

Cup and cap products in intersection (co)homology

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Abstract

We construct cup and cap products in intersection (co)homology with field coefficients. The existence of the cap product allows us to give a new proof of Poincaré duality in intersection (co)homology which is similar in spirit to the usual proof for ordinary (co)homology of manifolds.

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1 Introduction

Intersection homology is a basic tool in the study of singular spaces. It has important features in common with ordinary homology (excision, Mayer-Vietoris, intersection pairing, Poincaré duality with field coefficients) and important differences (restrictions on functoriality, failure of homotopy invariance, restrictions on Poincaré duality with integer coefficients). An important difference is the fact that the Alexander-Whitney map does not induce a map of intersection chains (because if a simplex satisfies the relevant allowability condition there is no reason for its front and back faces to do so). Because of this, it has long been thought that there is no reasonable way to define cup and cap products in intersection (co)homology. In this paper, we use a different method to construct cup and cap products (with field coefficients) with the usual properties, and we use the cap product to give a new proof of Poincaré duality for intersection (co)homology with field coefficients.

We give applications and extensions of these results in [18] and [17]. In [18] we show that our Poincaré duality isomorphism agrees with that obtained by sheaf-theoretic methods in [20] and that our cup product is Poincaré dual to the intersection pairing of [20]. We also prove that the de Rham isomorphism of [5] takes the wedge product of intersection differential forms to the cup product of intersection cochains. In [17] we give a new construction of the symmetric signature for Witt spaces (which responds to a question raised in [1]).

In future work, we plan to use the cup product (and the underlying structure on cochains) as a starting point for developing an “intersection” version of rational homotopy theory (see Remark 1.2 below).

Our basic strategy for constructing cup and cap products is to replace the Alexander-Whitney map by a combination of the cross product and the (geometric) diagonal map. To illustrate this, we explain how it works in ordinary homology. For a field F , the cross

product gives an isomorphism (where the tensor is over F)

$$H_*(X; F) \otimes H_*(Y; F) \rightarrow H_*(X \times Y; F),$$

and we can use this isomorphism to construct the algebraic diagonal map

$$\bar{d}: H_*(X; F) \rightarrow H_*(X; F) \otimes H_*(X; F)$$

as the composite

$$H_*(X; F) \xrightarrow{d} H_*(X \times X; F) \xleftarrow{\cong} H_*(X; F) \otimes H_*(X; F),$$

where d is the geometric diagonal. The evaluation map induces an isomorphism

$$H^*(X; F) \rightarrow \text{Hom}(H_*(X; F), F),$$

and we define the cup product of cohomology classes α and β by

$$(\alpha \cup \beta)(x) = (\alpha \otimes \beta)(\bar{d}(x)), \quad x \in H_*(X; F).$$

The fact that the cup product is associative, commutative, and unital follows easily from the corresponding properties of the cross product. Similarly, we define the cap product by

$$\alpha \cap x = (1 \otimes \alpha)(\bar{d}(x)).$$

In order to carry out the analogous constructions in intersection homology, we need to know that the cross product gives an isomorphism on intersection homology (with suitable perversities) and that the geometric diagonal map induces a map of intersection homology (with suitable perversities). The first of these facts is Theorem 3.1 and the second is Proposition 4.2.

Here is an outline of the paper. In Section 2, we establish terminology and notations for stratified pseudomanifolds and intersection homology. (We allow strata of codimension one and completely general perversities, which means that intersection homology is not independent of the stratification in general.) In Section 3, we state the Künneth theorem for intersection homology, which is the basic tool in our work. In Section 4, we construct the algebraic diagonal map, cup product, and cap product in intersection (co)homology and show that they have the expected properties. In Section 5, we show that an orientation of an n -dimensional stratified pseudomanifold X determines a fundamental class in $I^{\bar{0}}H_n(X, X - K; R)$ for each compact K and each ring R . In Section 6, we show that cap product with the fundamental class induces a Poincaré duality isomorphism

$$I_{\bar{p}}H_c^i(X; F) \rightarrow I^{\bar{q}}H_{n-i}(X; F)$$

when \bar{p} and \bar{q} are complementary perversities. In Section 7, we extend our results to stratified pseudomanifolds with boundary. The proofs in Sections 5 through 7 follow the general outline of the corresponding proofs in [21], but the details are more intricate.

Remark 1.1 (Signs). We include a sign in the Poincaré duality isomorphism (see [14, Section 4.1]). Except for this we follow the signs in [8], which means that we use the Koszul convention everywhere except in the definition of the coboundary on cochains (see Remark 4.10 below).

Remark 1.2. For a space X , the method given above for constructing the cup product in $H^*(X)$ also leads to a Leinster partial commutative algebra structure (see [25], [27, Section 9]) on the singular cochains $S^*(X)$. This structure can be rectified over \mathbb{Q} ([32]) to give a commutative algebra structure on a cochain complex naturally quasi-isomorphic to $S^*(X; \mathbb{Q})$; thus we obtain a functor from spaces to commutative DGA's over \mathbb{Q} which solves the “commutative cochain problem” ([31]) and represents the rational homotopy type of X ([26]). We expect that our work will lead in a similar way to a model for a pseudomanifold X which is part of an “intersection” version of rational homotopy theory (that is, a Quillen equivalence between a certain model category of filtered spaces and a model category of “perverse” commutative algebras [22]).

Remark 1.3. In [3, Section 7], Markus Banagl constructed a cup product

$$I_{\bar{p}}H^*(X; \mathbb{Q}) \otimes I_{\bar{q}}H^*(X; \mathbb{Q}) \rightarrow I_{\bar{q}}H^*(X; \mathbb{Q})$$

for certain pairs of perversities \bar{p}, \bar{q} (namely for classical perversities satisfying $\bar{p}(k) + \bar{p}(l) \leq \bar{p}(k+l) \leq \bar{p}(k) + \bar{p}(l) + 2$ for all k, l and $\bar{q}(k) + k \leq \bar{p}(2k)$ for all k). We show in Appendix D that this cup product agrees with ours (up to sign) for all such pairs \bar{p}, \bar{q} . Banagl's construction is similar to ours, except that the Künneth theorem he uses is the one in [7] (which is a special case of that in [13]; see [13, Corollary 3.6]). He does not consider the cap product.

2 Background

We begin with a brief review of basic definitions. Subsection 2.1 reviews the definition of stratified pseudomanifold. Subsection 2.2 reviews singular intersection homology with general perversities as defined in [16, 15]. Other standard sources for more classical versions of intersection homology include [19, 20, 4, 24, 2, 23, 12].

2.1 Stratified pseudomanifolds

We use the definition of stratified pseudomanifold in [20], except that we allow strata of codimension one. Before giving the definition we need some background.

For a space W we define the *open cone* $c(W)$ by $c(W) = ([0, 1) \times W)/(0 \times W)$ (we put the $[0, 1)$ factor first so that our signs will be consistent with the usual definition of the algebraic mapping cone). Note that $c(\emptyset)$ is a point.

If Y is a filtered space

$$Y = Y^n \supseteq Y^{n-1} \supseteq \dots \supseteq Y^0 \supseteq Y^{-1} = \emptyset,$$

we define $c(Y)$ to be the filtered space with $(c(Y))^i = c(Y^{i-1})$ for $i \geq 0$ and $(c(Y))^{-1} = \emptyset$.

The definition of stratified pseudomanifold is now given by induction on the dimension.

Definition 2.1. A 0-dimensional stratified pseudomanifold X is a discrete set of points with the trivial filtration $X = X^0 \supseteq X^{-1} = \emptyset$.

An n -dimensional (*topological*) *stratified pseudomanifold* X is a paracompact Hausdorff space together with a filtration by closed subsets

$$X = X^n \supseteq X^{n-1} \supseteq X^{n-2} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset$$

such that

1. $X - X^{n-1}$ is dense in X , and
2. for each point $x \in X^i - X^{i-1}$, there exists a neighborhood U of x for which there is a *compact* $n - i - 1$ dimensional stratified pseudomanifold L and a homeomorphism

$$\phi : \mathbb{R}^i \times cL \rightarrow U$$

that takes $\mathbb{R}^i \times c(L^{j-1})$ onto $X^{i+j} \cap U$. A neighborhood U with this property is called *distinguished* and L is called a *link* of x .

The X^i are called *skeleta*. We write X_i for $X^i - X^{i-1}$; this is an i -manifold that may be empty. We refer to the connected components of the various X_i as *strata*¹. If a stratum Z is a subset of X_n it is called a *regular stratum*; otherwise it is called a *singular stratum*. The *depth* of a stratified pseudomanifold is the number of distinct skeleta it possesses minus one.

We note that this definition of stratified pseudomanifolds is slightly more general than the one in common usage [19], as it is common to assume that $X^{n-1} = X^{n-2}$. We will not make that assumption here, but when we do assume $X^{n-1} = X^{n-2}$, intersection homology with Goresky-MacPherson perversities is known to be a topological invariant; in particular, it is invariant under choice of stratification (see [20], [4], [23]). Examples of pseudomanifolds include irreducible complex algebraic and analytic varieties (see [4, Section IV]).

If L and L' are links of points in the same stratum then there is a stratified homotopy equivalence between them (see, e.g., [11]), and therefore they have the same intersection homology by Appendix A. Because of this, we will sometimes refer to “the link” of a stratum instead of “a link” of a point in the stratum.

2.2 Singular intersection homology with general perversities.

Definition 2.2. Let X be a stratified pseudomanifold. A *perversity* on X is a function $\bar{p} : \{\text{strata of } X\} \rightarrow \mathbb{Z}$ with $\bar{p}(Z) = 0$ if Z is a regular stratum.

This is a much more general definition than that in [19, 20]; on the rare occasions when we want to refer to perversities as defined in [19, 20] we will call them “classical perversities.”

Besides being interesting in their own right, general perversities are required in our work because of their role in the Künneth theorem (Theorem 3.1 below).

¹This terminology agrees with some sources, but is slightly different from others, including our own past work, which would refer to X_i as the stratum and what we call strata as “stratum components.”

In the literature, there are several non-equivalent definitions of intersection homology with general perversities. We will use the version in [16, 15] (which is equivalent to that in [30]). The reason for this choice is that it gives the most useful version of the “cone formula” (Proposition 2.3 below).

As motivation for the definition, recall that the singular chain group $S_i(X; G)$ of a space X with coefficients in an abelian group G consists of finite sums $\sum g_j \sigma_j$, where each $g_j \in G$ and each σ_j is a map $\sigma_j : \Delta^i \rightarrow X$ of the standard i -simplex to X . The boundary is given by $\partial \sum g_j \sigma_j = \sum g_j \partial \sigma_j = \sum_{j,k} (-1)^k g_j \partial_k \sigma_j$. If instead \mathcal{G} is a local coefficient system of abelian groups, then an element of $S_i(X; \mathcal{G})$ is again a sum $\sum g_j \sigma_j$, where now g_j is a lift of σ_j to \mathcal{G} or, equivalently, a section of the coefficient system $\sigma_j^* \mathcal{G}$ over Δ^i . The boundary map becomes $\partial \sum g_j \sigma_j = \sum_{j,k} (-1)^k g_j|_{\partial_k \sigma_j} \partial_k \sigma_j$; in other words, the restriction of the “coefficient” g_j to the boundary piece $\partial_k \sigma_j$ is the restriction of the section over Δ^i to $\partial_k \Delta^i$. If the system \mathcal{G} is constant, then we recover $S_i(X; G)$.

If X is a stratified pseudomanifold we make a slight adjustment. Suppose \mathcal{G} is a local coefficient system defined on $X - X^{n-1}$. Let $C_i(X; \mathcal{G})$ again consist of chains $\sum g_j \sigma_j$, where now g_j is a section of $(\sigma_j|_{\sigma_j^{-1}(X - X^{n-1})})^* \mathcal{G}$ over $\sigma_j^{-1}(X - X^{n-1})$. Note that if $\sigma_j^{-1}(X - X^{n-1})$ is empty then the sections of $(\sigma_j|_{\sigma_j^{-1}(X - X^{n-1})})^* \mathcal{G}$ form the trivial group (because there’s exactly one map from the empty set to any set). The differential is given by the same formula as in the previous paragraph, with restrictions to boundaries $\partial_k \Delta^i$ being trivial if σ_j maps $\partial_k \Delta^i$ into X^{n-1} . Even when we have a globally defined coefficient system, such as the constant system G , we continue to let ${}^2C_i(X; \mathcal{G})$ denote $C_i(X; \mathcal{G}|_{X - X^{n-1}})$.

Now given a stratified pseudomanifold X , a general perversity \bar{p} , and a local coefficient system \mathcal{G} on $X - X^{n-1}$, we define the *intersection chain complex* $I^{\bar{p}}C_*(X; \mathcal{G})$ as a subcomplex of $C_*(X; \mathcal{G})$ as follows. An i -simplex $\sigma : \Delta^i \rightarrow X$ in $C_i(X)$ is *allowable* if

$$\sigma^{-1}(Z) \subset \{i - \text{codim}(Z) + \bar{p}(Z) \text{ skeleton of } \Delta^i\}$$

for any singular stratum Z of X . The chain $\xi \in C_i(X; \mathcal{G})$ is allowable if each simplex with non-zero coefficient in ξ or in $\partial \xi$ is allowable. $I^{\bar{p}}C_*(X; \mathcal{G})$ is the complex of allowable chains.

The associated homology theory is denoted $I^{\bar{p}}H_*(X; \mathcal{G})$ and called *intersection homology*. Relative intersection homology is defined similarly.

If \bar{p} is a perversity in the sense of Goresky-MacPherson [19] and X has no strata of codimension one, then $I^{\bar{p}}H_*(X; \mathcal{G})$ is isomorphic to the intersection homology groups $I^{\bar{p}}H_*(X; \mathcal{G})$ of Goresky-MacPherson [19, 20].

Even with general perversities, many of the basic properties of singular intersection homology established in [23] and [12] hold with little or no change to the proofs, such as excision and Mayer-Vietoris sequences. Intersection homology is also invariant under properly formulated stratified versions of homotopy equivalences. Proof of this folk result for Goresky-MacPherson perversities is written down in [10]; the slightly more elaborate details necessary for general perversities are provided below in Appendix A.

Intersection homology with general perversities can also be formulated sheaf theoretically; see [15, 16] for more details.

²In the first-named author’s prior work, this would have been denoted $C_i(X; \mathcal{G}_0)$.

Cone formula. General perversity intersection homology satisfies the following cone formula, which generalizes that in [19, 20] (but it differs from King's formula in [23]); see [15, Proposition 2.1] and [12, Proposition 2.18]. We state it with constant coefficients, which is all that we require.

Proposition 2.3. *Let L be an $n - 1$ dimensional stratified pseudomanifold, and let G be an abelian group. Let cL be the cone on L , with vertex v and stratified so that $(cL)^0 = v$ and $(cL)^i = c(L^{i-1})$ for $i > 0$. Then*

$$I^{\bar{p}}H_i(cL; G) \cong \begin{cases} 0, & i \geq n - 1 - \bar{p}(\{v\}), \\ I^{\bar{p}}H_i(L; G), & i < n - 1 - \bar{p}(\{v\}), \end{cases}$$

where the isomorphism in the second case is induced by any inclusion $\{t\} \times L \hookrightarrow ([0, 1) \times L)/(0 \times L) = cL$ with $t \neq 0$.

Therefore, also

$$I^{\bar{p}}H_i(cL, L; G) \cong \begin{cases} I^{\bar{p}}H_{i-1}(L; G), & i \geq n - \bar{p}(\{v\}), \\ 0, & i < n - \bar{p}(\{v\}). \end{cases}$$

3 The Künneth theorem for intersection homology

Let X and Y be stratified pseudomanifolds, and let F be a field. We stratify $X \times Y$ in the obvious way: for any strata $Z \subset X$ and $S \subset Y$, $Z \times S$ is a stratum of $X \times Y$.

By [13, page 382], the cross product (where the tensor is over F)

$$C_*(X; F) \otimes C_*(Y; F) \rightarrow C_*(X \times Y; F)$$

restricts to give a map

$$I^{\bar{p}}C_*(X; F) \otimes I^{\bar{q}}C_*(Y; F) \rightarrow I^Q C_*(X \times Y; F)$$

provided that $Q(Z \times S) \geq \bar{p}(Z) + \bar{q}(S)$ for all strata $Z \subset X$, $S \subset Y$.

We can now state the Künneth theorem:

Theorem 3.1. *Let \bar{p} and \bar{q} be perversities on X and Y , and define a perversity $Q_{\bar{p}, \bar{q}}$ on $X \times Y$ by*

$$Q_{\bar{p}, \bar{q}}(Z \times S) = \begin{cases} \bar{p}(Z) + \bar{q}(S) + 2, & Z, S \text{ both singular strata,} \\ \bar{p}(Z), & S \text{ a regular stratum and } Z \text{ singular,} \\ \bar{q}(S), & Z \text{ a regular stratum and } S \text{ singular,} \\ 0, & Z, S \text{ both regular strata.} \end{cases}$$

Then the cross product induces an isomorphism

$$I^{\bar{p}}H_*(X; F) \otimes I^{\bar{q}}H_*(Y; F) \rightarrow I^Q H_*(X \times Y; F).$$

This is a somewhat sharper form of the main result of [13]. We show in Appendix B how to deduce it from the results of [13].

Remark 3.2. In fact there are other choices of Q that give isomorphisms, as explained in [13], but this is the right choice for our purposes because of its compatibility with the diagonal map; see Proposition 4.2.

There is also a relative version of the Künneth theorem.

Theorem 3.3. *Let X and Y be stratified pseudomanifolds with open subsets $A \subset X, B \subset Y$. The cross product induces an isomorphism*

$$I^{\bar{p}}H_*(X, A; F) \otimes I^{\bar{q}}H_*(Y, B; F) \rightarrow I^{Q_{\bar{p}, \bar{q}}}H_*(X \times Y, (A \times Y) \cup (X \times B); F).$$

The proof is given in Appendix B.

