TWISTED ALEXANDER POLYNOMIALS DETECT FIBERED 3–MANIFOLDS

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Abstract. A classical result in knot theory says that for a fibered knot the Alexander polynomial is monic and that the degree equals twice the genus of the knot. This result has been generalized by various authors to twisted Alexander polynomials and fibered 3–manifolds. In this paper we show that the conditions on twisted Alexander polynomials are not only necessary but also sufficient for a 3–manifold to be fibered. By previous work of the authors this result implies that if a manifold of the form $S^1 \times N^3$ admits a symplectic structure, then N fibers over S^1. In fact we will completely determine the symplectic cone of $S^1 \times N$ in terms of the fibered faces of the Thurston norm ball of N.

1. Introduction

1.1. **Twisted Alexander polynomials and fibered 3–manifolds.** Let N be a compact, connected, oriented 3–manifold with empty or toroidal boundary. Given a nontrivial class $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ we say that (N,ϕ) *fibers over* S^1 if there exists a fibration $f : N \to S^1$ such that the induced map $f_* : \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ agrees with ϕ. Stated otherwise, the homotopy class in $[N,S^1] = H^1(N;\mathbb{Z})$ identified by ϕ can be represented by a fibration.

It is a classical result in knot theory that if a knot $K \subset S^3$ is fibered, then the Alexander polynomial is monic (i.e. the top coefficient equals ± 1), and the degree of the Alexander polynomial equals twice the genus of the knot. This result has been generalized in various directions by several authors (e.g. [McM02, Ch03, GKM05, FK06, Ki07]) to show that twisted Alexander polynomials give necessary conditions for (N,ϕ) to fiber.

To formulate this kind of result more precisely we have to introduce some definitions. Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$. Given (N,ϕ) the *Thurston norm* of ϕ (cf. [Th86]) is defined as

$$||\phi||_T = \min\{\chi_-(S) \,|\, S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$.

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In the following we assume that $\phi \in H^1(N;\mathbb{Z})$ is non–trivial. Let $\alpha : \pi_1(N) \to G$ be a homomorphism to a finite group. We have the permutation representation $\pi_1(N) \to \mathrm{Aut}(\mathbb{Z}[G])$ given by left multiplication, which we also denote by α. We can therefore consider the twisted Alexander polynomial $\Delta^\alpha_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$, whose definition is detailed in Section 2.3. We denote by ϕ_α the restriction of $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ to $\mathrm{Ker}(\alpha)$. Note that ϕ_α is necessarily non–trivial. We denote by $\mathrm{div}\phi_\alpha \in \mathbb{N}$ the divisibility of ϕ_α, i.e.

$$\mathrm{div}\phi_\alpha = \max\{n \in \mathbb{N} \mid \phi_\alpha = n\psi \text{ for some } \psi : \mathrm{Ker}(\alpha) \to \mathbb{Z}\}.$$

We can now formulate the following theorem which appears as [FK06, Theorem 1.3 and Remark p. 938].

Theorem 1.1. *Let $N \neq S^1 \times S^2, S^1 \times D^2$ be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ a nontrivial class. If (N,ϕ) fibers over S^1, then for any homomorphism $\alpha : \pi_1(N) \to G$ to a finite group the twisted Alexander polynomial $\Delta^\alpha_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and*

$$deg(\Delta^\alpha_{N,\phi}) = |G|\, \|\phi\|_T + (1 + b_3(N))\, div\phi_\alpha.$$

It is well known that in general the constraint of monicness and degree for the ordinary Alexander polynomial falls short from characterizing fibered 3–manifolds. The main result of this paper is to show that on the other hand the collection of all twisted Alexander polynomials does detect fiberedness, i.e. the converse of Theorem 1.1 holds true:

Theorem 1.2. *Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ a nontrivial class. If for any homomorphism $\alpha : \pi_1(N) \to G$ to a finite group the twisted Alexander polynomial $\Delta^\alpha_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and*

$$deg(\Delta^\alpha_{N,\phi}) = |G|\, \|\phi\|_T + (1 + b_3(N))\, div\phi_\alpha$$

holds, then (N,ϕ) fibers over S^1.

Note that alternatively it is possible to rephrase this statement in terms of Alexander polynomials of the finite regular covers of N, using the fact that $\Delta^\alpha_{N,\phi} = \Delta_{\tilde{N},p^*(\phi)}$ (cf. [FV08a]), where $p : \tilde{N} \to N$ is the cover of N determined by $\mathrm{Ker}(\alpha)$.

Note that this theorem asserts that twisted Alexander polynomials detect whether (N,ϕ) fibers under the assumption that $||\phi||_T$ is known; while it is known that twisted Alexander polynomials give lower bounds (cf. [FK06, Theorem 1.1]), it is still an open question whether twisted Alexander polynomials determine the Thurston norm.

In the case where ϕ has trivial Thurston norm, this result is proven in [FV08b], using subgroup separability. Here, following a different route (see Section 1.3 for a summary of the proof), we prove the general case.

1.2. **Symplectic 4–manifolds and twisted Alexander polynomials.** In 1976 Thurston [Th76] showed that if a closed 3–manifold N admits a fibration over S^1, then $S^1 \times N$ admits a symplectic structure, i.e. a closed, nondegenerate 2–form ω. It is natural to ask whether the converse to this statement holds true. In its simplest form, we can state this problem in the following way:

Conjecture 1.3. *Let N be a closed 3–manifold. If $S^1 \times N$ is symplectic, then there exists a $\phi \in H^1(N;\mathbb{Z})$ such that (N,ϕ) fibers over S^1.*

Interest in this question was motivated by Taubes' results in the study of Seiberg-Witten invariants of symplectic 4–manifolds (see [Ta94, Ta95]), that gave initial evidence to an affirmative solution of this conjecture. In the special case where N is obtained via 0–surgery along a knot in S^3, this question appears also in [Kr98, Question 7.11]. Over the last ten years evidence for this conjecture was given by various authors [Kr98, CM00, Et01, McC01, Vi03].

In [FV08a] the authors initiated a project relating Conjecture 1.3 to the study of twisted Alexander polynomials. The outcome of that investigation is that if $S^1 \times N$ is symplectic, then the twisted Alexander polynomials of N behave like twisted Alexander polynomials of a fibered 3–manifold. More precisely, the following holds (cf. [FV08a, Theorem 4.4]):

Theorem 1.4. *Let N be an irreducible closed 3–manifold and ω a symplectic structure on $S^1 \times N$ such that ω represents an integral cohomology class. Let $\phi \in H^1(N;\mathbb{Z})$ be the Künneth component of $[\omega] \in H^2(S^1 \times N;\mathbb{Z})$. Then for any homomorphism $\alpha : \pi_1(N) \to G$ to a finite group the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and*

$$\mathit{deg}(\Delta^{\alpha}_{N,\phi}) = |G|\, \|\phi\|_T + 2\,\mathit{div}\phi_{\alpha}.$$

Note that it follows from McCarthy's work [McC01] (see also Lemma 7.1) and Perelman's proof of the geometrization conjecture (cf. e.g. [MT07]) that if $S^1 \times N$ is symplectic, then N is prime, i.e. either irreducible or $S^1 \times S^2$. The proof of Theorem 1.4 relies heavily on the results of [Kr98] and [Vi03], which in turn build on results of Taubes [Ta94, Ta95] and Donaldson [Do96].

As the symplectic condition is open, the assumption that a symplectic manifold admits an integral symplectic form is not restrictive. Therefore, combining Theorem 1.2 with Theorem 1.4, we deduce that Conjecture 1.3 holds true. In fact, in light of [FV07, Theorems 7.1 and 7.2], we have the following more refined statement:

Theorem 1.5. *Let N be a closed oriented 3–manifold. Then given $\Omega \in H^2(S^1 \times N;\mathbb{R})$ the following are equivalent:*

1. *Ω can be represented by a symplectic structure;*
2. *Ω can be represented by a symplectic structure which is S^1–invariant;*
3. *$\Omega^2 > 0$ and the Künneth component $\phi \in H^1(N;\mathbb{R})$ of Ω lies in the open cone on a fibered face of the Thurston norm ball of N.*

Note that the theorem allows us in particular to completely determine the symplectic cone of a manifold of the form $S^1 \times N$ in terms of the fibered cones of N.

Combined with the results of [FV07, FV08a], Theorem 1.2 shows in particular that the collection of the Seiberg-Witten invariants of all finite covers of $S^1 \times N$ determines whether $S^1 \times N$ is symplectic or not. In particular, we have the following corollary (we refer to [Vi99, Vi03] for the notation and the formulation in the case that $b_1(N) = 1$).

Corollary 1.6. *Let N be a closed 3–manifold with $b_1(N) > 1$. Then given a spin^c structure $K \in H^2(S^1 \times N; \mathbb{Z})$ there exists a symplectic structure representing a cohomology class $\Omega \in H^2(S^1 \times N; \mathbb{R})$ with canonical class K if and only if the following conditions hold:*

(1) *$K \cdot \phi = \|\phi\|_T$, where $\phi \in H^1(N; \mathbb{R})$ is the Künneth component of Ω,*

and for any regular finite cover $p : \tilde{N} \to N$

(2) *$SW_{S^1 \times \tilde{N}}(p^*(K)) = 1$,*

(3) *for any Seiberg–Witten basic class $\kappa \in H^2(S^1 \times \tilde{N}; \mathbb{Z})$ we have*

$$|p_*(\kappa) \cdot \phi| \leq deg(p)\ K \cdot \phi,$$

(where p_* is the transfer map) and the latter equality holds if and only if $\kappa = \pm p^* K$.

(Note that, under the hypotheses of the Corollary, all basic classes of $S^1 \times \tilde{N}$ are the pull–back of elements of $H^2(\tilde{N}; \mathbb{Z})$.)

Remark. A different approach to Conjecture 1.3 involves a deeper investigation of the consequence of the symplectic condition on $S^1 \times N$, that goes beyond the information encoded in Theorem 1.4. A major breakthrough in this direction has recently been obtained by Kutluhan and Taubes ([KT09]). They show that if N is a 3–manifold such that $S^1 \times N$ is symplectic, under some cohomological assumption on the symplectic form, then the Monopole Floer homology of N behaves like the Monopole Floer homology of a fibered 3–manifold. On the other hand it is known, due to the work of Ghiggini, Kronheimer and Mrowka, and Ni that Monopole Floer homology detects fibered 3–manifolds ([Gh08, Ni09, KM08, Ni08]). The combination of the above results proves in particular Conjecture 1.3 in the case that $b_1(N) = 1$.

1.3. **Fibered 3–manifolds and finite solvable groups: outline of the proof.** In this subsection we will outline the strategy of the proof of Theorem 1.2. It is useful to introduce the following definition.

Definition. Let N be a 3–manifold with empty or toroidal boundary, and let $\phi \in H^1(N; \mathbb{Z})$ be a nontrivial class. We say that (N, ϕ) satisfies Condition $(*)$ if for any homomorphism $\alpha : \pi_1(N) \to G$ to a finite group the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and

$$\deg(\Delta^{\alpha}_{N,\phi}) = |G|\, \|\phi\|_T + (1 + b_3(N))\mathrm{div}\phi_{\alpha}.$$

It is well–known (see [McC01] for the closed case, and 7.1 for the general case) that Condition $(*)$ implies, using geometrization, that N is prime, so we can restrict ourself to the case where N is irreducible.

Note that McMullen [McM02] showed that, when the class ϕ is primitive, the condition $\Delta_{N,\phi} \neq 0$ implies that there exists a connected Thurston norm minimizing surface Σ dual to ϕ. It is well–known that to prove Theorem 1.2 it is sufficient to consider a primitive ϕ, and we will assume that in the following. Denote $M = N \setminus \nu\Sigma$; the boundary of M contains two copies $\Sigma^{\pm}$ of Σ and throughout the paper we denote the inclusion induced maps $\Sigma \to \Sigma^{\pm} \to M$ by $\iota_{\pm}$.

By Stallings' theorem [St62] the surface Σ is a fiber of a fibration $N \to S^1$ if and only if $\iota_{\pm} : \pi_1(\Sigma) \to \pi_1(M)$ are isomorphisms. Hence to prove Theorem 1.2 we need to show that if (N, ϕ) satisfies Condition $(*)$, then the monomorphisms $\iota_{\pm} : \pi_1(\Sigma) \to \pi_1(M)$ are in fact isomorphisms. Using purely group theoretic arguments we are not able to show directly that Condition $(*)$ implies the desired isomorphism; however, we have the following result:

Proposition 1.7. *Assume that (N, ϕ) satisfies Condition $(*)$ and that ϕ is primitive. Let $\Sigma \subset N$ be a connected Thurston norm minimizing surface dual to ϕ and let ι be either of the two inclusion maps of Σ into $M = N \setminus \nu\Sigma$. Then $\iota : \pi_1(\Sigma) \to \pi_1(M)$ induces an isomorphism of the prosolvable completions.*

We refer to Section 2.4 for information regarding group completions. Proposition 1.7 translates the information from Condition $(*)$ into information regarding the maps $\iota_{\pm} : \pi_1(\Sigma) \to \pi_1(M)$. From a purely group theoretic point of view it is a difficult problem to decide whether a homomorphism which gives rise to an isomorphism of prosolvable completions has to be an isomorphism itself (cf. [Gr70], [BG04], [AHKS07] and also Lemma 4.7). But in our 3–dimensional setting we can use a recent result of Agol [Ag08] to prove the following theorem.

Theorem 1.8. *Let N be an irreducible 3–manifold with empty or toroidal boundary. Let $\Sigma \subset N$ be a connected Thurston norm minimizing surface. We write $M = N \setminus \nu\Sigma$. Assume the following hold:*

(1) *the inclusion induced maps $\iota_{\pm} : \pi_1(\Sigma) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions, and*
(2) *$\pi_1(M)$ is residually finite solvable,*

then $\iota_{\pm} : \pi_1(\Sigma) \to \pi_1(M)$ are isomorphisms, hence $M = \Sigma \times I$.

In light of Proposition 1.7, the remaining obstacle for the proof of Theorem 1.2 is the condition in Theorem 1.8 that $\pi_1(M)$ has to be residually finite solvable. It is well–known that linear groups (and hence in particular hyperbolic 3–manifolds groups) are virtually residually p for all but finitely many primes p (cf. e.g. [We73, Theorem 4.7] or [LS03, Window 7, Proposition 9]), in particular they are residually finite solvable. Thurston conjectured that 3–manifold groups in general are linear (cf. [Ki, Problem 3.33]), but this is still an open problem. Using the recent proof of the geometrization conjecture (cf. e.g. [MT07]) we will prove the following result, which will be enough for our purposes.

Theorem 1.9. *Let N be a closed prime 3–manifold. Then for all but finitely many primes p there exists a finite cover N' of N such that the fundamental group of any component of the JSJ decomposition of N' is residually a p–group.*

We can now deduce Theorem 1.2 as follows: We first show in Lemmas 7.1 and 7.2 that it suffices to show Theorem 1.2 for closed prime 3–manifolds. Theorem 1.2 in that situation now follows from combining Theorems 1.7, 1.8 and 1.9 with a more technical theorem which allows us to treat the various JSJ pieces separately (cf. Theorem 6.4).

Added in proof: In a very recent paper ([AF10]) Matthias Aschenbrenner and the first author showed that any 3–manifold group is virtually residually p. This simplifies the proof of Theorem 1.2 as outlined in [FV10].

This paper is structured as follows. In Section 2 we recall the definition of twisted Alexander polynomials and some basics regarding completions of groups. In Section 3 we will prove Proposition 1.7 and in Section 4 we give the proof of Theorem 1.8. In Section 5 we prove Theorem 1.9 and in Section 6 we provide the proof for Theorem 6.4. Finally in Section 7 we complete the proof of Theorem 1.2.

Conventions and notations. Throughout the paper, unless otherwise stated, we will assume that all manifolds are oriented and connected, and all homology and cohomology groups have integer coefficients. Furthermore all surfaces are assumed to be properly embedded and all spaces are compact and connected, unless it says explicitly otherwise. The derived series of a group G is defined inductively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

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2. Preliminaries: Twisted invariants and completions of groups

2.1. **Twisted homology.** Let X be a CW–complex with base point x_0. Let R be a commutative ring, V a module over R and $\alpha : \pi_1(X, x_0) \to \mathrm{Aut}_R(V)$ a representation. Let $\tilde{X}$ be the universal cover of X. Note that $\pi_1(X, x_0)$ acts on the left on $\tilde{X}$ as group of deck transformations. The cellular chain groups $C_*(\tilde{X})$ are in a natural way right $\pi_1(X)$–modules, with the right action on $C_*(\tilde{X})$ defined via $\sigma \cdot g := g^{-1}\sigma$, for $\sigma \in C_*(\tilde{X})$. We can form by tensoring the chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} V$, which is a complex of R–modules. Now define $H_i(X;V) := H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} V)$. The isomorphism type of the R–module $H_i(X;V)$ does not depend on the choice of the base point, in fact it only depends on the homotopy type of X and the isomorphism type of the representation.

In this paper we will also frequently consider twisted homology for a finitely generated group Γ; its definition can be reduced to the one above by looking at the twisted homology of the Eilenberg-Maclane space $K(\Gamma, 1)$.

The most common type of presentation we consider in this paper is as follows: Let X be a topological space, $\alpha : \pi_1(X) \to G$ a homomorphism to a group G and $H \subset G$ a subgroup of finite index. Then we get a natural action of $\pi_1(X)$ on $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}[G/H])$ by left–multiplication, which gives rise to the homology groups $H_i(X;\mathbb{Z}[G/H])$.

We will now study the $\mathbb{Z}[\pi_1(X)]$–module $\mathbb{Z}[G/H]$ in more detail. We write $C := \alpha(\pi_1(X))$. Consider the set of double cosets $C\backslash G/H$. By definition $g, g' \in G$ represent the same equivalence class if and only if there exist $c, c' \in C$ and $h, h' \in H$ such that $cgh = c'g'h'$. Note that $g_1, \dots, g_k \in G$ are a complete set of representatives of $C\backslash G/H$ if and only if G is the disjoint union of $Cg_1H, \dots, Cg_kH$. The first part of the following lemma is an immediate consequence of [Br94, II.5.2.], the second part follows either from Shapiro's lemma or a straightforward calculation.

Lemma 2.1. *Let $g_1, \dots, g_k \in G$ be a set of representatives for the equivalence classes $C\backslash G/H$. For $i = 1, \dots, k$ write $\tilde{C}_i = C \cap g_iHg_i^{-1}$. We then have the following isomorphisms of left $\mathbb{Z}[C]$–modules:*

$$\mathbb{Z}[G/H] \cong \bigoplus_{i=1}^{k} \mathbb{Z}[C/\tilde{C}_i].$$

In particular $H_0(X;\mathbb{Z}[G/H])$ is a free abelian group of rank $k = |C\backslash G/H|$.

2.2. **Induced maps on low dimensional homology groups.** In this section we will give criteria when maps between groups give rise to isomorphisms between low dimensional twisted homology groups. We start out with a study of the induced maps on 0–th twisted homology groups.

Lemma 2.2. *Let $\varphi : A \to B$ be a monomorphism of finitely generated groups. Suppose that B is a subgroup of a group π and let $\tilde{\pi} \subset \pi$ be a subgroup of finite index. Let $g_1, \dots, g_k \in \pi$ be a set of representatives for the equivalence classes $B\backslash \pi/\tilde{\pi}$. For $i = 1, \dots, k$ we write $\tilde{B}_i = B \cap g_i\tilde{\pi}g_i^{-1}$ and $\tilde{A}_i = \varphi^{-1}(\tilde{B}_i)$. Then*

$$\varphi_* : H_0(A;\mathbb{Z}[\pi/\tilde{\pi}]) \to H_0(B;\mathbb{Z}[\pi/\tilde{\pi}])$$

is an epimorphism of free abelian groups and it is an isomorphism if and only if $\varphi : A/\tilde{A}_i \to B/\tilde{B}_i$ is a bijection for any i.

