and we let $W \subset \mathbb{F}^n$ be the orthogonal complement of V. In particular $V \oplus W = \mathbb{F}^n$. Note that V is non–trivial by assumption and note that W is non–trivial since $\Gamma = \mathrm{Ker}(\phi)$ is non–trivial and since α (and hence α^*) is non–trivial when restricted to Γ by our assumption.

Note that with respect to the decomposition $\mathbb{F}^n = V \oplus W$ we have

$$\alpha^*(g) = \begin{pmatrix} \mathrm{id} & * \\ 0 & * \end{pmatrix}$$

for any $g \in \Gamma$. In particular there exist maps $A : \pi \to \mathrm{Hom}(W, V)$ and $B : \pi \to \mathrm{End}(W)$ such that

$$\alpha^*(g) = \begin{pmatrix} \mathrm{id} & A(g) \\ 0 & B(g) \end{pmatrix}$$

for any $g \in \Gamma$. We now write

$$\alpha^*(\mu) = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$$

with $C \in \mathrm{End}(V), D \in \mathrm{Hom}(W, V), E \in \mathrm{Hom}(V, W), F \in \mathrm{End}(W)$. For any $g \in \Gamma$ we have $g\mu = \mu(\mu^{-1}g\mu)$ with $\mu^{-1}g\mu \in \Gamma$. In particular we have

$$\begin{pmatrix} \mathrm{id} & A(g) \\ 0 & B(g) \end{pmatrix} \cdot \begin{pmatrix} C & D \\ E & F \end{pmatrix} = \begin{pmatrix} C & D \\ E & F \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id} & A(\mu^{-1}g\mu) \\ 0 & B(\mu^{-1}g\mu) \end{pmatrix}$$

for any $g \in \Gamma$. Considering the first block column we see that

$$\begin{pmatrix} A(g)E \\ B(g)E \end{pmatrix} = \begin{pmatrix} 0 \\ E \end{pmatrix}.$$

If $E : V \to W$ was non–trivial, then there would exist $v \in V$ such that $w = E(v)$ is non–trivial. Given any $g \in \Gamma$ we then have

$$\begin{aligned} \alpha^*(g)(w) &= \begin{pmatrix} \mathrm{id} & A(g) \\ 0 & B(g) \end{pmatrix} \begin{pmatrix} 0 \\ w \end{pmatrix} &&= \begin{pmatrix} \mathrm{id} & A(g) \\ 0 & B(g) \end{pmatrix} \begin{pmatrix} 0 \\ Ev \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & A(g)E \\ 0 & B(g)E \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} &&= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ Ev \end{pmatrix} &&= w. \end{aligned}$$

But this is not possible by the definition of V. We therefore conclude that E is the zero homomorphism, hence the representation α^* restricts to a representation of V. It is now straightforward to see that α preserves W. Since W is neither zero nor all of $\mathbb{F}^n$ this shows that α is reducible. $\square$

2.4. **Twisted torsion of cyclic covers.** Let $K \subset S^3$ be an oriented knot. We write $X = X_K$ and $\phi = \phi_K$. Given m we denote by X_m the m-fold cyclic cover of X. Throughout this section let $\alpha : \pi_1(X_K) \to \mathrm{SL}(k, \mathbb{C})$ be an *even-dimensional* representation. [3] We denote by α_m the representation $\pi_1(X_m) \to \pi_1(X) \xrightarrow{\alpha} \mathrm{SL}(k, \mathbb{C})$.

[3][S] We have to decide at some point whether we only want to consider the 2-dimensional case, or whether we want to consider all the representations of $SL(2, \mathbb{C})$, (for each $m \geq 2$ there exists one irreducible representation of $SL(2, \mathbb{C})$, for example the adjoint representation is the 3-dimensional). Let me explain the pros and cons of doing the general case, as I see it:
Several of the results we use (most importantly Menal-Ferrer-Porti) have been proved for the general case, in particular we would be able to define a sequence of canonical twisted polynomials $\mathcal{T}_K^m(t)$ which are all non-zero. The case of the adjoint representation has been studied in detail by Dubois-Yamaguchi. The result of Müller (regarding torsion for closed 3-manifolds) suggests that the values $\mathcal{T}_K^m(1)$ determine the hyperbolic volume.

In this section we will relate the $\mathbb{C}(t)$-valued torsion $\tau(K,\alpha)=\tau(X,\alpha\otimes\phi)$ to the $\mathbb{C}$-valued torsions $\tau(X_m,\alpha_m)$. Note that the assumption that α is an even-dimensional representation implies that $\tau(K,\alpha)$ is well-defined up to multiplication by an even power of t and that $\tau(X_m,\alpha_m)\in\mathbb{C}$ is well-defined without any indeterminacy.

We now have the following theorem. (We refer to [DY09, Corollary 27] for a related result.)

Theorem 2.6. *Let $\alpha:\pi\to SL(k,\mathbb{C})$ be an even-dimensional representation. We write $f(t)=\tau(K,\alpha)$. Let $m\in\mathbb{N}$. Suppose that $H_0^{\alpha_m}(X_m;\mathbb{C}^k)=0$. Then $f(e^{2\pi ij/m})$ is defined for $j=1,\dots,m$ and*

$$\prod_{j=1}^{m} f(e^{2\pi ij/m})=\tau(X_m,\alpha_m).$$

Note that the left hand side of Theorem 2.6 is a well-defined complex number since $\tau(K,\alpha)$ is well-defined up to multiplication by an *even* power of t.

Proof. [4] Before we delve into the proof we first introduce several definitions which will be of use later. Given two representations $\beta:\pi\to \mathrm{GL}(k,\mathbb{C})$ and $\gamma:\pi\to\mathrm{GL}(l,\mathbb{C})$ we denote by $\beta\otimes\gamma$ the resulting tensor representation $\pi\to\mathrm{Aut}(\mathbb{C}^k\otimes\mathbb{C}^l)=\mathrm{GL}(kl,\mathbb{C})$. Given $\zeta\in\mathbb{C}^*$ we denote by γ_ζ the representation $\pi_1(X)\to\mathrm{GL}(1,\mathbb{C})$ given by sending the meridian to ζ. We denote by γ_m the representation $\pi_1(X)\to\mathrm{Aut}(\mathbb{C}[\mathbb{Z}/m])\cong\mathrm{GL}(m,\mathbb{C})$ where the first map is the regular representation corresponding to the epimorphism $\pi\to\mathbb{Z}/m$. Denote by $\xi_1,\dots,\xi_m$ the m-th roots of unity. Note that $\bigoplus_{i=1}^m\zeta_{\xi_i}$ and ζ_m are equivalent representations of π.

We write $\pi=\pi_1(X)$. We pick a CW-structure for X with one 0-cell a, 1-cells $b_1,\dots,b_{l+1}$ and 2-cells $c_1,\dots,c_l$. We then pick lifts $\tilde{a},\tilde{b}_1,\dots,\tilde{b}_{l+1},\tilde{c}_1,\dots,\tilde{c}_l$ to the universal cover $\widetilde{X}$. These form an ordered basis $\mathcal{B}$ for $C_*(\widetilde{X})$ as a free $\mathbb{Z}[\pi]$-module chain complex.

We now consider X_m. We write $\pi_m:=\pi_1(X_m)$. We can identify the universal cover of X_m with $\widetilde{X}$. Now pick an element $\mu\in\pi$ which represents the oriented meridian. Then $\mu^j\tilde{a},\mu^j\tilde{b}_1,\dots,\mu^j\tilde{b}_{l+1},\mu^j\tilde{c}_1,\dots,\mu^j\tilde{c}_l, j=0,\dots,m-1$ form a basis for $C_*(\widetilde{X})$ as a free $\mathbb{Z}[\pi_m]$-module chain complex.

So what's not to like? The problem is that Theorem 2.6 and Theorem 2.7 get tricky. More precisely, in the case of an odd-dimensional representation the torsion is well-defined only up to sign. To deal with this one has to work with sign-refined torsion (this involves the orientation of the knot). To get a precise (i.e. not just up to sign) equality in Theorem 2.6 gets rather tricky. Note for example that $\tau(K,\alpha)$, even in the best of all cases, is only well-defined up to multiplication by a power of t, but the left hand side is not invariant under multiplication of $\tau(K,\alpha)$ by t. Presumably all this can be dealt with, but the proofs would be significantly more delicate and less readable

[4] In principle the lemma is straightforward, what turns it into a delicate dance is that we want to get the signs right, and that we want to make sure we never divide by zero!