4 The diagonal map, cup product, and cap product

4.1 The diagonal map

In this subsection, we define the algebraic diagonal map using the method described in the introduction. The first step is to show that the geometric diagonal map induces a map of intersection chains for suitable perversities.

First we need some notation. Recall that the *top perversity* \bar{t} is defined by

$$\bar{t}(Z) = \begin{cases} 0, & \text{if } Z \text{ is regular,} \\ \text{codim}(Z) - 2, & \text{if } Z \text{ is singular.} \end{cases}$$

Definition 4.1. Let \bar{p} be a perversity. Define the *dual perversity* $D\bar{p}$ by

$$D\bar{p}(Z) = \bar{t}(Z) - \bar{p}(Z).$$

With the notation of Theorem 3.1, let us write $Q_{\bar{p}, \bar{q}, \bar{r}}$ for $Q_{Q_{\bar{p}, \bar{q}}, \bar{r}}$ (which is equal to $Q_{\bar{p}, Q_{\bar{q}, \bar{r}}}$).

Proposition 4.2. *Let $d : X \rightarrow X \times X$ be the diagonal and let G be an abelian group.*

1. *If $D\bar{r}(Z) \geq D\bar{p}(Z) + D\bar{q}(Z)$ for each stratum Z of X then d induces a map*

$$d : I^{\bar{r}}C_*(X; G) \rightarrow I^{Q_{\bar{p}, \bar{q}}}C_*(X \times X; G).$$

2. *If $D\bar{s}(Z) \geq D\bar{q}(Z) + D\bar{r}(Z)$ for each stratum Z of X then $1 \times d$ induces a map*

$$1 \times d : I^{Q_{\bar{p}, \bar{s}}}C_*(X \times X; G) \rightarrow I^{Q_{\bar{p}, \bar{q}, \bar{r}}}C_*(X \times X \times X; G).$$

3. *If $D\bar{s}(Z) \geq D\bar{p}(Z) + D\bar{q}(Z)$ for each stratum Z of X , then $d \times 1$ induces a map*

$$d \times 1 : I^{Q_{\bar{s}, \bar{r}}}C_*(X \times X; G) \rightarrow I^{Q_{\bar{p}, \bar{q}, \bar{r}}}C_*(X \times X \times X; G).$$

Proof. We prove the first part, the other two are similar. A chain ξ is in $I^{\bar{r}}C_i(X; G)$ if for any simplex σ of ξ and any singular stratum Z of X , $\sigma^{-1}(Z)$ is contained in the $i - \text{codim}(Z) + \bar{r}(Z)$ skeleton of the model simplex Δ^i . Now the only singular strata of $X \times X$ which intersect the image of d have the form $Z \times Z$, where Z is a singular stratum of X , so the chain $d(\xi)$ will be in $I^{Q_{\bar{p}, \bar{q}}}C_i(X \times X; G)$ if each $(d\sigma)^{-1}(Z \times Z)$ is contained in the $i - \text{codim}(Z \times Z) + Q_{\bar{p}, \bar{q}}(Z \times Z)$ skeleton of the model simplex Δ^i . For this it suffices to have

$$i - \text{codim}(Z) + \bar{r}(Z) \leq i - \text{codim}(Z \times Z) + Q_{\bar{p}, \bar{q}}(Z \times Z) = i - 2 \text{codim}(Z) + \bar{p}(Z) + \bar{q}(Z) + 2,$$

and this is equivalent to the condition in the hypothesis. \square

Now we can define the algebraic diagonal map.

Definition 4.3. If $D\bar{r} \geq \bar{D}p + \bar{D}q$ let

$$\bar{d} : I^{\bar{r}}H_*(X; F) \rightarrow I^{\bar{p}}H_*(X; F) \otimes I^{\bar{q}}H_*(X; F)$$

be the composite

$$I^{\bar{r}}H_*(X; F) \xrightarrow{d} I^{Q_{\bar{p}, \bar{q}}}H_*(X \times X; F) \xleftarrow{\cong} I^{\bar{p}}H_*(X; F) \otimes I^{\bar{q}}H_*(X; F),$$

where the second map is the Künneth isomorphism (Theorem 3.1).

In the remainder of this subsection we show that the algebraic diagonal map has the expected properties.

Note that \bar{d} is a natural map due to the naturality of the cross product.

Proposition 4.4 (Coassociativity). *Suppose that $D\bar{s} \geq D\bar{u} + D\bar{r}$, $D\bar{s} \geq D\bar{p} + D\bar{v}$, $D\bar{u} \geq D\bar{p} + D\bar{q}$ and $D\bar{v} \geq D\bar{q} + D\bar{r}$. Then the following diagram commutes*

$$\begin{array}{ccc} I^{\bar{s}}H_*(X; F) & \xrightarrow{\bar{d}} & I^{\bar{u}}H_*(X; F) \otimes I^{\bar{r}}H_*(X; F) \\ \bar{d} \downarrow & & \downarrow \bar{d} \otimes 1 \\ I^{\bar{p}}H_*(X; F) \otimes I^{\bar{v}}H_*(X; F) & \xrightarrow{1 \otimes \bar{d}} & I^{\bar{p}}H_*(X; F) \otimes I^{\bar{q}}H_*(X; F) \otimes I^{\bar{r}}H_*(X; F). \end{array}$$

Proof. Consider the following diagram (with coefficients left tacit):

$$\begin{array}{ccccc} I^{\bar{s}}C_*(X) & \xrightarrow{d} & I^{Q_{\bar{u}, \bar{r}}}C_*(X \times X) & \xleftarrow{\text{q.i.}} & I^{\bar{u}}C_*(X) \otimes I^{\bar{r}}C_*(X) \\ \downarrow d & & \downarrow d \times 1 & & \downarrow d \otimes 1 \\ I^{Q_{\bar{p}, \bar{v}}}C_*(X \times X) & \xrightarrow{1 \times d} & I^{Q_{\bar{p}, \bar{q}, \bar{r}}}C_*(X \times X \times X) & \xleftarrow{\text{q.i.}} & I^{Q_{\bar{p}, \bar{q}}}C_*(X \times X) \otimes I^{\bar{r}}C_*(X) \\ \uparrow \text{q.i.} & & \uparrow \text{q.i.} & & \uparrow \text{q.i.} \\ I^{\bar{p}}C_*(X) \otimes I^{\bar{v}}C_*(X) & \xrightarrow{1 \otimes d} & I^{\bar{p}}C_*(X) \otimes I^{Q_{\bar{q}, \bar{r}}}C_*(X \times X) & \xleftarrow{\text{q.i.}} & I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(X) \otimes I^{\bar{r}}C_*(X) \end{array}$$

Here the arrows $1 \times d$ and $d \times 1$ exist by parts 2 and 3 of Proposition 4.2. The arrows marked q.i. are induced by the cross product and are quasi-isomorphisms by Theorem 3.1. The upper left square obviously commutes, the upper right and lower left squares commute by naturality of the cross product, and the lower right square commutes by associativity of the cross product. The result follows from this. \square

Proposition 4.5 (Cocommutativity). *If $D\bar{r} \geq D\bar{p} + D\bar{q}$ then the following diagram commutes.*

$$\begin{array}{ccc} I^{\bar{r}}H_*(X; F) & \xrightarrow{\bar{d}} & I^{\bar{q}}H_*(X; F) \otimes I^{\bar{p}}H_*(X; F) \\ & \searrow \bar{d} & \downarrow \cong \\ & & I^{\bar{p}}H_*(X; F) \otimes I^{\bar{q}}H_*(X; F). \end{array}$$

\square

As background for our next result, note that for any \bar{q} and any abelian group G there is an augmentation $\varepsilon : I^{\bar{q}}H_*(X; G) \rightarrow G$ that takes a 0-chain to the sum of its coefficients and all other chains to 0. Also note that $D\bar{t}$ is identically 0, so for every \bar{p} there is an algebraic diagonal map

$$\bar{d} : I^{\bar{p}}H_*(X; F) \rightarrow I^{\bar{t}}H_*(X; F) \otimes I^{\bar{p}}H_*(X; F).$$

Proposition 4.6 (Counital property). *For any \bar{p} , the composite*

$$I^{\bar{p}}H_*(X; F) \xrightarrow{\bar{d}} I^{\bar{t}}H_*(X; F) \otimes I^{\bar{p}}H_*(X; F) \xrightarrow{\varepsilon \otimes 1} F \otimes I^{\bar{p}}H_*(X; F) \cong I^{\bar{p}}H_*(X; F)$$

is the identity.

Proof. First observe that (by an easy argument using the definition of allowable chain) the projection $p_2 : X \times X \rightarrow X$ induces a map

$$I^{Q_{\bar{t}, \bar{p}}}H_*(X \times X; F) \rightarrow I^{\bar{p}}H_*(X; F).$$

Now it suffices to observe that the following diagram commutes.

$$\begin{array}{ccccc} I^{\bar{p}}H_*(X; F) & \xrightarrow{d} & I^{Q_{\bar{t}, \bar{p}}}H_*(X \times X) & \xleftarrow{\cong} & I^{\bar{t}}H_*(X; F) \otimes I^{\bar{p}}H_*(X; F) \\ & \searrow = & \downarrow p_2 & & \downarrow \varepsilon \otimes 1 \\ & & I^{\bar{p}}H_*(X; F) & \xleftarrow{\cong} & F \otimes I^{\bar{p}}H_*(X; F) \end{array}$$

The commutativity of the square follows easily from the fact that the cross product is induced by the chain-level shuffle product [8, Exercise VI.12.26(2)]. \square

The results of this subsection also have relative forms. Suppose A and B are open subsets of X and that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Then there is an algebraic diagonal

$$\bar{d} : I^{\bar{r}}H_*(X, A \cup B; F) \rightarrow I^{\bar{p}}H_*(X, A; F) \otimes I^{\bar{q}}H_*(X, B; F),$$

and the obvious generalizations of the preceding results hold. Moreover, we have the following proposition.

Proposition 4.7. *Let A be an open subset of X and let $i : A \rightarrow X$ be the inclusion. Then the diagram*

$$\begin{array}{ccc} I^{\bar{r}}H_*(X, A; F) & \xrightarrow{\bar{d}} & I^{\bar{p}}H_*(X, A; F) \otimes I^{\bar{q}}H_*(X; F) \\ \partial \downarrow & & \downarrow \partial \otimes 1 \\ I^{\bar{r}}H_*(A; F) & \xrightarrow{\bar{d}} I^{\bar{p}}H_*(A; F) \otimes I^{\bar{q}}H_*(A; F) \xrightarrow{1 \otimes i} & I^{\bar{p}}H_*(A; F) \otimes I^{\bar{q}}H_*(X; F) \end{array}$$

commutes.

Proof. This follows from the commutativity of the diagrams

$$\begin{array}{ccc} I^{\bar{r}}H_*(X, A; F) & \xrightarrow{d} & I^{Q_{\bar{p}, \bar{q}}}H_*(X \times X, A \times X; F) \\ \partial \downarrow & & \downarrow \partial \\ I^{\bar{r}}H_*(A; F) & \xrightarrow{d} I^{Q_{\bar{p}, \bar{q}}}H_*(A \times A; F) \xrightarrow{1 \times i} & I^{Q_{\bar{p}, \bar{q}}}H_*(A \times X; F), \end{array}$$

which commutes by the naturality of ∂ ,

$$\begin{array}{ccc} I^{Q_{\bar{p}, \bar{q}}}H_*(X \times X, A \times X; F) & \xleftarrow{\cong} & I^{\bar{p}}H_*(X, A; F) \otimes I^{\bar{q}}H_*(X; F) \\ \partial \downarrow & & \downarrow \partial \otimes 1 \\ I^{Q_{\bar{p}, \bar{q}}}H_*(A \times X; F) & \xleftarrow{\cong} & I^{\bar{p}}H_*(A; F) \otimes I^{\bar{q}}H_*(X; F), \end{array}$$

which commutes because the cross product is a chain map, and

$$\begin{array}{ccccc} I^{Q_{\bar{p}, \bar{q}}}H_*(A \times A; F) & \xrightarrow{1 \times i} & I^{Q_{\bar{p}, \bar{q}}}H_*(A \times X; F) & \xleftarrow{\cong} & I^{\bar{p}}H_*(A; F) \otimes I^{\bar{q}}H_*(X; F) \\ & \searrow \cong & & \nearrow 1 \otimes i & \\ & & I^{\bar{p}}H_*(A; F) \otimes I^{\bar{q}}H_*(A; F), & & \end{array}$$

which commutes by naturality of the cross product. \square

4.2 Cochains and the cup product

We begin by defining intersection cochains and intersection cohomology with field coefficients.

Definition 4.8. Define $I_{\bar{p}}C^*(X; F)$ to be $\text{Hom}_F(I^{\bar{p}}C_*(X; F), F)$ and $I_{\bar{p}}H^*(X; F)$ to be $H^*(I_{\bar{p}}C^*(X; F))$. Similarly for the relative groups: $I_{\bar{p}}C^*(X, A; F)$ is $\text{Hom}_F(I^{\bar{p}}C_*(X, A; F), F)$ and $I_{\bar{p}}H^*(X, A; F)$ is $H^*(I_{\bar{p}}C^*(X, A; F))$.

Remark 4.9. Because F is a field we have

$$I_{\bar{p}}H^*(X; F) \cong \text{Hom}_F(I^{\bar{p}}H_*(X; F), F)$$

and

$$I_{\bar{p}}H^*(X, A; F) \cong \text{Hom}_F(I^{\bar{p}}H_*(X, A; F), F).$$

Remark 4.10. We will typically write α for a cochain and x for a chain. We follow Dold's convention for the differential of a cochain (see [8, Remark VI.10.28]; note that this differs slightly from the Koszul convention):

$$(\delta\alpha)(x) = -(-1)^{|\alpha|}\alpha(\partial x).$$

This convention is necessary in order for the evaluation map to be a chain map.

Definition 4.11. If $D\bar{s} \geq D\bar{p} + D\bar{q}$, we define the *cup product* in intersection cohomology

$$\smile: I_{\bar{p}}H^*(X; F) \otimes I_{\bar{q}}H^*(X; F) \rightarrow I_{\bar{s}}H^*(X; F)$$

by

$$(\alpha \smile \beta)(x) = (\alpha \otimes \beta)\bar{d}(x).$$

Explicitly, if $d(x) = \sum y_i \times z_i$, then $(\alpha \smile \beta)(x) = \sum (-1)^{|\beta||y_i|}\alpha(y_i)\beta(z_i)$.

As immediate consequences of Propositions 4.4 and 4.5 we have.

Proposition 4.12 (Associativity). *Let $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ be perversities such that $D\bar{s} \geq D\bar{p} + D\bar{q} + D\bar{r}$. Let $\alpha \in I_{\bar{p}}H^*(X; F)$, $\beta \in I_{\bar{q}}H^*(X; F)$, and $\gamma \in I_{\bar{r}}H^*(X; F)$. Then*

$$(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$$

in $I_{\bar{s}}H^*(X; F)$.

Proposition 4.13 (Commutativity). *Let $\bar{p}, \bar{q}, \bar{s}$ be perversities such that $D\bar{s} \geq D\bar{p} + D\bar{q}$. Let $\alpha \in I_{\bar{p}}H^*(X; F)$, $\beta \in I_{\bar{q}}H^*(X; F)$. Then*

$$\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$$

in $I_{\bar{s}}H^*(X; F)$.

4.3 The cap product

Definition 4.14. If $D\bar{r} \geq D\bar{p} + D\bar{q}$ and A, B are open subsets of X , we define the *cap product*

$$\frown: I_{\bar{q}}H^i(X, B; F) \otimes I^{\bar{r}}H_j(X, A \cup B; F) \rightarrow I^{\bar{p}}H_{j-i}(X, A; F)$$

by

$$\alpha \frown x = (1 \otimes \alpha)\bar{d}(x).$$

Explicitly, if $d(x) = \sum y_i \times z_i$, then $\alpha \frown x = \sum (-1)^{|\alpha||y_i|}\alpha(z_i)y_i$.

Remark 4.15. This definition is modeled on [8, Section VII.12]. The reason Dold has $1 \otimes \alpha$ instead of $\alpha \otimes 1$ in the definition is so that the cap product will make the chains a left module over the cochains (in accordance with the fact that α is on the left in the symbol $\alpha \frown x$).

In the remainder of this subsection we show that the cap product has the expected properties. We begin with the analogue of [8, VII.12.6].

Proposition 4.16. *Suppose $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let A, B, X', A', B' be open subsets of X with $A' \subset X' \cap A$ and $B' \subset X' \cap B$. Let $i : (X'; A', B') \rightarrow (X; A, B)$ be the inclusion map of triads. Let $\alpha \in I_{\bar{q}}H^k(X, B; F)$ and $x \in I^{\bar{r}}H_j(X', A' \cup B'; F)$. Then*

$$\alpha \frown i_*x = i_*((i^*\alpha) \frown x)$$

in $I^{\bar{p}}H_{j-k}(X, A; F)$.

Proof.

$$\begin{aligned} \alpha \frown i_*x &= (1 \otimes \alpha)\bar{d}(i_*x) \\ &= (1 \otimes \alpha)(i_* \otimes i_*)\bar{d}(x) \\ &= (i_* \otimes i^*\alpha)\bar{d}(x) \\ &= i_*((i^*\alpha) \frown x). \end{aligned}$$

□

Proposition 4.17. *Suppose $D\bar{r} \geq D\bar{p} + D\bar{q} + D\bar{u}$. Let $\alpha \in I_{\bar{p}}H^i(X; F)$, $\beta \in I_{\bar{q}}H^j(X; F)$, and $x \in I^{\bar{r}}H_k(X; F)$. Then*

$$(\alpha \smile \beta) \frown x = \alpha \frown (\beta \frown x)$$

in $I^{\bar{u}}H_{k-i-j}(X; F)$.