Proof. It is well–known that the induced map on 0–th twisted homology groups is always surjective (cf. e.g. [HS97, Section 6]) and by Lemma 2.1 both groups are free abelian groups. Now note that without loss of generality we can assume that $A \subset B$ and that φ is the inclusion map. It follows from Lemma 2.1 that $H_0(B;\mathbb{Z}[\pi/\tilde{\pi}])$ is a free abelian group of rank $k = |B\backslash\pi/\tilde{\pi}|$. By the same Lemma we also have

$$\mathbb{Z}[\pi/\tilde{\pi}] \cong \bigoplus_{i=1}^{k} \mathbb{Z}[B/\tilde{B}_i]$$

as left $\mathbb{Z}[B]$–modules and hence also as left $\mathbb{Z}[A]$–modules. By applying Lemma 2.1 to the $\mathbb{Z}[A]$–modules $\mathbb{Z}[B/\tilde{B}_i]$ we see that $H_0(A;\mathbb{Z}[\pi/\tilde{\pi}])$ is a free abelian group of rank k if and only if $|A\backslash B/\tilde{B}_i| = 1$ for any i. It is straightforward to see that this is equivalent to $A/\tilde{A}_i \to B/\tilde{B}_i$ being a bijection for any i. □

We will several times make use of the following corollary.

Corollary 2.3. *Let $\varphi : A \to B$ be a monomorphism of finitely generated groups. Let $\beta : B \to G$ be a homomorphism to a finite group. Then*

$$\varphi_* : H_0(A;\mathbb{Z}[G]) \to H_0(B;\mathbb{Z}[G])$$

is an epimorphism of free abelian groups and it is an isomorphism if and only if

$$Im\{A \to B \to G\} = Im\{B \to G\}.$$

Proof. Let $\pi' = B \times G$ and $\tilde{\pi}' = B$. We can then apply Lemma 2.2 to $A' = A, B' = \{(g,\beta(g)) \,|\, g \in B\} \subset \pi'$ and $\varphi'(a) = (\varphi(a),\beta(\varphi(a)), a \in A'$. It is straightforward to verify that the desired equivalence of statements follows. □

We now turn to the question when group homomorphisms induce isomorphisms of the 0–th and the first twisted homology groups at the same time.

Lemma 2.4. *Let $\varphi : A \to B$ be a monomorphism of finitely generated groups. Suppose that B is a subgroup of a group π and let $\tilde{\pi} \subset \pi$ be a subgroup of finite index. Let $g_1,\dots,g_k \in \pi$ be a set of representatives for the equivalence classes $B\backslash\pi/\tilde{\pi}$. For $i = 1,\dots,k$ we write $\tilde{B}_i = B \cap g_i\tilde{\pi}g_i^{-1}$ and $\tilde{A}_i = \varphi^{-1}(\tilde{B}_i)$. Then*

$$\varphi_* : H_i(A;\mathbb{Z}[\pi/\tilde{\pi}]) \to H_i(B;\mathbb{Z}[\pi/\tilde{\pi}])$$

is an isomorphism for $i = 0$ and $i = 1$ if and only if the following two conditions are satisfied:

1. *$\varphi : A/\tilde{A}_i \to B/\tilde{B}_i$ is a bijection for any i,*
2. *$\varphi : A/[\tilde{A}_i,\tilde{A}_i] \to B/[\tilde{B}_i,\tilde{B}_i]$ is a bijection for any i.*

Proof. Without loss of generality we can assume that $A \subset B$ and that φ is the inclusion map. By Lemmas 2.1 and 2.2 it suffices to show for any i the following: If $A/\tilde{A}_i \to B/\tilde{B}_i$ is a bijection, then the map $H_1(A;\mathbb{Z}[B/\tilde{B}_i]) \to H_1(B;\mathbb{Z}[B/\tilde{B}_i])$ is an isomorphism if and only if $\varphi : A/[\tilde{A}_i,\tilde{A}_i] \to B/[\tilde{B}_i,\tilde{B}_i]$ is a bijection.

Using the above and using Shapiro's Lemma we can identify

$$\begin{array}{rcll} H_1(A;\mathbb{Z}[B/\tilde{B}_i]) = H_1(A;\mathbb{Z}[A/\tilde{A}_i]) &=& \tilde{A}_i/[\tilde{A}_i,\tilde{A}_i] & \quad\text{and} \\ H_1(B;\mathbb{Z}[B/\tilde{B}_i]) &=& \tilde{B}_i/[\tilde{B}_i,\tilde{B}_i]. & \end{array}$$

Note that $A/\tilde{A}_i, B/\tilde{B}_i, A/[\tilde{A}_i,\tilde{A}_i]$ and $B/[\tilde{B}_i,\tilde{B}_i]$ are in general not groups, but we can view them as pointed sets. We now consider the following commutative diagram of exact sequences of pointed sets:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(A;\mathbb{Z}[B/\tilde{B}_i]) & \longrightarrow & A/[\tilde{A}_i,\tilde{A}_i] & \longrightarrow & A/\tilde{A}_i & \longrightarrow & 1 \\ & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\ 0 & \longrightarrow & H_1(B;\mathbb{Z}[B/\tilde{B}_i]) & \longrightarrow & B/[\tilde{B}_i,\tilde{B}_i] & \longrightarrow & B/\tilde{B}_i & \longrightarrow & 1. \end{array}$$

Recall that the map on the right is a bijection. It now follows from the 5–lemma for exact sequences of pointed sets that the middle map is a bijection if and only if the left hand map is a bijection. □

We will several times make use of the following corollary which can be deduced from Lemma 2.4 the same way as Corollary 2.3 is deduced from Lemma 2.2.

Corollary 2.5. *Let $\varphi : A \to B$ be a monomorphism of finitely generated groups, and assume we are given a homomorphism $\beta : B \to G$ to a finite group G. Then*

$$\varphi_* : H_i(A;\mathbb{Z}[G]) \to H_i(B;\mathbb{Z}[G]), \quad i = 0,1$$

is an isomorphism if and only if the following two conditions hold:

(1) *$Im\{A \to B \to G\} = Im\{B \to G\}$,*
(2) *φ induces an isomorphism*

$$A/[Ker(\beta\circ\varphi), Ker(\beta\circ\varphi)] \to B/[Ker(\beta), Ker(\beta)].$$

Under extra conditions we can also give a criterion for a map between groups to induce an isomorphism of second homology groups.

Lemma 2.6. *Let $\varphi : A \to B$ be a homomorphism between two groups such that $X = K(A,1)$ and $Y = K(B,1)$ are finite 2–complexes with vanishing Euler characteristic. Let $\beta : B \to G$ be a homomorphism to a finite group such that*

$$\varphi_* : H_i(A;\mathbb{Z}[G]) \to H_i(B;\mathbb{Z}[G]), \quad i = 0,1$$

is an isomorphism, then

$$\varphi_* : H_2(A;\mathbb{Z}[G]) \to H_2(B;\mathbb{Z}[G])$$

is also an isomorphism.

Proof. We can and will view X as a subcomplex of Y. It suffices to show that $H_2(Y,X;\mathbb{Z}[G]) = 0$. Note that our assumption implies that $H_i(Y,X;\mathbb{Z}[G]) = 0$ for $i = 0,1$. Now note that $H_2(Y,X;\mathbb{Z}[G])$ is a submodule of $C_2(Y,X;\mathbb{Z}[G])$, in particular $H_2(Y,X;\mathbb{Z}[G])$ is a free $\mathbb{Z}$–module. We therefore only have to show that $\operatorname{rank} H_2(Y,X;\mathbb{Z}[G]) = 0$. Now note that

$$\begin{aligned}\operatorname{rank} H_2(Y,X;\mathbb{Z}[G]) &= \operatorname{rank} H_2(Y,X;\mathbb{Z}[G]) - \operatorname{rank} H_1(Y,X;\mathbb{Z}[G]) + \operatorname{rank} H_0(Y,X;\mathbb{Z}[G]) \\ &= |G|\chi(Y,X) \\ &= |G|(\chi(Y)-\chi(X)) \\ &= 0.\end{aligned}$$

□

We conclude this section with the following lemma.

Lemma 2.7. *Let $\varphi : A \to B$ be a homomorphism. Let $\hat{B} \subset \tilde{B} \subset B$ be two subgroups. Suppose that $\hat{B} \subset B$ is normal. We write $\hat{A} := \varphi^{-1}(\hat{B})$ and $\tilde{A} := \varphi^{-1}(\tilde{B})$. Assume that*

$$\varphi : A/\hat{A} \to B/\hat{B} \text{ and } \varphi : A/[\hat{A},\hat{A}] \to B/[\hat{B},\hat{B}]$$

are bijections, then

$$\varphi : A/\tilde{A} \to B/\tilde{B} \text{ and } \varphi : A/[\tilde{A},\tilde{A}] \to B/[\tilde{B},\tilde{B}]$$

are also bijections.

Proof. In the following let $n = 0$ or $n = 1$. Suppose that $\varphi : A/\hat{A}^{(n)} \to B/\hat{B}^{(n)}$ is a bijection. Note that $\hat{A}^{(n)} \subset A$ and $\hat{B}^{(n)} \subset B$ are normal, in particular $\varphi : A/\hat{A}^{(n)} \to B/\hat{B}^{(n)}$ is in fact an isomorphism. We have to show that $\varphi : A/\tilde{A}^{(n)} \to B/\tilde{B}^{(n)}$ is a bijection.

Claim. The map φ induces a bijection $\tilde{A}^{(n)}/\hat{A}^{(n)} \to \tilde{B}^{(n)}/\hat{B}^{(n)}$.

We write $\overline{A} := A/\hat{A}^{(n)}$, $\overline{B} := B/\hat{B}^{(n)}$ and we denote by $\overline{\varphi} : \overline{A} \to \overline{B}$ the induced map which by assumption is an isomorphism. We denote by $\overline{H}$ the subgroup $\tilde{B}/\hat{B}^{(n)} \subset \overline{B}$. Note that $\overline{\varphi}$ restricts to isomorphisms $\overline{\varphi}^{-1}(\overline{H}) \to \overline{H}$ and $\overline{\varphi}^{-1}(\overline{H}^{(n)}) \to \overline{H}^{(n)}$. Since $\overline{\varphi}^{-1}$ is an isomorphism it follows that $\big(\overline{\varphi}^{-1}(\overline{H})\big)^{(n)} = \overline{\varphi}^{-1}\big(\overline{H}^{(n)}\big)$. Now recall that $\overline{H} = \tilde{B}/\hat{B}^{(n)}$, hence $\overline{H}^{(n)} = \tilde{B}^{(n)}/\hat{B}^{(n)}$. We clearly have $\overline{\varphi}^{-1}(\overline{H}) = \tilde{A}/\hat{A}^{(n)}$ and therefore $\overline{\varphi}^{-1}(\overline{H})^{(n)} = \tilde{A}^{(n)}/\hat{A}^{(n)}$. This shows that the isomorphism $\overline{\varphi} : \overline{\varphi}^{-1}\big(\overline{H}^{(n)}\big) \to \overline{H}^{(n)}$ is precisely the desired isomorphism $\tilde{A}^{(n)}/\hat{A}^{(n)} \to \tilde{B}^{(n)}/\hat{B}^{(n)}$. This concludes the proof of the claim.

Now consider the following commutative diagram of short exact sequences of pointed sets:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{A}^{(n)}/\hat{A}^{(n)} & \longrightarrow & A/\hat{A}^{(n)} & \longrightarrow & A/\tilde{A}^{(n)} & \longrightarrow & 1 \\ & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\ 1 & \longrightarrow & \tilde{B}^{(n)}/\hat{B}^{(n)} & \longrightarrow & B/\hat{B}^{(n)} & \longrightarrow & B/\tilde{B}^{(n)} & \longrightarrow & 1. \end{array}$$

The middle vertical map is a bijection by assumption and we just verified that the vertical map on the left is a bijection. It now follows from the 5–Lemma for exact sequences of pointed sets that the vertical map on the right is also a bijection. □

2.3. **Twisted Alexander polynomials.** In this section we are going to recall the definition of twisted Alexander polynomials. These were introduced, for the case of knots, by Xiao-Song Lin in 1990 (published in [Li01]), and his definition was later generalized to 3–manifolds by Wada [Wa94], Kirk–Livingston [KL99] and Cha [Ch03].

Let N be a compact manifold. Let R be a commutative, Noetherian unique factorization domain (in our applications $R = \mathbb{Z}$ or $R = \mathbb{F}_p$, the finite field with p elements) and V a finite free R–module Let $\alpha : \pi_1(N) \to \operatorname{Aut}_R(V)$ a representation and let $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ a nontrivial element. We write $V \otimes_R R[t^{\pm 1}] =: V[t^{\pm 1}]$. Then α and ϕ give rise to a representation $\alpha \otimes \phi : \pi_1(N) \to \operatorname{Aut}_{R[t^{\pm 1}]}(V[t^{\pm 1}])$ as follows:

$$\big((\alpha \otimes \phi)(g)\big)(v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)}p),$$

where $g \in \pi_1(N), v \otimes p \in V \otimes_R R[t^{\pm 1}] = V[t^{\pm 1}]$.

Note that N is homotopy equivalent to a finite CW–complex, which, by abuse of notation, we also denote by N. Then we consider $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} V[t^{\pm 1}]$ which is a complex of finitely generated $R[t^{\pm 1}]$–modules. Since $R[t^{\pm 1}]$ is Noetherian it follows that for any i the $R[t^{\pm 1}]$–module $H_i(N; V[t^{\pm 1}])$ is a finitely presented $R[t^{\pm 1}]$–module. This means $H_i(N; V[t^{\pm 1}])$ has a free $R[t^{\pm 1}]$–resolution

$$R[t^{\pm 1}]^{r_i} \xrightarrow{S_i} R[t^{\pm 1}]^{s_i} \to H_i(N; V[t^{\pm 1}]) \to 0.$$

Without loss of generality we can assume that $r_i \geq s_i$.

Definition. The *i–th twisted Alexander polynomial* of (N, α, ϕ) is defined to be the order of the $R[t^{\pm 1}]$–module $H_i(N; V[t^{\pm 1}])$, i.e. the greatest common divisor (which exists since $R[t^{\pm 1}]$ is a UFD as well) of the $s_i \times s_i$ minors of the $s_i \times r_i$–matrix S_i. It is denoted by $\Delta^{\alpha}_{N,\phi,i} \in R[t^{\pm 1}]$.

Note that $\Delta^{\alpha}_{N,\phi,i} \in R[t^{\pm 1}]$ is well–defined up to a unit in $R[t^{\pm 1}]$, i.e. up to an element of the form rt^i where r is a unit in R and $i \in \mathbb{Z}$. We say that $f \in R[t^{\pm 1}]$ is *monic* if its top coefficient is a unit in R. Given a nontrivial $f = \sum_{i=r}^{s} a_s t^i$ with $a_r \neq 0, a_s \neq 0$ we write $\deg f = s - r$. For $f = 0$ we write $\deg(f) = -\infty$. Note that $\deg \Delta^{\alpha}_{N,\phi,i}$ is well–defined.

We now write $\pi = \pi_1(N)$. If we are given a homomorphism $\alpha : \pi \to G$ to a finite group, then this gives rise to a finite dimensional representation of π, that we will denote by $\alpha : \pi \to \mathrm{Aut}_R(R[G])$ as well. In the case that we have a finite index subgroup $\tilde{\pi} \subset \pi$ we get a finite dimensional representation $\pi \to \mathrm{Aut}_R(R[\pi/\tilde{\pi}])$ given by left–multiplication. When $R = \mathbb{Z}$, the resulting twisted Alexander polynomials will be denoted by $\Delta^{\pi/\tilde{\pi}}_{N,\phi,i} \in \mathbb{Z}[t^{\pm1}]$, while for $R = \mathbb{F}_p$ we will use the notation $\Delta^{\pi/\tilde{\pi},p}_{N,\phi,i} \in \mathbb{F}_p[t^{\pm1}]$. See [FV08a] for the relation between these polynomials.

Finally, in the case that $\alpha : \pi \to \mathrm{GL}(1, \mathbb{Z})$ is the trivial representation we drop the α from the notation, and in the case that $i = 1$ we drop the subscript "$,1$" from the notation.

We summarize some of the main properties of twisted Alexander polynomials in the following lemma. It is a consequence of [FV08a, Lemma 3.3 and 3.4] and [FK06, Proposition 2.5].

Lemma 2.8. *Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ nontrivial and $\tilde{\pi} \subset \pi := \pi_1(N)$ a finite index subgroup. Denote by $\phi_{\tilde{\pi}}$ the restriction of ϕ to $\tilde{\pi}$, then the following hold:*

(1) $\Delta^{\pi/\tilde{\pi}}_{N,\phi,0} = (1 - t^{div\,\phi_{\tilde{\pi}}})$,
(2) *if* $\Delta^{\pi/\tilde{\pi}}_{N,\phi,1} \neq 0$, *then* $\Delta^{\pi/\tilde{\pi}}_{N,\phi,2} = (1 - t^{div\,\phi_{\tilde{\pi}}})^{b_3(N)}$,
(3) $\Delta^{\pi/\tilde{\pi}}_{N,\phi,i} = 1$ *for any* $i \geq 3$.

Assume we also have a subgroup π' with $\tilde{\pi} \subset \pi' \subset \pi$. Denote the covering of N corresponding to π' by N' and denote by ϕ' the restriction of ϕ to π', then

$$\Delta^{\pi/\tilde{\pi}}_{N,\phi,i} = \Delta^{\pi'/\tilde{\pi}}_{N',\phi',i}$$

for any i. Finally note that the statements of the lemma also hold for the polynomial $\Delta^{\pi/\tilde{\pi},p}_{N,\phi,i} \in \mathbb{F}_p[t^{\pm1}]$.

We also recall the following well–known result (cf. e.g. [Tu01]).

Lemma 2.9. *Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ nontrivial and $\tilde{\pi} \subset \pi := \pi_1(N)$ a finite index subgroup. Then given i the following are equivalent:*

(1) $\Delta^{\pi/\tilde{\pi}}_{N,\phi,i} \neq 0$,
(2) $H_i(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}])$ *is* $\mathbb{Z}[t^{\pm1}]$*–torsion,*
(3) $H_i(N;\mathbb{Q}[\pi/\tilde{\pi}][t^{\pm1}])$ *is* $\mathbb{Q}[t^{\pm1}]$*–torsion,*
(4) *the rank of the abelian group* $H_i(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}])$ *is finite.*

In fact if any of the four conditions holds, then

$$deg\Delta^{\pi/\tilde{\pi}}_{N,\phi,i} = rank\, H_i(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}]) = dim\, H_i(N;\mathbb{Q}[\pi/\tilde{\pi}][t^{\pm1}]).$$

2.4. **Completions of groups.** Throughout the paper it is convenient to use the language of completions of groups. Although the proof of Theorem 1.2 does not explicitly require this terminology, group completions are the natural framework for these results. We recall here the definitions and some basic facts, we refer to [LS03, Window 4] and [Wi98, RZ00] for proofs and for more information.

Let $\mathcal{C}$ be a variety of groups (cf. [RZ00, p. 20] for the definition). Examples of varieties of pertinence to this paper are given by any one of the following:

(1) finite groups;
(2) p–groups for a prime p;
(3) the variety $\mathcal{FS}(n)$ of finite solvable groups of derived length at most n;
(4) the variety $\mathcal{FS}$ of finite solvable groups.

In the following we equip a finitely generated group A with its *pro–$\mathcal{C}$ topology*, this topology is the translation invariant topology uniquely defined by taking as a fundamental system of neighborhoods of the identity the collection of all normal subgroups of A such that the quotient lies in $\mathcal{C}$. Note that in particular all groups in $\mathcal{C}$ are endowed with the discrete topology.

Given a group A denote by $\hat{A}_{\mathcal{C}}$ its pro–$\mathcal{C}$ completion, i.e. the inverse limit

$$\hat{A}_{\mathcal{C}} = \varprojlim A/A_i$$

where A_i runs through the inverse system determined by the collection of all normal subgroups of A such that $A/A_i \in \mathcal{C}$. Then $\hat{A}_{\mathcal{C}}$, which we can view as a subgroup of the direct product of all A/A_i, inherits a natural topology. Henceforth by homomorphisms between groups we will mean a homomorphism which is continuous with respect to the above topologies. Using the standard convention we refer to the pro–$\mathcal{FS}$ completion of a group as the prosolvable completion.