Given $i = 0, \dots, m-1$ we now denote by g_i the element of $\mathbb{Z}/m$ represented by i. In the following v stands for a vector in $\mathbb{C}^k$. The maps

$$\begin{array}{rcl} \mu^j \tilde{a} \otimes v & \mapsto & \tilde{a} \otimes g_j \otimes v \\ \mu^j \tilde{b}_i \otimes v & \mapsto & \tilde{b}_i \otimes g_j \otimes v \\ \mu^j \tilde{c}_i \otimes v & \mapsto & \tilde{c}_i \otimes g_j \otimes v \end{array}$$

give raise to an isomorphism of based chain complexes

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_m]} \mathbb{C}^k \to C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{C}[\mathbb{Z}/m] \otimes \mathbb{C}^k.$$

In particular we conclude that $\tau(X_m, \alpha_m) = \tau(X, \alpha \otimes \gamma_m)$. (Note that both are well-defined, independent of the ordering of the cells and independent of the choice of lifts of the cells.)

Since $\bigoplus_{i=1}^m \zeta_{\xi_i}$ and ζ_m are equivalent representations of π we obtain that

$$\tau(X, \alpha \otimes \gamma_m, \mathcal{B}) = \tau(X, \alpha \otimes \oplus_{i=1}^m \zeta_{\xi_i}, \mathcal{B}) = \prod_{i=1}^m \tau(X, \alpha \otimes \gamma_{\xi_i}, \mathcal{B}). \tag{1}$$

(Note that the terms on the left hand side are independent of the choice of $\mathcal{B}$, but each of the m terms on the right hand side does depend on the choice of $\mathcal{B}$ since $\alpha \otimes \gamma_{\xi_i}$ is no longer a special linear representation.) Note that arguments similar to the above also show that

$$\bigoplus_{i=1}^m H_0^{\alpha\otimes\gamma_{\xi_i}}(X; \mathbb{C}^k) \cong H_0^{\alpha\otimes\gamma_m}(X; \mathbb{C}^k \otimes \mathbb{C}^m) \cong H_0^{\alpha_m}(X_m; \mathbb{C}^k). \tag{2}$$

We now write $g(t) = \tau(X, \alpha \otimes \phi, \mathcal{B})$. Note that $g(t)$ is a representative of $g(t)$. By (1) it suffices to show that $\tau(X, \alpha \otimes \gamma_\xi, \mathcal{B}) = g(\xi)$ for any m-th root of unity. Note that by (2) we have $H_0^{\alpha\otimes\gamma_\xi}(X; \mathbb{C}^k) \subset H_0^{\alpha_m}(X_m; \mathbb{C}^k)$, but the latter is zero by assumption. Now consider the chain complex $C_*(X) \otimes_{\mathbb{Z}[\pi]} \mathbb{C}^k \otimes \mathbb{C}(t)$. Using the above basis $\mathcal{B}$ and using any basis for $\mathbb{C}^k$ we can view this as a based $\mathbb{C}(t)$-chain complex. We denote by $B_2(t)$ and $B_1(t)$ the matrices corresponding to the boundary maps ∂_2 and ∂_1. Note that $B_2(t)$ and $B_1(t)$ are in fact defined over $\mathbb{C}[t^{\pm 1}]$. Also note that $B_2(\xi)$ and $B_1(\xi)$ denote the matrices corresponding to the boundary maps of the based chain complex $C_*(X) \otimes_{\mathbb{Z}[\pi]} (\mathbb{C}^k \otimes \mathbb{C})_{\alpha\otimes\gamma_\xi}$. Now consider the $m \times (l+1)m$-matrix $B_1(\xi)$. Since $H_0^{\alpha\otimes\gamma_\xi}(X; \mathbb{C}^k) = 0$ it follows that $B_1(\xi)$ has rank m. We can thus find m columns of $B_1(\xi)$ such that the corresponding $m \times m$-matrix is invertible. We now denote by $A_1(t)$ the $m \times m$-matrix given by picking out these m columns from $B_1(t)$, and we denote by $A_2(t)$ the $ml \times ml$-matrix given by deleting the corresponding rows of $B_2(t)$.

It now follows from [Tu01, Theorem 2.2] that

$$\begin{array}{rcl} \tau(X, \alpha \otimes \phi, \mathcal{B}) & = & \det(A_2(t)) \det(A_1(t))^{-1} \in \mathbb{C}(t), \text{ and} \\ \tau(X, \alpha \otimes \gamma_\xi, \mathcal{B}) & = & \det(A_2(\xi)) \det(A_1(\xi))^{-1} \in \mathbb{C}. \end{array}$$

(Note that both denominators are non-zero.) It follows that

$$\tau(X, \alpha \otimes \phi, \mathcal{B})(\xi) = \tau(X, \alpha \otimes \gamma_\xi, \mathcal{B}).$$

This conclues the proof of the theorem. $\square$

Given an even-dimensional representation the theorem in particular says that the $\mathbb{C}(t)$-valued torsion $\tau(K, \alpha) = \tau(X, \alpha \otimes \phi)$ determines the $\mathbb{C}$-valued torsions $\tau(X_m, \alpha_m)$ for any m. In order to state and prove a partial converse to the previous lemma we need to introduce a few more definitions. We say that $p(t), q(t) \in \mathbb{C}[t^{\pm 1}]$ are *equivalent*, written $p(t) \sim q(t)$, if $p(t) = t^n q(t)$ for some $n \in \mathbb{Z}$. We say that $p(t) \in \mathbb{C}[t^{\pm 1}]$ is *palindromic* if $p(t) \sim p(t^{-1})$.

Theorem 2.7. *Let K be a knot and $\alpha : \pi_1(X) \to SL(k, \mathbb{C})$ an even-dimensional representation such that $\tau(X, \alpha)$ is a palindromic polynomial. If $\tau(X_m, \alpha_m)$ is non-zero for any m, then the torsions $\tau(X_m, \alpha_m)$ determine $\tau(X, \alpha) \in \mathbb{C}[t^{\pm 1}]$.*

This theorem is a consequence of Theorem 2.6 and a theorem of Fried. In order to state the theorem of Fried we will need the following definition. Let $p = a \prod_{j=1}^{d}(t - \lambda_i) \in \mathbb{C}[t^{\pm 1}]$ be a polynomial. Given n we denote by $r_n(p)$ the resultant of $t^n - 1$ and p, i.e.

$$r_n(p) = a^d \prod_{l=1}^{d}(\lambda_l^n - 1) = a^d \prod_{j=1}^{n}\prod_{l=1}^{d}(\lambda_l - e^{2\pi i j/n}) = (-1)^{dn} \prod_{j=1}^{n} p(e^{2\pi i j/n}).$$

Note that $r_n(p)$ is non-zero for any n if and only if no root of unity is a zero of p.

The following theorem was stated and proved by Fried for real polynomials.

Theorem 2.8. *Let $p = p(t), p' = p'(t) \in \mathbb{C}[t^{\pm 1}]$ be palindromic Laurent polynomials. Suppose that $r_n(p) = r_n(p')$ for all n and suppose that $r_n(p) = r_n(p')$ is non-zero for any n. Then $p \sim p'$.*

We provide a proof of the theorem which for the most part follows closely Fried's argument. We chose to include the proof for the convenience of the reader and to verify that the proof carries over to complex polynomials.

Note that if $\tau(X_m, \alpha_m)$ is non-zero, then in particular $H_0^{\alpha_m}(X_m; C^k) = 0$. We now see that Theorem 2.8 together with Theorem 2.6 implies Theorem 2.7.

Proof. We write $p = a \prod_{i=1}^{d}(t - \lambda_i)$ and $p' = a' \prod_{i=1}^{d'}(t - \lambda_i')$. Note that p, p' being palindromic implies that if z is a zero, then z^{-1} is also a zero. We can thus without loss of generality assume that $\lambda_i = \lambda_{d-i}^{-1}$ and $\lambda_i' = (\lambda_{d'-i}')^{-1}$ for any i. We denote by $H \subset \mathbb{C}^\times$ the group generated by $\lambda_1, \dots, \lambda_d, \lambda_1', \dots, \lambda_d'$.