Proof. First we observe that the perversity condition ensures that both sides of the equation are defined. Now we have

$$\begin{aligned} (\alpha \smile \beta) \frown x &= (1 \otimes (\alpha \smile \beta))\bar{d}(x) \\ &= (1 \otimes \alpha \otimes \beta)(1 \otimes \bar{d})\bar{d}(x) \\ &= (1 \otimes \alpha \otimes \beta)(\bar{d} \otimes 1)\bar{d}(x) \text{ by Proposition 4.4} \\ &= (1 \otimes \alpha)\bar{d}((1 \otimes \beta)\bar{d}(x)) \\ &= \alpha \frown (\beta \frown x). \end{aligned}$$

□

For our next result, note that (because $D\bar{t}$ is identically 0) there is a cap product $I_{\bar{p}}H^i(X, A; F) \otimes I^{\bar{p}}H_j(X, A; F) \rightarrow I^{\bar{t}}H_{j-i}(X; F)$.

Proposition 4.18. *Let A be an open subset of X . Let $\alpha \in I_{\bar{p}}H^i(X, A; F)$ and $x \in I^{\bar{p}}H_i(X, A; F)$. Then the image of $\alpha \frown x$ under the augmentation $\varepsilon : I^{\bar{t}}H_0(X; F) \rightarrow F$ is $\alpha(x)$.*

Proof.

$$\begin{aligned} \varepsilon(\alpha \frown x) &= \varepsilon(1 \otimes \alpha)\bar{d}(x) \\ &= \alpha(\varepsilon \otimes 1)\bar{d}(x) \\ &= \alpha(x) \text{ by the relative version of 4.6.} \end{aligned}$$

□

Proposition 4.19. *Suppose $D\bar{r} \geq D\bar{p} + D\bar{q}$, and let $i : A \hookrightarrow X$ be the inclusion of an open subset.*

1. *Let $\alpha \in I_{\bar{q}}H^k(X; F)$ and $x \in I^{\bar{r}}H_j(X, A; F)$. Then*

$$\partial(\alpha \frown x) = (-1)^{|\alpha|}(i^*\alpha) \frown (\partial x)$$

in $I^{\bar{p}}H_{j-k-1}(A; F)$, where ∂ is the connecting homomorphism.

2. *Let $\alpha \in I_{\bar{q}}H^k(A; F)$ and $x \in I^{\bar{r}}H_j(X, A; F)$. Then*

$$\delta(\alpha) \frown x = -(-1)^{|\alpha|}i_*(\alpha \frown \partial x)$$

in $I^{\bar{p}}H_{j-k}(X; F)$, where ∂ and δ are the connecting homomorphisms.

Proof. We prove part 2; part 1 is similar.

$$\begin{aligned} \delta(\alpha) \frown x &= (1 \otimes \delta(\alpha))\bar{d}(x) \\ &= -(-1)^{|\alpha|}(1 \otimes \alpha)(1 \otimes \partial)\bar{d}(x) \\ &= -(-1)^{|\alpha|}(1 \otimes \alpha)(i_* \otimes 1)\bar{d}(\partial x) \text{ by Proposition 4.7 and the relative version of 4.5} \\ &= -(-1)^{|\alpha|}i_*(\alpha \frown \partial x) \end{aligned}$$

□

We conclude this subsection with a fact which that be needed at one point in Section 6. First observe that if M is a nonsingular manifold with trivial stratification the cross product induces a map

$$H_*(M; F) \otimes I^{\bar{p}}H_*(X; F) \rightarrow I^{\bar{p}}H_*(M \times X; F)$$

for any perversity \bar{p} . This map is an isomorphism by [13, Corollary 3.7], and we define the cohomology cross product

$$\times : H^*(M; F) \otimes I_{\bar{p}}H^*(X; F) \rightarrow I_{\bar{p}}H^*(M \times X; F)$$

to be the composite

$$\begin{aligned} H^*(M; F) \otimes I_{\bar{p}}H^*(X; F) &\cong \text{Hom}_F(H_*(M; F), F) \otimes \text{Hom}_F(I^{\bar{p}}H_*(X; F), F) \rightarrow \\ &\text{Hom}_F(H_*(M; F) \otimes I^{\bar{p}}H_*(X; F), F) \cong \text{Hom}_F(I^{\bar{p}}H_*(M \times X; F), F) \cong I_{\bar{p}}H^*(M \times X; F). \end{aligned}$$

Remark 4.20. Note that the second map in this composite, and therefore the entire composite, is an isomorphism whenever either $H_*(M; F)$ or $I^{\bar{p}}H_*(X; F)$ is finitely generated.

Proposition 4.21. *Suppose $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let $\alpha \in H^*(M; F)$, $x \in H_*(M; F)$, $\beta \in I_{\bar{q}}H^*(X; F)$, and $y \in I^{\bar{r}}H_*(X; F)$. Then*

$$(\alpha \times \beta) \frown (x \times y) = (-1)^{|\beta||x|}(\alpha \frown x) \times (\beta \frown y)$$

in $I^{\bar{p}}H_(M \times X; F)$.*

Proof. This is a straightforward consequence of the definitions and the commutativity of (the outside of) the following diagram (where the F coefficients are tacit and Q denotes $Q_{\bar{p},\bar{q}}$).

$$\begin{array}{ccccc}
H_*(M) \otimes I^{\bar{r}}H_*(X) & \xrightarrow{\bar{d} \otimes \bar{d}} & H_*(M) \otimes H_*(M) \otimes I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(X) \\
\downarrow \times & \searrow d \otimes d & \swarrow \times \otimes \times & \downarrow \cong \\
& H_*(M \times M) \otimes I^Q H_*(X \times X) & H_*(M) \otimes I^{\bar{p}}H_*(X) \otimes H_*(M) \otimes I^{\bar{q}}H_*(X) \\
& \downarrow \times & \downarrow \times \otimes \times \\
& I^Q H_*(M \times X \times M \times X) & \\
\downarrow \times & \nearrow d & \nwarrow \times & \downarrow \times \otimes \times \\
I^{\bar{r}}H_*(M \times X) & \xrightarrow{\bar{d}} & I^{\bar{p}}H_*(M \times X) \otimes I^{\bar{q}}H_*(M \times X)
\end{array}$$

This diagram commutes by the definition of \bar{d} and the naturality, associativity, and commutativity properties of the cross product. \square

5 Fundamental classes

Recall that the basic homological theory of oriented n -manifolds has three parts: the calculation of $H_*(M, M - \{x\})$ and the construction of the local orientation class; the construction of the fundamental class in $H_*(M, M - K)$ for K compact; and the calculation of $H_i(M)$ for $i \geq n$. In this section we show that all of these have analogues for stratified pseudomanifolds using the 0 perversity:

Definition 5.1. $\bar{0}$ is the perversity which is 0 for all strata.

We begin with an overview of the main results, which will be proved in later subsections.

Let R be a ring, and let X be an n -dimensional stratified pseudomanifold. As usual, we do not assume that X is compact or connected and we allow strata of codimension one.

Let X_n denote $X - X^{n-1}$. Recall the following definition from [20, Section 5].

Definition 5.2. An R -orientation of X is an R -orientation of the manifold X_n .

Our first goal is to understand $I^{\bar{0}}H_*(X, X - \{x\}; R)$ (assuming X is R -oriented).

To begin with we note that for $x \in X_n$ we have $I^{\bar{0}}H_*(X, X - \{x\}; R) \cong H_*(X_n, X_n - \{x\}; R)$ by excision. In particular the usual local orientation class in $H_n(X_n, X_n - \{x\}; R)$ determines a local orientation class $o_x \in I^{\bar{0}}H_n(X, X - \{x\}; R)$.

Next we consider the case when X is *normal* (that is, when its links are connected).³ The following proposition generalizes a standard result for R -oriented manifolds.

Proposition 5.3. *Let X be a normal R -oriented n -dimensional stratified pseudomanifold.*

³This differs from the definition of normal given in [20, Section 5.6] but is equivalent in the cases considered there.

1. For all $x \in X$ and all $i \neq n$, $I^{\bar{0}}H_i(X, X - \{x\}; R) = 0$.
2. The sheaf generated by the presheaf $U \rightarrow I^{\bar{0}}H_n(X, X - \bar{U}; R)$ is constant, and there is a unique global section s whose value at each $x \in X_n$ is o_x .
3. For all $x \in X$, $I^{\bar{0}}H_n(X, X - \{x\}; R)$ is the free R -module generated by $s(x)$.

Definition 5.4. Let X be a normal R -oriented stratified pseudomanifold, and let $x \in X$. Define the local orientation class $o_x \in I^{\bar{0}}H_n(X, X - \{x\}; R)$ to be $s(x)$.

Now we recall that Padilla [28] constructs a *normalization* $\pi : \hat{X} \rightarrow X$ for each stratified pseudomanifold X . Here \hat{X} is normal and π has the properties given in [28, Definition 2.2]; in particular π is a finite-to-one map which induces a homeomorphism from $\hat{X} - \hat{X}^{n-1}$ to $X - X^{n-1}$. Padilla shows that the normalization is unique up to isomorphism (that is, up to isomorphism there is a unique \hat{X} and π satisfying [28, Definition 2.2]).

Proposition 5.5. Let X be an R -oriented n -dimensional stratified pseudomanifold, not necessarily normal. Give \hat{X} the R -orientation induced by π . Let $x \in X$.

1. For all $i \neq n$, $I^{\bar{0}}H_i(X, X - \{x\}; R) = 0$.
2. $I^{\bar{0}}H_n(X, X - \{x\}; R)$ is the free R -module generated by the set $\{\pi_*(o_y) \mid y \in \pi^{-1}(x)\}$.

Definition 5.6. Let X be an R -oriented stratified pseudomanifold and give \hat{X} the R -orientation induced by π . For $x \in X$, define the local orientation class $o_x \in I^{\bar{0}}H_n(X, X - \{x\}; R)$ to be

$$\sum_{y \in \pi^{-1}(x)} \pi_*(o_y).$$

This is consistent with Definition 5.4 because for normal X , π is the identity map.

Our next result constructs the fundamental class.

Theorem 5.7. Let X be an R -oriented stratified pseudomanifold. For each compact $K \subset X$, there is a unique $\Gamma_K \in I^{\bar{0}}H_n(X, X - K; R)$ that restricts to o_x for each $x \in K$.

Definition 5.8. Define the *fundamental class* of X over K to be Γ_K .

Remark 5.9. For later use we note that if $K \subset K'$ then the map $I^{\bar{0}}H_n(X, X - K'; R) \rightarrow I^{\bar{0}}H_n(X, X - K; R)$ takes $\Gamma_{K'}$ to Γ_K .

Our next result describes $I^{\bar{0}}H_i(X; R) = 0$ for $i \geq n$ when X is compact. Note that if Z is a regular stratum of X then the closure \bar{Z} (with the induced filtration) is a stratified pseudomanifold; this follows from a straightforward induction over the depth of X .

Theorem 5.10. Let X be a compact R -oriented n -dimensional stratified pseudomanifold.

1. $I^{\bar{0}}H_i(X; R) = 0$ for $i > n$.

2. The natural map

$$\bigoplus_Z I^{\bar{0}}H_n(\bar{Z}; R) \rightarrow I^{\bar{0}}H_n(X; R)$$

is an isomorphism, where the sum is taken over the regular strata of X .

3. If Z is a regular stratum of X , then $I^{\bar{0}}H_n(\bar{Z}; R)$ is the free R -module generated by the fundamental class of \bar{Z} .

Remark 5.11. If X is connected and normal then X has only one regular stratum, by [28, Lemma 2.1]. So in this case we have $I^{\bar{0}}H_n(X; R) \cong R$. (This also follows from the second proposition in [20, Section 5.6], but that result does not give the relation with the local orientation classes.)

Here is an outline of the rest of the section. We prove Proposition 5.3 and Theorems 5.7 and 5.10 (assuming X is normal) in Subsections 5.1 and 5.2. We deduce Proposition 5.5 and the general case of Theorems 5.7 and 5.10 in Subsection 5.3. In Subsection 5.4, we give some further properties of the fundamental class, and in Subsection 5.5, we show that if X is compact and has no strata of codimension one then Γ_X is independent of the stratification.

Remark 5.12. In this section we focus attention on the perversity $\bar{0}$ because this is where the fundamental class needed for our duality results lives, but much of our work is also valid for other perversities and even for ordinary homology:

1. For any nonnegative perversity \bar{p} , Propositions 5.3 and 5.5 and Theorems 5.7 and 5.10 all hold with $I^{\bar{0}}H_*$ replaced by $I^{\bar{p}}H_*$, except that it is not true in general that $I^{\bar{p}}H_*(X, X - \{x\}; R) = 0$ for $* < n$. The proofs are exactly the same.
2. If X is normal and has no codimension one strata then Proposition 5.3 and Theorems 5.7 and 5.10 hold with $I^{\bar{0}}H_*$ replaced by ordinary homology H_* , except that it is not true that $H_*(X, X - \{x\}; R) = 0$ for $* < n$. Again, the proofs are exactly the same.

5.1 The orientation sheaf

Our goal in this subsection will be to prove Proposition 5.3, while in the next subsection we prove Theorems 5.7 and 5.10 under the assumption that X is normal. The general plan of the proofs is the same as in the classical case when X is a manifold; see e.g. [21, Theorem 3.26] for a recent reference. However, there are some technical issues that need to be overcome due to the lack of homogeneity of X .

The proofs of the proposition and the theorems proceed by a simultaneous induction on the depth of X . Note that if the depth of X is 0, then X is a manifold, and all results follow from the classical manifold theory. In the remainder of this subsection, we prove Proposition 5.3 under the assumption that Proposition 5.3 and Theorems 5.7 and 5.10 have been proven for normal stratified pseudomanifolds of depth less than that of X . In the next section, we will then use Proposition 5.3 to prove Theorems 5.7 and 5.10 at the depth of X .

Proof of Proposition 5.3. Recall we let $X_i = X^i - X_{i-1}$.

We first show that for any $x \in X$, $I^{\bar{0}}H_n(X, X - \{x\}; R) \cong R$ and $I^{\bar{0}}H_i(X, X - \{x\}; R) = 0$ for $i \neq n$. This is trivial for $x \in X_n$. For $x \in X_{n-k}$, we may assume that x has a distinguished neighborhood of the form $N \cong \mathbb{R}^{n-k} \times cL^{k-1}$. By excision (see [12]), $I^{\bar{0}}H_i(X, X - \{x\}; R) \cong I^{\bar{0}}H_i(N, N - \{x\}; R)$, and by the Künneth theorem [23] with the unfiltered $(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - 0)$, this is isomorphic to $I^{\bar{0}}H_{i-(n-k)}(cL, cL - \{x\}; R)$. By the cone formula, this is $I^{\bar{0}}H_{i-(n-k)-1}(L; R)$ for $i - (n - k) > k - 1$ (i.e. for $i \geq n$) and 0 otherwise. The link L is compact by the definition of a stratified pseudomanifold, it is connected since X is normal, and it has depth less than that of X . So by induction, $I^{\bar{0}}H_{k-1}(L; R) = 0$ for $i > k - 1$ and, given an R -orientation of L (which we shall find in a moment), we have $I^{\bar{0}}H_{k-1}(L; R) \cong R$ with a preferred generator Γ_L representing the local orientation class. It follows that $I^{\bar{0}}H_i(X, X - \{x\}; R) = 0$ for $i \neq n$ and $I^{\bar{0}}H_n(X, X - \{x\}; R) \cong R$ for any $x \in X$.

The $I^{\bar{0}}H_n(X, X - \{x\}; R)$ are the stalks of the sheaf \mathcal{O}^X generated by the presheaf $U \rightarrow I^{\bar{0}}H_n(X, X - \bar{U}; R)$. We next show this is a locally constant sheaf. Certainly it is a locally-constant sheaf over X_n by manifold theory. So assume by induction hypothesis that this sheaf is locally-constant over $X - X^{n-k}$. Let $x \in X_{n-k}$, and again choose a distinguished neighborhood $N \cong \mathbb{R}^{n-k} \times cL$. To appropriately orient L , we assume that L is embedded in cL as some $b \times L$ with $b \in (0, 1)$, and we use that for any choice $z \in N \cap X_n$, there is a local orientation class $o_z \in I^{\bar{0}}H_n(X, X - \{z\}; R)$, determined by the orientation of X . In particular, let $z \in L \cap X_n \subset N$, which we can write as $z = (0, b, c)$ for $0 \in \mathbb{R}^{n-k}$, $b \in (0, 1)$ (along the cone line), and $c \in L - L^{k-2}$. Then $I^{\bar{0}}H_n(X, X - \{z\}; R) \cong H_{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\}; \mathbb{Z}) \otimes H_1((0, 1), (0, 1) - \{b\}; \mathbb{Z}) \otimes I^{\bar{0}}H_{k-1}(L, L - \{c\}; R)$. Choosing the canonical generators of $H_{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\}; \mathbb{Z})$ and $H_1((0, 1), (0, 1) - \{b\}; \mathbb{Z})$, the local orientation class of $I^{\bar{0}}H_n(X, X - \{z\}; R)$ thus determines a local orientation class of $I^{\bar{0}}H_{k-1}(L, L - \{c\}; R)$. Since $c \in L - L^{k-2}$ was arbitrary but the generators of $H_{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\}; \mathbb{Z})$ and $H_1((0, 1), (0, 1) - \{b\}; \mathbb{Z})$ are fixed and we know the generator of $I^{\bar{0}}H_n(X, X - \{z\}; R)$ remains constant over $L \cap X_n$, this determines a fixed R -orientation of L and hence a choice of Γ_L .