Note that by the assumption that $\mathcal{C}$ is a variety, the pro–$\mathcal{C}$ completion is a covariant functor, i.e. given $\varphi : A \to B$ we get an induced homomorphism $\hat{\varphi} : \hat{A}_{\mathcal{C}} \to \hat{B}_{\mathcal{C}}$.

A group A is called *residually* $\mathcal{C}$ if for any nontrivial $g \in A$ there exists a homomorphism $\alpha : A \to G$ where $G \in \mathcal{C}$ such that $\alpha(g) \neq e$. It is easily seen that A is residually $\mathcal{C}$ if and only if the map $A \to \hat{A}_{\mathcal{C}}$ is injective. In particular, if we are given a homomorphism $\varphi : A \to B$ between residually $\mathcal{C}$ groups A, B such that $\hat{\varphi} : \hat{A}_{\mathcal{C}} \to \hat{B}_{\mathcal{C}}$ is an injection, then it follows from the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ \hat{A}_{\mathcal{C}} & \xrightarrow{\hat{\varphi}} & \hat{B}_{\mathcal{C}} \end{array}$$

that φ is injective as well.

The following well–known lemma gives sufficient and necessary conditions for a homomorphism $\varphi : A \to B$ to induce an isomorphism of pro–$\mathcal{C}$ completions.

Lemma 2.10. *Let $\mathcal{C}$ be a variety of groups and assume that there is a homomorphism $\varphi : A \to B$. Then the following are equivalent:*

1. $\hat{\varphi} : \hat{A}_{\mathcal{C}} \to \hat{B}_{\mathcal{C}}$ *is an isomorphism,*
2. *for any $G \in \mathcal{C}$ the induced map*

$$\varphi^* : Hom(B, G) \to Hom(A, G)$$

is a bijection.

We also note the following well–known lemma.

Lemma 2.11. *Let $\mathcal{C}$ be an extension–closed variety and let $\varphi : A \to B$ a homomorphism of finitely generated groups which induces an isomorphism of pro–$\mathcal{C}$ completions. Then for any homomorphism $\beta : B \to G$ to a $\mathcal{C}$–group the restriction of φ to $Ker(\beta \circ \varphi) \to Ker(\beta)$ induces an isomorphism of pro–$\mathcal{C}$ completions.*

When a homomorphism $\varphi : A \to B$ of finitely generated groups induces an isomorphism of their pro–$\mathcal{C}$ completions, then we have a relation of the twisted homology with coefficients determined by $\mathcal{C}$–groups. More precisely, we have the following.

Lemma 2.12. *Let $\mathcal{C}$ be a variety of groups and let $\varphi : A \to B$ be a homomorphism of finitely generated groups which induces an isomorphism of pro–$\mathcal{C}$ completions. Then for any homomorphism $\beta : B \to G$ to a $\mathcal{C}$–group the map $\varphi_* : H_0(A; \mathbb{Z}[G]) \to H_0(B; \mathbb{Z}[G])$ is an isomorphism. Furthermore, if $\mathcal{C}$ is an extension–closed variety containing all finite abelian groups, the map $\varphi_* : H_1(A; \mathbb{Z}[G]) \to H_1(B; \mathbb{Z}[G])$ is an isomorphism.*

Proof. Observe that, by Corollary 2.3, the first part of the statement is equivalent to the assertion that, for any element $\beta \in \operatorname{Hom}(B, G)$,

$$\operatorname{Im}\{\beta \circ \varphi : A \to G\} = \operatorname{Im}\{\beta : B \to G\}.$$

Without loss of generality, we can reduce the proof of this isomorphism to the case where β is surjective. Denote $\alpha = \beta \circ \varphi \in \operatorname{Hom}(A, G)$. Assume to the contrary that $\alpha(A) \subsetneq G$; then $\alpha \in \operatorname{Hom}(A, \alpha(A)) \subset \operatorname{Hom}(A, G)$ and as $\alpha(A) \in \mathcal{C}$ there exists by hypothesis a map $\beta' \in \operatorname{Hom}(B, \alpha(A)) \subset \operatorname{Hom}(B, G)$ such that $\alpha = \beta' \circ \varphi$. Now the two maps $\beta, \beta' \in \operatorname{Hom}(B, G)$ (that must differ as they have different image) induce the same map $\alpha \in \operatorname{Hom}(A, G)$, contradicting the bijectivity of $\operatorname{Hom}(B, G)$ and $\operatorname{Hom}(A, G)$.

We now turn to the proof of the second part of the statement. Let $\beta : B \to G$ a homomorphism to a $\mathcal{C}$–group. Again, without loss of generality, we can assume that $\beta : B \to G$ is surjective. Note that by the above the homomorphism $\beta \circ \varphi : A \to G$ is surjective as well. We now write $B' = \operatorname{Ker}(\beta)$ and $A' = \operatorname{Ker}(\beta \circ \varphi)$. By Shapiro's

Lemma, we have the commutative diagram

$$\begin{array}{ccc} H_1(A';\mathbb{Z}) & \xrightarrow{\cong} & H_1(A;\mathbb{Z}[G]) \\ \downarrow & & \downarrow \\ H_1(B';\mathbb{Z}) & \xrightarrow{\cong} & H_1(B;\mathbb{Z}[G]). \end{array}$$

The claim amounts therefore to showing that the map $\varphi_* : H_1(A';\mathbb{Z}) \to H_1(B';\mathbb{Z})$ is an isomorphism. As A and B are finitely generated, A' and B' are finitely generated as well. When $\mathcal{C}$ is extension closed, and contains all finite abelian groups, Lemma 2.11 asserts that the map φ induces a bijection between $\mathrm{Hom}(B',\Gamma)$ and $\mathrm{Hom}(A',\Gamma)$ for any finite abelian group Γ; the desired isomorphism easily follows. □

3. Monic twisted Alexander polynomials and solvable groups

The aim of this section is to prove Proposition 1.7.

3.1. **Preliminary results.** We will often make use of the following proposition (cf. [McM02, Section 4 and Proposition 6.1]).

Proposition 3.1. *Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ a primitive class. If $\Delta_{N,\phi} \neq 0$, then there exists a connected Thurston norm minimizing surface Σ dual to ϕ.*

Given a connected oriented surface $\Sigma \subset N$ we will adopt the following conventions for the rest of the paper. We choose a neighborhood $\Sigma \times [-1,1] \subset N$ and write $\nu\Sigma = \Sigma \times (-1,1)$. Let $M := N \setminus \nu\Sigma$; we will write $\Sigma^{\pm} = \Sigma \times \{\pm 1\} \subset \partial M$, and we will denote the inclusion induced maps $\Sigma \to \Sigma^{\pm} \subset M$ by $\iota_{\pm}$.

We pick a base point in M and endow N with the same base point. Also, we pick a base point for Σ and endow $\Sigma^{\pm}$ with the corresponding base points. With these choices made, we will write $A = \pi_1(\Sigma)$ and $B = \pi_1(M)$. We also pick paths in M connecting the base point of M with the base points of Σ^- and Σ^+. We now have inclusion induced maps $\iota_{\pm} : A \to B$ for either inclusion of Σ in M and, using the constant path, a map $\pi_1(M) \to \pi_1(N)$. Under the assumption that Σ is incompressible (in particular, whenever Σ is Thurston norm minimizing) these maps are injective. Since M and N have the same base point we can view B canonically as a subgroup of $\pi_1(N)$.

Before we state the first proposition we have to introduce a few more definitions. Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ a nontrivial class. Let $\tilde{\pi} \subset \pi$ be a finite index subgroup. We denote by $\phi_{\tilde{\pi}}$ the restriction of $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi,\mathbb{Z})$ to $\tilde{\pi}$. Note that $\phi_{\tilde{\pi}}$ is necessarily non–trivial. We say that $\tilde{\pi} \subset \pi$ has Property (M) if the twisted Alexander polynomial $\Delta_{N,\phi}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$ is monic and if

$$\deg(\Delta_{N,\phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}]\, \|\phi\|_T + (1 + b_3(N))\mathrm{div}\phi_{\tilde{\pi}}$$

holds. Note that a pair (N,ϕ) satisfies Condition $(*)$ if and only if Property (M) is satisfied by all normal subgroups of $\pi_1(N)$.

The following proposition is the key tool for translating information on twisted Alexander polynomials into information on the maps $\iota_\pm : A \to B$. The proposition is well known in the classical case. In the case of normal subgroups a proof for the 'only if' direction of the proposition is given by combining [FV08a, Section 8] with [FV08b, Section 4].

Proposition 3.2. *Let N be a 3–manifold with empty or toroidal boundary with $N \neq S^1\times D^2, N\neq S^1\times S^2$. Let $\phi \in H^1(N;\mathbb{Z})$ a primitive class which is dual to a connected Thurston norm minimizing surface Σ. Let $\tilde{\pi}\subset\pi$ be a finite index subgroup. Then $\tilde{\pi}$ has Property (M) if and only if the maps $\iota_\pm : H_i(A;\mathbb{Z}[\pi/\tilde{\pi}]) \to H_i(B;\mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms for $i=0,1$.*

Proof. Let $R=\mathbb{Z}$ or $R=\mathbb{F}_p$ with p a prime. We have canonical isomorphisms $H_i(\Sigma;R[\pi/\tilde{\pi}])\cong H_i(A;R[\pi/\tilde{\pi}])$ and $H_i(M;R[\pi/\tilde{\pi}])\cong H_i(B;R[\pi/\tilde{\pi}])$ for $i=0,1$. It follows from [FK06, Proposition 3.2] that splitting N along Σ gives rise to the following Mayer–Vietoris type exact sequence

$$\begin{array}{l} \dots \qquad\qquad \to H_2(N;R[\pi/\tilde{\pi}][t^{\pm1}]) \\ \to H_1(A;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}] \xrightarrow{t\iota_+-\iota_-} H_1(B;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}] \to H_1(N;R[\pi/\tilde{\pi}][t^{\pm1}]) \to \\ \to H_0(A;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}] \xrightarrow{t\iota_+-\iota_-} H_0(B;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}] \to H_0(N;R[\pi/\tilde{\pi}][t^{\pm1}]) \to 0. \end{array}$$

which we refer to as the Mayer–Vietoris sequence of (N,Σ) with $R[\pi/\tilde{\pi}][t^{\pm1}]$–coefficients. First note that by Shapiro's lemma the groups $H_i(A;R[\pi/\tilde{\pi}])$ are the i–th homology with R–coefficients of a (possibly) disconnected surface. It follows that $H_i(A;R[\pi/\tilde{\pi}])$ is a free R–module, in particular the $R[t^{\pm1}]$–modules $H_i(A;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}]$ are free $R[t^{\pm1}]$–modules. We will several times make use of the observation that if $H_i(N;R[\pi/\tilde{\pi}][t^{\pm1}])$ is $R[t^{\pm1}]$–torsion, then the map $H_i(N;R[\pi/\tilde{\pi}][t^{\pm1}])\to H_{i-1}(A;R[\pi/\tilde{\pi}])\otimes R[t^{\pm1}]$ is necessarily zero.

We first assume that $\tilde{\pi}$ has Property (M). Since $\Delta_{N,\phi}^{\pi/\tilde{\pi}}\neq 0$ we have that the module $H_1(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}])\otimes_{\mathbb{Z}[t^{\pm1}]}\mathbb{Q}(t)$ is trivial. Note that by Lemma 2.9 we have that $H_0(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}])$ is also $\mathbb{Z}[t^{\pm1}]$–torsion. We now consider the Mayer–Vietoris sequence of (N,Σ) with $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm1}]$–coefficients. Tensoring the exact sequence with $\mathbb{Q}(t)$ we see that

$$\begin{array}{rcl} \operatorname{rank}_{\mathbb{Z}}(H_0(A;\mathbb{Z}[\pi/\tilde{\pi}])) &=& \operatorname{rank}_{\mathbb{Q}(t)}(H_0(A;\mathbb{Z}[\pi/\tilde{\pi}])\otimes_{\mathbb{Z}}\mathbb{Q}(t)) \\ &=& \operatorname{rank}_{\mathbb{Q}(t)}(H_0(B;\mathbb{Z}[\pi/\tilde{\pi}])\otimes_{\mathbb{Z}}\mathbb{Q}(t)) = \operatorname{rank}_{\mathbb{Z}}(H_0(B;\mathbb{Z}[\pi/\tilde{\pi}])). \end{array}$$

Using this observation and using Lemma 2.2 we see that the maps $\iota_\pm : H_0(A;\mathbb{Z}[\pi/\tilde{\pi}])\to H_0(B;\mathbb{Z}[\pi/\tilde{\pi}])$ are epimorphism between free abelian groups of the same rank. Hence the maps are in fact isomorphisms.

In order to prove that the maps $\iota_\pm : H_1(A;\mathbb{Z}[\pi/\tilde{\pi}])\to H_1(B;\mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms we first consider the following claim.

Claim. $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}])$ and $H_1(B;\mathbb{Z}[\pi/\tilde{\pi}])$ are free abelian groups of the same rank.

Let p be a prime. We consider the Mayer–Vietoris sequence of (N,Σ) with $\mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}]$–coefficients. Denote by $\Delta_{N,\phi}^{\pi/\tilde{\pi},p} \in \mathbb{F}_p[t^{\pm 1}]$ the twisted Alexander polynomial with coefficients in $\mathbb{F}_p$. It follows from $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ monic and from [FV08a, Proposition 6.1] that $\Delta_{N,\phi}^{\pi/\tilde{\pi},p} \neq 0 \in \mathbb{F}_p[t^{\pm 1}]$. Furthermore by Lemma 2.8 we have that $\Delta_{N,\phi,2}^{\pi/\tilde{\pi},p} \neq 0 \in \mathbb{F}_p[t^{\pm 1}]$. In particular $H_i(N;\mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}])$ is $\mathbb{F}_p[t^{\pm 1}]$–torsion for $i=1,2$. It follows from the fact that $H_i(A;\mathbb{F}_p[\pi/\tilde{\pi}]) \otimes_{\mathbb{F}_p} \mathbb{F}_p[t^{\pm 1}]$ is a free $\mathbb{F}_p[t^{\pm 1}]$–module and the above observation that $H_i(N;\mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}])$ is $\mathbb{F}_p[t^{\pm 1}]$–torsion for $i=1,2$ that the Mayer–Vietoris sequence gives rise to the following short exact sequence

$$0 \to H_1(A;\mathbb{F}_p[\pi/\tilde{\pi}])\otimes\mathbb{F}_p[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B;\mathbb{F}_p[\pi/\tilde{\pi}])\otimes\mathbb{F}_p[t^{\pm 1}] \to H_1(N;\mathbb{F}_p[\pi/\tilde{\pi}][t^{\pm 1}]) \to 0.$$

Tensoring with $\mathbb{F}_p(t)$ we see that in particular $H_1(A;\mathbb{F}_p[\pi/\tilde{\pi}]) \cong H_1(B;\mathbb{F}_p[\pi/\tilde{\pi}])$ as $\mathbb{F}_p$–vector spaces. The homology group $H_0(A;\mathbb{Z}[\pi/\tilde{\pi}])$ is $\mathbb{Z}$–torsion free. It follows from the universal coefficient theorem applied to the complex of $\mathbb{Z}$–modules $C_*(\tilde{\Sigma}) \otimes_{\mathbb{Z}[A]} \mathbb{Z}[\pi/\tilde{\pi}]$ that

$$H_1(A;\mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H_1(A;\mathbb{F}_p[\pi/\tilde{\pi}])$$

for every prime p. The same statement holds for B. Combining our results we see that $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $H_1(B;\mathbb{Z}[\pi/\tilde{\pi}]) \otimes_{\mathbb{Z}} \mathbb{F}_p$ are isomorphic for any prime p. Since $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}])$ is free abelian it follows that $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}]) \cong H_1(B;\mathbb{Z}[\pi/\tilde{\pi}])$. This completes the proof of the claim.

In the following we equip the free $\mathbb{Z}$–modules $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}])$ and $H_1(B;\mathbb{Z}[\pi/\tilde{\pi}])$ with a choice of basis. We now study the Mayer–Vietoris sequence for (N,Σ) with $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]$–coefficients. Using an argument similar to the above we see that it gives rise to the following exact sequence

$$H_1(A;\mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B;\mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \to H_1(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) \to 0.$$

Since $H_1(A;\mathbb{Z}[\pi/\tilde{\pi}])$ and $H_1(B;\mathbb{Z}[\pi/\tilde{\pi}])$ are free abelian groups of the same rank it follows that the above exact sequence is a resolution of $H_1(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$ by free $\mathbb{Z}[t^{\pm 1}]$–modules and that $\Delta_{N,\phi}^{\pi/\tilde{\pi}} = \det(t\iota_+ - \iota_-)$. Recall that Property (M) states in particular that

$$\deg\Delta_{N,\phi}^{\pi/\tilde{\pi}} = |\pi/\tilde{\pi}|\,\|\phi\|_T + (1+b_3(N))\operatorname{div}\phi_{\tilde{\pi}}. \tag{1}$$

Recall that we assumed that $N \neq S^1\times D^2$ and $N \neq S^1\times S^2$, in particular $\chi(\Sigma)\leq 0$ and therefore $-\chi(\Sigma) = ||\phi||_T$. Writing $b_i = \operatorname{rank}_{\mathbb{Z}}(H_i(\Sigma;\mathbb{Z}[\pi/\tilde{\pi}])) = \operatorname{rank}_{\mathbb{Z}}(H_i(A;\mathbb{Z}[\pi/\tilde{\pi}]))$ a standard Euler characteristic argument now shows that

$$-b_0 + b_1 - b_2 = -|\pi/\tilde{\pi}|\chi(\Sigma) = |\pi/\tilde{\pi}|\cdot||\phi||_T.$$

By [FK06, Lemma 2.2] we have $b_i = \deg\Delta_{N,\phi,i}^{\pi/\tilde{\pi}}$ for $i=0$ and $i=2$. We also have $\deg\Delta_{N,\phi,0}^{\pi/\tilde{\pi}} = \operatorname{div}\phi_{\pi/\tilde{\pi}}$ and $\deg\Delta_{N,\phi,2}^{\pi/\tilde{\pi}} = b_3(N)\operatorname{div}\phi_{\tilde{\pi}}$ by Lemma 2.8. Combining these facts with (1) we conclude that $\deg\Delta_{N,\phi}^{\pi/\tilde{\pi}} = b_1$. So we now have $\deg(\det(t\iota_+ - \iota_-)) = b_1$.

Since ι_+ and ι_- are $b_1 \times b_1$ matrices over $\mathbb{Z}$ it now follows that $\det(\iota_+)$ equals the top coefficient of $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$, which by Property (M) equals ± 1. By the symmetry of twisted Alexander polynomials we have that the bottom coefficient of $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ also equals ± 1, we deduce that $\det(\iota_-) = \pm 1$. This shows that $\iota_+, \iota_- : H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \to H_1(B; \mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms. We thus showed that if $\tilde{\pi}$ has Property (M), then the maps $\iota : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \to H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms for $i = 0, 1$.

Now assume that we are given a finite index subgroup $\tilde{\pi} \subset \pi$ such that the maps $\iota_\pm : H_i(A; \mathbb{Z}[\pi/\tilde{\pi}]) \to H_i(B; \mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms for $i = 0, 1$. It follows from the assumption that $\iota_\pm : H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \to H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$ are isomorphisms that the map

$$H_0(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_0(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

is injective. In particular the Mayer–Vietoris sequence of (N, Σ) with $\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]$–coefficients gives rise to the following exact sequence

$$H_1(A; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \otimes \mathbb{Z}[t^{\pm 1}] \to H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) \to 0.$$

As above $H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$ is a free abelian group and by our assumption $H_1(B; \mathbb{Z}[\pi/\tilde{\pi}]) \cong H_1(A; \mathbb{Z}[\pi/\tilde{\pi}])$ is also free abelian. In particular the above exact sequence defines a presentation for $H_1(N; \mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}])$ and we deduce that

$$\Delta^{\pi/\tilde{\pi}}_{N,\phi} = \det(t\iota_+ - \iota_-).$$

Since ι_- and ι_+ are isomorphisms it follows that $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ is monic of degree b_1. An argument similar to the above now shows that

$$\deg \Delta^{\pi/\tilde{\pi}}_{N,\phi} = |\pi/\tilde{\pi}| \, \|\phi\|_T + (1 + b_3(N)) \mathrm{div} \phi_{\tilde{\pi}}.$$

□

3.2. **Finite solvable quotients.** Given a solvable group S we denote by $\ell(S)$ its derived length, i.e. the length of the shortest decomposition into abelian groups. Put differently, $\ell(S)$ is the minimal number such that $S^{(\ell(S))} = \{e\}$. Note that $\ell(S) = 0$ if and only if $S = \{e\}$.