We write $r_n = r_n(p) = r_n(p')$. We also write $V = (\mathbb{Z}/2)^d$ and given $v = (v_1, \dots, v_d) \in V$ we define

$$\mu_v := a \cdot (-1)^{\#\{i \mid v_i = 1\}} \cdot \prod_{i\, with\, v_i = 0} \lambda_i.$$

Note that

$$r_n = a^n \prod_{i=1}^{d} (\lambda_i^n - 1) = a^n(\lambda_1^n - 1) \cdot \dots \cdot (\lambda_d^n - 1) = \sum_{v \in V} \mu_v^n.$$

We now see that

$$\begin{array}{rclcl} B(t) & := & \exp\left(\sum\limits_{n=1}^{\infty} r_n \frac{t^n}{n}\right) & = & \exp\left(\sum\limits_{n=1}^{\infty} \sum\limits_{v \in V} \mu_v^n \frac{t^n}{n}\right) \\ & = & \exp\left(\sum\limits_{v \in V} \sum\limits_{n=1}^{\infty} \mu_v^n \frac{t^n}{n}\right) & = & \prod\limits_{v \in V} \exp\left(\sum\limits_{n=1}^{\infty} \frac{\mu_v t^n}{n}\right) \\ & = & \prod\limits_{v \in V} (1 - \mu_v t). & & \end{array}$$

(Here the last equality follows from the power series expansion of the logarithm.) We thus see that the r_n determine $B(t) = \prod_{v \in V}(1 - \mu_v t)$. In particular the r_n determine the set μ_v (with multiplicities). Note that in the group ring $\mathbb{Z}[H]$ we have the following equality:

$$\sum_{v \in V} [\mu_v] = a \prod_{i=1}^{d} ([\lambda_i] - 1).$$

(Here, and in the remainder of the proof, given $z \in H$ we denote by $[z]$ the corresponding element $1 \cdot z$ in the group ring $\mathbb{Z}[H]$).

We can do a similar discussion as above for p'. We then see that the assumption that $r_n(p) = r_n(p')$ for any n implies that

$$a \prod_{i=1}^{d} ([\lambda_i] - 1) = a' \prod_{i=1}^{d'} ([\lambda'_i] - 1) \in \mathbb{C}[H].$$

It follows immediately that $a = a'$.

Note that the assumption that $r_n(p) = r_n(p')$ is non-zero for any n implies that no λ_i and no λ'_i is a root of unity. The following claim thus implies the theorem.

Claim. Let $\lambda_1, \dots, \lambda_d \in \mathbb{C}^\times$ and $\lambda'_1, \dots, \lambda_{d'} \in \mathbb{C}^\times$ which are not zeros of unity. Suppose that $\lambda_i = \lambda_{d-i}^{-1}$ and $\lambda'_i = (\lambda'_{d'-i})^{-1}$ for any i. If

$$\prod_{i=1}^{d} ([\lambda_i] - 1) = \prod_{i=1}^{d'} ([\lambda'_i] - 1) \in \mathbb{C}[H],$$

then $\{\lambda_1, \dots, \lambda_d\} = \{\lambda'_1, \dots, \lambda'_{d'}\}$ as sets (with multiplicities).

We refer to [Fr88, p. 126] for a detailed proof. In the following we just give an outline of a proof. First note that since H is a finitely generated abelian group we can find a splitting $H = F \times T$ where T is a torsion group and F a free abelian group. We now write $\lambda_i = f_i c_i$ where $f_i \in F, c_i \in T$ and $\lambda'_i = f'_i c'_i$ where $f'_i \in F, c'_i \in T$. Note that the assumption that $\lambda_1, \dots, \lambda_d \in \mathbb{C}^\times$ and $\lambda'_1, \dots, \lambda'_d \in \mathbb{C}^\times$ are not zeros of unity implies that the f_i, f'_i are non-trivial. We thus obtain the following equality

$$\prod_{i=1}^{d}(c_i[f_i]-1) = \prod_{i=1}^{d}(c'_i[f'_i]-1) \in \mathbb{C}[F].$$

Note that $\mathbb{C}[F]$ is a UFD. It now follows (fairly) easily that the set $\{\lambda_i\}$ agrees with the set $\{\lambda'_i\}$. This concludes the proof. $\square$

Question 2.9. *We just saw that under mild assumptions the torsions of the cyclic covers determine the $\mathbb{C}(t)$-valued torsion. This process is rather indirect though, and it would be very interesting if one could 'directly' read off the degree and the top coefficient of $\tau(K,\alpha)$ from the torsions of the cyclic covers.*

3. Torsion of hyperbolic knots

3.1. **The discrete and faithful** $\mathrm{SL}(2,\mathbb{C})$ **representations.** Let $K \subset S^3$ be an oriented knot. Throughout this section we write $\pi = \pi_K := \pi_1(X_K)$ and take the base point on the boundary torus ∂X_K. Let $\mu = \mu_K \in \pi_K$ be meridian, i.e. a simple closed curve in ∂X_K which is null-homologous in νK and $\mathrm{lk}(\mu, K) = +1$. Recall that $\phi_K : \pi_K \to \mathbb{Z}$ is the map that sends μ_K to one; when K is understood, we just write $\phi = \phi_K$.

Now assume that $M = S^3 \setminus K \cong \mathrm{int}(X_K)$ has a complete hyperbolic structure. The manifold M inherits an orientation from S^3, and so its universal cover $\widetilde{M}$ can be identified with $\mathbb{H}^3$ by an *orientation preserving* isometry. This identification is unique up to the action of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$, and the action of π_K on $\widetilde{M} = \mathbb{H}^3$ gives the *holonomy representation* $\overline{\alpha}\colon \pi_K \to \mathrm{PSL}(2,\mathbb{C})$, which is unique up to conjugation.

Remark. By Mostow-Prasad rigidity, the complete hyperbolic structure on M is unique. Thus $\overline{\alpha}$ is determined, up to conjugacy, solely by the knot K (sans orientation). A subtle point is that there are actually *two* conjugacy classes of discrete faithful representations $\pi_K \to \mathrm{PSL}(2,\mathbb{C})$; the other one corresponds to reversing the orientation of S^3 (not K) or equivalently complex-conjugating the entries of the image matrices.

To define the torsion, we want a representation into $\mathrm{SL}(2,\mathbb{C})$ rather than $\mathrm{PSL}(2,\mathbb{C})$. Thurston proved that $\overline{\alpha}$ always lifts to a representation $\alpha\colon \pi_K \to \mathrm{SL}(2,\mathbb{C})$, see [Th97] and [Sh02, Section 1.6] for details. In fact, there are exactly two such lifts, the other being $g \mapsto (-1)^{\phi(g)}\alpha(g)$; the point being that any other lift has the form $g \mapsto \epsilon(g)\alpha(g)$ for some homomorphism $\epsilon\colon \pi_K \to \{\pm 1\}$, i.e. some element of $H^1(M;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Now $\overline{\alpha}(\mu)$ is parabolic, and so $\mathrm{tr}(\alpha(\mu)) = \pm 2$; arbitrarily, we focus on the lift where the trace is 2 and call it the *distinguished representation.* This representation is determined, up conjugacy, solely by K (sans orientation). (We explain below the simple change that results if we instead required the trace to -2.)

3.2. **The hyperbolic torsion.** Consider an oriented hyperbolic knot K, and let $\alpha_K\colon \pi_K \to \mathrm{SL}(2,\mathbb{C})$ be a distinguished representation. As an initial definition, the *hyperbolic torsion* of K is $\mathcal{T}_K(t) = \tau(X_K, \alpha_K \otimes \phi_K)$. This is a polynomial in $\mathbb{C}[t^{\pm 1}]$ by Theorem 2.3 (cf. also [KM05, Theorem 3.1]) which is well–defined up to multiplication by an element of the form t^k for $k \in \mathbb{Z}$ by Lemma 2.2. (Below we refine things so that $\mathcal{T}_K(t)$ becomes simply an element of $\mathbb{C}[t^{\pm 1}]$.) We begin with a key basic property.