Now, having chosen the R -orientation for L and a corresponding fundamental class Γ_L , a more careful look at the usual Künneth and cone formula arguments show that $I^{\bar{0}}H_n(N, N - \{x\}; R) \cong R$ is generated by $[\eta] \times \bar{c}\Gamma_L$, where η is a chain representing the local orientation class of $H_{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - 0; \mathbb{Z})$ and $\bar{c}\Gamma_L$ is the singular chain cone on Γ_L . More explicitly, if we let ξ stand for a specific intersection chain representing the class Γ_L , continuing to consider L as the subset $\{b\} \times L \subset cL$, then $\bar{c}\Gamma_L$ is represented by the chain $\bar{c}\xi$ formed by extending each simplex σ in ξ to the singular cone simplex $[v, \sigma]$, where v represents the cone point of cL .

Now, if we assume x lives at $0 \times v \in N \cong \mathbb{R}^{n-k} \times cL$, where 0 is the origin in \mathbb{R}^{n-k} and v is the cone point of cL , then we can take a smaller neighborhood N' of x with $N' \cong B_\delta \times \bar{c}_\epsilon L$, where B_δ is the ball of radius δ about the origin in \mathbb{R}^{n-k} and $\bar{c}_\epsilon L$ is $([0, \epsilon] \times L) / \sim$ within the cone $([0, 1] \times L) / \sim$ (where \sim collapses $L \times 0$ to a point). We choose δ so that the image of η in $H_{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{a\}; \mathbb{Z})$ is a generator for all $a \in B_\delta$, and we let $\epsilon < b$, where again we have embedded L in cL at distance b from the vertex. With these choices, the chain $[\eta] \times \bar{c}\Gamma_L$ not only generates $I^{\bar{0}}H_n(N, N - \{x\}; R) \cong I^{\bar{0}}H_n(X, X - \{x\}; R) \cong R$, but it also restricts to the local orientation class $o_z \in I^{\bar{0}}H_n(X, X - \{z\}; R)$ for any $z \in N' \cap X_n$.

Next we observe that $I^{\bar{0}}H_n(X, X - N'; R) \cong I^{\bar{0}}H_n(X, X - \{x\}; R)$ via the inclusion map since $X - N'$ is stratified homotopy equivalent to $X - \{x\}$ (see Appendix A), and furthermore, $[\eta] \times \bar{c}\Gamma_L$ generates both groups. Similarly, it generates $I^{\bar{0}}H_n(X, X - \{x'\}; R)$ for any other $x' \in N' \cap X_{n-k}$. This is enough to guarantee that our orientation sheaf is locally constant along $N' \cap X_{n-k}$. But now also if z is any point in the top stratum of N' , we have seen that $o_z \in I^{\bar{0}}H_n(X, X - \{z\}; R) \cong R$ is also represented by $[\eta] \times \bar{c}\Gamma_L$ (and in a way that preserves the choice of orientation). But then by the induction hypotheses, this chain must also restrict to $o_{z'} \in I^{\bar{0}}H_n(X, X - \{z'\}; R) \cong R$ for any $z' \in N' - N' \cap X_{n-k}$. So \mathcal{O}^X is constant on N' , and for $x' \in N' \cap X_{n-k}$, we can now let $o_{x'} \in I^{\bar{0}}H_n(X, X - \{x'\}; R)$ be the image of $[\eta] \cap \bar{c}\Gamma_L$.

It now follows by induction that \mathcal{O}^X must be locally constant. Furthermore, over sufficiently small distinguished neighborhoods N' , we have found local sections that restrict to o_x for each $x \in N' \cap X_n$. If U, V are any two such open sets of X with corresponding local sections s_U, s_V , then we see that s_U and s_V must therefore agree on $U \cap V \cap X_n$. But it follows from the local constancy that they must therefore also agree on all of $U \cap V$. Therefore, we can piece together the local sections that agree with the local orientation classes into a global section, and it follows that \mathcal{O}^X is constant. \square

5.2 Fundamental classes and global computations

In this section, we provide the combined proofs of Theorems 5.7 and 5.10 assuming X is normal and assuming Proposition 5.3 up through the depth of X . In order to prove Theorem 5.10 we need a somewhat stronger version of Theorem 5.7:

Proposition 5.13. *Suppose X is a normal n -dimensional R -oriented stratified pseudomanifold. Then for any compact $K \subset X$, $I^{\bar{0}}H_i(X, X - K; R) = 0$ for $i > n$, and there is a unique class $\Gamma_K \in I^{\bar{0}}H_n(X, X - K; R)$ such that for any $x \in K$, the image of Γ_K in X is the local orientation class $o_x \in I^{\bar{0}}H_n(X, X - \{x\}; R)$. Furthermore, if $\eta \in I^{\bar{0}}H_n(X, X - K; R)$ is such that the image of η in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ is zero for all $x \in K$, then $\eta = 0$.*

Proof of Proposition 5.13. We first observe that if the proposition is true for compact sets K, K' , and $K \cap K'$, then it is true for $K \cup K'$. This follows from a straightforward Mayer-Vietoris argument exactly as it does for manifolds (see [21, Lemma 3.27]). Also analogously to the manifold case in [21], we can reduce to the situation where X has the form of a distinguished neighborhood $X \cong \mathbb{R}^{n-k} \times cL^{k-1}$. To see this, we note that any compact $K \subset X$ can be written as the union of finitely many compact sets $K = K_1 \cup \dots \cup K_m$ with each K_i contained in such a distinguished neighborhood in X . This can be seen by a covering argument using the compactness of K . Now, notice that $(K_1 \cup \dots \cup K_{m-1}) \cap K_m = (K_1 \cap K_m) \cup \dots \cup (K_{m-1} \cap K_m)$ is also a union of $m - 1$ compact sets each contained in a distinguished neighborhood, so the Mayer-Vietoris argument and an induction on m reduces matters to the base case of a single K inside a distinguished neighborhood. Then by excision, we can assume that $X = \mathbb{R}^{n-k} \times cL^{k-1}$.

The next step in the classical manifold setting, as discussed in [21], would be to consider the case where K is a finite union of convex sets in \mathbb{R}^n . This is not available to us in an obvious way. However, let us define a compact set in $X = \mathbb{R}^{n-k} \times cL^{k-1}$ to be PM-convex (“PM” for pseudomanifold) if either

1. it has the form $C \times ([0, b] \times L) / \sim$, where C is a convex set in \mathbb{R}^{n-k} , and $([0, b] \times L) / \sim$ is part of the cone on L , including the vertex, or
2. it has the form $C \times [a, b] \times A$, where C is a convex set in \mathbb{R}^{n-k} , A is a compact subset of L , and $[a, b]$ is an interval along the cone line with $a > 0$.

It is clear that the intersection of any two PM-convex sets is also a PM-convex set, so another Mayer-Vietoris argument and induction allows us to reduce to the case of a single PM-convex set. For PM-convex sets of the second type, we can use an excision argument to cut $\mathbb{R}^{n-k} \times \{v\}$ (where v is the cone vertex) out of $I^{\bar{0}}H_*(X, X - K; R)$ and then appeal to an induction on depth. For a PM-convex set K of the first type, computations exactly as in the proof of Proposition 5.3 show that there is a class Γ_K with the desired restrictions to $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for each $x \in K$. Now, by stratified homotopy equivalence (see Appendix A), $I^{\bar{0}}H_*(X, X - K; R) \cong I^{\bar{0}}H_*(X, X - \{x\}; R)$ for any $x \in C \times v$, and by the usual computations then $I^{\bar{0}}H_*(X, X - K; R) \cong I^{\bar{0}}H_*(X, X - \{x\}; R) \cong I^{\bar{0}}H_{*-n-k-1}(L; R)$, which is 0 for $* > n$ and R for $* = n$. Therefore, any other element of $I^{\bar{0}}H_n(X, X - K; R)$ that is not Γ_K cannot yield the correct generator of $I^{\bar{0}}H_n(X, X - \{x\}; R)$ upon restriction, and so Γ_K is unique. Furthermore, we see that if $\eta \in I^{\bar{0}}H_n(X, X - K; R)$ restricts to 0 in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for any $x \in C \times v \subset K$ then $\eta = 0$ (and so certainly $\eta = 0$ if η goes to 0 in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for every $x \in K$).

Next, we consider an arbitrary compact K in $\mathbb{R}^{n-k} \times cL^{k-1}$ and again follow the general idea from [21]. For the existence of a Γ_K , let Γ_K be the image in $I^{\bar{0}}H_n(X, X - K; R)$ of any Γ_D , where D is any PM-convex set sufficiently large to contain K . It is clear that such a D exists using that our space has the form $\mathbb{R}^{n-k} \times cL^{k-1}$. By applying the results of the preceding paragraph for the PM-convex case, we see that Γ_K has the desired properties. To show that Γ_K is unique, suppose that $\Gamma'_K \in I^{\bar{0}}H_n(X, X - K; R)$ is another class with the desired properties. Suppose z is a relative cycle representing $\Gamma_K - \Gamma'_K \in I^{\bar{0}}H_n(X, X - K; R)$. Let $|\partial z|$ be the support of ∂z , which lies in $X - K$. Since $|\partial z|$ is also compact, we can cover K by a finite number of sufficiently small PM-convex sets that do not intersect $|\partial z|$. Let P denote the union of these PM-convex sets. The relative cycle z defines an element $\alpha \in I^{\bar{0}}H_n(X, X - P; R)$ that maps by inclusion to $\Gamma_K - \Gamma'_K \in I^{\bar{0}}H_i(X, X - K; R)$. So α is 0 in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for all $x \in K$. This implies that the image of α is also 0 in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for all x in P . To see this, note that any $x \in P$ is in the same PM-convex set, say Q , as some $y \in K$. But then by the preceding paragraph, $I^{\bar{0}}H_n(X, X - \{x\}; R) \cong I^{\bar{0}}H_n(X, X - Q; R) \cong I^{\bar{0}}H_n(X, X - \{y\}; R)$. But now we must have $\alpha = 0$ in $I^{\bar{0}}H_n(X, X - P; R)$ since P is a finite union of PM-convex sets. Hence $\Gamma_K - \Gamma'_K = 0$. This implies the uniqueness of the class Γ_K . The same arguments show that if $\eta \in I^{\bar{0}}H_n(X, X - K; R)$ goes to 0 in $I^{\bar{0}}H_n(X, X - \{x\}; R)$ for each $x \in K$, then $\eta = 0$.

Finally, to see that $I^{\bar{0}}H_i(X, X - K; R) = 0$ for $i > n$, again let z be a relative cycle representing an element $\xi \in I^{\bar{0}}H_i(X, X - K; R)$, $i > n$. We can form P exactly as in the preceding paragraph. Once again, the relative cycle z defines an element $\alpha \in I^{\bar{0}}H_i(X, X - P; R)$ that maps by inclusion to $\xi \in I^{\bar{0}}H_i(X, X - K; R)$. But now if $i > n$, then by the preceding results, $\alpha = 0$ since P is PM-convex. Thus also $\xi = 0$. \square

For X normal, we can now complete the proof of Theorems 5.7 and 5.10. Theorem 5.7 follows directly from Proposition 5.13, as well as the first part of Theorem 5.10 by taking $K = X$ if X is compact.

We need to see that if X is compact and connected then $I^{\bar{0}}H_n(X; R) \cong R$, generated by Γ_X . For any $x \in X$, we have a homomorphism $I^{\bar{0}}H_n(X; R) \rightarrow I^{\bar{0}}H_n(X, X - \{x\}; R) \cong R$, which we know is surjective, sending Γ_X onto a local orientation class, by Proposition 5.13. On the other hand, suppose that $\eta \in I^{\bar{0}}H_n(X; R)$ goes to $0 \in I^{\bar{0}}H_n(X, X - \{x\}; R)$ for some $x \in X$. Since $x \rightarrow \text{im}(\eta) \in I^{\bar{0}}H_n(X, X - \{x\}; R)$ is a section of the locally-constant orientation sheaf, this implies that $\text{im}(\eta) = 0 \in I^{\bar{0}}H_n(X, X - \{x\}; R)$ for all $x \in X$ (since X is connected). But then $\eta = 0$ by Proposition 5.13. Thus for any $x \in X$, the homomorphism $I^{\bar{0}}H_n(X; R) \rightarrow I^{\bar{0}}H_n(X, X - \{x\}; R) \cong R$ is an isomorphism. If X has multiple compact normal connected components, the rest of the theorem follows by noting that $I^{\bar{0}}H_n(X; R)$ is the direct sum over the connected components and by piecing together the results for the individual components. \square

5.3 5.5, 5.7 and 5.10 for general pseudomanifolds

Proof of Proposition 5.5. If X is not necessarily normal, let $\pi : \hat{X} \rightarrow X$ be its normalization. By Lemma C.1 of the Appendix C (which extends results well-known for Goresky-MacPherson perversities), π induces isomorphisms $I^{\bar{0}}H_*(\hat{X}, \hat{X} - \pi^{-1}(\bar{U}); R) \rightarrow I^{\bar{0}}H_*(X, X - \bar{U}; R)$. Proposition 5.3 then implies that $I^{\bar{0}}H_i(X, X - \{x\}; R) = 0$ for $i \neq n$ and that $\mathcal{O}^X \cong \pi_*\mathcal{O}^{\hat{X}}$. We thus obtain our desired global section of \mathcal{O}^X from the preferred global section of $\mathcal{O}^{\hat{X}}$ using the general sheaf theory fact $\Gamma(X; \pi_*\mathcal{O}^{\hat{X}}) = \Gamma(\hat{X}; \mathcal{O}^{\hat{X}})$. The formula for $I^{\bar{0}}H_n(X, X - \{x\}; R)$ also follows from basic sheaf theory. \square

Finally, we prove Theorems 5.7 and 5.10 for X not necessarily normal.

Proof. If X is not necessarily normal, once again there is a map $\pi : \hat{X} \rightarrow X$ such that \hat{X} is normal, π restricts to a homeomorphism from $\hat{X} - \hat{X}^{n-1}$ to $X_n = X - X^{n-1}$, and π induces an isomorphism on intersection homology by Lemma C.1. In addition, the number of connected components of \hat{X} is equal⁴ to the number of connected components of $X - X^{n-1}$. Notice that if Z is such a component of $X - X^{n-1}$, π restricts to a normalization map from the closure of $\hat{Z} := \pi^{-1}(Z)$ in \hat{X} to the closure of Z in X , which is also a stratified pseudomanifold (as follows from a local argument via an induction on depth). Also, if K is a compact subset

⁴Since $\hat{X} - \hat{X}^{n-1} \cong X - X^{n-1}$ and $\hat{X} - \hat{X}^{n-1}$ is dense in \hat{X} , the number of connected components of \hat{X} is less than or equal to the number of components of $X - X^{n-1}$. But each connected normal stratified pseudomanifold has only one regular stratum [28, Lemma 2.1], so this must in fact be an equality.

of X , then $\hat{K} = \pi^{-1}(K)$ is compact since π is proper, and π restricts to the normalization $\hat{X} - \hat{K} \rightarrow X - K$ (see Proposition 2.5 and Theorem 2.6 of [28]).

So π also induces isomorphisms $I^0 H_*(\hat{X}, \hat{X} - \hat{K}; R) \rightarrow I^0 H_*(X, X - K; R)$, and since \hat{X} is normal, our preceding results yield a unique fundamental class $\hat{\Gamma}_K \in I^0 H_n(\hat{X}, \hat{X} - \hat{K}; R)$ and show that $I^0 H_i(\hat{X}, \hat{X} - \hat{K}; R) = 0$ for $i > n$. But the isomorphism π on intersection homology then shows that $I^0 H_i(X, X - K; R) = 0$ for $i > n$ and provides a class $\pi \hat{\Gamma}_K \in I^0 H_n(X, X - K; R)$. Let us see that $\pi \hat{\Gamma}_K$ has the desired properties for it to be Γ_K .

By taking sufficiently small distinguished neighborhoods around $x \in X$, letting $\pi^{-1}(x) = \{y_1, \dots, y_m\}$, and excising, we have the following commutative diagram (coefficients tacit):

$$\begin{array}{ccccccc}
I^0 H_n(\hat{X}, \hat{X} - \hat{K}) & \rightarrow & I^0 H_n(\hat{X}, \hat{X} - \pi^{-1}(x)) & \xleftarrow{\cong} & I^0 H_n(\hat{N}, \hat{N} - \pi^{-1}(x)) & \cong \oplus_{i=1}^m I^0 H_n(\hat{N}_i, \hat{N}_i - y_i) & \xrightarrow{\cong} \oplus_{i=1}^m R \\
\downarrow \pi \cong & & \downarrow \pi \cong & & \downarrow \pi \cong & & \downarrow \cong \\
I^0 H_n(X, X - K) & \longrightarrow & I^0 H_n(X, X - x) & \xleftarrow{\cong} & I^0 H_n(N, N - x) & \xrightarrow{\cong} & \oplus_{i=1}^m R.
\end{array} \tag{1}$$

So by the definition of the local orientation class o_x , we see that the image of $\pi \hat{\Gamma}_K$ in $I^0 H_n(X, X - \{x\}; R)$ is precisely o_x since $\hat{\Gamma}_K$ restricts to the local orientation class in each $I^0 H_n(X, X - \{y_i\}; R)$. Restricting this same argument to \bar{Z} and $cl(\hat{Z})$ demonstrates that the image of $\Gamma_{cl(\hat{Z})}$ must be $\Gamma_{\bar{Z}}$.

To see that Γ_K is unique, let Γ'_K be another class with the desired properties. Then Γ_K and Γ'_K each correspond to unique elements $\hat{\Gamma}_K$ and $\hat{\Gamma}'_K$ in $I^0 H_n(\hat{X}, \hat{X} - \hat{K}; R)$. But now using diagram (1) again, we see that $\hat{\Gamma}_K$ and $\hat{\Gamma}'_K$ must each restrict to the local orientation class of $I^0 H_n(\hat{X}, \hat{X} - \{y\}; R)$ for each $y \in \hat{K}$ or else their images in $I^0 H_n(X, X - \{x\}; R)$ will not be correct. But by the uniqueness for normal stratified pseudomanifolds, which we obtained in Proposition 5.13, this implies $\hat{\Gamma}_K = \hat{\Gamma}'_K$, and so $\Gamma_K = \Gamma'_K$.