For sake of comprehension, we briefly recall the notation. We are considering a 3–manifold N with empty or toroidal boundary, and we fix a primitive class $\phi \in H^1(N; \mathbb{Z})$. We denote by Σ a connected Thurston norm minimizing surface dual to ϕ, and write $A = \pi_1(\Sigma)$ and $B = \pi_1(M)$ (with $M = N \setminus \nu\Sigma$) and we denote the two inclusion induced maps $A \to B$ with $\iota_\pm$. We also write $\pi = \pi_1(N)$. Note that $\pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1} \rangle$.

Given $n \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{S}(n)$ the statement that for any finite solvable group S with $\ell(S) \leq n$ we have that for $\iota = \iota_-, \iota_+$ the map

$$\iota^* : \mathrm{Hom}(B, S) \to \mathrm{Hom}(A, S)$$

is a bijection. This is equivalent by Lemma 2.10 to assert that $\iota : A \to B$ induces an isomorphism of pro–$\mathcal{FS}(n)$ completions. Recall that by Corollary 2.3 and Lemma

2.12 statement $\mathcal{S}(n)$ implies then that for any homomorphism $\beta : B \to S$ to a finite solvable group S with $\ell(S) \le n$ we have $\operatorname{Im}\{\beta \circ \iota : A \to B \to S\} = \operatorname{Im}\{\beta : B \to S\}$.

Our goal is to show that $\mathcal{S}(n)$ holds for all n. We will show this by induction on n. For the induction argument we use the following auxiliary statement: Given $n \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{H}(n)$ the statement that for any homomorphism $\beta : B \to S$ where S is finite solvable with $\ell(S) \le n$ we have that for $\iota = \iota_-, \iota_+$ the homomorphism

$$\iota_* : H_1(A; \mathbb{Z}[S]) \to H_1(B; \mathbb{Z}[S])$$

is an isomorphism.

In the next two sections we will prove the following two propositions:

Proposition 3.3. *If $\mathcal{H}(n)$ and $\mathcal{S}(n)$ hold, then $\mathcal{S}(n+1)$ holds as well.*

Proposition 3.4. *Assume that (N, ϕ) satisfies Condition $(*)$. If $\mathcal{S}(n)$ holds, then $\mathcal{H}(n)$ holds as well.*

We can now prove the following corollary, which amounts to Proposition 1.7.

Corollary 3.5. *Assume that (N, ϕ) satisfies Condition $(*)$ and that ϕ is primitive. Let $\Sigma \subset N$ be a connected Thurston norm minimizing surface dual to ϕ and let $\iota : A \to B$ be one of the two injections. Then for any finite solvable group G the map*

$$Hom(B, G) \xrightarrow{\iota^*} Hom(A, G)$$

is a bijection, i.e. $\iota : A \to B$ induces an isomorphism of prosolvable completions.

Proof. The condition $\mathcal{S}(0)$ holds by *fiat*. It follows from Proposition 3.2 applied to the trivial group that if (N, ϕ) satisfies Condition $(*)$, then $\iota_\pm : H_1(A; \mathbb{Z}) \to H_1(B; \mathbb{Z})$ are isomorphisms, i.e. $\mathcal{H}(0)$ holds. The combination of Propositions 3.3 and 3.4 then shows that $\mathcal{H}(n)$ and $\mathcal{S}(n)$ hold for all n. The corollary is now immediate. □

3.3. **Proof of Proposition 3.3.** In this section we will prove Proposition 3.3. Let $\iota = \iota_-$ or $\iota = \iota_+$. Since $\mathcal{S}(n)$ holds we only have to consider the case of G a finite solvable group with $\ell(G) = n+1$. By definition G fits into a short exact sequence

$$1 \to I \to G \to S \to 1,$$

where $I = G^{(n)}$ is finite abelian and $S = G/G^{(n)}$ finite solvable with $\ell(S) = n$.

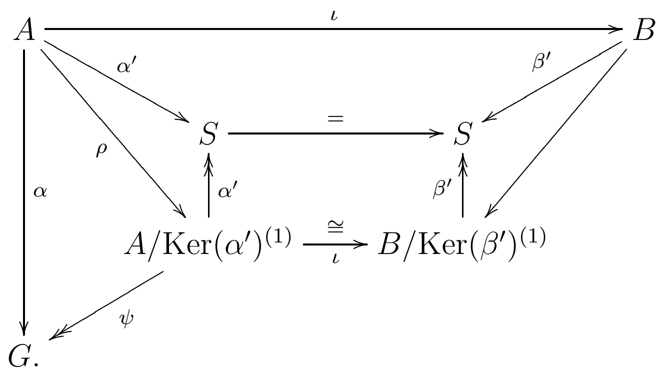
We will construct a map $\Phi : \operatorname{Hom}(A, G) \to \operatorname{Hom}(B, G)$ which is an inverse to $\iota^* : \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G)$. Let $\alpha : A \to G$ be a homomorphism. Without loss of generality we can assume that α is an epimorphism. Denote $A \xrightarrow{\alpha} G \to S$ by α' and denote the map $A \to A/\operatorname{Ker}(\alpha')^{(1)}$ by ρ. Note that α sends $\operatorname{Ker}(\alpha')$ to the abelian group I, hence α vanishes on $\operatorname{Ker}(\alpha')^{(1)}$. This shows that α factors through ρ, in particular $\alpha = \psi \circ \rho$ for some $\psi : A/\operatorname{Ker}(\alpha')^{(1)} \to G$.

Recall that $\ell(S) = n$, therefore by $\mathcal{S}(n)$ we have that $\alpha' : A \to S$ equals $\iota^*(\beta')$ for some $\beta' : B \to S$. By Lemma 2.12, $\mathcal{S}(n)$ guarantees that $i_* : H_0(A; \mathbb{Z}[S]) \to H_0(B; \mathbb{Z}[S])$ is an isomorphism; on the other hand $\mathcal{H}(n)$ asserts that $i_* : H_1(A; \mathbb{Z}[S]) \to$

$H_1(B;\mathbb{Z}[S])$ is an isomorphism as well. By Corollary 2.5 this implies that ι induces an isomorphism

$$\iota : A/\mathrm{Ker}(\alpha')^{(1)} \xrightarrow{\cong} B/\mathrm{Ker}(\beta')^{(1)}.$$

The various homomorphisms can be summarized in the following commutative diagram:



Now we define $\Phi(\alpha) \in \mathrm{Hom}(B,G)$ to be the homomorphism

$$B \to B/\mathrm{Ker}(\beta')^{(1)} \xrightarrow{\iota^{-1}} A/\mathrm{Ker}(\alpha')^{(1)} \xrightarrow{\psi} G.$$

It is now straightforward to check that Φ and ι^* are inverses to each other.

3.4. **Proof of Proposition 3.4.** In this section we will prove Proposition 3.4. So let $\beta : B \to S$ be a homomorphism to a finite solvable group S with $\ell(S) \le n$. If β extends to $\pi_1(N)$, $\mathcal{H}(n)$ will follow immediately from Proposition 3.2. In general β though will not extend; however using $\mathcal{S}(n)$ we will construct a homomorphism $\pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1}\rangle \to G$ to a finite group G 'which contains $\beta : B \to S$' to get the required isomorphism.

We first need some notation. Given groups C and H we define

$$C(H) = \bigcap_{\gamma \in Hom(C,H)} \mathrm{Ker}(\gamma).$$

We summarize a few properties of $C(H) \subset C$ in the following lemma.

Lemma 3.6. *Let C be a finitely generated group. Then the subgroup $C(H) \subset C$ has the following properties:*

1. *$C(H) \subset C$ is normal and characteristic.*
2. *If H is finite and solvable, then $C/C(H)$ is finite and solvable with $\ell(C/C(H)) \le \ell(H)$.*

Proof. Statement (1) is immediate. To prove the rest, consider the injection

$$C/C(H) = C/\bigcap_{\gamma \in Hom(C,H)} \mathrm{Ker}(\gamma) \to \prod_{\gamma \in Hom(C,H)} C/\mathrm{Ker}(\gamma).$$

If H is finite, then $\operatorname{Hom}(C,H)$ is a finite set (since C is finitely generated), hence $C/C(H)$ is finite. If H is furthermore solvable, then for any $\gamma \in \operatorname{Hom}(C,H)$ the groups $C/\operatorname{Ker}(\gamma)$ are solvable, hence $C/C(H)$ is solvable as well. Moreover for any $\gamma \in \operatorname{Hom}(C,H)$ we have $\ell(C/\operatorname{Ker}(\gamma)) \le \ell(H)$. We therefore get

$$\ell(C/C(H)) \le \max_{\gamma \in Hom(C,H)} \ell(C/\operatorname{Ker}(\gamma)) \le \ell(H).$$

□

We will also need the following group homomorphism extension lemma.

Lemma 3.7. *Assume that $\mathcal{S}(n)$ holds and that S is a finite solvable group with $\ell(S) \le n$. Let $\beta : B \to S$ be a homomorphism.*

Then there exists a $k \in \mathbb{N}$, a semidirect product $\mathbb{Z}/k \ltimes B/B(S)$ and a homomorphism

$$\pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1} \rangle \to \mathbb{Z}/k \ltimes B/B(S)$$

which extends $B \to B/B(S)$, i.e. we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B/B(S) & \longrightarrow & \mathbb{Z}/k \ltimes B/B(S) & \longrightarrow & \mathbb{Z}/k & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & & & \\ & & B & \longrightarrow & \pi. & & & & \end{array}$$

Proof. Assume that $\mathcal{S}(n)$ holds and that S is a finite solvable group with $\ell(S) \le n$. Let $\beta : B \to S$ be a homomorphism. We denote the projection map $B \to B/B(S)$ by ρ.

Claim. There exists an automorphism $\gamma : B/B(S) \to B/B(S)$ such that $\rho(\iota_+(a)) = \gamma(\rho(\iota_-(a)))$ for all $a \in A$.

Let $\iota = \iota_-$ or $\iota = \iota_+$. By Lemma 3.6 we know that $B/B(S)$ is finite solvable with $\ell(B/B(S)) \le n$. It follows from $\mathcal{S}(n)$ that

$$\iota_* : A/\operatorname{Ker}\{A \xrightarrow{\iota} B \xrightarrow{\rho} B/B(S)\} \to B/B(S)$$

is an isomorphism. On the other hand it is also a straightforward consequence of $\mathcal{S}(n)$ that

$$\operatorname{Ker}\{A \xrightarrow{\iota} B \xrightarrow{\rho} B/B(S)\} = A(S).$$

Combining these two observations we see that ι gives rise to an isomorphism $\iota_* : A/A(S) \to B/B(S)$. We now take $\gamma := \iota_{+*} \circ (\iota_{-*})^{-1}$. This concludes the proof of the claim.

We now write $H = B/B(S)$. It is now straightforward to verify that

$$\begin{array}{rcl} \pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1} \rangle & \to & \mathbb{Z} \ltimes H = \langle H, t | H = t\gamma(H)t^{-1} \rangle \\ b & \mapsto & \rho(b), \;\; b \in B, \\ t & \mapsto & t \end{array}$$

defines a homomorphism. Since $H = B/B(S)$ is a finite group it follows that the automorphism γ has finite order k, in particular the projection map $\mathbb{Z} \ltimes B/B(S) \to \mathbb{Z}/k \ltimes B/B(S)$ is a homomorphism. Clearly the resulting homomorphism $\pi \to \mathbb{Z}/k \ltimes B/B(S)$ has all the required properties. □

We are in position now to prove Proposition 3.4.

Proof of Proposition 3.4. Recall that we assume that (N, ϕ) satisfies Condition $(*)$ and that $\mathcal{S}(n)$ holds. We have to show that $\mathcal{H}(n)$ holds as well. So let $\beta : B \to S$ be a homomorphism to a finite solvable group S with $\ell(S) \leq n$. We have to show that for $\iota = \iota_-, \iota_+$ the homomorphism

$$\iota_* : H_1(A; \mathbb{Z}[S]) \to H_1(B; \mathbb{Z}[S])$$

is an isomorphism. Without loss of generality we can assume that β is surjective. Recall that $\mathcal{S}(n)$ implies that $\beta \circ \iota : A \to S$ is surjective as well.

We now apply Lemma 3.7 to find a homomorphism

$$\pi = \langle B, t | \iota_-(A) = t\iota_+(A)t^{-1} \rangle \to \mathbb{Z}/k \ltimes B/B(S)$$

which extends $B \to B/B(S)$. Note that

(2)
$$\begin{array}{rcl} \operatorname{Ker}\{\gamma : B \to \pi \to \mathbb{Z}/k \ltimes B/B(S)\} & = & \operatorname{Ker}\{B \to B/B(S)\} \\ \operatorname{Ker}\{\gamma \circ \iota : A \to B \to \pi \to \mathbb{Z}/k \ltimes B/B(S)\} & = & \operatorname{Ker}\{\iota : A \to B \to B/B(S)\}. \end{array}$$

We let

$$\begin{array}{rcl} \hat{B} & = & \operatorname{Ker}\{B \to B/B(S)\}, \\ \tilde{B} & = & \operatorname{Ker}(\beta). \end{array}$$

Clearly $\hat{B} \subset \tilde{B}$ by the definition of $B/B(S)$. We also write $\hat{A} = \iota^{-1}(\hat{B})$ and $\tilde{A} = \iota^{-1}(\tilde{B})$. We now consider the epimorphism $\pi_1(N) = \pi \to \mathbb{Z}/k \ltimes B/B(S)$. By Condition $(*)$, Equation (2), Proposition 3.2 and Corollaries 2.5 and 2.5 it follows that the maps

$$\iota : A/\hat{A} \to B/\hat{B} \text{ and } \iota : A/[\hat{A}, \hat{A}] \to B/[\hat{B}, \hat{B}]$$

are isomorphisms. It now follows from Lemma 2.7 and Corollary 2.5 that the maps

$$\iota : A/\tilde{A} \to B/\tilde{B} \text{ and } \iota : A/[\tilde{A}, \tilde{A}] \to B/[\tilde{B}, \tilde{B}]$$

are also isomorphisms. □

4. A product criterion

In this section we will apply a theorem of Agol to prove a criterion for a manifold to be a product, which complements Proposition 1.7.

In order to state our result, we first recall the definition of a sutured manifold (cf. [Ga83, Definition 2.6] or [CC03, p. 364]). A *sutured manifold* (M, γ) is a compact oriented 3–manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$

and tori $T(\gamma)$. Furthermore, the structure of a sutured manifold consists of the following choices of orientations:

(1) For each $A \in A(\gamma)$ a choice of a simple closed, oriented curve in A (called *suture*) such that A is the tubular neighborhood of the curve, and
(2) the choice of an orientation for each component of $\partial M \setminus A(\gamma)$.

The orientations must be compatible, i.e. the orientation of the components of $\partial M \setminus A(\gamma)$ must be coherent with the orientations of the sutures.

Given a sutured manifold (M, γ) we define $R_+(\gamma)$ as the components of $\overline{\partial M \setminus \gamma}$ where the orientation agrees with the orientation induced by M on ∂M, and $R_-(\gamma)$ as the components of $\overline{\partial M \setminus \gamma}$ where the two orientations disagree. We define also $R(\gamma) = R_+(\gamma) \cup R_-(\gamma)$.

A sutured manifold (M, γ) is called *taut* if M is irreducible and if each component of $R(\gamma)$ is incompressible and Thurston norm–minimizing in $H_2(M, \gamma; \mathbb{Z})$ (we refer to [Sc89] for information regarding the Thurston norm on sutured manifolds).

An example of a taut sutured manifold is given by taking an oriented surface Σ and considering $\Sigma \times I$ with sutures given by the annuli $\partial\Sigma \times I$. The sutures are oriented by the orientation of $\partial\Sigma$. We can pick orientations such that $R_-(\gamma) = \Sigma \times 0$ and $R_+(\gamma) = \Sigma \times 1$. If a sutured manifold (M, γ) is diffeomorphic (as a sutured manifold) to such a product then we say that (M, γ) is a *product sutured manifold*.

Another example of a taut sutured manifold comes from considering an oriented incompressible Thurston norm minimizing surface $\Sigma \subset N$ in an irreducible 3–manifold with empty or toroidal boundary. We let $(M, \gamma) = (N \setminus \nu\Sigma, \partial N \cap (N \setminus \nu\Sigma))$. With appropriate orientations (M, γ) is a taut sutured manifold such that $R_-(\gamma) = \Sigma^-$ and $R_+(\gamma) = \Sigma^+$.

The following theorem immediately implies Theorem 1.8.

Theorem 4.1. *Assume we have a taut sutured manifold (M, γ) which has the following properties:*

(1) *$R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each, and the inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions,*
(2) *$\pi_1(M)$ is residually finite solvable,*

then (M, γ) is a product sutured manifold.

The key ingredient to the proof of Theorem 4.1 is a result of Agol's [Ag08] which we recall in Section 4.1. We will then provide the proof for Theorem 4.1 in Sections 4.2 and 4.3.

Remark. (1) It is an immediate consequence of 'peripheral subgroup separability' [LN91] that the theorem holds under the assumption that the inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective *profinite* completions. It is not clear how the approach of [LN91] can be adapted to prove Theorem 4.1.

(2) It is also interesting to compare Theorem 4.1 with a result of Grothendieck. In [Gr70, Section 3.1] Grothendieck proves that if $\varphi : A \to B$ is a homomorphism between finitely presented, residually finite groups which induces an isomorphism of the profinite completions, and if A is arithmetic (e.g. a surface group), then φ is an isomorphism. It is an interesting question whether Theorem 4.1 can be proved using purely group theoretic arguments. We refer to [AHKS07] for more information regarding this question.

4.1. **Agol's theorem.** Before we can state Agol's result we have to introduce more definitions. A group G is called *residually finite* $\mathbb{Q}$*–solvable* or *RFRS* if there exists a filtration of groups $G = G_0 \supset G_1 \supset G_2 \ldots$ such that the following hold:

(1) $\cap_i G_i = \{1\}$,
(2) G_i is a normal, finite index subgroup of G for any i,
(3) for any i the map $G_i \to G_i/G_{i+1}$ factors through $G_i \to H_1(G_i; \mathbb{Z})/\text{torsion}$.

Note that RFRS groups are in particular residually finite solvable, but the RFRS condition is considerably stronger than being residually finite solvable. The notion of an RFRS group was introduced by Agol [Ag08], we refer to Agol's paper for more information on RFRS groups.

Given a sutured manifold (M, γ) the double DM_γ is defined to be the double of M along $R(\gamma)$, i.e. $DM_\gamma = M \cup_{R(\gamma)} M$. Note that the annuli $A(\gamma)$ give rise to toroidal boundary components of DM_γ. We denote by $r : DM_\gamma \to M$ the retraction map given by 'folding' the two copies of M along $R(\gamma)$.

We are now in a position to state Agol's result. The theorem as stated here is clearly implicit in the proof of [Ag08, Theorem 6.1].