Proposition 3.1. *For an oriented hyperbolic knot K, the hyperbolic torsion $\mathcal{T}_K(\xi)$ is non-zero for any root of unity ξ.*

For 2-bridge knots and $\xi = \pm 1$ this proposition is also a consequence of the work of Hirasawa-Murasugi [HM08] and Silver-Williams [SW09d].

Proof. We write $\alpha = \alpha_K, X = X_K$ and $\phi = \phi_K$. Suppose that ξ is an m-th root of unity. We denote by X_m the m-fold cyclic cover of X. We denote by α_m the representation $\pi_1(X_m) \to \pi_1(X) \xrightarrow{\alpha} \mathrm{SL}(2,\mathbb{C})$. Note that $\alpha_m : \pi_1(X_m) \to SL(2,\mathbb{C})$ is a lift of the discrete and faithful representation of the hyperbolic 3-manifold X_m.

By the work of Menal–Ferrer and Porti [MP09, Theorem 0.4] (which builds on work of Raghunathan [Ra65]) we have that $H_*^{\alpha_m}(X_m, \mathbb{C}^2) = 0$, or equivalently, $\tau(X_m, \alpha_m)$ is non-zero. It now follows from Theorem 2.6 that

$$\prod_{k=1}^{m} \mathcal{T}(e^{2\pi i j/m}) = \tau(X_m, \alpha_m).$$

□

We now have the following proposition:

Proposition 3.2. *For an oriented hyperbolic knot K, the invariant $\mathcal{T}_K(t)$ is a non-zero palindromic polynomial of even degree, which is well-defined up to multiplication by elements of the form t^k for $k \in \mathbb{Z}$.*

Proof. First, $\mathcal{T}_K(t)$ is non-zero as Proposition 3.1 implies that $\mathcal{T}_K(1) \neq 0$. By [HSW09, Corollary 3.4] there exists $\varrho \in \{-1, 1\}$ and $k \in \mathbb{Z}$ such that

$$\mathcal{T}_K(t^{-1}) = \varrho t^k \mathcal{T}_K(t).$$

Since $\mathcal{T}_K(1) \neq 0$, it follows that $\varrho = 1$, i.e. $\mathcal{T}_K(t)$ is palindromic.

To see that $\mathcal{T}_K(t)$ has even degree d, fix a representative $p(t) = a_d t^d + \cdots + a_1 t + a_0$ for $\mathcal{T}_K(t)$, where a_k and a_0 are non-zero. As $p(t)$ is palindromic, we have $p(t^{-1}) = t^k p(t)$ for some k; since $p(t^{-1}) = a_0 + a_1 t^{-1} + \cdots + a_d t^{-d}$, this forces $k = d$. Taking $t = -1$ then gives $p(-1) = (-1)^d p(-1)$. Since $p(-1) \neq 0$ by Proposition 3.1, this forces $(-1)^d = 1$, i.e. d is even. □

In light of Proposition 3.2, we henceforth resolve the ambiguity of the hyperbolic torsion by insisting that it be *symmetric*, i.e. $\mathcal{T}_K(t) = \mathcal{T}_K(t^{-1})$. Thus $\mathcal{T}_K$ is now a well-defined element of $\mathbb{C}[t^{\pm 1}]$. We now prove Theorem 1.1 from the introduction which summarizes some of the properties of $\mathcal{T}_K(t)$.

Theorem 1.1. *Let K be an oriented hyperbolic knot in S^3. Then $\mathcal{T}_K(t)$ has the following properties:*

1. *$\mathcal{T}_K(t)$ lies in $\mathbb{C}[t^{\pm 1}]$,*
2. *$\mathcal{T}_K(\xi)$ is non-zero for any root of unity ξ,*
3. *$\mathcal{T}_K(t) = \mathcal{T}_K(t^{-1})$,*
4. *$\mathcal{T}_K(t)$ is independent of the orientation of K,*
5. *if K^* denotes the mirror image of K, then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, where the coefficients of the latter polynomial are the complex conjugates of those of $\mathcal{T}_K$.*
6. *if K is amphichiral, i.e. if $K = K^*$, then $\mathcal{T}_K(t)$ is a real polynomial.*

Proof. Assertions (1–3) are immediate from Propositions 3.1 and 3.2 and our choice of normalization for $\mathcal{T}_K(t)$. For (4), we noted above that the distinguished representation α does not depend on the orientation of K. Thus changing the orientation simply reverses the orientation of the meridian μ, which replaces ϕ with $-\phi$, and correspondingly replaces t by t^{-1} when computing the torsion. By (3), this substitution does not change $\mathcal{T}_K(t)$, proving (4). For (5), taking the mirror image replaces α with $\overline{\alpha}$ where $\overline{\alpha}(g)$ is the matrix which is the complex conjugate of $\alpha(g)$. It also reverses the meridian, but that is negligible by (4), and so we can regard ϕ as unchanged. Thus $\mathcal{T}_{K^*}(t) = \tau(X_K, \overline{\alpha} \otimes \phi) = \overline{\tau(X_k, \alpha \otimes \phi)} = \overline{\mathcal{T}_K(t)}$ as claimed. Finally, claim (6) follows immediately from (5). □

Remark. When choosing our distinguished representation, we arbitrarily chose the lift $\alpha\colon \pi \to \mathrm{SL}(2,\mathbb{C})$ where $\mathrm{tr}(\alpha(\mu)) = 2$. As discussed, the other lift β is given by $g \mapsto (-1)^{\phi(g)}\alpha(g)$. Note that given $g \in \pi$ we have

$$\big((\beta \otimes \phi)(g)\big)(t) = \beta(g) \cdot t^{\phi(g)} = \alpha(g) \cdot (-1)^{\phi(g)} \cdot t^{\phi(g)} = \alpha(g) \cdot (-t)^{\phi(g)} = \big((\alpha \otimes \phi)(g)\big)(-t).$$

The definition of torsion gives

$$\tau(X_K, \beta \otimes \phi_K)(t) = \tau(X_K, \alpha \otimes \phi_K)(-t) = \mathcal{T}_K(-t)$$

and thus using β instead of α would simply replace t by $-t$.

Proposition 3.3. *Let K be an oriented hyperbolic knot in S^3. Denote by $\alpha : \pi_K \to SL(2,\mathbb{C})$ the preferred lift of the canonical representation. Then $\mathcal{T}_K(t)$ is determined by the torsions $\tau(X_m, \alpha_m) \in \mathbb{C}$.*

Proof. This proposition is an immediate consequence of the definitions, of the fact that $\mathcal{T}_K(t)$ is palindromic, and of Theorem 2.7, once we convinced ourselves that $H_0^{\alpha_m}(X_m; \mathbb{C}^2) = 0$ for any m. But the vanishing of these twisted homology groups is an easy consequence of the fact that the α_m are irreducible non-trivial representations. □

4. Calculation of twisted torsion using Fox calculus

Let $K \subset S^3$ be an oriented knot. Let $\alpha : \pi_K \to SL(2,\mathbb{C})$ be a representation. We write $\pi := \pi_1(X_K)$ and $\phi = \phi_K$. Recall that given $g \in \pi$ we define $(\alpha \otimes \phi)(g) := \alpha(g) \cdot t^{\phi(g)} \in \mathrm{GL}(2, \mathbb{C}[t^{\pm 1}])$ and we now extend the group homomorphism $\alpha \otimes \phi : \pi \to \mathrm{GL}(2, \mathbb{C}[t^{\pm 1}])$ to a ring homomorphism $\mathbb{Z}[\pi] \to M(2, \mathbb{C}[t^{\pm 1}])$ which we also denote by $\alpha \otimes \phi$. Finally given an $k \times l$–matrix $A = (a_{ij})$ over $\mathbb{Z}[\pi]$ we denote by $(\alpha \otimes \phi)(A)$ the $2k \times 2l$–matrix obtained from A by replacing each entry a_{ij} by the 2×2–matrix $(\alpha \otimes \phi)(a_{ij})$.