This completes the proof of Theorem 5.7.

If X_n is connected and we take $K = X$, then Theorem 5.10 follows immediately from the isomorphism $I^0 H_*(\hat{X}; R) \cong I^0 H_*(X; R)$.

If X_n is not connected, let \bar{Z} be the closure of Z in X , let $cl(\hat{Z})$ be the closure of \hat{Z} in \hat{X} , and notice \hat{X} is the disjoint union of the connected components $\hat{X} = \coprod cl(\hat{Z})$. So then

$$\begin{array}{ccc}
\oplus_Z I^0 H_n(cl(\hat{Z}); R) & \xrightarrow{\cong} & I^0 H_n(\hat{X}; R) \\
\downarrow \pi \cong & & \downarrow \pi \cong \\
\oplus_Z I^0 H_n(\bar{Z}; R) & \xrightarrow{\cong} & I^0 H_n(X; R).
\end{array} \tag{2}$$

The top map is an isomorphism because the spaces $cl(\hat{Z})$ are disjoint. The vertical maps are normalization isomorphisms, and it follows that the bottom is an isomorphism. Thus it follows from the normal case that $I^0 H_i(\hat{X}; R) = 0$ for $i > n$ and that $I^0 H_n(\hat{X}; R) \cong R^m$, where m is the number of connected components of $X - X^{n-1}$.

The remainder of the Theorem 5.10 follows since our earlier arguments imply that the image of $\Gamma_{cl(\hat{Z})} \in I^0 H_n(cl(\hat{Z}); R)$ under the normalization $\pi|_{cl(\hat{Z})} : cl(\hat{Z}) \rightarrow \bar{Z}$ is $\Gamma_{\bar{Z}} \in I^0 H_n(\bar{Z}; R)$. \square

5.4 Corollaries and Complements

For later use we record some further properties of the fundamental class.

Corollary 5.14. *Suppose X is a compact connected normal n -dimensional R -oriented stratified pseudomanifold. If $\gamma \in I^{\bar{0}}H_n(X; R) \cong R$ is a generator, then γ is the fundamental class of X with respect to some orientation of X .*

Proof. If Γ_X is the fundamental class of X with respect to the given orientation, then clearly $\gamma = r\Gamma_X$ for some unit $r \in R$. Thus the image of γ in any $I^{\bar{0}}H_n(X, X - \{x\}; R)$ is r times the image of Γ_X in $I^{\bar{0}}H_n(X, X - \{x\}; R)$. Thus γ is the fundamental class corresponding to the global orientation section obtained from that corresponding to Γ_X by multiplication by r . \square

Corollary 5.15. *Suppose X is a compact connected n -dimensional R -oriented stratified pseudomanifold. Let $\{x_i\}$ be a collection of points of X , one in each connected component of $X - X^{n-1}$. If $\gamma \in I^{\bar{0}}H_n(X; R)$ restricts to $o_{x_i} \in I^{\bar{0}}H_n(X, X - \{x_i\}; R)$ for each x_i , then $\gamma = \Gamma_X$. More generally, given that X is orientable, any element of $I^{\bar{0}}H_n(X; R)$ that restricts to a generator of each $I^{\bar{0}}H_n(X, X - \{x_i\}; R)$ determines an orientation of X .*

Proof. First assume X is given an orientation and hence has a fundamental class Γ_X . We know Γ_X has the desired property. It is clear from Theorem 5.7 and diagram (2), that no other element of $I^{\bar{0}}H_n(X; R)$ can have this property, since, as in the proof of the preceding corollary, any other element of $I^{\bar{0}}H_n(X; R)$ would have to restrict to a different element of $I^{\bar{0}}H_n(X, X - \{x_i\}; R)$ for at least one of the x_i .

Conversely, an element of $I^{\bar{0}}H_n(X; R)$ that restricts to a generator of each $I^{\bar{0}}H_n(X, X - \{x_i\}; R)$ determines a local orientation at each x_i . But since X is orientable, any local orientation at one point of each regular stratum determines an orientation of X . \square

Our next result will be needed in [17]. It utilizes the definition of stratified homotopy equivalence given in Appendix A.

Corollary 5.16. *Suppose X and Y are compact n -dimensional stratified pseudomanifolds, that X is R -oriented, and that $f : X \rightarrow Y$ is a stratified homotopy equivalence. Then Y is orientable and f takes Γ_X to Γ_Y for some orientation of Y .*

Proof. By Padilla [28, Theorem 2.6], normalization is functorial, so we have a diagram

$$\begin{array}{ccc} I^{\bar{0}}H_n(\hat{X}; R) & \xrightarrow{\hat{f}} & I^{\bar{0}}H_n(\hat{Y}; R) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ I^{\bar{0}}H_n(X; R) & \xrightarrow{f} & I^{\bar{0}}H_n(Y; R). \end{array}$$

The bottom map is an isomorphism by the invariance of intersection homology under stratified homotopy (see Appendix A), and the vertical maps are isomorphisms by normalization.

Hence the top map is also an isomorphism. Borrowing the notation from above, since each connected component $\text{cl}(\hat{Z})$ of \hat{X} is a compact connected oriented normal stratified pseudomanifold, with its orientation coming from that of the stratum $Z \subset X$, we have each $I^0 H_n(\text{cl}(\hat{Z}); R) \cong R$, and the fundamental class $\Gamma_{\hat{X}}$ determined by Γ_X is a sum of generators of the separate $I^0 H_n(\text{cl}(\hat{Z}); R)$. Since X and Y are stratified homotopy equivalent, there is a bijection between connected components of $X - X^{n-1}$ and $Y - Y^{n-1}$, and f induces homotopy equivalences of these manifolds. It follows that Y must be orientable and that $\hat{f}(\Gamma_{\hat{Y}})$ must similarly be a sum of generators of the corresponding $I^0 H_n(\text{cl}(\hat{S}); R)$ for regular strata $S \subset Y$. By Corollary 5.14, these generators must be fundamental classes for the $\text{cl}(\hat{S})$ with respect to some orientation on \hat{Y} . This determines an orientation of Y by the homeomorphism $\hat{Y} - \hat{Y}^{n-1} \cong Y - Y^{n-1}$, and, it follows from the diagram that $f(\Gamma_X) = \pi_Y \hat{f}(\Gamma_{\hat{X}})$ is the corresponding fundamental class on Y . \square

We conclude this subsection with a fact that will be needed at one point in Section 6. First observe that if M is an R -oriented manifold and X is an R -oriented stratified pseudomanifold of dimension n there is a canonical R -orientation on $M \times X$ (namely the product orientation on $M \times X_n$; see [8, VIII.2.13]).

Proposition 5.17. *Let M be an R -oriented manifold and let X be an R -oriented stratified pseudomanifold. Let $K_1 \subset M$ and $K_2 \subset X$ be compact. Then $\Gamma_{K_1} \times \Gamma_{K_2} = \Gamma_{K_1 \times K_2}$ in $I^0 H_*(M \times X, M \times X - K_1 \times K_2)$.*

Proof. It suffices to show that for each $x \in M$ and $y \in X$ we have $o_{(x,y)} = o_x \times o_y$.

First suppose X is normal. Applying Proposition 5.3(2) to M , X and $M \times X$ gives sections s , t and u . Then $s \times t$ is a continuous section which agrees with u for regular points by [8, VIII.2.13] and hence for all points by the uniqueness property in Proposition 5.3(2). Thus we have

$$o_{(x,y)} = u(x, y) = s(x) \times t(y) = o_x \times o_y$$

for all $x \in M$ and $y \in X$.

For general X , let $\pi : \hat{X} \rightarrow X$ be a normalization. Then $1 \times \pi : M \times \hat{X} \rightarrow M \times X$ is also a normalization by [28, Example 2.3(3)], and if $x \in M$, $y \in X$ we have

$$\begin{aligned} o_x \times o_y &= o_x \times \left(\sum_{z \in \pi^{-1}(y)} \pi_*(o_z) \right) \\ &= \sum_{(x,z) \in (1 \times \pi)^{-1}(x,y)} (1 \times \pi)_*(o_{(x,z)}) \\ &= o_{(x,y)}. \end{aligned}$$

\square

Remark 5.18. The analogous fact is true for a product of two pseudomanifolds, but the proof is more involved (because one has to show that the product of two normalizations is a normalization).

5.5 Topological invariance of the fundamental class

In this subsection, we demonstrate a fact needed in [17]: the fundamental class of a compact oriented stratified pseudomanifold with no codimension one strata is an oriented homeomorphism invariant.

The proscription on codimension one strata is necessary because the group $I^{\bar{0}}H_n(X; R)$ itself depends on the stratification of X if codimension one strata are allowed. For example, let $S = S^1$ be the unit circle stratified trivially, and let S' be the circle stratified as $S^1 \supset \{x, y\}$, where x, y are any two distinct points of S^1 . Then simple computations reveal that $I^{\bar{0}}H_1(S; R) \cong H_1(S; R) \cong R$ but $I^{\bar{0}}H_1(S'; R) \cong H_1(S^1, \{x, y\}; R) \cong R \oplus R$.

For the remainder of this subsection, we limit discussion to stratified pseudomanifolds with no codimension one strata.

We recall from [23] that for any stratified pseudomanifold X there is an intrinsic coarsest “stratification” X^* of X (which is actually a CS-set, not a stratified pseudomanifold) that depends only on X as a topological space. The inclusion map $I^{\bar{0}}H_*(X; R) \rightarrow I^{\bar{0}}H_*(X^*; R)$ is an isomorphism, and hence if X' is a restratification of X (that is, the space X with an alternate pseudomanifold stratification) there is a canonical composite isomorphism

$$I^{\bar{0}}H_*(X; R) \xrightarrow{\cong} I^{\bar{0}}H_*(X^*; R) \xleftarrow{\cong} I^{\bar{0}}H_*(X'; R).$$

We first show that an orientation of X determines an orientation on each restratification of X and on X^* .

Lemma 5.19. *Let X be an oriented stratified pseudomanifold. Let X' be a restratification of X . Then X' has a unique orientation so that the induced orientations on $X_n \cap X'_n$ from X and X' agree. This remains true with X' replaced by X^* .*

Proof. The proof is the same for X' or X^* . Note that since X^* is coarser than X or X' , it must also have no codimension one strata.

Notice that $X_n \cap X'_n$ is dense in both X_n and X'_n since X_n and X'_n are each dense in X . Now, by definition, an orientation on X is an orientation on X_n , i.e. an isomorphism on X_n from the orientation R -bundle to the constant R -bundle. Just as in the proof of Borel [4, Lemma 4.11.a], the restriction of this isomorphism to the dense subset $X_n \cap X'_n$ extends uniquely to an isomorphism from the orientation R -bundle to the constant R -bundle on X'_n , using the equivalence of local systems with π_1 -modules on connected manifolds (in this case, components of X'_n), and that the fundamental group of a dense open set of a connected manifold surjects onto the fundamental group of the connected manifold. \square

Proposition 5.20. *If X is a compact R -oriented n -dimensional stratified pseudomanifold and X' is a restratification of X with the induced R -orientation then the canonical isomorphism $I^{\bar{0}}H_n(X; R) \cong I^{\bar{0}}H_n(X'; R)$ takes Γ_X to $\Gamma_{X'}$.*

Proof. This follows easily from Corollary 5.15, choosing points in $X_n \cap X'_n$. \square

Finally, we observe that the fundamental class is an oriented topological invariant. To see what this means, suppose X, Y are compact R -oriented n -dimensional stratified pseudomanifolds and that $f : X \rightarrow Y$ is a topological homeomorphism (not necessarily stratified). The

stratification of X induces a restratification Y' of Y with $(Y')^i = f(X^i)$ and the R -orientation of X induces an R -orientation of Y' .

Definition 5.21. Let X, Y be compact R -oriented n -dimensional stratified pseudomanifolds without codimension one strata. We will say that the topological homeomorphism $f : X \rightarrow Y$ is an *oriented homeomorphism* if the induced orientation on Y' is consistent with the given orientation of Y in the sense of Lemma 5.19.

The following corollary is now evident from the preceding results of this subsection:

Corollary 5.22. *If $f : X \rightarrow Y$ is an oriented homeomorphism of compact R -oriented n -dimensional stratified pseudomanifolds without codimension one strata, then f takes $\Gamma_X \in I^0 H_n(X; R)$ to $\Gamma_Y \in I^0 H_n(Y; R)$.*

6 Poincaré duality

Let F be a field. In this section all intersection homology and cohomology will have F coefficients.

We will show that cap product with the fundamental class induces a Poincaré duality isomorphism from compactly supported intersection cohomology to intersection homology.

Let X be an F -oriented stratified pseudomanifold of dimension n , possibly noncompact and possibly with codimension one strata. Let \bar{p} be a perversity.

Definition 6.1. The compactly supported intersection cohomology of X with perversity \bar{p} , denoted $I_{\bar{p}} H_c^*(X; F)$, is defined to be

$$\varinjlim I_{\bar{p}} H^*(X, X - K; F),$$

where K ranges over all compact subsets of X .

Let $\bar{q} = \bar{t} - \bar{p}$.

For each compact $K \subset X$ we define

$$\mathcal{D}_K : I_{\bar{p}} H^*(X, X - K; F) \rightarrow I^{\bar{q}} H_{n-*}(X; F)$$

by

$$\mathcal{D}_K(\alpha) = (-1)^{|\alpha|n}(\alpha \frown \Gamma_K).$$

Remark 6.2. For the sign $(-1)^{|\alpha|n}$, which does not appear in the literature, see [14, Section 4.1], where this sign is introduced to make the duality map a chain map of appropriate degree.

Next we observe that the \mathcal{D}_K are consistent as K varies. Let $K \subset K'$ and let $j : X - K' \hookrightarrow X - K$ be the inclusion. Then for $\alpha \in I_{\bar{p}} H^*(X, X - K)$ we have

$$\begin{aligned} \mathcal{D}_{K'}(j^* \alpha) &= (-1)^{|\alpha|n}((j^* \alpha) \frown \Gamma'_{K'}) \\ &= (-1)^{|\alpha|n}(\alpha \frown (j_* \Gamma'_{K'})) \quad \text{by Proposition 4.16} \\ &= (-1)^{|\alpha|n}(\alpha \frown \Gamma_K) \quad \text{by Remark 5.9} \\ &= \mathcal{D}_K(\alpha). \end{aligned}$$

Now we define

$$\mathcal{D} : I_{\bar{p}}H_c^*(X; F) \rightarrow I^{\bar{q}}H_{n-*}(X; F)$$

to be

$$\varinjlim \mathcal{D}_K.$$

Theorem 6.3 (Poincaré duality). *Let F be a field. Let X be an n -dimensional F -oriented stratified pseudomanifold, possibly noncompact and possibly with codimension one strata, and let $\bar{p} + \bar{q} = \bar{t}$. Then $\mathcal{D} : I_{\bar{p}}H_c^i(X; F) \rightarrow I^{\bar{q}}H_{n-i}(X; F)$ is an isomorphism.*

Proof. We argue by induction on the depth of X . When X has depth 0, X is a manifold and the result is classical. So we assume now that X has positive depth and that the theorem has been proven on stratified pseudomanifolds of depth less than that of X .

Lemma 6.4. *If the conclusion of Theorem 6.3 holds for the compact F -oriented stratified $k-1$ pseudomanifold L , then it holds for cL .*

For the proof we need some notation, which will also be used later.

Notation 6.5. For $0 < r < 1$ let c_rL denote the image of $[0, r] \times L$ in $cL = ([0, 1] \times L) / \sim$.

Proof. We orient cL consistently with the product $(0, 1) \times (L - L^{k-2}) \cong cL - (cL)^{k-1}$. Let v denote the vertex of cL .

We are free to choose any cofinal collection of compact sets, so we choose K to have the form c_rL , $0 < r < 1$.

Fix such a K . We claim that the map

$$\mathcal{D}_K : I_{\bar{p}}H^i(cL, cL - K; F) \rightarrow I^{\bar{q}}H_{k-i}(cL; F)$$

is already an isomorphism (before passage to the direct limit).

Let $b \in (r, 1)$ and let $j : L \rightarrow cL$ take x to (b, x) . Then j is a stratified homotopy equivalence $L \rightarrow cL - K$, so Appendix A implies that, for every \bar{r} , j_* is an isomorphism $I^{\bar{r}}H_*(L; F) \rightarrow I^{\bar{r}}H_*(cL - K; F)$ and j induces an isomorphism $I^{\bar{r}}H_*(cL, L; F) \rightarrow I^{\bar{r}}H_*(cL, cL - K; F)$.

Now if $i < k - \bar{p}(\{v\})$ then Proposition 2.3 and Remark 4.9 imply that the the domain and range of \mathcal{D}_K are both 0 and \mathcal{D}_K is vacuously an isomorphism.

So let $i \geq k - \bar{p}(\{v\})$ and consider the following diagram

$$\begin{array}{ccccc} I_{\bar{p}}H^i(cL, cL - K; F) & \xleftarrow{\delta} & I_{\bar{p}}H^{i-1}(cL - K; F) & \xrightarrow{j_* \cong} & I_{\bar{p}}H^{i-1}(L; F) \\ \mathcal{D}_K \downarrow & & \downarrow \cdot \frown j_* \Gamma_L & & \downarrow \cdot \frown \Gamma_L \\ I^{\bar{q}}H_{k-i}(cL; F) & \xleftarrow{\text{inc}} & I^{\bar{q}}H_{k-i}(cL - K; F) & \xleftarrow{j_* \cong} & I^{\bar{q}}H_{k-i}(L; F). \end{array}$$

The right vertical arrow is an isomorphism by hypothesis (since L is compact), and the right square commutes up to sign by Proposition 4.16, so the middle vertical arrow is an isomorphism. Proposition 2.3 and Remark 4.9 imply that the horizontal arrows in the left

square are isomorphisms so it suffices to show that the left square commutes up to sign. For this it suffices, by Proposition 4.19(2), to show that $j_*\Gamma_L = \partial\Gamma_K$. This in turn follows from the fact, shown in the proof of Proposition 5.3, that $\bar{c}\Gamma_L = \Gamma_K$ (in that proof it was assumed that X is normal, but the relevant part of the argument holds more generally). \square

Lemma 6.6. *If the conclusion of Theorem 6.3 holds for the compact F -oriented stratified $k-1$ pseudomanifold L , then it holds for $M \times cL$, where M is an F -oriented unstratified $n-k$ manifold and we use the product stratification and the product orientation.*

Proof. For convenience, let $M \times cL = Y$.