Theorem 4.2. *Let (M, γ) be a connected, taut sutured manifold such that $\pi_1(M)$ satisfies property RFRS. Write $W = DM_\gamma$. Then there exists an epimorphism $\alpha : \pi_1(M) \to S$ to a finite solvable group, such that in the covering $p : \widetilde{W} \to W$ corresponding to $\alpha \circ r_* : \pi_1(W) \to S$ the pull back of the class $[R_-(\gamma)] \in H_2(W, \partial W; \mathbb{Z})$ lies on the closure of the cone over a fibered face of $\widetilde{W}$.*

Note that $[R_+(\gamma)] = \pm[R_-(\gamma)]$ in $H_2(W, \partial W; \mathbb{Z})$, i.e. $[R_-(\gamma)]$ is a fibered class if and only if $[R_+(\gamma)]$ is a fibered class. In case that $\widetilde{W}$ has vanishing Thurston norm, then we adopt the usual convention that by the fibered face we actually mean $H^1(\widetilde{W}, \mathbb{R}) \setminus \{0\}$.

4.2. **Proof of Theorem 4.1.** From now on assume we have a taut sutured manifold (M, γ) with the following properties:

(1) $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each and the inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions.
(2) $\pi_1(M)$ is residually finite solvable.

Since Theorem 4.1 is obvious in the case $M = S^2 \times [0,1]$ we will henceforth assume that $M \neq S^2 \times [0,1]$.

Our main tool in proving Theorem 4.1 is Theorem 4.2. In order to apply it we need the following claim.

Claim. The group $\pi_1(M)$ is RFRS.

Proof. By assumption the group $\pi_1(M)$ is residually finite solvable. This means that we can find a sequence $\pi_1(M) = B_0 \supset B_1 \supset B_2 \dots$ with the following properties:

(1) $\cap_i B_i = \{1\}$,
(2) B_i is a normal, finite index subgroup of $\pi_1(M)$ for any i,
(3) for any i the map $B_i \to B_i/B_{i+1}$ factors through $B_i \to H_1(B_i;\mathbb{Z})$.

It remains to show that $B_i \to B_i/B_{i+1}$ factors through $H_1(B_i;\mathbb{Z})/\text{torsion}$. In fact we claim that $H_1(B_i;\mathbb{Z})$ is torsion–free. Indeed, first note that by Shapiro's lemma $H_1(B_i;\mathbb{Z}) \cong H_1(B;\mathbb{Z}[B/B_i]) \cong H_1(M;\mathbb{Z}[B/B_i])$. Furthermore, by Lemma 2.12 we have

$$H_1(\Sigma^-;\mathbb{Z}[B/B_i]) \xrightarrow{\cong} H_1(M;\mathbb{Z}[B/B_i]),$$

but the first group is clearly torsion–free as it is the homology of a finite cover of a surface. □

In the following we write $W = DM_\gamma$. By the above claim we can apply Theorem 4.2 which says that there exists an epimorphism $\alpha : \pi_1(M) \to S$ to a finite solvable group, such that in the covering $p : \widetilde{W} \to W$ corresponding to $\alpha \circ r_* : \pi_1(W) \to S$ the pull back of the class $[R_-(\gamma)] = [\Sigma^-] \in H_2(W, \partial W;\mathbb{Z})$ lies on the closure of the cone over of a fibered face of $\widetilde{W}$.

Note that we can view $\widetilde{W}$ as the double of the cover $(\widetilde{M},\tilde{\gamma})$ of (M,γ) induced by $\alpha : \pi_1(M) \to S$. We summarize the main properties of $\tilde{\Sigma}^\pm$ and $\widetilde{W}$ in the following lemma.

Lemma 4.3. (1) $\tilde{\Sigma}^\pm := p^{-1}(\Sigma^\pm)$ *are connected surfaces,*
(2) *the inclusion induced maps* $\pi_1(\tilde{\Sigma}^\pm) \to \pi_1(\widetilde{M})$ *give rise to isomorphisms of prosolvable completions,*
(3) *if* $\tilde{\Sigma}^-$ *is the fiber of a fibration* $\widetilde{W} = D\widetilde{M}_{\tilde{\gamma}} \to S^1$, *then* $\widetilde{M}$ *is a product over* $\tilde{\Sigma}^-$,
(4) M *is a product over* Σ^- *if and only if* $\widetilde{M}$ *is a product over* $\tilde{\Sigma}^-$.

Proof. First note that it follows from Lemma 2.12 and Corollary 2.3 and the assumption that $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions that $\pi_1(\Sigma^\pm) \to \pi_1(M) \to S$ is surjective, i.e. the preimages $\tilde{\Sigma}^\pm := p^{-1}(\Sigma^\pm)$ are connected. The second claim follows from Lemma 2.11 since the maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of their prosolvable completions.

For the third claim consider the following commutative diagram

$$\begin{array}{ccc} \pi_1(\tilde{\Sigma}^-) & \longrightarrow & \pi_1(\widetilde{W} \setminus \nu\tilde{\Sigma}^-) \\ & \searrow \quad \nearrow & \\ & \pi_1(\widetilde{M}). & \end{array}$$

If $\tilde{\Sigma}^-$ is the fiber of a fibration $D\widetilde{M}_{\tilde{\gamma}} \to S^1$, then the top map in the above commutative diagram is an isomorphism. We can think of $\widetilde{W} \setminus \nu\tilde{\Sigma}^-$ as $\widetilde{M} \cup_{\tilde{\Sigma}^+} \widetilde{M}$. It is now clear that the lower two maps are injective. But then the lower left map also has to be an isomorphism, i.e. $\widetilde{M}$ is a product over $\tilde{\Sigma}^-$. The last claim is well–known, it is for example a consequence of [He76, Theorem 10.5]. □

Using the above lemma it is now clear that the following lemma implies Theorem 4.1.

Lemma 4.4. *Let (M, γ) be a taut sutured manifold such that $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each. Assume the following hold:*

(A) *The inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions.*

(B) *The class in $H^1(DM_\gamma; \mathbb{Z})$ represented by the surface Σ^- lies on the closure of the cone over a fibered face of DM_γ.*

Then Σ^- is the fiber of a fibration $DM_\gamma \to S^1$.

In the following we write $W = DM_\gamma$. Note that we have a canonical involution τ on W with fix point set $R(\gamma)$. From now on we think of $W = DM_\gamma$ as $M \cup_{R(\gamma)} \tau(M)$.

Our main tool in the proof of Lemma 4.4 will be the interplay between the Thurston norm and McMullen's Alexander norm [McM02]. Recall that given a 3–manifold V with $b_1(V) \geq 2$ the Alexander norm $|| - ||_A : H^1(V; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ has the following properties:

(a) The Alexander norm ball is dual to the Newton polyhedron defined by the symmetrized Alexander polynomial $\Delta_V \in \mathbb{Z}[H_1(V; \mathbb{Z})/\text{torsion}]$.

(b) The Alexander norm ball is a (possibly noncompact) polyhedron with finitely many faces.

(c) For any $\phi \in H^1(V; \mathbb{R})$ we have $||\phi||_A \leq ||\phi||_T$, and equality holds for fibered classes.

(d) Let $C \subset H^1(V; \mathbb{R})$ be a fibered cone, i.e. the cone on a fibered face of the Thurston norm ball, then C is contained in the cone on the interior of a top–dimensional face of the Alexander norm ball.

(e) Let $C_1, C_2 \subset H^1(V; \mathbb{R})$ be fibered cones which are contained in the same cone on the interior of a top–dimensional face of the Alexander norm ball, then $C_1 = C_2$.

Our assumption that the induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions implies that $W = DM_\gamma$ 'looks algebraically' the same as $\Sigma^- \times S^1$. More precisely, we have the following lemma which we will prove in Section 4.3.

Lemma 4.5. *Let (M, γ) be a taut sutured manifold with the property that $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each. Assume that (A) holds. Then the following hold:*

(1) *There exists an isomorphism*

$$f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \to H_2(W, \partial W; \mathbb{R})$$

such that $f(1, 0) = [\Sigma^-]$ and such that $\tau(f(r, h)) = f(r, -h)$.

(2) *The class $\phi = PD(\Sigma^-) \in H^1(W; \mathbb{Z})$ lies in the cone D on the interior of a top–dimensional face of the Alexander norm ball.*

Note that (1) implies in particular that $b_1(W) \geq 2$. Assuming this lemma we are now in a position to prove Lemma 4.4.

Proof of Lemma 4.4. Let (M, γ) be a taut sutured manifold with the property that $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each. Assume that (A) and (B) hold.

By Lemma 4.5 there exists a cone $D \subset H^1(W; \mathbb{R})$ on the interior of a top–dimensional face of the Alexander norm ball which contains $\phi = PD([\Sigma^-])$. We denote the map

$$\mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \xrightarrow{f} H_2(W, \partial W; \mathbb{R}) \xrightarrow{PD} H^1(W; \mathbb{R})$$

by Φ.

By (B) we can find $h \in H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R})$ such that $\Phi(1, h)$ and $\Phi(1, -h)$ lie in D and such that $\Phi(1, h)$ lies in the cone C on a fibered face F of the Thurston norm ball. Note that the $\tau_* : H^1(W; \mathbb{R}) \to H^1(W; \mathbb{R})$ sends fibered classes to fibered classes and preserves the Thurston norm. In particular $\tau(\Phi(1, h)) = \Phi(1, -h)$ is fibered as well and it lies in $\tau(C)$ which is the cone on the fibered face $\tau(F)$ of the Thurston norm ball. Recall that $\tau(\Phi(1, h)) = \Phi(1, -h)$ lies in D, it follows from Property (d) of the Alexander norm that $\tau(C) \subset D$. We then use (e) to conclude that $C = \tau(C)$. In particular $\Phi(1, h)$ and $\Phi(1, -h)$ lie in C. Since C is convex it follows that $\phi = \Phi(1, 0) \in C$ i.e. ϕ is a fiber class. □

4.3. **Proof of Lemma 4.5.** By Lemma 2.12 the following lemma is just the first statement of Lemma 4.5.

Lemma 4.6. *Let (M, γ) be a taut sutured manifold with the property that $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each. Assume that $\iota_\pm : H_1(\Sigma^\pm; \mathbb{R}) \to H_1(M; \mathbb{R})$ are isomorphisms. Then there exists an isomorphism*

$$f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-; \mathbb{R}) \to H_2(W, \partial W; \mathbb{R})$$

such that $f(1, 0) = [\Sigma^-]$ and such that $\tau(f(b, c)) = f(b, -c)$.

Proof. We start out with the following two claims.

Claim. M has no toroidal sutures.

Proof. Denote the toroidal sutures by $T_1, \dots, T_n$. Recall that for any compact 3–manifold X we have $b_1(\partial X) \leq 2b_1(X)$. In our case it is easy to see that we have $b_1(\partial M) = b_1(\Sigma^-) + b_1(\Sigma^+) + \sum_{i=1}^n b_1(T_i) = 2b_1(\Sigma) + 2n$. On the other hand, since $H_1(\Sigma^\pm;\mathbb{R}) \to H_1(M;\mathbb{R})$ are isomorphisms we have $b_1(M) = b_1(\Sigma)$. It now follows from $b_1(\partial M) \leq 2b_1(M)$ that $n = 0$. □

Claim. The inclusion induced maps $H_1(\Sigma^\pm, \partial\Sigma^\pm;\mathbb{R}) \to H_1(M, A(\gamma);\mathbb{R})$ are isomorphisms.

Now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
H_1(\partial\Sigma^-;\mathbb{R}) & \longrightarrow & H_1(\Sigma^-;\mathbb{R}) & \longrightarrow & H_1(\Sigma^-, \partial\Sigma^-;\mathbb{R}) & \longrightarrow & H_0(\partial\Sigma^-;\mathbb{R}) & \longrightarrow & H_0(\Sigma^-;\mathbb{R}) \\
\downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
H_1(A(\gamma);\mathbb{R}) & \longrightarrow & H_1(M;\mathbb{R}) & \longrightarrow & H_1(M, A(\gamma);\mathbb{R}) & \longrightarrow & H_0(A(\gamma);\mathbb{R}) & \longrightarrow & H_0(M;\mathbb{R}).
\end{array}$$

Note that by the compatibility condition in the definition of sutured manifolds we have that for each component A of $A(\gamma)$ the subset $\partial A \cap \Sigma^- = A \cap \partial\Sigma^- \subset \partial A$ consists of exactly one boundary component of A. This implies that the maps $H_i(\partial\Sigma^- \cap A;\mathbb{R}) \to H_i(A;\mathbb{R})$ are isomorphisms. The claim now follows immediately from the above commutative diagram and from the assumption that $H_1(\Sigma^\pm;\mathbb{R}) \to H_1(M;\mathbb{R})$ are isomorphisms.

We now define

$$g : H_1(\Sigma^-, \partial\Sigma^-;\mathbb{R}) \to H_2(W, \partial W;\mathbb{R})$$

as follows: given an element $c \in H_1(\Sigma^-, \partial\Sigma^-;\mathbb{R})$ represent it by a chain c^-, since the maps $H_1(\Sigma^\pm, \partial\Sigma^\pm;\mathbb{R}) \to H_1(M, A(\gamma);\mathbb{R})$ are isomorphisms we can find a chain c^+ in Σ^+ such that $[c^-] = [c^+] \in H_2(M, A(\gamma);\mathbb{R})$. Now let d be a 2–chain in M such that $\partial d = c^- \cup -c^+$. Then define $g(c)$ to be the element in $H_2(W, \partial W;\mathbb{R})$ represented by the closed 2–chain $d \cup -\tau(d)$. It is easy to verify that g is a well–defined homomorphism. Note that $\partial W = A(\gamma) \cup \tau(A(\gamma))$ since W has no toroidal sutures. It is now straightforward to check, using a Mayer–Vietoris sequence, that the map

$$\begin{array}{rcl}
f : \mathbb{R} \oplus H_1(\Sigma^-, \partial\Sigma^-;\mathbb{R}) & \to & H_2(W, \partial W;\mathbb{R}) \\
(b \quad , \quad c) & \mapsto & b[\Sigma^-] + g(c)
\end{array}$$

is an isomorphism. Clearly $f(1, 0) = [\Sigma^-]$. It is also easy to verify that $\tau(f(b, c)) = f(b, -c)$. This shows that f has all the required properties. □

The second statement of Lemma 4.5 is more intricate. We start with the following lemma which in light of [Gr70], [BG04] and [AHKS07] has perhaps some independent interest.

Lemma 4.7. *Let $\varphi : A \to B$ be a homomorphism of finitely generated metabelian groups which induces an isomorphism of prosolvable completions. Then φ is also an isomorphism.*

Proof. We first show that $A \to B$ is an injection. We consider the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \widehat{A}_{\mathcal{FS}} & \longrightarrow & \widehat{B}_{\mathcal{FS}}. \end{array}$$

The vertical maps are injections since metabelian groups are residually finite (cf. [Ha59]). The bottom map is an isomorphism by assumption. It now follows that the top map is an injection.

Now suppose that the homomorphism $A \to B$ is not surjective. We identify $H_1(A;\mathbb{Z}) = H_1(A;\mathbb{Z}) \xrightarrow{\cong} H_1(B;\mathbb{Z}) = H_1(B;\mathbb{Z})$ via φ and refer to the group as H. Let $g' \in B \setminus \varphi(A)$. We can pick an $a \in A$ such that $\varphi(a)$ and g' represent the same element in H. Let $g = \varphi(a)^{-1}g'$. Then g represents the trivial element in H but g is also an element in $B \setminus \varphi(A)$.

We will show that there exists a homomorphism $\alpha : B \to G$ to a finite metabelian group such that α separates g from $\varphi(A)$, i.e. such that $\alpha(g) \notin \alpha(\varphi(A))$. This then immediately contradicts, via Lemma 2.10, our assumption that $\varphi : A \to B$ induces an isomorphism of prosolvable completions. Our construction of finding such α builds on some ideas of the proof of [LN91, Theorem 1].

We write $B_1 = B_2 = B$. We denote the inclusion maps $A \to B_i = B$ by φ_i. We let $C = B_1 *_A B_2$. It is straightforward to see that the homomorphisms $B_i \to C$ give rise to an isomorphism $H_1(B_i;\mathbb{Z}) = H \to H_1(C;\mathbb{Z})$. Denote by

$$s : B_1 *_A B_2 \to B_1 *_A B_2$$

the switching map, i.e. the map induced by $s(b) = b \in B_2$ for $b \in B_1$ and $s(b) = b \in B_1$ for $b \in B_2$. Note that s acts as the identity on $A \subset C$. Also note that s descends to a map $C/C^{(2)} \to C/C^{(2)}$ which we also denote by s. We now view g as an element in B_1 and hence as an element in C. Note that the fact that g represents the trivial element in H implies that g represents an element in $C^{(1)}/C^{(2)}$. We will first show that $s(g) \neq g \in C/C^{(2)}$. Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} H_1(A;\mathbb{Z}[H]) & \to & H_1(B_1;\mathbb{Z}[H]) \oplus H_1(B_2;\mathbb{Z}[H]) & \to & H_1(C;\mathbb{Z}[H]) & \to & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow \\ A^{(1)} & \to & B_1^{(1)} \times B_2^{(1)} & \to & C^{(1)}/C^{(2)} & \to & 0. \\ h & \mapsto & (\varphi_1(h), \varphi_2(h)^{-1}) & & & & \end{array}$$

Since $g \in B_1^{(1)} \setminus \varphi(A^{(1)})$ it follows that (g, g^{-1}) does not lie in the image of $A^{(1)}$ in $B_1^{(1)} \times B_2^{(1)}$. It therefore follows from the above diagram that $gs(g)^{-1} \neq e \in C^{(1)}/C^{(2)}$.

Note that $C/C^{(2)}$ is metabelian, and hence by [Ha59] residually finite. We can therefore find an epimorphism $\alpha : C/C^{(2)} \to G$ onto a finite group G (which is necessarily metabelian) such that $\alpha(gs(g)^{-1}) \neq e$. Now consider $\beta : C/C^{(2)} \to G \times G$ given by $\beta(h) = (\alpha(h), \alpha(s(h)))$. Then clearly $\beta(g) \notin \beta(A) \subset \{(g,g)\,|\,g \in G\}$. The restriction of β to $B = B_1$ now clearly separates g from A. □

Corollary 4.8. *Let $\varphi : A \to B$ be a homomorphism of finitely generated groups which induces an isomorphism of prosolvable completions. Then the induced map $A/A^{(2)} \to B/B^{(2)}$ is an isomorphism.*

Proof. It follows immediately from Lemma 2.10 that φ induces an isomorphism of the prosolvable completions of the metabelian groups $A/A^{(2)}$ and $B/B^{(2)}$. It now follows from Lemma 4.7 that the induced map $A/A^{(2)} \to B/B^{(2)}$ is an isomorphism. □

We now turn to the proof of the second claim of Lemma 4.5. For the remainder of this section let (M, γ) be a taut sutured manifold with the property that $R_\pm(\gamma)$ consist of one component $\Sigma^\pm$ each. We assume that (A) holds, i.e. the inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the respective prosolvable completions. We have to show that the class $\phi = PD(\Sigma^-) \in H^1(W;\mathbb{Z})$ lies in the cone on the interior of a top–dimensional face of the Alexander norm ball.

For the remainder of the section we pick a base point $x^- \in \Sigma^-$ and a base point $x^+ \in \Sigma^+$. We endow W, M and $\tau(M)$ with the base point x^-. Furthermore we pick a path γ in M connecting $x^- \in \Sigma^-$ to $x^+ \in \Sigma^+$.

Our goal is to understand the Alexander norm ball of W. In order to do this we first have to study $H = H_1(W;\mathbb{Z})$. Let t denote the element in H represented by the closed path $\gamma \cup -\tau(\gamma)$. It follows from a straightforward Mayer–Vietoris sequence argument that we have an isomorphism

$$\begin{array}{rcccl} f : H_1(\Sigma^-;\mathbb{Z}) & \oplus & \langle t\rangle & \to & H_1(W;\mathbb{Z}) \\ (\,b & , & t^k\,) & \mapsto & \iota(b) + kt. \end{array}$$

In particular H is torsion–free. We write $F = H_1(\Sigma^-;\mathbb{Z})$. We use f to identify H with $F \times \langle t\rangle$ and to identify $\mathbb{Z}[H]$ with $\mathbb{Z}[F][t^{\pm 1}]$.