Let $F = \langle x_1, \dots, x_k \rangle$ be the free group on k generators. By Fox (cf. [Fo53, Fo54, CF63] and see also [Ha05, Section 6]) there exists for each $i = 1, \dots, k$ a unique map, called *Fox derivative*,

$$\frac{\partial}{\partial x_i} : F \to \mathbb{Z}[F]$$

with the following two properties:

$$\begin{array}{rcl} \frac{\partial x_j}{\partial x_i} & = & \delta_{ij}, \\ \frac{\partial (uv)}{\partial x_i} & = & \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}, \text{ for any } u, v \in F. \end{array}$$

The following proposition now allows for efficient calculation of the hyperbolic torsion of a knot.

Proposition 4.1. *Let K be a hyperbolic knot. Let*

$$\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation of deficiency one (e.g. a Wirtinger presentation). We denote by $A = (a_{ij})$ the $(k-1) \times k$–matrix given by $a_{ij} = \frac{\partial r_i}{\partial x_j}$. Let $i \in \{1, \dots, k\}$ with $\phi(x_i) \neq 0$. Denote by A_i the matrix obtained from A by deleting the i–th column. Let

$$p(t) = \frac{\det((\alpha \otimes \phi)(A_i))}{\det((\alpha \otimes \phi)(x_i - 1))}. \tag{3}$$

Then $p(t) = \tau(K, \alpha) \in \mathbb{C}[t^{\pm 1}]$ (up to multiplication by a power of t).

If $\alpha : \pi_K \to SL(2,\mathbb{C})$ is a distinguished representation of type $(1,1)$, then we obtain $\mathcal{T}_K(t)$ from symmetrizing the polynomial $p(t)$.

Remark. Note that if $\phi(x_i) \neq 0$, then $\det((\alpha \otimes \phi)(x_i - 1)) \neq 0$, in fact $\det((\alpha \otimes \phi)(x_i - 1)) = \det((\alpha \otimes \phi)(x_i) - \mathrm{id}_2)$ is a polynomial of degree 2. In particular the right hand side of (3) is defined. The right hand side of (3) is also known as Wada's invariant [Wa94]. Note that in the literature Wada's invariant is often referred to as twisted Alexander polynomial.

Proof. Kitano [Ki96] proved that $p(t) = \tau(K, \alpha)$ in the case that the presentation of π is a Wirtinger presentation. The general case of any deficiency one presentation can be proved as in the proof of Theorem 3.1 of [GKM05]. □

5. Twisted torsion and the character variety of a knot

Let $K \subset S^3$ be an oriented knot. We write $\pi = \pi_K$ and we fix a Wirtinger presentation
$$\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle.$$
We write
$$R(K) := \mathrm{Hom}(\pi, SL(2,\mathbb{C})).$$
It is well-known that $R(K)$ can be equipped canonically with the structure of a complex variety, and we refer to it as the *representation variety* of K. Given $\alpha \in R(K)$ we consider the map
$$\begin{array}{rccc} \chi_\alpha : & \pi & \to & \mathbb{C} \\ & g & \mapsto & \mathrm{tr}(\alpha(g)). \end{array}$$
We refer to χ_α as the *character* of α. We denote by $X(K) \subset \mathrm{Maps}(\pi, \mathbb{C})$ the set of all characters. The set $X(K)$ can also be endowed with the structure of a variety in such a way that the natural map
$$\begin{array}{rccc} p : & R(K) & \to & X(K) \\ & \alpha & \mapsto & \chi_\alpha \end{array}$$
is an algebraic map with the property that $U \in R(K)$ is open if and only if $p(U) \in X(K)$ is open. We refer to the classic paper of Culler and Shalen [CS83] and to the survey article of Shalen [Sh02] for more information and full details.

Recall that given $\alpha \in R(K)$ we defined a twisted torsion invariant $\tau(K,\alpha) \in \mathbb{C}(t)$ which by Lemma 2.2 is well-defined up to multiplication by an element of the form $t^k, k \in \mathbb{Z}$. Given $p(t), q(t) \in \mathbb{C}[t^{\pm 1}]$ we write $p(t) \doteq q(t)$ if there exists $k \in \mathbb{Z}$ with $p(t) = t^k q(t)$. We can now formulate the following lemma.

Lemma 5.1. *Let $\alpha, \beta \in R(K)$ be representations which represent the same point in the character variety. Then $\tau(K,\alpha) \doteq \tau(K,\beta)$.*

Proof. Let $\alpha, \beta \in R(K)$ be representations which represent the same point in the character variety. First assume that α or β is an irreducible representation. It follows from [CS83, Proposition 1.5.2] that α and β are equivalent representations. It now follows from Lemma 2.2 that $\tau(K,\alpha) \doteq \tau(K,\beta)$.

Now suppose that α and β are reducible representations. We can thus find matrices P and Q and homomorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2 : \pi \to GL(1,\mathbb{C})$ such that
$$\alpha(g) = P \begin{pmatrix} \alpha_1(g) & * \\ 0 & \alpha_2(g) \end{pmatrix} P^{-1} \text{ and } \beta(g) = Q \begin{pmatrix} \beta_1(g) & * \\ 0 & \beta_2(g) \end{pmatrix} Q^{-1}$$
for any $g \in \pi$. It now follows from standard arguments and from Lemma 2.2 that
$$\tau(K,\alpha) \doteq \tau(K,\alpha_1) \cdot \tau(K,\alpha_2) \text{ and } \tau(K,\beta) \doteq \tau(K,\beta_1) \cdot \tau(K,\beta_2).$$
On the other hand standard arguments show that $\chi_\alpha = \chi_\beta$ implies that $\{\alpha_1, \alpha_2\} = \{\beta_1, \beta_2\} \subset \mathrm{Hom}(\pi, GL(1,\mathbb{C}))$. Combining these results it now follows immediately that $\tau(K,\alpha) \doteq \tau(K,\beta)$. $\square$

Given $\chi \in X(K)$ we now define

$$\tau(K,\chi) = \tau(K,\alpha) \in \mathbb{C}(t)$$

where α is any representation with $\chi_\alpha = \chi$. By Lemmas 2.2 and 5.1 $\tau(K,\chi)$ is well-defined up to multiplication by an element of the form $t^k, k \in \mathbb{Z}$.

Given $f(t) \in \mathbb{C}(t)$ with $f(t) = \frac{p(t)}{q(t)}$ where $p(t), q(t) \in \mathbb{C}[t^{\pm 1}]$ we now define $\deg(f(t)) = \deg(p(t)) - \deg(q(t))$. Furthermore we say that $f(t)$ is *monic* if we can find monic polynomials $p(t), q(t) \in \mathbb{C}[t^{\pm 1}]$ with $f(t) = \frac{p(t)}{q(t)}$. The following is the main result of this section:

Theorem 5.2. *Let $K \subset S^3$ be an oriented knot.*

(1) *The set*
$$M := \{\chi \in X(K) \,|\, \tau(K,\chi) \text{ is monic } \} \subset X(K)$$
is Zariski closed.

(2) *The set*
$$D := \{\chi \in X(K) \,|\, deg\, \tau(K,\chi) = 2x(K)\} \subset X(K)$$
is Zariski open.

Proof. Let $K \subset S^3$ be an oriented knot. We define

$$\begin{array}{rcl} \hat{M} & := & \{\alpha \in R(K) \,|\, \tau(K,\alpha) \text{ is monic } \} \subset R(K) \\ \hat{D} & := & \{\alpha \in R(K) \,|\, \deg \tau(K,\alpha) = 2x(K)\} \subset R(K). \end{array}$$

By the discussion of the map $p : R(K) \to X(K)$ it suffices to show that $\hat{M} \subset R(K)$ is closed and that $\hat{D} \subset R(K)$ is open.