Any compact set $K \subset Y$ is contained in the compact set $p_1(K) \times p_2(K)$, where p_1, p_2 are the respective projections to M and to cL . So the compact sets of the form $K_1 \times K_2 \subset Y$ are cofinal among all compact sets. Furthermore, since compact sets of the form $c_r L$ (in the notation of the proof of Lemma 6.4) are cofinal among compact sets in cL , compact sets of the form $K_1 \times c_r L$ are cofinal among the compact sets of Y . Therefore, to prove the lemma, it suffices to show that the direct limit of the maps

$$\cdot \cap \Gamma_{K_1 \times c_r L} : I_{\bar{p}} H^i(Y, Y - (K_1 \times c_r L); F) \rightarrow I^{\bar{q}} H_{n-i}(Y; F)$$

is an isomorphism.

Now consider the following diagram.

$$\begin{array}{ccc} I_{\bar{p}} H^*(Y, Y - K_1 \times c_r L; F) & \xleftarrow{c} & H^*(M, M - K_1; F) \otimes I_{\bar{p}} H^*(cL, cL - c_r L; F) \\ \downarrow \cdot \cap \Gamma_{K_1 \times c_r L} & & \downarrow (\cdot \cap \Gamma_{K_1}) \otimes (\cdot \cap \Gamma_{c_r L}) \\ I^{\bar{q}} H_*(Y; F) & \xleftarrow{\times} & H_*(M) \otimes I^{\bar{q}} H_*(cL). \end{array}$$

Here the map c is defined by $c(\alpha \otimes \beta) = (-1)^{|\beta|(n-k)}(\alpha \times \beta)$ (recall that the cohomology cross product was defined just before Proposition 4.21). The diagram commutes by Proposition 5.17 and the relative version of Proposition 4.21. The lower horizontal arrow is an isomorphism by Theorem 3.1 (using perversity $\bar{0}$ for the M factor) and the upper horizontal arrow is an isomorphism by the relative version of Remark 4.20; note that $I_{\bar{p}} H^*(cL, cL - c_r L; F)$ is finitely generated because L is compact. The right hand vertical arrow induces an isomorphism after passage to the direct limit by [21, Theorem 3.35] and Lemma 6.4. It follows that the left hand vertical arrow induces an isomorphism after passage to the direct limit as required. \square

We can now complete the proof of Poincaré duality on X with a Zorn's Lemma argument, as in the proof of manifold duality in Hatcher [21, Proof of Theorem 3.35]. By the induction assumption, any space of depth less than that of X satisfies the conclusion of the theorem. In particular, it is true on $X - X^m$ where X^m is the smallest non-empty skeleton of X . Let \mathcal{U} denote the set of open sets of X containing $X - X^m$ and on which \mathcal{D} is an isomorphism; \mathcal{U} is partially ordered by inclusion. Suppose S is a totally ordered subset of \mathcal{U} , and let $W = \cup_{U \in S} U$. For $U_a \subset U_b$ elements of S , there is a natural map

$I_{\bar{p}}H_c^i(U_a; F) \rightarrow I_{\bar{p}}H_c^i(U_b; F)$ since an element of $I_{\bar{p}}H_c^i(U_a; F)$ is represented by an element of $I_{\bar{p}}H^i(U_a, U_a - K; F)$ for some compact K and then $I_{\bar{p}}H^i(U_a, U_a - K; F) \cong I_{\bar{p}}H^i(U_b, U_b - K; F)$ by excision. Furthermore, we then see that $\varinjlim_{U \in S} I_{\bar{p}}H_c^i(U; F) \cong I_{\bar{p}}H_c^i(W; F)$. Of course also $I^{\bar{q}}H_{n-i}(W; F) \cong \varinjlim_{U \in S} I^{\bar{q}}H_{n-i}(U; F)$, and it follows that $\mathcal{D} : I_{\bar{p}}H_c^i(W; F) \rightarrow I^{\bar{q}}H_{n-i}(W; F)$ is the direct limit of duality isomorphisms and hence an isomorphism.

Therefore, each totally ordered set in \mathcal{U} has a maximal element, and by Zorn's lemma, there is a largest open $U \subset X$ such that U contains $X - X^m$ and duality holds on U . If $U = X$ we are done. Suppose $U \neq X$, and let $x \in X - U$. Then $x \in X - X^{n-k}$ for some $k \geq 1$, and x is contained in a distinguished neighborhood N homeomorphic to $\mathbb{R}^{n-k} \times cL^{k-1}$. From now on we write X_{n-k} for $X^{n-k} - X^{n-k-1}$. Proceeding as in the proof of [23, Proposition 8], let $V = U \cap N$. Since this set is open (and so is $U \cap N \cap X_{n-k}$ in X_{n-k}), we can shrink the cL factors in N to obtain an open neighborhood W of $U \cap N \cap X_{n-k}$ in $U \cap N = V$ such that W is homeomorphic to $(U \cap N \cap X_{n-k}) \times cL$.

Now we have the following diagram, in which the rows are Mayer-Vietoris sequences:

$$\begin{array}{ccccccc}
\longrightarrow & I_{\bar{p}}H_c^i(W - W \cap X_{n-k}; F) & \longrightarrow & I_{\bar{p}}H_c^i(W; F) \oplus I_{\bar{p}}H_c^i(V - V \cap X_{n-k}; F) & \longrightarrow & I_{\bar{p}}H_c^i(V; F) & \longrightarrow \\
& \downarrow \mathcal{D} & & \downarrow \mathcal{D} \oplus (-\mathcal{D}) & & \downarrow \mathcal{D} & \\
\longrightarrow & I^{\bar{q}}H_{n-i}(W - W \cap X_{n-k}; F) & \longrightarrow & I^{\bar{q}}H_{n-i}(W; F) \oplus I^{\bar{q}}H_{n-i}(V - V \cap X_{n-k}; F) & \longrightarrow & I^{\bar{q}}H_{n-i}(V; F) & \longrightarrow .
\end{array} \tag{3}$$

The diagram commutes up to sign by Proposition 6.7 in subsection 6.1.

The left hand vertical map and the second summand of the middle map are isomorphisms by the induction hypothesis on depth. The first summand of the middle map is an isomorphism by Lemma 6.6. Hence the right hand map is an isomorphism by the five lemma.

Now we can plug this into the Mayer-Vietoris diagram

$$\begin{array}{ccccccc}
\longrightarrow & I_{\bar{p}}H_c^i(V; F) & \longrightarrow & I_{\bar{p}}H_c^i(U; F) \oplus I_{\bar{p}}H_c^i(N; F) & \longrightarrow & I_{\bar{p}}H_c^i(U \cup N; F) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & I^{\bar{q}}H_{n-i}(V; F) & \longrightarrow & I^{\bar{q}}H_{n-i}(U; F) \oplus I^{\bar{q}}H_{n-i}(N; F) & \longrightarrow & I^{\bar{q}}H_{n-i}(U \cup N; F) & \longrightarrow ,
\end{array}$$

and we conclude similarly that duality holds on $U \cup N$, contradicting the maximality of U . Hence we must have $U = X$ and duality holds on X .

Note: if we assume that X^{n-1} is second countable, then rather than resort to Zorn's lemma, we could instead use the same diagrams to perform an induction, starting with $X - X^{n-1}$ and then taking unions one at a time with members of a countable covering of X^{n-1} by distinguished neighborhoods.

□

6.1 Commutativity of Diagram (3)

In this subsection all intersection chain groups and intersection homology groups have F -coefficients, which will not be included in the notation. Our goal is to prove the following

analogue of Lemma 3.36 of [21].

Proposition 6.7. *Let X be an F -oriented stratified pseudomanifold. Let U and V be open subsets of X with $X = U \cup V$. Let $\bar{p} + \bar{q} = \bar{t}$. Then the following diagram, in which the rows are Mayer-Vietoris sequences, commutes up to sign.*

$$\begin{array}{ccccccc}
\longrightarrow & I_{\bar{p}}H_c^i(U \cap V) & \longrightarrow & I_{\bar{p}}H_c^i(U) \oplus I_{\bar{p}}H_c^i(V) & \longrightarrow & I_{\bar{p}}H_c^i(X) & \longrightarrow I_{\bar{p}}H_c^{i+1}(U \cap V) \longrightarrow \\
& \downarrow \mathcal{D} & & \downarrow \mathcal{D} \oplus (-\mathcal{D}) & & \downarrow \mathcal{D} & \downarrow \mathcal{D} \\
\longrightarrow & I^{\bar{q}}H_{n-i}(U \cap V) & \longrightarrow & I^{\bar{q}}H_{n-i}(U) \oplus I^{\bar{q}}H_{n-i}(V) & \longrightarrow & I^{\bar{q}}H_{n-i}(X) & \longrightarrow I^{\bar{q}}H_{n-i-1}(U \cap V) \longrightarrow .
\end{array} \tag{4}$$

Our proof will follow the general strategy of [21] (but with our sign conventions). As in [21], the commutativity up to sign of the three squares shown in diagram (4) is an easy consequence of the three parts of the following lemma.

Lemma 6.8. *Let K and L be compact subsets of U and V . The following diagrams commute.*

1.

$$\begin{array}{ccc}
I_{\bar{p}}H^k(X, X - K \cap L) & \longrightarrow & I_{\bar{p}}H^k(X, X - K) \oplus I_{\bar{p}}H^k(X, X - L) \\
\downarrow & & \downarrow \\
I_{\bar{p}}H^k(U \cap V, U \cap V - K \cap L) & & I_{\bar{p}}H^k(U, U - K) \oplus I_{\bar{p}}H^k(V, V - K) \\
\downarrow \frown \Gamma_{K \cap L} & & \downarrow \frown \Gamma_K \oplus -(\frown \Gamma_L) \\
I^{\bar{q}}H_{n-k}(U \cap V) & \longrightarrow & I^{\bar{q}}H_{n-k}(U) \oplus I^{\bar{q}}H_{n-k}(V)
\end{array}$$

2.

$$\begin{array}{ccc}
I_{\bar{p}}H^k(X, X - K) \oplus I_{\bar{p}}H^k(X, X - L) & \longrightarrow & I_{\bar{p}}H^k(X, X - K \cup L) \\
\downarrow & & \downarrow \\
I_{\bar{p}}H^k(U, U - K) \oplus I_{\bar{p}}H^k(V, V - K) & & \\
\downarrow \frown \Gamma_K \oplus -(\frown \Gamma_L) & & \downarrow \frown \Gamma_{K \cup L} \\
I^{\bar{q}}H_{n-k}(U) \oplus I^{\bar{q}}H_{n-k}(V) & \longrightarrow & I^{\bar{q}}H_{n-k}(X)
\end{array}$$

3.

$$\begin{array}{ccc}
I_{\bar{p}}H^k(X, X - K \cup L) & \xrightarrow{\delta} & I_{\bar{p}}H^{k+1}(X, X - K \cap L) \longrightarrow I_{\bar{p}}H^{k+1}(U \cap V, U \cap V - K \cap L) \\
\downarrow \frown \Gamma_{K \cup L} & & \downarrow \frown \Gamma_{K \cap L} \\
I^{\bar{q}}H_{n-k}(X) & \xrightarrow{\partial} & I^{\bar{q}}H_{n-k-1}(U \cap V).
\end{array} \tag{5}$$

In the remainder of this section we prove Lemma 6.8.

For part 1, it suffices to consider the two summands on the right hand side separately. We will verify commutativity for the first summand; the second is similar. Consider the following diagram, where all unmarked arrows are induced by inclusions.

$$\begin{array}{ccccc}
I_{\bar{p}}H^k(X, X - K \cap L) & \xrightarrow{\quad} & I_{\bar{p}}H^k(X, X - K) & & \\
\downarrow & \searrow & \downarrow & & \\
I_{\bar{p}}H^k(U \cap V, U \cap V - K \cap L) & \xleftarrow{\quad} & I_{\bar{p}}H^k(U, U - K \cap L) & \xrightarrow{\quad} & I_{\bar{p}}H^k(U, U - K) \\
\downarrow \cap \Gamma_{K \cap L} & & \searrow \cap \Gamma_{K \cap L} & & \downarrow \cap \Gamma_K \\
I^{\bar{q}}H_{n-k}(U \cap V) & \xrightarrow{\quad} & I^{\bar{q}}H_{n-k}(U) & &
\end{array}$$

Here the upper half obviously commutes, and the lower half commutes by Proposition 4.16 (using the fact that the inclusion $(U, U - K) \rightarrow (U, U - K \cap L)$ takes Γ_K to $\Gamma_{K \cap L}$).

For part 2, it again suffices to work one summand at a time. For the first summand, consider the following diagram.

$$\begin{array}{ccc}
I_{\bar{p}}H^k(X, X - K) & \xrightarrow{\quad} & I_{\bar{p}}H^k(X, X - K \cup L) \\
\downarrow & \searrow \cap \Gamma_K & \downarrow \cap \Gamma_{K \cup L} \\
I_{\bar{p}}H^k(U, U - K) & & \\
\downarrow \cap \Gamma_K & & \\
I^{\bar{q}}H_{n-k}(U) & \xrightarrow{\quad} & I^{\bar{q}}H_{n-k}(X)
\end{array}$$

Both triangles commute by Proposition 4.16, using the fact that the inclusion $(X, X - K \cup L) \rightarrow (X, X - K)$ takes $\Gamma_{K \cup L}$ to Γ_K .

Next we prove part 3. We need a lemma which will be proved at the end of this subsection.

Lemma 6.9. *There exist chains*

$$\begin{aligned}
\beta_{U-L} &\in I^{\bar{p}}C_*(U - L) \otimes I^{\bar{q}}C_*(U - L, U - K \cup L), \\
\beta_{U \cap V} &\in I^{\bar{p}}C_*(U \cap V) \otimes I^{\bar{q}}C_*(U \cap V, U \cap V - K \cup L)
\end{aligned}$$

and

$$\beta_{V-K} \in I^{\bar{p}}C_*(V - K) \otimes I^{\bar{q}}C_*(V - K, V - K \cup L)$$

such that $\beta_{U-L} + \beta_{U \cap V} + \beta_{V-K}$ represents $\bar{d}(\Gamma_{K \cup L}) \in I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(X, X - K \cup L)$.

The inclusion $(X, X - K \cup L) \rightarrow (X, X - K \cap L)$ takes $\Gamma_{K \cup L}$ to $\Gamma_{K \cap L}$, so the image of $\beta_{U-L} + \beta_{U \cap V} + \beta_{V-K}$ in $I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(X, X - K \cap L)$ represents $\bar{d}(\Gamma_{K \cap L})$. But this image is just $\beta_{U \cap V}$, since the other two terms map to 0 in $I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(X, X - K \cap L)$. Thus $\beta_{U \cap V}$ represents the class $\bar{d}(\Gamma_{K \cap L})$ in $I^{\bar{p}}H_*(U \cap V) \otimes I^{\bar{q}}H_*(U \cap V, U \cap V - K \cup L)$.

Now let $\varphi \in I_{\bar{p}}C^k(X, X - K \cup L)$ be a cocycle; we want to calculate the image of $[\varphi]$ for the two ways of going around the diagram (5). Let A and B denote $X - K$ and $X - L$, so that $\varphi \in I_{\bar{p}}C^k(X, A \cap B)$.

As in [21], we have $\delta[\varphi] = [\delta\varphi_A]$, where $\varphi = \varphi_A - \varphi_B$ with $\varphi_A \in I_{\bar{p}}C^k(X, A)$ and $\varphi_B \in I_{\bar{p}}C^k(X, B)$. Continuing on to $I^{\bar{q}}H_{n-k-1}(U \cap V)$ we obtain $[(1 \otimes \delta\varphi_A)(\beta_{U \cap V})]$, which is the same as

$$(-1)^{k+1}[(1 \otimes \varphi_A)(\partial\beta_{U \cap V})]$$

because $\partial((1 \otimes \varphi_A)(\beta_{U \cap V})) = (1 \otimes \delta\varphi_A)(\beta_{U \cap V}) + (-1)^k(1 \otimes \varphi_A)(\partial\beta_{U \cap V})$.

Going around the diagram (5) the other way, let β denote $\beta_{U-L} + \beta_{U \cap V} + \beta_{V-K}$. $[\varphi]$ first maps to $[(1 \otimes \varphi)(\beta)]$. To apply the Mayer-Vietoris boundary ∂ to this, we first write $(1 \otimes \varphi)(\beta)$ as a sum of a chain in U and a chain in V :

$$(1 \otimes \varphi)(\beta) = (1 \otimes \varphi)(\beta_{U-L}) + ((1 \otimes \varphi)(\beta_{U \cap V}) + (1 \otimes \varphi)(\beta_{V-K})).$$

Then we take the boundary of the first of these two terms, obtaining the homology class $[\partial(1 \otimes \varphi)(\beta_{U-L})]$. To compare this to $(-1)^{k+1}[(1 \otimes \varphi_A)(\partial\beta_{U \cap V})]$, we have

$$\begin{aligned} \partial(1 \otimes \varphi)(\beta_{U-L}) &= (-1)^k(1 \otimes \varphi)(\partial\beta_{U-L}) && \text{since } \delta\varphi = 0 \\ &= (-1)^k(1 \otimes \varphi_A)(\partial\beta_{U-L}) && \text{since } (1 \otimes \varphi_B)(\partial\beta_{U-L}) = 0, \varphi_B \text{ being} \\ & && \text{zero on chains in } B = X - L \\ &= (-1)^{k+1}(1 \otimes \varphi_A)(\partial\beta_{U \cap V}), \end{aligned}$$

where the last equality comes from the fact that $\partial(\beta_{U-L}) + \partial(\beta_{U \cap V}) = \partial(\beta) - \partial(\beta_{V-K})$ and φ_A vanishes on chains in $V - K \subset A$.