We now consider the Alexander module $H_1(W;\mathbb{Z}[H])$. Recall that $H_1(W;\mathbb{Z}[H])$ is the homology of the covering of W corresponding to $\pi_1(W) \to H_1(W;\mathbb{Z}) = H$ together with the $\mathbb{Z}[H]$–module structure given by deck transformations.

In the following claim we compare W with $\Sigma \times S^1$. We also write $F = H_1(\Sigma;\mathbb{Z})$ and we can identify $H_1(\Sigma \times S^1;\mathbb{Z})$ with $H = F \times \langle t\rangle$. In particular we identify $H_1(\Sigma \times S^1;\mathbb{Z})$ with $H_1(W;\mathbb{Z})$. With these identifications we can now state the following lemma.

Lemma 4.9. *The $\mathbb{Z}[H]$–module $H_1(W;\mathbb{Z}[H])$ is isomorphic to the $\mathbb{Z}[H]$–module $H_1(\Sigma \times S^1;\mathbb{Z}[H])$.*

Proof. In the following we identify Σ with $\Sigma^- \subset W$. We denote by X the result of gluing M and $\tau(M)$ along $\Sigma = \Sigma^-$. Note that we have two canonical maps $r : \Sigma^+ \to M \to X$ and $s : \Sigma^+ \to \tau(M) \to X$. We furthermore denote the canonical inclusion maps $\Sigma \to M, \Sigma \to \tau(M)$ and $\Sigma = \Sigma^- \to X$ by i. Throughout this proof we denote by i, r, s the induced maps on solvable quotients as well.

Claim A. The map $i : \pi_1(\Sigma) \to \pi_1(X)$ gives rise to an isomorphism $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \to \pi_1(X)/\pi_1(X)^{(2)}$.

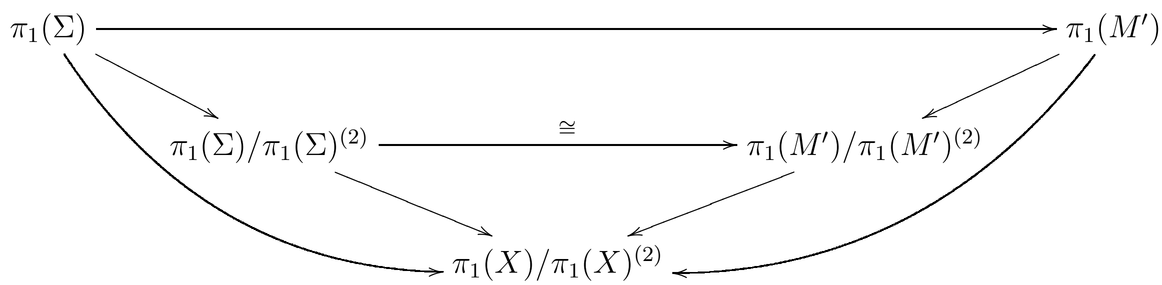
In the following let M' be either M or $\tau(M)$. Recall that we assume that $\pi_1(\Sigma) \to \pi_1(M')$ gives rise to isomorphisms of the prosolvable completions. It now follows from Corollary 4.8 that $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \to \pi_1(M'/\pi_1(M')^{(2)}$ is an isomorphism. Now let $g : \pi_1(X) = \pi_1(M \cup_\Sigma \tau(M)) \to \pi_1(M))$ be the 'folding map'. Note that

$$\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \xrightarrow{i} \pi_1(X)/\pi_1(X)^{(2)} \xrightarrow{g} \pi_1(M)/\pi_1(M)^{(2)} \xleftarrow{\cong} \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$$

is the identity map. In particular $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \xrightarrow{i} \pi_1(X)/\pi_1(X)^{(2)}$ is injective. On the other hand it follows from the van Kampen theorem that

$$\pi_1(X) = \pi_1(M) *_{\pi_1(\Sigma)} \pi_1(M'),$$

in particular $\pi_1(X)/\pi_1(X)^{(2)}$ is generated by the images of $\pi_1(M)$ and $\pi_1(\tau(M))$ in $\pi_1(X)/\pi_1(X)^{(2)}$. But it follows immediately from the following commutative diagram



that image of $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$ in $\pi_1(X)/\pi_1(X)^{(2)}$ also generates the group. This concludes the proof of the claim A.

Claim B. For any $g \in \pi_1(\Sigma^+)/\pi_1(\Sigma^+)$ we have

$$r(g) = s(g) \in \pi_1(X)/\pi_1(X)^{(2)}$$

Denote by $\tau : X = M \cup_\Sigma \tau(M) \to X = M \cup_\Sigma \tau(M)$ the map given by switching the two copies of M. Clearly $r(g) = \tau_*(s(g))$. But τ_* acts trivially on image of $\pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}$ in $\pi_1(X)/\pi_1(X)^{(2)}$. By the above claim this means that τ_* acts trivially on $\pi_1(X)/\pi_1(X)^{(2)}$. This concludes the proof of the claim.

We now view W as the result of gluing the two copies of Σ^+ in ∂X by the identity map. First note that by the van Kampen theorem we have

$$\pi_1(W) = \langle t, \pi_1(X) \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+)\rangle.$$

Note that by Claim B the obvious assignments give rise to a well–defined map

$$\pi_1(W) = \langle t, \pi_1(X) \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+)\rangle \to \langle t\rangle \times \pi_1(X)/\pi_1(X)^{(2)}.$$

Since $\pi_1(X)/\pi_1(X)^{(2)}$ is metabelian this map descends to a map

$$\Phi : \langle t, \pi_1(X) \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+)\rangle/(\dots)^{(2)} \to \langle t\rangle \times \pi_1(X)/\pi_1(X)^{(2)}.$$

Claim C. The map $\Phi : \pi_1(W)/\pi_1(W)^{(2)} \to \langle t\rangle \times \pi_1(X)/\pi_1(X)^{(2)}$ is an isomorphism.

We denote by Ψ the following map:

$$\begin{array}{rcl} \langle t\rangle \times \pi_1(X)/\pi_1(X)^{(2)} & \to & \langle t, \pi_1(X)/\pi_1(X)^{(2)} \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+)\rangle \\ & = & \langle t, \pi_1(X) \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+), \pi_1(X)^{(2)}\rangle \\ & \to & \langle t, \pi_1(X) \,|\, ts(g)t^{-1} = r(g), g \in \pi_1(\Sigma^+)\rangle/(\dots)^{(2)}. \end{array}$$

Clearly Ψ is surjective and we have $\Phi \circ \Psi = \mathrm{id}$. It follows that Φ is an isomorphism. This concludes the proof of the claim.

Finally note that we have a canonical isomorphism

$$\pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} = \langle t\rangle \times \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)}.$$

It now follows from the above discussion that we have an isomorphism

$$\begin{array}{rcl} \pi_1(W)/\pi_1(W)^{(2)} & \xrightarrow{\Phi} & \langle t\rangle \times \pi_1(X)/\pi_1(X)^{(2)} \\ & \cong & \langle t\rangle \times \pi_1(\Sigma)/\pi_1(\Sigma)^{(2)} \\ & = & \pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} \end{array}$$

which we again denote by Φ. Note that under the abelianization the map Φ descends to the above identification $H_1(\Sigma \times S^1; \mathbb{Z}) = H = H_1(W; \mathbb{Z})$. We now get the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H_1(W; \mathbb{Z}[H]) & \to & \pi_1(W)/\pi_1(W)^{(2)} & \to & H := H_1(W; \mathbb{Z}) & \to & 0 \\ & & \downarrow & & \downarrow \Phi & & \downarrow = & & \\ 0 & \to & H_1(\Sigma \times S^1; \mathbb{Z}[H]) & \to & \pi_1(\Sigma \times S^1)/\pi_1(\Sigma \times S^1)^{(2)} & \to & H_1(\Sigma \times S^1; \mathbb{Z}) & \to & 0. \end{array}$$

The lemma is now immediate. □

We are now ready to prove the second statement of Lemma 4.5. Note that the isomorphism of Alexander modules implies that the Alexander polynomials Δ_W and $\Delta_{\Sigma \times S^1}$ agree in $\mathbb{Z}[H]$. It is well–known that $\Delta_{\Sigma \times S^1} = (t-1)^{||\phi||_T} \in \mathbb{Z}[H] = \mathbb{Z}[F][t^{\pm 1}]$. Recall that we are interested in $\phi = PD([\Sigma])$, and that ϕ as an element in $\mathrm{Hom}(H, \mathbb{Z}) = \mathrm{Hom}(F \times \langle t\rangle), \mathbb{Z})$ is given by $\phi(t) = 1, \phi|_F = 0$. It is now obvious from $\Delta_W = \Delta_{\Sigma \times S^1} = (t-1)^{||\phi||_T}$ that ϕ lies in the interior of a top–dimensional face of the Alexander norm ball of W. This concludes the proof of the second statement of Lemma 4.5 modulo the proof of the claim.

5. Residual properties of 3–manifold groups

Proposition 1.7 and Theorem 4.1 are almost enough to deduce Theorem 1.2, but we still have to deal with the assumption in Theorem 4.1 that $\pi_1(W)$ has to be residually finite solvable.

Using well–known arguments (see Section 7 for details) one can easily see that Proposition 1.7 and Theorem 4.1 imply Theorem 1.2 for 3–manifolds N which have virtually residually finite solvable fundamental groups. Here we say that a group π has virtually a property if a finite index subgroup of π has this property. It seems reasonable to conjecture that all 3–manifold groups are virtually residually finite solvable. For example linear groups (and hence fundamental groups of hyperbolic 3–manifolds and Seifert fibered spaces) are virtually residually finite solvable and (virtually) fibered 3–manifold groups are easily seen to be (virtually) residually finite solvable.

It is not known though whether all 3–manifold groups are linear. In the case of 3–manifolds with non–trivial JSJ decomposition we therefore use a slightly different route to deduce Theorem 1.2 from Proposition 1.7 and Theorem 4.1. In Lemmas 7.1 and 7.2 we first show that it suffices in the proof of Theorem 1.2 to consider closed prime 3–manifolds. In this section we will show that given a closed prime 3–manifold N, there exists a finite cover N' of N such that all pieces of the JSJ decomposition of N' have residually finite solvable fundamental groups (Theorem 5.1). Finally in Section 6 we will prove a result which allows us in the proof of Theorem 1.2 to work with each JSJ piece separately (Theorem 6.4).

5.1. **Statement of the theorem.** We first recall some definitions. Let p be a prime. A p–group is a group such that the order of the group is a power of p. Note that any p–group is in particular finite solvable. A group π is called *residually p* if for any nontrivial $g \in \pi$ there exists a homomorphism $\alpha : \pi \to P$ to a p–group such that $\alpha(g) \neq e$. A residually p group is evidently also residually finite solvable.

For the reader's convenience we recall the statement of Theorem 1.9 which we will prove in this section.

Theorem 5.1. *Let N be a closed irreducible 3–manifold. Then for all but finitely many primes p there exists a finite cover N' of N such that the fundamental group of any JSJ component of N' is residually p.*

Remark.
1. Note that this theorem relies on the geometrization results of Thurston and Perelman.
2. A slight modification of our proof shows that the statement of the theorem also holds for irreducible 3–manifolds with toroidal boundary.

5.2. **Proof of Theorem 5.1.** The proof of the theorem combines in a straightforward way ideas from the proof that finitely generated subgroups of $\mathrm{GL}(n, \mathbb{C})$ are virtually residually p for all but finitely many primes p (cf. e.g. [We73, Theorem 4.7] or [LS03, Window 7, Proposition 9]) with ideas from the proof that 3–manifold groups are residually finite (cf. [He87]). Since all technical results can be found in either [We73] or [He87], and in order to save space, we only give an outline of the proof by referring heavily to [We73] and [He87].

In the following recall that given a positive integer n there exists a unique characteristic subgroup of $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(\text{torus})$ of index n^2, namely $n(\mathbb{Z} \oplus \mathbb{Z})$.

Definition 5.2. Let N be a 3-manifold which is either closed or has toroidal boundary. Given a prime p we say that a subgroup $\Gamma \subset \pi_1(N)$ has Property (p) if it satisfies the following two conditions:

1. (1) Γ is residually p, and
2. (2) for any torus $T \subset \partial N$ the group $\Gamma \cap \pi_1(T)$ is the unique characteristic subgroup of $\pi_1(T)$ of index p^2.

Proposition 5.3. *Let N be a compact orientable 3-manifold with empty or toroidal boundary such that the interior has a complete hyperbolic structure of finite volume. Then for all but finitely many primes p there exists a finite index subgroup of $\pi_1(N)$ which has Property (p).*

Proof. This theorem is essentially a straightforward combination of [He87, Lemma 4.1] with the proof that finitely generated linear groups are virtually residually p. We will use throughout the notation of the proof of [He87, Lemma 4.1]. First we pick a finitely generated subring $A \subset \mathbb{C}$ as in [He87, Proof of Lemma 4.1]. In particular we can assume that $\pi_1(N) \subset \mathrm{SL}(2, A)$ where $A \subset \mathbb{C}$. We pick a prime p and a maximal ideal $\mathfrak{m} \subset A$ as in [He87, p. 391]. We then have in particular that $\mathrm{char}(A/\mathfrak{m}) = p$. For $i \geq 1$ we now let $\Gamma_i = \mathrm{Ker}\{\pi_1(N) \to \mathrm{SL}(2, A/\mathfrak{m}^i) \times H/p^iH\}$ where $H = H_1(N; \mathbb{Z})/\text{torsion}$. We claim that $\Gamma_1 \subset \pi_1(N)$ is a finite index subgroup which has Property (p). Clearly Γ_1 is of finite index in $\pi_1(N)$ and by [He87, p. 391] the subgroup Γ_1 also satisfies condition (2). The proof that finitely generated linear groups are virtually residually p (cf. [We73, Theorem 4.7] or [LS03, Window 7, Proposition 9]) then shows immediately that all the groups $\Gamma_1/\Gamma_i, i \geq 1$ are p–groups and that $\cap_{i=1}^{\infty}\Gamma_i = \{1\}$. In particular Γ_1 is residually p. □

Proposition 5.4. *Let N be a Seifert fibered space. Then for all but finitely many primes p, there exists a finite index subgroup of $\pi_1(N)$ which has Property (p).*

Proof. If N is a closed Seifert fibered space, then it is well–known that $\pi_1(N)$ is linear, and the proposition immediately follows from the fact that linear groups are virtually residually p for almost all primes p.

Now consider the case that N has boundary. It is well–known (cf. for example [Ha01, Lemma 6] and see also [He87, p. 391]) that there exists a finite cover $q : N' \to N$ with the following two properties:

1. (1) $N' = S^1 \times F$ for some surface F,
2. (2) for any torus $T \subset \partial N$ the group $\pi_1(N') \cap \pi_1(T)$ is the unique characteristic subgroup of $\pi_1(T)$ of index p^2.

We now write $\Gamma := \pi_1(N') \subset \pi_1(N)$. The group Γ is residually p since free groups are residually p. It now follows from (2) that Γ has the required properties. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. Let N be a closed irreducible 3–manifold. Let $N_1, \dots, N_r$ be the JSJ components. For all but finitely many primes p we can by Propositions 5.3 and 5.4 find finite index subgroups $\Gamma_i \subset \pi_1(N_i)$ for $i = 1, \dots, r$ which have Property (p). We denote by N_i' the cover of N_i corresponding to Γ_i.

By the second condition of Property (p) the intersections of the subgroups $\Gamma_i, i = 1, \dots, r$ with the fundamental group of any torus of the JSJ decomposition coincide. We can therefore appeal to [He87, Theorem 2.2] to find a finite cover N' of N such that any component in the JSJ decomposition of N' is homeomorphic to some $N_i', i \in \{1, \dots, r\}$. Recall that $\pi_1(N_i') = \Gamma_i$ is residually p for any i, hence the cover N' of N has the desired properties. □

6. The JSJ decomposition and prosolvable completions

Let N be a closed 3–manifold and let $\phi \in H^1(N; \mathbb{Z})$ primitive with $||\phi||_T > 0$. If (N, ϕ) fibers, and if $\Sigma \subset N$ is a surface dual to ϕ which is the fiber of the fibration, then it is well–known (cf. e.g. [EN85]) that the JSJ tori of N cut the product $N \setminus \nu\Sigma \cong \Sigma \times [0, 1]$ into smaller products.

If (N, ϕ) satisfies Condition $(*)$, and if $\Sigma \subset N$ is a connected Thurston norm minimizing surface dual to ϕ, then we will see in Lemma 6.3 and Theorem 6.4 that the JSJ tori of N cut the manifold $N \setminus \nu\Sigma$ into smaller pieces which look like products 'on the level of prosolvable completions'. This result will play an important role in the proof of Theorem 1.2 as it allows us to work with each JSJ piece separately.

6.1. **The statement of the theorem.** Throughout this section let N be a closed irreducible 3–manifold. Furthermore let $\phi \in H^1(N; \mathbb{Z})$ be a primitive class which is dual to a *connected* Thurston norm minimizing surface. (Recall that by Proposition 3.1 this is in particular the case if (N, ϕ) satisfies Condition $(*)$.) Finally we assume that $||\phi||_T > 0$.

We now fix once and for all embedded tori $T_1, \dots, T_r \subset N$ which give the JSJ decomposition of N. (Recall that the $T_1, \dots, T_r$ are unique up to reordering and isotopy.)

We will make several times use of the following well–known observations:

Lemma 6.1. *Let $\Sigma \subset N$ be an incompressible surface in general position with the JSJ torus $T_i, i \in 1, \dots, r$. Let c be a component of $\Sigma \cap T_i$. Then c represents a non–trivial element in $\pi_1(T_i)$ if and only if c represents a non–trivial element in $\pi_1(\Sigma)$.*

Lemma 6.2. *There exists an embedded Thurston norm minimizing surface $\Sigma \subset N$ dual to ϕ with the following three properties:*

(1) *Σ is connected,*

(2) *the tori* $T_i, i = 1, \ldots, r$ *and the surface* Σ *are in general position,*
(3) *any component of* $\Sigma \cap T_i, i = 1, \ldots, r$ *represents a nontrivial element in* $\pi_1(T_i)$*.*

Now, among all surfaces dual to ϕ satisfying the properties of the lemma we pick a surface Σ which minimizes the number $\sum_{i=1}^r b_0(\Sigma \cap T_i)$.

Given Σ we can and will fix a tubular neighborhood $\Sigma \times [-1, 1] \subset N$ such that the tori $T_i, i = 1, \ldots, r$ and the surface $\Sigma \times t$ are in general position for any $t \in [-1, 1]$. We from now on write $M = N \setminus \Sigma \times (-1, 1)$ and $\Sigma^{\pm} = \Sigma \times \pm 1$.

We denote the components of N cut along $T_1, \ldots, T_r$ by $N_1, \ldots, N_s$. Let $\{A_1, \ldots, A_m\}$ be the set of components of the intersection of the tori $T_1 \cup \cdots \cup T_r$ with M. Note that the surfaces $A_i \subset M, i = 1, \ldots, m$ are properly embedded since we assumed that the tori T_i and the surfaces $\Sigma^{\pm} = \Sigma \times \pm 1$ are in general position. We also let $\{M_1, \ldots, M_n\}$ be the set of components of the intersection of N_i with M for $i = 1, \ldots, s$. Put differently, $M_1, \ldots, M_n$ are the components of M cut along $A_1, \ldots, A_m$. For $i = 1, \ldots, n$ we furthermore write $\Sigma_i^{\pm} = M_i \cap \Sigma^{\pm}$.

Let $i \in \{1, \ldots, m\}$. If the surface A_i is an annulus, then we say that A_i *connects* Σ^- *and* Σ^+ if one boundary component of A_i lies on Σ^- and the other boundary component lies on Σ^+. The following lemma will be proved in Section 6.2

Lemma 6.3. *Assume that* (N, ϕ) *satisfies Condition* $(*)$*, then for* $i = 1, \ldots, m$ *the surface* A_i *is an annulus which connects* Σ^- *and* Σ^+*.*

We can now formulate the main theorem of this section. The proof will be given in Sections 6.2, 6.3 and 6.4.