We denote by $A = (a_{ij})$ the $(k-1) \times k$–matrix given by $a_{ij} = \frac{\partial r_i}{\partial x_j}$. Let $i \in \{1, \dots, k\}$. Note that $\phi(x_i) = 1$ since we picked a Wirtinger presentation. Denote by A_i the matrix obtained from A by deleting the i–th column. Given $\alpha \in R(K)$ we now define

$$\begin{array}{rcl} p_\alpha(t) & := & \det((\alpha \otimes \phi)(A_i)) \\ q_\alpha(t) & := & \det((\alpha \otimes \phi)(x_i - 1)) = \det(\alpha(x_i)t - \mathrm{id}_2). \end{array}$$

By Proposition 4.1 we have

$$\tau(K,\alpha) \doteq \frac{p_\alpha(t)}{q_\alpha(t)}.$$

Note that $q_\alpha(t)$ is monic and of degree two for any α. It therefore follows that

$$\begin{array}{rcl} \hat{M} & = & \{\alpha \in R(K) \,|\, p_\alpha(t) \text{ is monic } \} \\ \hat{D} & = & \{\alpha \in R(K) \,|\, \deg p_\alpha(t) = 2x(K) + 2\}. \end{array}$$

Claim. There exists an $N \in \mathbb{N}$ such that

$$q_\alpha(t) \in \bigoplus_{i=-N}^{N} \mathbb{C} \cdot t^i.$$

Indeed, a straightforward argument using the definition of Fox derivatives shows that such an N is given by

$$2(k-1)\max\{\ell(r_i),\dots,\ell(r_{k-1})\}$$

where $\ell(r_i)$ denotes the length of the word r_i in the generators $x_1,\dots,x_k$.

We now denote by q the map

$$\begin{array}{rcl} R(K) & \to & \bigoplus_{i=-N}^{N} \mathbb{C}\cdot t^i = \mathbb{C}^{2N+1} \\ \alpha & \mapsto & q_\alpha(t). \end{array}$$

It is clear that q is an algebraic map.

Given $i\in\{1,\dots,2N-1\}$ and $j\in\{1,\dots,2N-1-i\}$ we write

$$V_{i,j}=\{(\underbrace{0,\dots,0}_{i},1,a_1,\dots,a_j,1,\underbrace{0,\dots,0}_{2N-1-i-j}\,|\,a_1,\dots,a_j\in\mathbb{C}\}\subset\mathbb{C}^{2N+1}.$$

Note that

$$\hat{M}=\bigcup_{i,j} q^{-1}(V_{i,j})$$

Since q is an algebraic map and since $V_{i,j}\subset\mathbb{C}^{2N+1}$ is Zariski closed for any i,j it follows that $\hat{M}\subset R(K)$ is Zariski closed.

Given $i\in\{1,\dots,2N-1-2x(K)\}$ we now write

$$W_i=\{(a_1,\dots,a_{2N+1}\,|\,a_i\neq 0 \text{ and } a_{i+2x(K)+2}\neq 0\}\subset\mathbb{C}^{2N+1}.$$

Note that q_α lies in some W_i if and only if $\deg(q_\alpha(t))>2x(K)+2$. Now note that by [FK06, Theorem 1.1] we always have $\deg(\tau(K,\alpha))\le 2x(K)$, in particular for any α we have $\deg(q_\alpha(t))\le 2x(K)+2$. It now follows that

$$\hat{M}=\bigcup_{i} q^{-1}(W_i)$$

Since q is an algebraic map and since $W_i\subset\mathbb{C}^{2N+1}$ is Zariski open for any i it follows that $\hat{M}\subset R(K)$ is Zariski closed.

□

6. Calculations

In this section we will study the hyperbolic torsion of hyperbolic knots with up to twelve crossings. Note that considering only hyperbolic knots is not very restrictive, indeed, by [HTW98] any knot with up to twelve crossings is either a torus knot or a hyperbolic knot. In fact the only torus knots with at most twelve crossings are the following:

$$3_1, 5_1, 7_1, 8_{19}, 9_1, 10_{124}, 11_{367}.$$

Also note that torus knots are well-understood, it is well-known that they are fibered and chiral. The next torus knots are $T(13,2)=13_{4878}$? and $T(7,3)=14_{?}$??. There are two 13 crossing knots which are satellite knots, namely the $(2,1)$ cables of the left hand trefoil and the right hand trefoil. In the knot census these knots are 13_{9465} and 13_{9517}. There are two 14 crossing knots which are satellite knots, namely the Whitehead doubles of the left hand trefoil and the right hand trefoil.

In this section we will first give a detailed calculation of the hyperbolic torsion for the Conway knot and the Kinoshita-Terasaka knot. We will then show that the hyperbolic torsion detects the genus, fiberedness and chirality for all hyperbolic knots up to twelve crossings.

Regarding the calculations we have to make a disclaimer: We believe that all calculations are precise up to at least eight digits. We did not make a serious error analysis though, and the results should therefore be taken with a (hopefully very small) grain of salt.

6.1. **Calculations: Conway knot and Kinoshita–Terasaka knot.** We now consider the Conway knot 11_{401} and the Kinoshita-Terasaka knot 11_{409} in details. These two knots are mutants and the Alexander polynomial in both cases is trivial, i.e. equal to one. On the other hand, the genus of the Conway knot 11_{401} is equal to three and the genus of the Kinoshita-Terasaka knot 11_{409} is equal to two.

For the Conway knot we calculate

$$\begin{array}{rllll}\mathcal{T}_{K_{401}}(t) \approx & (4.8952-0.0992i)t^5 & +(-15.6857+0.2976i)t^4 & +(23.1036+0.0784i)t^3 \\ & +(-26.9416-4.8451i)t^2 & +(38.3835+24.4943i)t & +(-43.3240-44.0806i) \\ & +(38.3835+24.4943i)t^{-1} & +(-26.9416-4.8451i)t^{-2} & +(23.1036+0.0784i)t^{-3} \\ & +(-15.6857+0.2976i)t^{-4} & +(4.8952-0.0992i)t^{-5}. & \end{array}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{array}{rllll}\mathcal{T}_{K_{409}}(t) \approx & (4.4179+0.3760i)t^3 & +(-22.9416-4.8451i)t^2 & +(61.9644+24.0974i)t \\ & +(-82.6954-43.4854i) & +(61.9644+24.0974i)t^{-1} & +(-22.9416-4.8451i)t^{-2} \\ & +(4.4179+0.3760i)t^{-3}. & & \end{array}$$

Note that Conjectures 1.4 and 1.5 hold for both knots. Also note that the polynomials are not real, reflecting the fact that they are chiral. Finally note that the polynomials are different, which implies that $\mathcal{T}_K(t)$ is not invariant under mutation.

Remark. It is interesting to study the evaluations at $t=1$ and $t=-1$. We calculate

$$\begin{array}{lclcl}\mathcal{T}_{K_{401}}(1) & \approx & \mathcal{T}_{K_{409}}(1) & \approx & 4.186003-4.228629i \\ \mathcal{T}_{K_{401}}(-1) & \approx & \mathcal{T}_{K_{409}}(-1) & \approx & 261.3432+102.1226i.\end{array}$$

This calculation reflects our conjecture that the invariants $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are unchanged under mutation. On the other hand we have

$$\begin{array}{lcl}\mathcal{T}_{K_{401}}(i) & \approx & 33.7952-20.8122i \\ \mathcal{T}_{K_{401}}(i) & \approx & -33.7952+36.8122i.\end{array}$$

This suggests that the evaluation of $\mathcal{T}_K(\xi)$ is not a mutation invariant if $\xi \neq \pm 1$. As a curiosity we point out that among all hyperbolic eleven crossing knots the Conway knot and the Kinoshita–Terasaka knot have the largest value for the invariant $|\mathcal{T}_K(1)|$. This does not hold though for the evaluation at $t = -1$.

Remark. It was shown in [FK06, Section 5] that twisted Alexander polynomials corresponding to representations over finite fields detect the genus of all knots with up to twelve crossings. For example we found in [FK06] a representation $\alpha : \pi_1(X_{K_{401}}) \to \mathrm{GL}(4, \mathbb{F}_{13})$ such that the corresponding torsion $\tau(K_{401}, \alpha) \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 14. From Theorem 1.3 we then obtain the inequality

$$\mathrm{genus}(K_{401}) \geq \frac{1}{8}\deg(\tau(K_{401}, \alpha)) + \frac{1}{2} = 2.25.$$

In particular this representation showed that $\mathrm{genus}(K_{401}) \geq 3$ since the genus is of course an integer. The calculations using the discrete and faithful $SL(2, \mathbb{C})$ representations are in some sense more satisfactory since they give the equality $\mathrm{genus}(K_{401}) = \frac{1}{4}(\deg(\mathcal{T}_{K_{401}}(t))) + \frac{1}{2}$ on the nose, and not just after rounding up to integers.