This concludes the proof of Lemma 6.8. \square

It remains to prove Lemma 6.9.

Let \mathcal{C} be the category with objects $U - L$, $U \cap V$, $V - K$ and their intersections and with morphisms the inclusion maps. It suffices to show that $\bar{d}(\Gamma_{K \cup L})$ is in the image of the map

$$\kappa : \varinjlim_{W \in \mathcal{C}} I^{\bar{p}}H_*(W) \otimes I^{\bar{q}}H_*(W, W - K \cup L) \rightarrow I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(X, X - K \cup L).$$

Let Y denote the subspace

$$((U - L) \times (U - L)) \cup ((U \cap V) \times (U \cap V)) \cup ((V - K) \times (V - K))$$

of $X \times X$ and consider the commutative diagram

$$\begin{array}{ccccc} I^{\bar{p}}H_*(X, X - K \cup L) & \xrightarrow{d} & I^{Q_{\bar{p}, \bar{q}}}H_*(X \times X, X \times (X - K \cup L)) & \xleftarrow{\cong} & I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(X, X - K \cup L) \\ & \searrow d & \uparrow & & \uparrow \kappa \\ & & I^{Q_{\bar{p}, \bar{q}}}H_*(Y, Y - (X \times (K \cup L))) & & \\ & & \uparrow \lambda & & \\ & & H_*(\varinjlim_{W \in \mathcal{C}} I^{Q_{\bar{p}, \bar{q}}}C_*(W \times W, W \times (W - K \cup L))) & \xleftarrow{\mu} & H_*(\varinjlim_{W \in \mathcal{C}} I^{\bar{p}}C_*(W) \otimes I^{\bar{q}}C_*(W, W - K \cup L)). \end{array}$$

$\bar{d}(\Gamma_{K \cup L})$ is the image of $\Gamma_{K \cup L}$ along the top row. The map λ is an isomorphism by [17, Proposition 6.1.1], so to show that $\bar{d}(\Gamma_{K \cup L})$ is in the image of κ it suffices to show that the map μ is an isomorphism.

Let us write W_1 , W_2 and W_3 for $U - L$, $U \cap V$ and $V - K$ respectively. Let \mathcal{C}' be the subcategory of \mathcal{C} with objects W_1 , W_2 and $W_1 \cap W_2$, and let \mathcal{C}'' be the subcategory of \mathcal{C} with objects $W_1 \cap W_3$, $W_2 \cap W_3$ and $W_1 \cap W_2 \cap W_3$. For any functor F from \mathcal{C} to chain complexes, $\varinjlim_{W \in \mathcal{C}} F(W)$ can be written as an iterated pushout: it is the pushout of the diagram

$$\begin{array}{ccc} \varinjlim_{W \in \mathcal{C}''} F(W) & \longrightarrow & F(W_3) \\ \downarrow & & \\ \varinjlim_{W \in \mathcal{C}'} F(W) & & \end{array}$$

and $\varinjlim_{W \in \mathcal{C}''} F(W)$ and $\varinjlim_{W \in \mathcal{C}'} F(W)$ are also pushouts.

Next recall that, if

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is a pushout diagram of chain complexes for which $A \rightarrow B \oplus C$ is a monomorphism, there is a Mayer-Vietoris sequence

$$\cdots \rightarrow H_i A \rightarrow H_i B \oplus H_i C \rightarrow H_i D \rightarrow H_{i-1} A \rightarrow \cdots$$

Combining this with Theorem 3.3 and the five lemma, we see that the map

$$\varinjlim_{W \in \mathcal{C}'} I^{\bar{p}} C_*(W) \otimes I^{\bar{q}} C_*(W, W - K \cup L) \rightarrow \varinjlim_{W \in \mathcal{C}'} I^{Q_{\bar{p}, \bar{q}}} C_*(W \times W, W \times (W - K \cup L)),$$

and the analogous map with \mathcal{C}' replaced by \mathcal{C}'' , are quasi-isomorphisms. Now one further application of the Mayer-Vietoris sequence, Theorem 3.3, and the five lemma shows that μ is an isomorphism as required. \square

7 Stratified pseudomanifolds-with-boundary and Lefschetz duality

In subsection 7.1, we give the definition of stratified pseudomanifold-with-boundary; we call these ∂ -stratified pseudomanifolds, following Dold's use of ∂ -manifold to mean manifold with boundary [8, Definition VIII.1.9]. In subsection 7.2, we show that a compact ∂ -stratified pseudomanifold has a fundamental class, and in subsection 7.3 we show that cap product with the fundamental class induces a Lefschetz duality isomorphism.

7.1 ∂ -stratified pseudomanifolds

Definition 7.1. An n -dimensional ∂ -stratified pseudomanifold is a pair (X, B) together with a filtration on X such that

1. $X - B$, with the induced filtration, is an n -dimensional stratified pseudomanifold (in the sense of Section 2.1),
2. B , with the induced filtration, is an $n - 1$ dimensional stratified pseudomanifold (in the sense of Section 2.1),
3. B has an *open collar neighborhood* in X , that is, a neighborhood N with a homeomorphism of filtered spaces $N \rightarrow B \times [0, 1)$ (where $[0, 1)$ is given the trivial filtration) that takes B to $B \times \{0\}$.

B is called the *boundary* of X and denoted ∂X .

We will often abuse notation by referring to the “ ∂ -stratified pseudomanifold X ,” leaving B tacit.

Note that a stratified pseudomanifold X (as defined in Section 2.1) is a ∂ -stratified pseudomanifold with $\partial X = \emptyset$.

Definition 7.2. The *strata* of a ∂ -stratified pseudomanifold X are the components of the spaces $X^i - X^{i-1}$.

Our next result shows that when there are no codimension one strata ∂X is a topological invariant.

Proposition 7.3. *Let (X, B) and (X', B') be ∂ -stratified pseudomanifolds of dimension n with no codimension one strata, and let $h : X \rightarrow X'$ be a homeomorphism (which is not required to be filtration preserving). Then h takes B to B' .*

Proof. It suffices to show that h takes the union of the regular strata of B to B' , since the regular strata are dense in B and B' is closed. So let x be in a regular stratum of B and suppose that $h(x)$ is not in B' . Then there is a Euclidean neighborhood E of x in B such that $h(E) \subset X' - B'$. The existence of an open collar neighborhood of B shows that the local homology group $H_n(X, X - \{y\})$ is 0 for each $y \in E$, so by topological invariance of homology $h(E)$ must be contained in the singular set S of $X' - B'$.

Next we use the dimension theory of [6, Section II.16]. We will use the fact that each skeleton of a pseudomanifold (and in particular the singular set) is locally compact.

$\dim_{\mathbb{Z}} E$ (as defined in [6, Definition II.16.6]) is $n-1$ by [6, Corollary II.16.28], so $\dim_{\mathbb{Z}} h(E)$ is also $n-1$, and by [6, Theorem II.16.8] (using the fact that S is locally compact) this implies that $\dim_{\mathbb{Z}} S$ is $\geq n-1$. To obtain a contradiction it suffices to show that $\dim_{\mathbb{Z}}$ of the i -skeleton of a pseudomanifold is $\leq i$ (a fact that doesn't seem to be written down explicitly in the literature).

So let Y be a pseudomanifold and assume by induction that $\dim_{\mathbb{Z}} Y^i \leq i$ for some i . Let c denote the family of compact supports and let $\dim_{c, \mathbb{Z}}$ be as in [6, Definition 16.3]. Then $\dim_{\mathbb{Z}}$ is equal to $\dim_{c, \mathbb{Z}}$ for any locally compact space by [6, Definition II.16.6]. Since Y^i is a closed subset of Y^{i+1} and $Y^{i+1} - Y^i$ is a (possibly empty) $(i+1)$ -manifold, [6, Exercise II.11 and Corollary II.16.28] imply that $\dim_{c, \mathbb{Z}} Y^{i+1}$ is $\leq i+1$ as required. \square

Proposition 7.3 is not true if codimension one strata are allowed, as shown by the following example.

Example 7.4. Let M be a paracompact n -manifold with boundary (in the classical sense), and let P be its boundary.

1. Suppose we filter M trivially so that M itself is the only non-empty stratum. Then (M, P) is a ∂ -stratified pseudomanifold. Note that all the conditions of Definition 7.1 are fulfilled: $M - P$ is an n -manifold, P is an $n - 1$ manifold, and P is collared in M by classical manifold theory (see [21, Proposition 3.42]).
2. On the other hand, suppose X is the filtered space $M \supset P$. Then it is easy to check that (X, \emptyset) is a ∂ -stratified pseudomanifold; that is, X is a stratified pseudomanifold in the sense of Section 2.1. With this filtration, we cannot have $\partial X = P$ because condition (3) of Definition 7.1 would not be satisfied.

Remark 7.5. All of the intersection homology machinery developed in Sections 2–4 of this paper applies immediately to ∂ -stratified pseudomanifolds.

7.2 Fundamental classes of ∂ -stratified pseudomanifolds

Definition 7.6. An R -orientation of a ∂ -stratified pseudomanifold X is an R -orientation of $X - \partial X$.

Given an R -orientation of X and a point $x \in X - \partial X$, Definition 5.6 gives a local orientation class $o_x \in I^{\bar{0}}H_n(X - \partial X, X - \{x\} - \partial X; R)$. We will denote the image of this class under the inclusion map $I^{\bar{0}}H_n(X - \partial X, X - \{x\} \cup \partial X; R) \rightarrow I^{\bar{0}}H_n(X, X - \{x\}; R)$ by o'_x .

Proposition 7.7. *Let X be a compact R -oriented ∂ -stratified pseudomanifold of dimension n . There is a unique class $\Gamma_X \in I^{\bar{0}}H_n(X, \partial X; R)$ that restricts to o'_x for every $x \in X - \partial X$.*

Proof. Let N be an open collar neighborhood of ∂X . Theorem 5.7 gives a fundamental class Γ_{X-N} in $I^{\bar{0}}H_n(X - \partial X, N - \partial X; R)$. Let Γ_X be the image of Γ_{X-N} under the composite

$$I^{\bar{0}}H_n(X - \partial X, N - \partial X; R) \rightarrow I^{\bar{0}}H_n(X, N; R) \xleftarrow{\cong} I^{\bar{0}}H_n(X, \partial X; R),$$

where the second map (which is induced by inclusion) is an isomorphism by a stratified homotopy equivalence (see Appendix A). It is easy to check that Γ_X is independent of N , using the fact that the intersection of two open collar neighborhoods contains another. If $x \in X - \partial X$, the fact that Γ_X restricts to o'_x follows from the fact that there is an N not containing x . Uniqueness follows from the uniqueness property in Theorem 5.7 and the fact that the maps $I^{\bar{0}}H_n(X - \partial X, X - \{x\} \cup \partial X; R) \rightarrow I^{\bar{0}}H_n(X, X - \{x\}; R)$ and $I^{\bar{0}}H_n(X - \partial X, N - \partial X; R) \rightarrow I^{\bar{0}}H_n(X, N; R)$ are isomorphisms by excision. \square

Γ_X will be called the *fundamental class* of X .

Remark 7.8. Corollary 5.22 has an analogue for ∂ -stratified pseudomanifolds. We will not give details here because this fact is not needed for our work.

We conclude this section with a result that will be needed in [17].

First we observe that an R -orientation of X induces an R -orientation of ∂X , because the union of the regular strata of X and the regular strata of ∂X is a (nonsingular) R -oriented ∂ -manifold.

Proposition 7.9. *Let X be a compact R -oriented ∂ -stratified pseudomanifold of dimension n , and give ∂X the induced orientation. Then the map*

$$\partial : I^{\bar{0}}H_n(X, \partial X; R) \rightarrow I^{\bar{0}}H_{n-1}(\partial X; R)$$

takes Γ_X to $\Gamma_{\partial X}$.

Proof. By Corollary 5.15, it suffices to show that $\partial\Gamma_X$ restricts to the local orientation class in $I^{\bar{0}}H_{n-1}(\partial X, \partial X - \{x\}; R)$ for each x that's in a regular stratum of ∂X . So let x be such a point. Let E be a closed Euclidean ball around x in ∂X , and let E° be the interior of E . Let N be an open collar neighborhood of ∂X , and let M be the image of $E \times [0, 1/2]$ under the homeomorphism $\partial X \times [0, 1) \rightarrow N$; then M is a (nonsingular) ∂ -manifold and the R -orientation of X restricts to an R -orientation of M . Let M° denote the interior of M . Now consider the following commutative diagram (where the R coefficients are tacit).

$$\begin{array}{ccccccc}
I^{\bar{0}}H_n(X, \partial X) & \longrightarrow & I^{\bar{0}}H_n(X, X - M^\circ) & \xleftarrow{\cong} & I^{\bar{0}}H_n(M, \partial M) & & \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
& & I^{\bar{0}}H_{n-1}(X - M^\circ) & \xleftarrow{\quad} & I^{\bar{0}}H_{n-1}(\partial M) & & \\
& \nearrow & \downarrow & & \downarrow & & \\
I^{\bar{0}}H_{n-1}(\partial X) & \longrightarrow & I^{\bar{0}}H_{n-1}(X - M^\circ, X - (M^\circ \cup E^\circ)) & \xleftarrow{\cong} & I^{\bar{0}}H_{n-1}(\partial M, \partial M - E^\circ) & \longrightarrow & I^{\bar{0}}H_{n-1}(E^\circ, E^\circ - \{x\}).
\end{array}$$

Here the second arrows in the first and last rows (which are induced by inclusion) are isomorphisms by a combination of excision and stratified homotopy equivalence. It's straightforward to check that the lower composite is the usual restriction map $I^{\bar{0}}H_{n-1}(\partial X) \rightarrow I^{\bar{0}}H_{n-1}(E^\circ, E^\circ - \{x\})$, so it suffices to show that this composite takes $\partial\Gamma_X$ to the local orientation class at x . But it's straightforward to check that the upper composite takes Γ_X to Γ_M , and a standard fact in manifold theory (using the fact that $I^{\bar{0}}H_* = H_*$ for spaces with trivial stratification) says that the rightmost ∂ takes Γ_M to $\Gamma_{\partial M}$. Since $\Gamma_{\partial M}$ maps to the local orientation class at x the proof is complete. \square

7.3 Lefschetz duality

Theorem 7.10 (Lefschetz Duality). *Let F be a field, and let X be an n -dimensional compact ∂ -stratified pseudomanifold such that $X - \partial X$ is F -oriented. Suppose that $\bar{p} + \bar{q} = \bar{t}$. Then the cap product with Γ_X is an isomorphism $I_{\bar{p}}H^i(X, \partial X; F) \rightarrow I^{\bar{q}}H_{n-i}(X; F)$.*

Proof. We follow the strategy in [21]. Let N be an open collar of ∂X . Consider the following commutative diagram

$$\begin{array}{ccc}
I_{\bar{p}}H^i(X - \partial X, N - \partial X; F) & \xleftarrow{\cong} & I_{\bar{p}}H^i(X, N; F) \\
\downarrow (-1)^{in} \cdot \cap \Gamma_{X-N} & & \downarrow (-1)^{in} \cdot \cap \Gamma_X \\
I^{\bar{q}}H_{n-i}(X - \partial X; F) & \xrightarrow{\cong} & I^{\bar{q}}H_{n-i}(X; F).
\end{array}$$

The top isomorphism is by excision and stratified homotopy equivalence. The bottom isomorphism is also by stratified homotopy equivalence. If we take the direct limit of the diagram as N shrinks to ∂X , then $\varinjlim I_{\bar{p}}H^i(X, N; F) \cong I_{\bar{p}}H^i(X, \partial X; F)$ (in fact, all maps in the directed system obtained by retracting the collar are isomorphisms), while $\varinjlim I_{\bar{p}}H^i(X - \partial X, N - \partial X; F) \cong I_{\bar{p}}H_c^i(X - \partial X; F)$. So by Theorem 6.3, the left hand map becomes an isomorphism in the limit. It follows therefore that the right hand map also becomes an isomorphism in the limit, proving the theorem. \square

A Stratified maps, homotopy, and homotopy equivalence

The definition of “stratum preserving homotopy equivalence” given in [10, 29] needs to be modified a little in the context of general perversities. In this appendix we give the necessary details.

Let X and Y be ∂ -stratified pseudomanifolds, and assume that we are given perversities \bar{p}, \bar{q} on X and Y respectively.

Definition A.1. We will say that a map $f : X \rightarrow Y$ is *stratified with respect to \bar{p}, \bar{q}* if

1. the image of each stratum of X is contained in a single stratum of Y of the same codimension, i.e. if $Z' \subset Y$ is a stratum of codimension k , then $f^{-1}(Z')$ is a union of strata of X of codimension k ,
2. if the stratum $Z \subset X$ maps to the stratum $Z' \subset Y$, then $\bar{p}(Z) \leq \bar{q}(Z')$.

Note that if $f : X \rightarrow Y$ is an inclusion of an open subset, then f is always stratified with respect to any perversity \bar{q} on Y and its induced restriction to X (i.e. the perversity on X whose value on Z is defined to be $\bar{q}(Z')$ if $Z \subset Z'$).