Theorem 6.4. *Assume that for* $i = 1, \ldots, m$ *the surface* A_i *is an annulus which connects* Σ^- *and* Σ^+*. Furthermore assume that the inclusion induced maps* $\pi_1(\Sigma^{\pm}) \to \pi_1(M)$ *give rise to an isomorphism of prosolvable completions. Then for* $i = 1, \ldots, n$ *the following hold:*

(1) *The surfaces* $\Sigma_i^{\pm}$ *are connected.*
(2) *Given* $j \in \{1, \ldots, n\}$ *with* $M_i \subset N_j$ *the inclusion induced map* $\pi_1(M_i) \to \pi_1(N_j)$ *is injective.*
(3) *The inclusion induced maps* $\pi_1(\Sigma_i^{\pm}) \to \pi_1(M_i)$ *give rise to isomorphisms of the respective prosolvable completions.*

We would like to remind the reader that at the beginning of the section we made the assumption that $||\phi||_T > 0$.

6.2. **Proof of Lemma 6.3.** We first recall the following theorem from an earlier paper (cf. [FV08b, Theorem 5.2]).

Theorem 6.5. *Let* Y *be a closed irreducible 3–manifold. Let* $\psi \in H^1(Y; \mathbb{Z})$ *a primitive class. Assume that for any epimorphism* $\alpha : \pi_1(Y) \to G$ *onto a finite group* G *the twisted Alexander polynomial* $\Delta^{\alpha}_{Y,\psi} \in \mathbb{Z}[t^{\pm 1}]$ *is nonzero. Let* $T \subset Y$ *be an incompressible embedded torus. Then either* $\psi|_T \in H^1(T; \mathbb{Z})$ *is nonzero, or* (Y, ψ) *fibers over* S^1 *with fiber* T*.*

With this theorem we are now able to prove Lemma 6.3. We use the notation from the previous section. So assume that (N, ϕ) is a pair which satisfies Condition $(*)$. In particular we have that $\Delta^{\alpha}_{N,\phi} \neq 0$ for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G. We can therefore apply Theorem 6.5 to the tori $T_1, \dots, T_r \subset N$ to conclude that either (N, ϕ) fibers over S^1 with toroidal fiber, or $\phi|_{T_i} \in H^1(T_i; \mathbb{Z})$ is nonzero for $i = 1, \dots, r$. Recall that we assumed that $||\phi||_T > 0$, we therefore only have to deal with the latter case. From $\phi|_{T_i} \in H^1(T_i; \mathbb{Z})$ nonzero we obtain that Σ (which is dual to ϕ) necessarily intersects T_i in at least one curve which is homologically essential on T_i. In fact by our assumption on Σ and $T_1, \dots, T_r$ any intersection curve $\Sigma \cap T_i \subset T_i$ is essential, in particular the components of T_i cut along Σ are indeed annuli.

In order to prove Lemma 6.3 it now remains to show that each of the annuli A_i connects Σ^- and Σ^+. So assume there exists an $i \in \{1, \dots, m\}$ such that the annulus A_i does not connect Σ^- and Σ^+. Without loss of generality we can assume that $\Sigma^+ \cap A_i = \emptyset$. We equip A_i with an orientation. Denote the two oriented components of ∂A_i by c and $-d$. By our assumption c and d lie in Σ^-, and they cobound the annulus $A_i \subset M$.

Now recall that by Proposition 3.2 our assumption that (N, ϕ) satisfies Condition $(*)$ implies in particular that $H_1(\Sigma^-; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is an isomorphism. Note that c, d are homologous in M via the annulus $A := A_i$, and since $H_1(\Sigma^-; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is an isomorphism we deduce that c and d are homologous in Σ^- as well. Since Σ^- is closed we can now find two subsurfaces $\Sigma_1, \Sigma_2 \subset \Sigma^-$ such that $\partial\Sigma_1 = -c \cup d$, and such that (with the orientations induced from Σ^-) the following hold: $\Sigma_1 \cup \Sigma_2 = \Sigma$, $\partial\Sigma_2 = c \cup -d$ and $\Sigma_1 \cap \Sigma_2 = c \cup d$. Note that possibly one of Σ_1 or Σ_2 is disconnected.

Claim. The surfaces $\overline{\Sigma}_1 = \Sigma_1 \cup A$ and $\overline{\Sigma}_2 = \Sigma_2 \cup -A$ are closed, orientable and connected. Furthermore, there exists a $j \in \{1, 2\}$ such that $\text{genus}(\overline{\Sigma}_j) = \text{genus}(\Sigma)$ and such that $\overline{\Sigma}_j$ is homologous to Σ in N.

Proof. It is clear that $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are closed, orientable and connected. We give $\overline{\Sigma}_k, k = 1, 2$ the orientation which restricts to the orientation of Σ_k. We therefore only have to show the second claim.

Recall that Condition $(*)$ implies that the inclusion induced maps $H_j(\Sigma^-; \mathbb{Z}) \to H_j(M; \mathbb{Z}), j = 0, 1$ are isomorphisms. It follows from Lemma 2.6 that we also have an isomorphism $H_2(\Sigma^-; \mathbb{Z}) \to H_2(M; \mathbb{Z})$, in particular $H_2(M; \mathbb{Z})$ is generated by $[\Sigma^-]$. Now note that $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ represent elements in $H_2(M; \mathbb{Z})$. We can write $[\overline{\Sigma}_k] = l_k[\Sigma^-], k = 1, 2$ for some $l_k \in \mathbb{Z}$. Note that $[\overline{\Sigma}_1] + [\overline{\Sigma}_2] = [\Sigma^-]$, i.e. $l_1 + l_2 = 1$.

Now let $k \in \{1, \dots, r\}$ such that $A_i \subset T_k$. Recall that we assume that any component of $\Sigma \cap T_k$ represents a nontrivial element in $\pi_1(T_k)$. By Lemma 6.1 any component of $\Sigma \cap T_k$ therefore also represents a nontrivial element in $\pi_1(\Sigma)$. In particular c and d do not bound disks on Σ, which in turn implies that $\chi(\Sigma_k) \leq 0, k = 1, 2$. It follows that

$$-\chi(\overline{\Sigma}_k) = -\chi((\Sigma^- \setminus \Sigma_{3-k}) \cup A) = -\chi(\Sigma) + \chi(\Sigma_{3-k}) \leq -\chi(\Sigma), \quad k = 1, 2. \tag{3}$$

On the other hand, by the linearity of the Thurston norm and the genus minimality of Σ we have

$$-\chi(\overline{\Sigma}_k) \geq -|l_k|\chi(\Sigma), \quad k=1,2. \tag{4}$$

Now recall our assumption that $\chi(\Sigma) = ||\phi||_T > 0$. It follows that $l_1 + l_2 = 1$ and the inequalities (3) and (4) can only be satisfied if there exists a j with $l_j = 1$ and $\chi(\overline{\Sigma}_j) = \chi(\Sigma)$. (Note that necessarily $l_{3-j} = 0$ and $\overline{\Sigma}_{3-j}$ is a torus.) □

Note that there exists a proper isotopy of $A \subset M$ to an annulus $A' \subset M$ such that $\partial A'$ lies entirely in Σ_j and such that A' is disjoint from all the other $A_j, j=1,\dots,r$. We then let $\Sigma_j' \subset \Sigma_j$ be the subsurface of Σ_j such that $\partial\Sigma_j' = \partial A'$. Clearly $\Sigma' := \Sigma_j' \cup -A'$ is isotopic to $\Sigma_j \cup -A$, in particular by the claim Σ' is a closed connected surface homologous to Σ in N with $\text{genus}(\Sigma') = \text{genus}(\Sigma)$ which satisfies all the properties listed in Lemma 6.2. On the other hand we evidently have $b_0(\Sigma' \cap T_j) \leq b_0(\Sigma) - 2$. Since we did not create any new intersections we in fact have $\sum_{i=1}^r b_0(\Sigma' \cap T_i) < \sum_{i=1}^r b_0(\Sigma \cap T_i)$. But this contradicts the minimality of $\sum_{i=1}^r b_0(\Sigma \cap T_i)$ in our choice of the surface Σ. We therefore showed that the assumption that A_i does not connect Σ^- and Σ^+ leads to a contradiction. This concludes the proof of Lemma 6.3.

6.3. **Preliminaries on the components $M_1,\dots,M_n$.** We continue with the notation from the previous sections. Using Lemma 6.3 we can now prove the following lemma, which in particular implies the first statement of Theorem 6.4.

Lemma 6.6. *Assume that the inclusion induced maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ give rise to isomorphisms of the prosolvable completions. Let $i \in \{1,\dots,n\}$. Then the following hold:*

1. *The surfaces $\Sigma_i^\pm$ are connected.*
2. *For any homomorphism $\alpha : \pi_1(M) \to S$ to a finite solvable group the inclusion maps induce isomorphisms*
$$H_j(\Sigma_i^\pm; \mathbb{Z}[S]) \to H_j(M_i; \mathbb{Z}[S])$$
for $j = 0,1$.

Proof. We first consider statement (1). Recall that $M_1,\dots,M_n$ are the components of M split along $A_1,\dots,A_m$. We therefore get the following commutative diagram of Mayer–Vietoris sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \bigoplus_{k=1}^m H_j(A_k \cap \Sigma^\pm;\mathbb{Z}) & \longrightarrow & \bigoplus_{l=1}^n H_j(M_l \cap \Sigma^\pm;\mathbb{Z}) & \longrightarrow & H_j(\Sigma^\pm;\mathbb{Z}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{k=1}^m H_j(A_k;\mathbb{Z}) & \longrightarrow & \bigoplus_{l=1}^n H_j(M_l;\mathbb{Z}) & \longrightarrow & H_j(M;\mathbb{Z}) & \longrightarrow \cdots.
\end{array}$$

Note that the vertical homomorphisms on the left are isomorphisms since by Lemma 6.3 for any $i = 1, \dots, m$ the A_i is an annulus which connects Σ^- and Σ^+, i.e. A_i is a product on $A_i \cap \Sigma^\pm$. Also note that the vertical homomorphisms on the right are isomorphisms for $j = 0, 1$ by Proposition 3.2 and for $j = 2$ by Lemma 2.6. We can now appeal to the 5–lemma to deduce that the middle homomorphisms are isomorphisms as well. But for any j the middle homomorphism is a direct sum of homomorphisms, it therefore follows in particular that the maps $H_j(\Sigma_i^\pm; \mathbb{Z}) \to H_j(M_i; \mathbb{Z}), j = 0, 1$ are isomorphisms for any $i \in \{1, \dots, n\}$. In particular $b_0(\Sigma_i^\pm) = b_0(M_i) = 1$, i.e. the surfaces $\Sigma_i^\pm$ are connected.

We now prove statement (2). Let $\alpha : \pi_1(M) \to S$ be a homomorphism to a finite solvable group. Recall that by Lemmas 2.12 and 2.6 we have that the inclusion induced maps $H_j(\Sigma^\pm; \mathbb{Z}[S]) \to H_j(M; \mathbb{Z}[S])$ are isomorphisms for $j = 0, 1, 2$. It now follows from the commutative diagram of Mayer–Vietoris sequences as above, but with $\mathbb{Z}[S]$–coefficients (cf. [FK06] for details) that

$$H_j(\Sigma_i^\pm; \mathbb{Z}[S]) \to H_j(M_i; \mathbb{Z}[S])$$

is an isomorphism for any $i \in \{1, \dots, n\}$ and $j = 0, 1$.

□

The following lemma in particular implies the second statement of Theorem 6.4.

Lemma 6.7. *For any pair (i, j) such that $M_i \subset N_j$ we have a commutative diagram of injective maps as follows:*

$$\begin{array}{ccccc} \pi_1(\Sigma_i^\pm) & \longrightarrow & \pi_1(M_i) & \longrightarrow & \pi_1(N_j) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(\Sigma^\pm) & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(N). \end{array}$$

Proof. First note that since Σ is incompressible we know that the two bottom maps are injective. Furthermore recall that N_j is a JSJ component of N, i.e. a component of the result of cutting N along incompressible tori, hence $\pi_1(N_j) \to \pi_1(N)$ is injective.

Claim. For any $k \in \{1, \dots, n\}$ the maps $\pi_1(\Sigma_k^\pm) \to \pi_1(\Sigma)$ are injective.

Let c be a component of $\Sigma \cap T_l$ for some $l \in \{1, \dots, r\}$. Recall that by our choice of tori $T_1, \dots, T_r$ the curve c represents a nontrivial element in $\pi_1(T_l)$. By Lemma 6.1 the curve c also represents a nontrivial element in $\pi_1(\Sigma)$. In particular none of the components of $\Sigma^\pm \setminus \Sigma_k^\pm$ are disks and therefore the maps $\pi_1(\Sigma_k^\pm) \to \pi_1(\Sigma^\pm)$ are injective. This concludes the proof of the claim.

Now let $K = \{k \in \{1, \dots, n\} \mid M_k \subset N_j\}$. It follows from the claim and the above commutative diagram that for any $k \in K$ the inclusion induced map $\pi_1(\Sigma_k) \to \pi_1(N_j)$ is injective, i.e. for any $k \in K$ the surface $\Sigma_k \subset N_j$ is incompressible. Since M_i is a component of cutting N_j along the incompressible surfaces $\Sigma_k^- \subset N_j, k \in K$ we have that $\pi_1(M_i) \to \pi_1(N_j)$ is injective.

By commutativity of the above diagram we now obtain that all other maps are injective as well. □

6.4. **The conclusion of the proof of Theorem 6.4.** In this section we will finally prove the third statement of Theorem 6.4. The main ingredient in the proof is the following result.

Proposition 6.8. *Let Σ be a closed surface and $\Sigma' \subset \Sigma$ a connected subsurface such that $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is injective. Let $\alpha : \pi_1(\Sigma') \to S$ be a homomorphism to a finite solvable group. Then there exists a homomorphism to a finite solvable group $\beta : \pi_1(\Sigma) \to T$ and a homomorphism $\pi : T' := Im\{\pi_1(\Sigma') \to T\} \to S$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
\pi_1(\Sigma') & & \longrightarrow & & \pi_1(\Sigma) \\
\downarrow \alpha & & \searrow & & \downarrow \beta \\
S & \xleftarrow{\pi} & T' & \longrightarrow & T.
\end{array}$$

Put differently, the prosolvable topology on $\pi_1(\Sigma')$ agrees with the topology on $\pi_1(\Sigma')$ induced from the prosolvable topology on $\pi_1(\Sigma)$.

Remark. Note that in general $H_1(\Sigma';\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z})$ is not injective, even if $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is an injection. In particular in general a homomorphism $\pi_1(\Sigma') \to S$ to an abelian group will not extend to a homomorphism from $\pi_1(\Sigma)$ to an abelian group. This shows that in general we can not take $T = S$ or T of the same solvability length as S in the above proposition.

Proof. The statement of the proposition is trivial if $\Sigma' = \Sigma$, we will therefore henceforth only consider the case that $\Sigma' \neq \Sigma$. Let $\alpha : \pi_1(\Sigma') \to S$ be a homomorphism to a finite solvable group. It suffices to show that there exists a homomorphism $\beta : \pi_1(\Sigma) \to T$ to a finite solvable group such that $\mathrm{Ker}(\beta) \cap \pi_1(\Sigma') \subset \mathrm{Ker}(\alpha)$.

Denote by $\Sigma_1, \dots, \Sigma_l$ the components of $\overline{\Sigma \setminus \Sigma'}$. Note that the condition that $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is injective is equivalent to saying that none of the subsurfaces $\Sigma_1, \dots, \Sigma_l$ is a disk.

It is straightforward to see that for each $j = 1, \dots, l$ we can find an annulus $A_j \in \mathrm{int}(\Sigma_j)$ such that $(\Sigma' \cup \Sigma_j) \setminus A_j$ is still connected.

Now let $\Sigma'' = \overline{\Sigma \setminus \cup_{j \in J} A_j}$. Clearly Σ'' is a connected surface with boundary. By assumption $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is injective. Since $\pi_1(\Sigma') \to \pi_1(\Sigma)$ factors through $\pi_1(\Sigma'')$ we see that Σ' is a subsurface of Σ'' such that $\pi_1(\Sigma') \to \pi_1(\Sigma'')$ is injective. Since Σ'' is a surface with boundary (contrary to Σ) this implies that $\pi_1(\Sigma')$ is in fact a free factor of $\pi_1(\Sigma'')$, i.e. we have an isomorphism $\gamma : \pi_1(\Sigma'') \xrightarrow{\cong} \pi_1(\Sigma') * F$ where F is a free group such that the map $\pi_1(\Sigma'') \xrightarrow{\gamma} \pi_1(\Sigma') * F \to \pi_1(\Sigma')$ splits the inclusion induced map $\pi_1(\Sigma') \to \pi_1(\Sigma'')$.

We now write $\pi := \pi_1(\Sigma'')$ and we denote by α'' the projection map $\pi \to \pi/\pi(S)$ (We refer to Section 3.4 for the definition and the properties of the characteristic

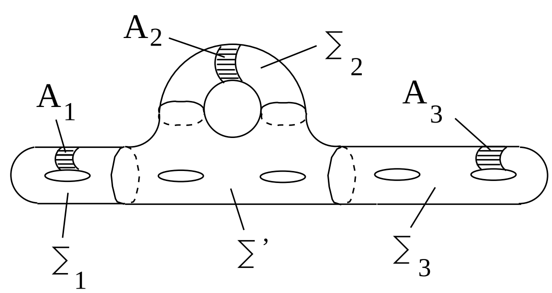


FIGURE 1. Surface $\Sigma' \subset \Sigma$ with annuli $A_i \subset \Sigma_i, i = 1, 2, 3$.

subgroup $\pi(S)$ of π). We can extend $\alpha : \pi_1(\Sigma') \to S$ to $\pi_1(\Sigma'') \xrightarrow{\gamma} \pi_1(\Sigma') * F \to \pi_1(\Sigma') \xrightarrow{\alpha} S$. It follows immediately that $\mathrm{Ker}(\alpha'') \cap \pi_1(\Sigma') \subset \mathrm{Ker}(\alpha)$.

We will now extend $\alpha'' : \pi_1(\Sigma'') \to \pi/\pi(S)$ to a homomorphism $\beta : \pi_1(\Sigma) \to \mathbb{Z}/n \ltimes \pi/\pi(S)$ where $1 \in \mathbb{Z}/n$ acts in an appropriate way on $\pi/\pi(S)$. In order to do this we will first study the relationship between $\pi_1(\Sigma'')$ and $\pi_1(\Sigma)$.

Evidently $\Sigma = \Sigma'' \cup \cup_{i=1}^k A_i$. We pick an orientation for Σ and give $A_1, \dots, A_k$ the induced orientations. We write $\partial A_i = -a_i \cup b_i, i = 1, \dots, k$ (see Figure 2). We

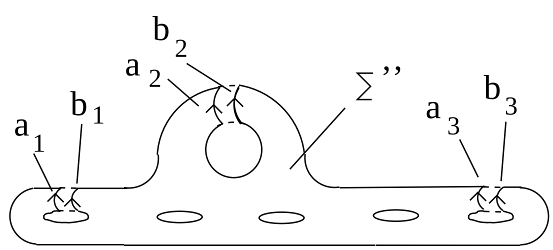


FIGURE 2. Surface $\Sigma'' \subset \Sigma$ with oriented boundary curves a_i, b_i.

now pick a base point for Σ''. We can find based curves $c_1, \dots, c_l, d_1, \dots, d_l$ and paths from the base point to the curves $a_1, \dots, a_k, b_1, \dots, b_k$ (and from now on we do not distinguish in the notation between curves and based curves) such that

$$\pi = \langle a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_l, d_1, \dots, d_l \,|\, a_1 \dots a_k b_k^{-1} \dots b_1^{-1} = [c_l, d_l] \dots [c_1, d_1] \rangle.$$

(See Figure 3 for an illustration.) By the van Kampen theorem we then have

$$\pi_1(\Sigma) = \langle \pi_1(\Sigma''), t_1, \dots, t_k \,|\, t_i a_i t_i^{-1} = b_i, i = 1, \dots, k \rangle.$$

Claim. There exists an automorphism $\varphi : \pi \to \pi$ such that $\varphi(a_i) = b_i$ and $\varphi(b_i) = a_i$ for any $i \in \{1, \dots, k\}$.