6.2. **Calculations: Genus and fiberedness for knots with up to twelve crossings.** There exist 36 knots with twelve crossings or less for which the ordinary Alexander polynomial does not detect the genus, i.e. for which $\mathrm{genus}(K) > \frac{1}{2}\deg\Delta_K(t)$ (cf. e.g. [CL09] or [St10]):

11_{401} 11_{409} 11_{412} 11_{434} 11_{440} 11_{464} 11_{519} 12_{1311} 12_{1316} 12_{1319}
12_{1339} 12_{1344} 12_{1351} 12_{1375} 12_{1412} 12_{1417} 12_{1420} 12_{1509} 12_{1519} 12_{1544}
12_{1545} 12_{1552} 12_{1555} 12_{1556} 12_{1581} 12_{1601} 12_{1609} 12_{1699} 12_{1718} 12_{1745}
12_{1807} 12_{1953} 12_{2038} 12_{2096} 12_{2100} 12_{2118}.

Among the hyperbolic knots there exist thirteen knots with up to 12 crossings which are not fibered but which are algebraically fibered, i.e. $\deg(\Delta_K) = 2\mathrm{genus}(K)$ and Δ_K is monic:

12_{1345} 12_{1498} 12_{1502} 12_{1546} 12_{1567} 12_{1670} 12_{1682} 12_{1752} 12_{1771} 12_{1823}
12_{1938} 12_{2089} 12_{2103}.

We computed $\mathcal{T}_K(t)$ for all hyperbolic knots with up to twelve crossings. For any knot K with at most thirteen crossings the absolute value of the $t^{x(K)}$ coefficient of $\mathcal{T}_K(t)$ is at least 0.00556. (The minimum value is attained by the chiral and alternating knot 12_{1287}.) This shows that even with an error term $\pm 10^{-8}$ the absolute value of the $t^{x(K)}$ coefficient of $\mathcal{T}_K(t)$ is non-zero for any knot with at most thirteen crossings. This shows that Conjecture 1.5 holds for all these knots.

A similar argument shows that Conjecture 1.4 holds for all these knots with at most thirteen crossings.

The genera for the 13 crossing knots were also independently determined by Alexander Stoimenow (see [St10]).

Finally, note that our calculations suggest that many fibered hyperbolic knots have the property that the second highest coefficient of $\mathcal{T}_K(t)$ is a real number. What is a good explanation for this phenomenon? Of course the second highest coefficient is just the sum of the eigenvalues of the monodromy of the twisted homology of the fiber.

6.3. **Calculations: Chirality.** There are 78 knots with at most thirteen crossings which are amphichiral (cf. [CL09], [HTW98] and [St10]):

4_1	6_3	8_3	8_9	8_{12}	8_{17}	8_{18}	10_{17}	10_{33}	10_{37}
10_{43}	10_{45}	10_{79}	10_{81}	10_{88}	10_{99}	10_{109}	10_{115}	10_{118}	10_{123}
12_4	12_{58}	12_{125}	12_{268}	12_{273}	12_{341}	12_{427}	12_{435}	12_{458}	12_{462}
12_{465}	12_{471}	12_{477}	12_{499}	12_{506}	12_{510}	12_{627}	12_{819}	12_{821}	12_{868}
12_{887}	12_{890}	12_{906}	12_{960}	12_{990}	12_{1008}	12_{1019}	12_{1039}	12_{1102}	12_{1105}
12_{1123}	12_{1124}	12_{1127}	12_{1152}	12_{1167}	12_{1188}	12_{1202}	12_{1209}	12_{1211}	12_{1218}
12_{1225}	12_{1229}	12_{1249}	12_{1251}	12_{1254}	12_{1260}	12_{1267}	12_{1269}	12_{1273}	12_{1275}
12_{1280}	12_{1281}	12_{1287}	12_{1288}	12_{1644}	12_{1750}	12_{1994}	12_{2161}.		

It is well-known that the Jones polynomial, the HOMFLY polynomial and the Kauffman polynomial can detect chirality. Among the knots with up to 10 crossings only the chirality of the knots 9_{42} and 10_{71} can not be detected this way (cf. [RGK94]). Our calculations show that if K is a hyperbolic knot with at most thirteen crossings, then K is amphichiral if and only if $\mathcal{T}_K(t)$ is a real polynomial.

Note that our calculations suggest that the evaluations at $t=1$ or $t=-1$ do not always detect chirality. For example the knot 10_{153} has the property that $\mathcal{T}_K(1)$ equals 4 up to about 10 digits, and the knot 10_{157} has the property that $\mathcal{T}_K(-1)$ equals 576 up to about 10 digits. In both cases though the chirality gets detected by evaluating $\mathcal{T}_{10_{153}}(-1)$ respectively $\mathcal{T}_{10_{157}}(1)$. We have not yet found a chiral knot where $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are real.

Finally we point out that for any amphichiral hyperbolic knot with at most thirteen crossings the top coefficient is always at least one. This begs the question whether this is the case in general.

6.4. **Calculations: The adjoint representation.** Let K be an oriented hyperbolic knot and let $\alpha : \pi_1(X_K) \to SL(2,\mathbb{C})$ be a distinguished representation of type $(1,1)$. We now consider the adjoint representation

$$\begin{array}{rcl} \alpha_{adj} : \pi_1(X_K) & \to & \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{C})) \\ g & \mapsto & A \mapsto \alpha(g)A\alpha^{-1} \end{array}$$

associated to α. It is well-known that this representation is also faithful and irreducible. The corresponding twisted torsion $\tau(K,\alpha_{adj}) \in \mathbb{C}[t^{\pm 1}]$ is well-defined up to multiplication by an element of the form $\pm t^l$. In fact using sign-refined torsion and using the orientation of the knot K we obtain an invariant $\mathcal{T}_K^{adj}(t)$ which is well-defined up to multiplication by an element of the form t^l. We refer to [DY09] for details on this construction and for further information on $\mathcal{T}_K^{adj}(t)$.

For the Conway knot we calculate that $\mathcal{T}^{adj}_{K_{401}}(t)$ equals approximately

$$\begin{array}{lll} (-0.2788 + 16.4072i)t^{13} & +(-3.9858 - 20.1706i)t^{12} & +(-4.2204 - 60.5497i)t^{11} \\ +(52.0953 + 134.5013i)t^{10} & +(-147.7856 - 46.07448i)t^{9} & +(897.2087 + 62.3265i)t^{8} \\ +(-2465.8556 - 1308.0110i)t^{7} & +(2465.8556 + 1308.0110i)t^{6} & +(-897.2087 - 62.3265i)t^{5} \\ +(147.7856 + 46.0745i)t^{4} & +(-52.0953 - 134.5013i)t^{3} & +(4.2204 + 60.5498i)t^{2} \\ +(3.9858 + 20.1706i)t & +(0.2788 - 16.4072i) & \end{array}$$

and for the Kinoshita-Terasaka knot we calculate that $\mathcal{T}^{adj}_{K_{409}}(t)$ equals approximately

$$\begin{array}{lll} (-0.7378 + 12.4047i)t^{7} & +(29.9408 - 56.5548i)t^{6} & +(-655.7823 - 173.0400i)t^{5} \\ +(2056.7509 + 1678.4875i)t^{4} & +(-2056.7509 - 1678.4875i)t^{3} & +(655.7823 + 173.0400i)t^{2} \\ +(-29.9408 + 56.5548i)t & +(0.7378 - 12.4047i). & \end{array}$$

Note $\dim \mathfrak{sl}(2,\mathbb{C}) = 3$, it now follows from Theorem [FK06, Theorem 1.1] that

$$x(K) \geq \frac{1}{3}\deg(\mathcal{T}^{adj}_K(t)).$$

Note that in the case of the Conway knot and the Kinoshita-Terasaka knot equality does *not* hold. This shows that in general the twisted torsion corresponding to an irreducible and faithful representation does not detect the genus of a knot.