An easy argument from the definitions shows that if $f : X \rightarrow Y$ is stratified and \mathcal{G} is a coefficient system on $Y - Y^{\dim(Y)-1}$, then $f_{\#} : I^{\bar{p}}C_*(X; f^*\mathcal{G}) \rightarrow I^{\bar{q}}C_*(Y; \mathcal{G})$ is well-defined and induces a map of intersection homology groups $f_* : I^{\bar{p}}H_*(X; f^*\mathcal{G}) \rightarrow I^{\bar{q}}H_*(Y; \mathcal{G})$.

Now stratify $X \times I$ by letting the strata have the form $Z \times I$, where Z is a stratum of X . This stratification induces a natural bijection $Z \leftrightarrow Z \times I$ between the singular strata of X and those of $X \times I$ and thus a natural bijection of perversities such that a perversity of X

corresponds to a perversity of $X \times I$ if the two take the same value on corresponding singular strata. In this case we will abuse notation and use the same symbol for both perversities.

We call $F : X \times I \rightarrow Y$ a *stratified homotopy (with respect to \bar{p}, \bar{q})* if F is a stratified map (with respect to \bar{p}, \bar{q}). In particular, the image under F of each stratum $Z \times I \subset X \times I$ is contained in a single stratum of Y (again compare [10, 29]). If $F : X \times I \rightarrow Y$ is a stratified homotopy, then $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$ are stratified maps $X \rightarrow Y$ and F induces a chain homotopy between the induced maps of intersection chains $f_\#$ and $g_\#$. The proof of this fact follows by the usual prism construction (see e.g. [21]). One checks that the necessary chains are all allowable as in the proof of Proposition 2.1 of [10], with some obvious changes necessary to account for the general perversities.

We call ∂ -stratified pseudomanifolds X, Y *stratified homotopy equivalent* if there is a homotopy equivalence $f : X \rightarrow Y$ with homotopy inverse $g : Y \rightarrow X$ such that f, g , and the respective homotopies from fg to id_Y and from gf to id_X all satisfy condition (1) of Definition A.1. The maps f and g are then deemed *stratified homotopy equivalences*. In this case, there must be a bijection between the strata of X and the strata of Y , and thus a bijection between perversities on X and perversities on Y . We often abuse notation and use a common symbol for the corresponding perversities. With respect to such corresponding perversities, f and g will be stratified maps, and the homotopies from fg to id_Y and from gf to id_X will be stratified homotopies.

Thus if $f : X \rightarrow Y$ is a stratified homotopy equivalence, it follows that $I^{\bar{p}}C_*(X; f^*\mathcal{G})$ is chain homotopy equivalent to $I^{\bar{p}}C_*(Y; \mathcal{G})$ and thus $I^{\bar{p}}H_*(X; f^*\mathcal{G}) \cong I^{\bar{p}}H_*(Y; \mathcal{G})$. In particular, any inclusion $X \times \{t\} \hookrightarrow X \times I$, where I is unfiltered and $X \times I$ is given the product filtration, induces $I^{\bar{p}}H_*(X \times \{t\}; \mathcal{G}|_{X \times \{t\}}) \cong I^{\bar{p}}H_*(X \times I; \mathcal{G})$.

B Proofs of Theorems 3.1 and 3.3

In this appendix, we provide some technical proofs concerning the intersection homology Künneth theorem of [13]. The notation is taken from [13]; we refer the reader there for discussion of the sheaves involved.

We first prove the following proposition, which implies Theorem 3.1.

Proposition B.1. *Let F be a field. Then the Künneth isomorphism of [13] is induced (up to sign) by the chain level cross product.*

Proof. Consider the following diagram (all coefficients are in F)

$$\begin{array}{ccc}
H_{n+m-*}(I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)) & \xrightarrow{\times} & H_{n+m-*}(I^{\mathcal{Q}_{\bar{p},\bar{q}}}(X \times Y)) \\
\downarrow & & \downarrow \cong \\
H^*(\Gamma_c(X \times Y; \mathcal{R}^*)) & \longrightarrow & H^*(\Gamma_c(X \times Y; \mathcal{I}^{\mathcal{Q}_{\bar{p},\bar{q}}}\mathcal{S}^*)) \\
\downarrow & & \downarrow \cong \\
H^*(\Gamma_c(X \times Y; \mathcal{I}^*)) & \xrightarrow{\cong} & H^*(\Gamma_c(X \times Y; \mathcal{J}^*)).
\end{array} \tag{6}$$

The sheaf \mathcal{R}^* is the sheaf defined in [13]; it is really just the sheaf $\pi_X^*(\mathcal{I}^{\bar{p}}\mathcal{S}_X^*) \otimes \pi_Y^*(\mathcal{I}^{\bar{q}}\mathcal{S}_Y^*)$. The top map is the chain cross product, which is allowable by [13]. The top vertical maps are induced by sheafification. The bottom vertical maps are induced by taking c -acyclic resolutions; therefore the diagram commutes (up to possible signs arising from the degree shifts in the upper vertical maps). The maps on the right are isomorphisms by the properties of the sheaf $\mathcal{I}^{\mathcal{Q}_{\bar{p},\bar{q}}}\mathcal{S}^*$, which is homotopically fine and generated by a monopresheaf that is conjunctive for coverings. The bottom isomorphism is the Künneth isomorphism of [13]. We want to show that \times is an isomorphism. It suffices to show that the composition on the left of the diagram is an isomorphism.

In fact, we know abstractly that $H_{n+m-*}(I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)) \cong \mathbb{H}^*(\Gamma_c(X \times Y; \mathcal{I}^*))$ by [13, Corollary 4.2]. But we need slightly more; we must show that the isomorphism is consistent with the left hand composition of the diagram here.

Let \mathcal{K}_X^* and \mathcal{K}_Y^* be injective resolutions of $\mathcal{I}^{\bar{p}}\mathcal{S}_X^*$ and $\mathcal{I}^{\bar{q}}\mathcal{S}_Y^*$, respectively. Then we have a diagram

$$\begin{array}{ccc}
H_{n+m-*}(I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)) & \xlongequal{\quad} & H_{n+m-*}(I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)) \\
\downarrow \cong & & \downarrow \cong \\
H^*(\Gamma_c(X; \mathcal{I}^{\bar{p}}\mathcal{S}_X^*) \otimes \Gamma_c(Y; \mathcal{I}^{\bar{q}}\mathcal{S}_Y^*)) & \xrightarrow{\cong} & H^*(\Gamma_c(X; \mathcal{K}_X^*) \otimes \Gamma_c(Y; \mathcal{K}_Y^*)) \\
\downarrow & & \downarrow \cong \\
H^*(\Gamma_c(X \times Y; \pi_X^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*) \otimes \pi_Y^*(\mathcal{I}^{\bar{q}}\mathcal{S}^*))) & \longrightarrow & H^*(\Gamma_c(X \times Y; \pi_X^*(\mathcal{K}_X^*) \otimes \pi_Y^*(\mathcal{K}_Y^*))).
\end{array}$$

The top left vertical map is induced by sheafification and is an isomorphism because $\mathcal{I}^{\bar{p}}\mathcal{S}_X^*$ and $\mathcal{I}^{\bar{q}}\mathcal{S}_Y^*$ are induced by appropriate monopresheaves that are conjunctive for covers. The middle and bottom horizontal maps are induced by the injective resolutions $\mathcal{I}^{\bar{p}}\mathcal{S}_X^* \rightarrow \mathcal{K}_X^*$ and $\mathcal{I}^{\bar{q}}\mathcal{S}_Y^* \rightarrow \mathcal{K}_Y^*$. The middle horizontal map is an isomorphism because $\mathcal{I}^{\bar{p}}\mathcal{S}_X^*$ and $\mathcal{I}^{\bar{q}}\mathcal{S}_Y^*$ are homotopically fine. We fill in the top right vertical arrow so that the top square commutes

by definition. The bottom vertical maps are defined and the bottom square commutes in the evident way (given a germ over $x \in X$ and a germ over $y \in Y$, this determines a germ of the tensor product of stalks over (x, y) ; see [6, Section II.15]). The composition of maps on the left is equivalent to the map $H_{n+m-*}(I^{\bar{p}}C_*(X) \otimes I^{\bar{q}}C_*(Y)) \rightarrow H^*(\Gamma_c(X \times Y; \mathcal{R}^*))$ of Diagram (6). Since \mathcal{K}_X^* and \mathcal{K}_Y^* are injective, and hence c -fine, and since X and Y are locally compact and Hausdorff, $\pi_X^*(\mathcal{K}_X^*) \otimes \pi_Y^*(\mathcal{K}_Y^*)$ is c -fine by [6, Exercise II.14 and page 494, fact (s)]. So the bottom map of the diagram is in fact induced by a c -fine resolution $\pi_X^*(\mathcal{I}^{\bar{p}}\mathcal{S}_X^*) \otimes \pi_Y^*(\mathcal{I}^{\bar{q}}\mathcal{S}_Y^*) \rightarrow \pi_X^*(\mathcal{K}_X^*) \otimes \pi_Y^*(\mathcal{K}_Y^*)$. Therefore we can let the bottom horizontal map here play the role of the bottom left map of Diagram (6). The bottom right vertical map is an isomorphism by [6, Proposition 15.1].

Note, we cannot conclude that either of the maps in the diagram not labeled as such are isomorphisms, but nonetheless, this is enough to show the composition along the left side of Diagram (6) is the desired isomorphism. \square

Next we prove the relative Künneth theorem (Proposition 3.3). We restate it for the convenience of the reader.

Theorem *Let X and Y be stratified pseudomanifolds with open subsets $A \subset X, B \subset Y$. The cross product induces an isomorphism*

$$I^{\bar{p}}H_*(X, A; F) \otimes I^{\bar{q}}H_*(Y, B; F) \rightarrow I^{Q_{\bar{p}, \bar{q}}}H_*(X \times Y, (A \times Y) \cup (X \times B); F).$$

Proof. Let Q denote $Q_{\bar{p}, \bar{q}}$. Consider the following diagram (where we leave the F coefficients tacit).

$$\begin{array}{ccccccc} \longrightarrow & I^{\bar{p}}H_*(A) \otimes I^{\bar{q}}H_*(Y) & \longrightarrow & I^{\bar{p}}H_*(X) \otimes I^{\bar{q}}H_*(Y) & \longrightarrow & I^{\bar{p}}H_*(X, A) \otimes I^{\bar{q}}H_*(Y) & \longrightarrow \\ & \downarrow \times & & \downarrow \times & & \downarrow \times & \\ \longrightarrow & I^Q H_*(A \times Y) & \longrightarrow & I^Q H_*(X \times Y) & \longrightarrow & I^Q H_*(X \times Y, A \times Y) & \longrightarrow . \end{array}$$

Both rows are exact; the top row is exact because we work over a field (so all modules are flat). The vertical maps are all induced by the chain cross product, and the diagram commutes up to sign (as can be seen by working with representative chains). So we have $I^{\bar{p}}H_*(X, A) \otimes I^{\bar{q}}H_*(Y) \cong I^Q H_*(X \times Y, A \times Y)$ by the five lemma.

Similarly, we now have the diagram

$$\begin{array}{ccccccc} \longrightarrow & I^{\bar{p}}H_*(X, A) \otimes I^{\bar{q}}H_*(B) & \longrightarrow & I^{\bar{p}}H_*(X, A) \otimes I^{\bar{q}}H_*(Y) & \longrightarrow & I^{\bar{p}}H_*(X, A) \otimes I^{\bar{q}}H_*(Y, B) & \longrightarrow \\ & \downarrow \times & & \downarrow \times & & \downarrow \times & \\ \longrightarrow & I^Q H_*(X \times B, A \times B) & \longrightarrow & I^Q H_*(X \times Y, A \times Y) & \longrightarrow & I^Q H_*(X \times Y, (A \times Y) \cup (X \times B)) & \longrightarrow . \end{array}$$

The top row is again exact by flatness. The bottom row is the long exact sequence associated to the short exact sequence

$$0 \longrightarrow I^Q C_*(X \times B, A \times B) \longrightarrow I^Q C_*(X \times Y, A \times Y) \longrightarrow I^{Q_{\bar{p}, \bar{q}}} C_*(X \times Y, (A \times Y) \cup (X \times B)) \longrightarrow 0,$$

which exists by some basic homological algebra.⁵ Again, commutativity follows from chain arguments, and the proposition now follows from the five lemma. \square

C Invariance of general perversity intersection homology under normalization

We provide here a theorem stating that intersection homology is preserved under normalization. For general background on normalizations see [28, 19].

Lemma C.1. *Let X be a stratified pseudomanifold, and let $\pi : \hat{X} \rightarrow X$ be its normalization. Then $\pi : I^{\bar{p}}H_*(\hat{X}; R) \rightarrow I^{\bar{p}}H_*(X; R)$ is an isomorphism.*

Proof. This is a standard fact for intersection homology with Goresky-MacPherson perversities and no codimension one strata. We briefly revisit the proof to show that it remains true in the more general setting. It is elementary to observe that π is well-defined as a homomorphism of intersection chains, and hence of intersection homology groups. The normalization map is proper (since all stratified pseudomanifolds have compact links by definition), so we can consider intersection homology either with closed or with compact supports.

By [15, Lemma 2.4], it is sufficient to consider perversities such that $\bar{p}(Z) \leq \text{codim}(Z) - 1$ for each singular stratum Z , for otherwise we get nothing new. This fact allows us mostly to reduce the proof to the usual one: if $\bar{p}(Z) \leq \text{codim}(Z) - 1$ for each singular Z , each simplex of each allowable chain ξ of $I^{\bar{p}}C_i(X; R)$ intersects X^{n-1} in at most the image of the $i - 1$ skeleton of the model simplex Δ^i . So for any such singular simplex σ in ξ , σ maps the interior of Δ^i into $X - X^{n-1}$. But this mapping of the interior can be lifted to \hat{X} , and continuity ensures that we can then lift all of σ to \hat{X} . This process generates a homomorphism $s : I^{\bar{p}}C_*(X; R) \rightarrow I^{\bar{p}}C_*(\hat{X}; R)$, and it is clear that s is an inverse of π . It only remains to check that s is a chain map. This is not difficult to see, recalling that any boundary simplices with support entirely in X^{n-1} are set automatically to 0. \square

D Comparison with the cup product of [3]

In this appendix we verify the claim in Remark 1.3.

First observe that for pairs \bar{p}, \bar{q} satisfying the conditions in Remark 1.3 we have $D\bar{q} \geq D\bar{p} + D\bar{p}$, so Definition 4.11 gives a cup product map

$$I_{\bar{p}}H^*(X; \mathbb{Q}) \otimes I_{\bar{q}}H^*(X; \mathbb{Q}) \rightarrow I_{\bar{q}}H^*(X; \mathbb{Q})$$

which we will show agrees up to sign with that constructed in [3, Section 7].

One of the ingredients in Banagl's construction is the "Eilenberg-Zilber type isomorphism"

$$I^{\bar{p}}C_*(X; \mathbb{Q}) \otimes I^{\bar{p}}C_*(Y; \mathbb{Q}) \rightarrow I^{\bar{p}}C_*(X \times Y; \mathbb{Q})$$

⁵This essentially comes from the intersection chain short exact Mayer-Vietoris sequence for the pairs $(X \times Y, A \times Y)$ and $(X \times B, X \times B)$, since $B \cap Y = B$, $Y \cup B = Y$, $A \times Y \cap X \times B = A \times B$.

((23) on page 175 of [3]).⁶ We will denote this map by E . The criterion given at the end of [13, Section 4.1] shows that, since $\bar{p}(k) + \bar{p}(l) \leq \bar{p}(k + l)$, the cross product also induces a map

$$\times : I^{\bar{p}}C_*(X; \mathbb{Q}) \otimes I^{\bar{p}}C_*(Y; \mathbb{Q}) \rightarrow I^{\bar{p}}C_*(X \times Y; \mathbb{Q}),$$

and we claim that (up to sign) this is the same as E . This follows from the uniqueness result [7, Proposition 2], using the fact that both E and \times are induced by maps of sheaves

$$\pi_X^* \mathcal{I}^{\bar{p}} \mathcal{S}_X^* \otimes \pi_Y^* \mathcal{I}^{\bar{p}} \mathcal{S}_Y^* \rightarrow \mathcal{I}^{\bar{p}} \mathcal{S}_{X \times Y}^*$$

(see the proofs of [3, Theorem 9.1] and Proposition B.1) which agree (up to sign) on $\pi_U^* \mathbb{Q}_U \otimes \pi_V^* \mathbb{Q}_V$.

Now consider the following diagram.

$$\begin{array}{ccccc} I^{\bar{q}}C_*(X; \mathbb{Q}) & \xrightarrow{d} & I^{\bar{p}}C_*(X \times X; \mathbb{Q}) & \xleftarrow[\cong]{E} & I^{\bar{p}}C_*(X; \mathbb{Q}) \otimes I^{\bar{p}}C_*(X; \mathbb{Q}) \\ & \searrow d & \downarrow & \swarrow \times & \\ & & I^{Q_{\bar{p}, \bar{p}}}C_*(X \times X; \mathbb{Q}) & & \end{array} \quad (7)$$

Here the two maps marked d are induced by the diagonal; the horizontal d is given by [3, Proposition 7.1] and the other d is given by Proposition 4.2.1. The vertical map exists because of the inequality $\bar{p}(k + l) \leq \bar{p}(k) + \bar{p}(l) + 2$. The left-hand triangle in diagram (7) obviously commutes and we have just seen that the right-hand triangle commutes up to sign.

The dual of the lower composite in diagram (7) is the cup product of Definition 4.11, so it suffices to show that the dual of the upper composite is the cup product of [3, Section 7]. This in turn is a straightforward consequence of the definition in [3] and Proposition IV.2.5 of [9].

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⁶Note that there is an implicit assumption in the constuction of this map that X and Y are orientable, since the orientation sheaf mentioned on line –12 of [3, page 163] is identified on page 175 of [3] with the constant sheaf \mathbb{Q}_U (resp., \mathbb{Q}_V). We leave it to the interested reader to work out the details in the non-orientable case.

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