Dubois and Yamaguchi [DY09] have shown that $\mathcal{T}^{adj}_K(t) \doteq -\mathcal{T}^{adj}_K(t^{-1})$. It follows that $\mathcal{T}^{adj}_K(1) = 0$ and that $\mathcal{T}^{adj}_K(t)$ has odd degree. It is also shown that $(\mathcal{T}^{adj}_K)'(1) \neq 0$ and it is conjectured that this number is invariant under mutation.

Question 6.1. *Does $\mathcal{T}^{adj}_K(t)$ detect fibered knots?*

6.5. **Computations which we should still do.** Here is my (S) wish list:

(1) compute $\mathcal{T}_K(t)$ for all knots up to 16 crossings
(2) check for fiberedness and genus for all knots up to 14 crossings (more would be nice but perhaps too much work)
(3) check for all these knots whether $\mathcal{T}_K(t)$ detects chirality (i.e. compare with Stoimenow's list [St10]). I think we so far did it up to 14 crossings
(4) check whether two knots have the same $\mathcal{T}_K(t)$ implies that they are mutants (again see [St10])
(5) check whether $|\mathcal{T}_K(-1)|$ is always larger than $|\mathcal{T}_K(1)|$.
(6) Check whether the top coefficient of a non-fibered chiral knot is always at least one.
(7) find out how many/which fibered knots have the property that the second highest coefficient is a real number (it seems like a non-trivial percentage has this property)
(8) it would be nice to do one precise calculation, e.g. find the representation of the Kinoshita-Terasaka knot and the Conway knot over the number field, and then do the calculations for $\mathcal{T}_K(t)$ and $\mathcal{T}^{adj}_K(t)$.

(9) compute $\mathcal{T}_K^{adj}(t)$ for enough examples to see whether it detects fiberedness (we know it does not detect the genus, at least, unless you are allowed to round up)
(10) Are there amphichiral knots such that $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are real numbers?
(11) compute the torsion for the canonical 4-dimensional representation of $SL(2,\mathbb{C})$ for the Conway knot and the Kinoshita-Terasaka knot. Do we get the genus? (i.e. the 2-dimensional representation gave the genus, the 3-dimensional one didn't, so do higher representations get worse or is it an even/odd thing?)

I also have an elementary mathematics question: Let $f(t), g(t)$ be complex polynomials such that for any n we have

$$\prod_{k=1}^{n} f(e^{2\pi ik/n}) = \prod_{k=1}^{n} g(e^{2\pi ik/n}).$$

Does this imply that $f(t) = g(t)$?

The question arises in our context as follows: is the invariant $\mathcal{T}_K(t)$ determined by the $\mathbb{C}$-valued torsions of the finite cyclic covers.

7. Appendix A: Connection to zeta functions

The goal is to write up a few facts about the relationship between our torsion and the analytic torsion. Right now that section is just a mess.

7.1. **Analytic torsion.** Let X be a compact oriented Riemannian manifold and let $\alpha : \pi_1(X) \to O(n)$ be an orthogonal representation. Then Ray and Singer [RS71] defined the analytic torsion $\tau^{an}(X, \alpha) \in \mathbb{R}$. Cheeger [Che77, Che79] and Müller [Mü78] showed independently that analytic torsion equals Reidemeister torsion.

Let X be a hyperbolic 3-manifold X which is either closed or has cusps. Given a geodesic γ we denote by $\ell(\gamma)$ its length. Recall that a closed geodesic is called *prime* if it can not be expressed as the multiple of a shorter closed geodesic. Let $\rho : \pi_1(X) \to \mathrm{SL}(n,\mathbb{C})$ be a representation let and $s \in \mathbb{C}$. The *Ruelle zeta function of* (X, ρ) is now defined as

$$R_\rho(s) := \prod \det\left(\mathrm{id} - \rho(\gamma)e^{-s\ell(\gamma)}\right)^{-1},$$

where γ runs over the set of all closed prime geodesics.

If X is closed and ρ an orthogonal representation $\rho : \pi_1(X) \to O(n)$ such that $H_*(X; \mathbb{R}^n) = 0$, then Fried [Fr86] showed that $R_\rho(s)$ extends meromorphically to $\mathbb{C}$ and

$$|R_\rho(0)| = \tau^{an}(X, \rho)^2,$$

where $\tau^{an}(X, \rho)$ denotes the analytic torsion introduced by Ray and Singer [RS71].

$$|R_\rho(0)| = \tau(X, \rho)^2.$$

(i.e. the Ruelle zeta function can be extended such that $R_\rho(0)$ is defined) and

$$R_\rho(0) = \tau^{an}(X, \rho) = \tau(X, \rho).$$

Here $\tau^{an}(X, \rho)$ is the analytic torsion of the pair (X, ρ) and $\tau(X, \rho)$ is the algebraic torsion defined earlier. Furthermore,

$$\frac{d}{ds}\|_{s=0} \ln R_\rho(s) = C \cdot \mathrm{volume}(X)$$

for a fixed constant C.

In [Par09] Jinsung Park extended some of Fried's results to the non-compact case. More precisely, if X is a complete hyperbolic 3-manifold which is non-compact, then Park showed that $R_\rho(s)$ has a meromorphic extension to the whole complex plane and that

$$R_\rho(0) = \tau^{an}(X, \rho).$$

It is conjectured that the following equality also holds:

$$\tau^{an}(X, \rho) = \tau(X, \rho).$$

Remark. Let K be a hyperbolic knot. Denote by X_n the n-fold cover of $X(K)$. Recall that the proof of Proposition 3.1 shows that the evaluation $\prod_{k=1}^{n} \mathcal{T}_K(e^{2\pi i k/n})$ equals $\tau(X_n, \alpha_{can})$. Does that imply that the invariants $\tau(X_n, \alpha_{can})$ determine $\mathcal{T}_K(t)$?

Remark. Franks [Fra81] gave a dynamical interpretation of the untwisted Alexander polynomial of a knot. The results of Franks, Fried and Park suggest that there should also exist a dynamical reinterpretation of twisted Alexander polynomials, in particular of the invariant $\mathcal{T}_K(t)$.

Can we show the mutation invariance using these geometric definitions?

7.2. **Representations of hyperbolic knots.** Throughout this section all hyperbolic 3-manifolds are understood to be topology finite, i.e. homeomorphic to the interior of a compact 3-manifold with empty or toroidal boundary.

Let V_m denote the vector space of symmetric tensors of rank m over $\mathbb{C}^2$. Put differently, V_m is the quotient of $V^{\otimes m}$ by the subspace generated by vectors of the form

$$a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots \otimes a_m - a_1 \otimes \cdots \otimes a_j \otimes \cdots \otimes a_i \otimes \cdots \otimes a_m.$$

Note that V_m is a vector space of dimension $m+1$, in fact a basis is given by the vectors $e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2$ where the number of e_1-terms ranges from 0 to m. We equip $\mathbb{C}^2$ with the non-singular form $(v, w) := \det(v\, w)$ and we extend it to a form on V_m by defining

$$(v_1 \otimes \cdots \otimes v_m, w_1 \otimes \cdots \otimes w_m) := \prod_{i=1}^{m} (v_i, w_j).$$

Then there exists a canonical representation $\rho_m : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(V_m)$ which is well-known to be the unique (up to isomorphism) irreducible $m+1$-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$. It is also well-known that $\det(\rho_m(A)) = 1$ for any $A \in \mathrm{SL}(2,\mathbb{C})$. Finally note that ρ_1 is just the identity, and note that ρ_2 is isomorphic to the adjoint representation $\rho_{adj} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{sl}(2,\mathbb{C}), A \mapsto (B \mapsto ABA^{-1})$.

Given an oriented hyperbolic 3-manifold together with a lift $\alpha : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ of the canonical representation $\pi_1(X) \to PSL(2,\mathbb{C})$ we denote by α_m the representation

$$\pi_1(X) \to \mathrm{SL}(2,\mathbb{C}) \xrightarrow{\rho_m} \mathrm{SAut}_{\mathbb{C}}(V_m) \cong \mathrm{SL}(m+1,\mathbb{C}).$$

The proof of the following theorem for the closed case can be found in [BW00, Theorem 6.7, Chapt. VII]

Theorem 7.1. *Let X be an oriented hyperbolic 3-manifold. Then $H_*^{\alpha_m}(X;\mathbb{C}^{m+1}) = 0$.*

So what non-degenerate form does this representation preserve?

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