Let Γ be the free group generated by $a_i, b_i, i = 1, \dots, k$ and $c_i, d_i, i = 1, \dots, l$ and consider the isomorphism $\varphi : \Gamma \to \Gamma$ defined by $\varphi(a_i) = b_i, \varphi(b_i) = a_i, i =$

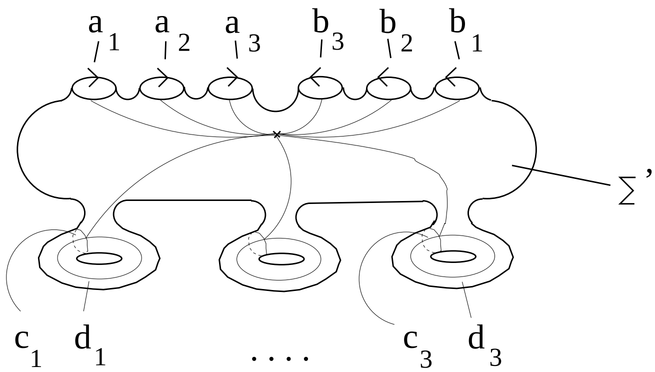


FIGURE 3. Surface $\Sigma'' \subset \Sigma$ with oriented based curves a_i, b_i, c_i, d_i.

$1, \dots, k$ and $\varphi(c_i) = d_{l+1-i}, \varphi(d_i) = c_{l+1-i}, i = 1, \dots, l$. In the following we write $w = [c_l, d_l] \dots [c_1, d_1]$ and we write $r = a_1 \dots a_k b_k^{-1} \dots b_1^{-1} \cdot [c_1, d_1]^{-1} \dots [c_l, d_l]^{-1}$ for the relator. Note that we have a canonical isomorphism $\pi \cong \Gamma / \langle\langle r \rangle\rangle$. We calculate

$$\begin{aligned} \varphi(r) &= \varphi\big(a_1 \dots a_k b_k^{-1} \dots b_1^{-1} \cdot [c_1, d_1]^{-1} \dots [c_l, d_l]^{-1}\big) \\ &= b_1 \dots b_k a_k^{-1} \dots a_1^{-1} \cdot [d_l, c_l]^{-1} \dots [d_1, c_1]^{-1} \\ &= b_1 \dots b_k a_k^{-1} \dots a_1^{-1} \cdot [c_l, d_l] \dots [c_1, d_1] \\ &= w^{-1} [c_l, d_l] \dots [c_1, d_1] b_1 \dots b_k a_k^{-1} \dots a_1^{-1} w \\ &= w^{-1} r^{-1} w. \end{aligned}$$

This shows that φ restricts to an automorphism of the subgroup of Γ normally generated by the relator r. In particular φ descends to an automorphism of π. This concludes the proof of the claim.

Recall that $\pi(S)$ is a characteristic subgroup of π, hence $\varphi : \pi \to \pi$ descends to an automorphism $\pi/\pi(S) \to \pi/\pi(S)$ which we again denote by φ. Furthermore recall that $\pi/\pi(S)$ is a finite solvable group. Since $\pi/\pi(S)$ is finite there exists $n > 0$ such that $\varphi^n : \pi/\pi(S) \to \pi/\pi(S)$ acts as the identity. We can therefore consider the semidirect product $\mathbb{Z}/n \ltimes \pi/\pi(S)$ where $1 \in \mathbb{Z}/n$ acts on $\pi/\pi(S)$ via φ.

It is now straightforward to check that the assignment

$$\begin{aligned} g &\mapsto (0, \alpha''(g)), \;\; g \in \pi_1(\Sigma''), \\ t_i &\mapsto (1, 0) \end{aligned}$$

defines a homomorphism

$$\pi_1(\Sigma) = \langle \pi_1(\Sigma''), t_1, \dots, t_k \,|\, t_i a_i t_i^{-1} = b_i, i = 1, \dots, k \rangle \to \mathbb{Z}/n \ltimes \pi/\pi(S)$$

which we denote by β. Clearly $\beta : \pi_1(\Sigma) \to \mathbb{Z}/n \ltimes \pi/\pi(S)$ restricts to $\pi_1(\Sigma'') \to \pi/\pi(S)$ and hence has the required properties. □

We can now prove the third statement of Theorem 6.4.

Proof of Theorem 6.4 (3). In light of Lemma 6.6 (together with Corollary 2.3) and Lemma 6.7 it suffices to show the following claim:

Claim. Let M be a 3–manifold and $\Sigma \subset \partial M$ such that $\pi_1(\Sigma) \to \pi_1(M)$ induces an isomorphism of prosolvable completions. Furthermore let $M' \subset M$ be a submanifold with the following properties:

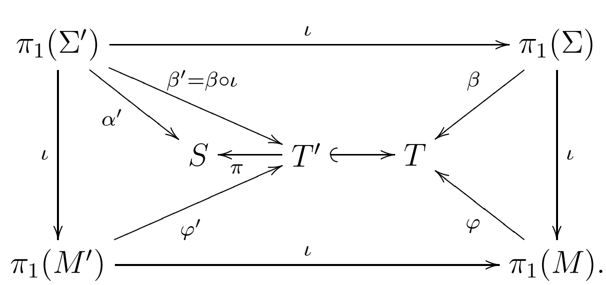
(A) $\Sigma' := \Sigma \cap M'$ is a connected subsurface of Σ',
(B) $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is injective, and
(C) for any homomorphism $\alpha : \pi_1(M) \to S$ to a finite solvable group the inclusion map induces isomorphisms
$$H_j(\Sigma'; \mathbb{Z}[S]) \to H_j(M'; \mathbb{Z}[S])$$
for $j = 0, 1$ and we have
$$\operatorname{Im}\{\pi_1(\Sigma') \to \pi_1(M) \xrightarrow{\alpha} S\} = \operatorname{Im}\{\pi_1(M') \to \pi_1(M) \xrightarrow{\alpha} S\}.$$

Then $\pi_1(\Sigma') \to \pi_1(M')$ induces an isomorphism of prosolvable completions.

By Lemma 2.10 we have to show that for any finite solvable group S the map
$$\iota^* : \operatorname{Hom}(\pi_1(M'), S) \to \operatorname{Hom}(\pi_1(\Sigma'), S)$$
is a bijection.

So let S be a finite solvable group. We first show that $\iota^* : \operatorname{Hom}(\pi_1(M'), S) \to \operatorname{Hom}(\pi_1(\Sigma'), S)$ is surjective. The various groups and maps in the proof are summarized in the diagram below. Assume we are given a homomorphism $\alpha' : \pi_1(\Sigma') \to S$. By (B) and Proposition 6.8 there exists a homomorphism $\beta : \pi_1(\Sigma) \to T$ to a finite solvable group and a homomorphism $\pi : \operatorname{Im}\{\pi_1(\Sigma') \to T\} \to S$ such that $\pi \circ (\beta \circ \iota) = \alpha'$. We write $T' = \operatorname{Im}\{\pi_1(\Sigma') \to T\}$ and $\beta' = \beta \circ \iota : \pi_1(\Sigma') \to T'$.

By our assumption that $\pi_1(\Sigma) \to \pi_1(M)$ induce isomorphisms of prosolvable completions and by Lemma 2.10 there exists a homomorphism $\varphi : \pi_1(M) \to T$ such that $\beta = \varphi \circ \iota$. By (C) we have
$$\operatorname{Im}\{\pi_1(M') \to \pi_1(M) \xrightarrow{\varphi} T\} = \operatorname{Im}\{\pi_1(\Sigma') \xrightarrow{\iota} \pi_1(M) \xrightarrow{\varphi} T\} = \operatorname{Im}\{\pi_1(\Sigma') \xrightarrow{\beta} T\} = T'.$$
Now denote the induced homomorphism $\pi_1(M') \to T'$ by φ'. Clearly $\varphi' \circ \iota = \beta'$. Hence $\alpha' = \pi \circ \beta' = (\pi \circ \varphi') \circ \iota$. This shows that $\iota^* : \operatorname{Hom}(\pi_1(M'), S) \to \operatorname{Hom}(\pi_1(\Sigma'), S)$ is surjective. The following diagram summarizes the homomorphisms in the proof of the previous claim:



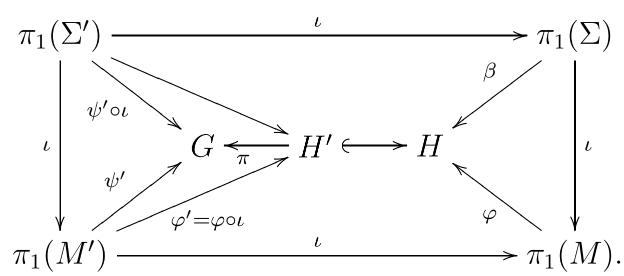
We now show that $\iota^* : \mathrm{Hom}(\pi_1(M'), S) \to \mathrm{Hom}(\pi_1(\Sigma'), S)$ is injective. Let $\alpha_1, \alpha_2 : \pi_1(M') \to S$ be two different homomorphisms. Let n be the maximal integer such that the homomorphisms $\pi_1(M') \to S \to S/S^{(n)}$ induced by α_1 and α_2 agree. We will show that the restriction to $\pi_1(\Sigma')$ of the maps $\pi_1(M') \to S \to S/S^{(n+1)}$ induced by α_1 and α_2 are different. Without loss of generality we can therefore assume that $S = S/S^{(n+1)}$.

We denote by ψ' the homomorphism $\pi_1(M') \to S \to S/S^{(n)} =: G$, induced by α_1 and α_2.

Claim. There exists a homomorphism $\varphi : \pi_1(M) \to H$ to a finite solvable group and a homomorphism $\pi : \mathrm{Im}\{\pi_1(M') \to \pi_1(M) \to H\} \to G$ such that $\psi' = \pi \circ (\varphi \circ \iota)$.

By (B) and Proposition 6.8 there exists a homomorphism $\beta : \pi_1(\Sigma) \to H$ to a finite solvable group H and a homomorphism $\pi : \mathrm{Im}\{\pi_1(\Sigma') \to H\} \to G$ such that $\pi' \circ (\beta \circ \iota) = \psi' \circ \iota$. By our assumption and by Lemma 2.10 there exists a homomorphism $\varphi : \pi_1(M) \to H$ such that $\beta = \varphi \circ \iota$. By (C) we have $\mathrm{Im}\{\pi_1(\Sigma') \to H\} = \mathrm{Im}\{\pi_1(M') \to H\} =: H'$. It is now clear that φ and π have the required properties. This concludes the proof of the claim.

The following diagram summarizes the homomorphisms in the proof of the previous claim:



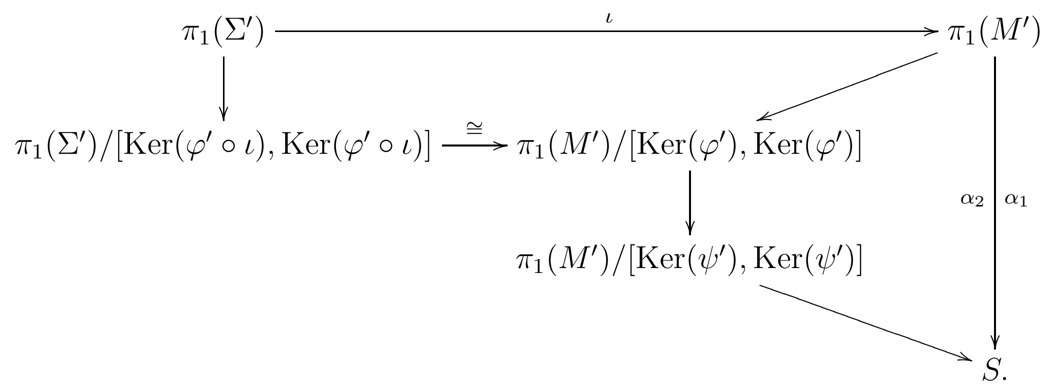
We now apply (C) and Corollary 2.5 to the case $A = \pi_1(\Sigma'), B = \pi_1(M')$ and $\varphi' : B \to H'$ to conclude that the inclusion map induces an isomorphism

$$\pi_1(\Sigma')/[\mathrm{Ker}(\varphi' \circ \iota), \mathrm{Ker}(\varphi' \circ \iota)] \to \pi_1(M')/[\mathrm{Ker}(\varphi'), \mathrm{Ker}(\varphi'].$$

We now consider the homomorphisms $\alpha_1, \alpha_2 : \pi_1(M') \to S = S/S^{(n+1)}$. First note that they factor through $\pi_1(M')/[\mathrm{Ker}(\psi'), \mathrm{Ker}(\psi')]$. Now note that $\mathrm{Ker}(\varphi') \subset \mathrm{Ker}(\psi') \subset \pi_1(M')$, in particular we have a surjection $\pi_1(M')/\mathrm{Ker}(\varphi') \to \pi_1(M')/\mathrm{Ker}(\psi')$ which gives rise to a surjection

$$\pi_1(M')/[\mathrm{Ker}(\varphi'), \mathrm{Ker}(\varphi')] \to \pi_1(M')/[\mathrm{Ker}(\psi'), \mathrm{Ker}(\psi')].$$

In particular α_1, α_2 factor through $\pi_1(M')/[\mathrm{Ker}(\varphi'), \mathrm{Ker}(\varphi')]$. We therefore obtain the following commutative diagram



It is now clear that $\alpha_1 \circ \iota$ and $\alpha_2 \circ \iota$ are different. This concludes the proof that $\iota^* : \mathrm{Hom}(\pi_1(M'), S) \to \mathrm{Hom}(\pi_1(\Sigma'), S)$ is injective. As we pointed out before, it now follows from Lemma 2.10 that $\iota : \pi_1(\Sigma') \to \pi_1(M')$ induces an isomorphism of prosolvable completions. □

7. The proof of Theorem 1.2

We start out with the following two results which allow us to reduce the proof of Theorem 1.2 to the case of closed prime 3–manifolds.

Lemma 7.1. *Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be nontrivial. If $\Delta^{\alpha}_{N,\phi}$ is nonzero for any homomorphism $\alpha : \pi_1(N) \to G$ to a finite group G, then N is prime.*

Note that the main idea for the proof of this lemma can already be found in [McC01].

Proof. Let N be a 3–manifold with empty or toroidal boundary which is not prime, i.e. $N = N_1 \# N_2$ with $N_1, N_2 \neq S^3$. We have to show that there exists a homomorphism $\alpha : \pi_1(N) \to G$ to a finite group such that $\Delta^{\alpha}_{N,\phi} = 0$. Recall that by Lemma 2.9 we have $\Delta^{\alpha}_{N,\phi} = 0$ if and only if $H_1(N;\mathbb{Q}[G][t^{\pm 1}])$ is not $\mathbb{Q}[t^{\pm 1}]$–torsion. Note that we can write $N = (N_1 \setminus \mathrm{int} D^3) \cup_{S^2} (N_2 \setminus \mathrm{int} D^3)$ and that $H_j(N_i \setminus \mathrm{int} D^3; \mathbb{Q}[t^{\pm 1}]) = H_j(N_i;\mathbb{Q}[t^{\pm 1}])$ for $j = 0, 1$ and $i = 1, 2$. The Mayer–Vietoris sequence corresponding to $N = (N_1 \setminus \mathrm{int} D^3) \cup_{S^2} (N_2 \setminus \mathrm{int} D^3)$ now gives rise to the following long exact sequence:

$$\begin{array}{ccccccc} H_1(S^2;\mathbb{Q}[t^{\pm 1}]) & \to & H_1(N_1;\mathbb{Q}[t^{\pm 1}]) \oplus H_1(N_2;\mathbb{Q}[t^{\pm 1}]) & \to & H_1(N;\mathbb{Q}[t^{\pm 1}]) & \to & \\ H_0(S^2;\mathbb{Q}[t^{\pm 1}]) & \to & H_0(N_1;\mathbb{Q}[t^{\pm 1}]) \oplus H_0(N_2;\mathbb{Q}[t^{\pm 1}]) & \to & H_0(N;\mathbb{Q}[t^{\pm 1}]) & \to & 0. \end{array}$$

A straightforward computation shows that $H_0(S^2;\mathbb{Q}[t^{\pm 1}]) = \mathbb{Q}[t^{\pm 1}]$ and $H_1(S^2;\mathbb{Q}[t^{\pm 1}]) = 0$.

First assume that $b_1(N_i) > 0$ for $i = 1, 2$. Denote by ϕ_i the restriction of $\phi : H_1(N;\mathbb{Q}) \to \mathbb{Q}$ to $H_1(N_i;\mathbb{Q})$. If ϕ_i is nontrivial for $i = 1$ and $i = 2$, then it follows from Lemma 2.2 and Lemma 2.9 that $H_0(N_i;\mathbb{Q}[t^{\pm 1}])$ is $\mathbb{Q}[t^{\pm 1}]$–torsion for $i = 1, 2$. On the other hand we have $H_0(S^2;\mathbb{Q}[t^{\pm 1}]) = \mathbb{Q}[t^{\pm 1}]$. It follows from the above Mayer–Vietoris sequence that $H_1(N;\mathbb{Q}[t^{\pm 1}])$ can not be $\mathbb{Q}[t^{\pm 1}]$–torsion. On the other hand, if ϕ_i is trivial for some $i \in \{1, 2\}$, then $H_1(N_i;\mathbb{Q}[t^{\pm 1}])$ is isomorphic to $H_1(N_i;\mathbb{Q}) \otimes \mathbb{Q}[t^{\pm 1}]$, in particular $H_1(N_i;\mathbb{Q}[t^{\pm 1}])$ is not $\mathbb{Q}[t^{\pm 1}]$–torsion, and using that $H_1(S^2;\mathbb{Q}[t^{\pm 1}]) = 0$ it follows again from the above Mayer–Vietoris sequence that $H_1(N;\mathbb{Q}[t^{\pm 1}])$ is not $\mathbb{Q}[t^{\pm 1}]$–torsion.

Now assume that either $b_1(N_1) = 0$ or $b_1(N_2) = 0$. Without loss of generality we can assume that $b_1(N_2) = 0$. Since $b_1(N) = b_1(N_1) + b_1(N_2)$ we have $b_1(N_1) > 0$. By the Geometrization Conjecture $\pi_1(N_2)$ is nontrivial and residually finite (cf. [Th82] and [He87]), in particular there exists an epimorphism $\alpha : \pi_1(N_2) \to G$ onto a nontrivial finite group G. Denote the homomorphism $\pi_1(N) = \pi_1(N_1) * \pi_1(N_2) \to \pi_1(N_2) \to G$ by α as well. Then by Lemma 2.8 we have

$$\Delta^{\alpha}_{N,\phi} = \Delta_{N_G,\phi_G}$$

where $p : N_G \to N$ is the cover of N corresponding to α and $\phi_G = p^*(\phi)$. But the prime decomposition of N_G has $|G|$ copies of N_1. By the argument above we now have that $\Delta_{N_G,\phi_G} = 0$, which implies that $\Delta^{\alpha}_{N,\phi} = \Delta_{N_G,\phi_G} = 0$. □

Lemma 7.2. *Let N be an irreducible 3–manifold with non–empty toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be nontrivial. Let $W = N \cup_{\partial N} N$ be the double of N along the boundary of N. Let $\Phi = p^*(\phi) \in H^1(W;\mathbb{Z})$ where $p : W \to N$ denotes the folding map. Then the following hold:*

1. *(W, Φ) fibers over S^1 if and only if (N, ϕ) fibers over S^1,*
2. *if (N, ϕ) satisfies Condition $(*)$, then (W, Φ) satisfies Condition $(*)$.*

In the proof of Lemma 7.2 we will make use of the following well–known lemma. We refer to [EN85, Theorem 4.2] and [Ro74] for the first statement, and to [EN85, p. 33] for the second statement.

Lemma 7.3. *Let Y be a closed 3–manifold. Let $T \subset Y$ be a union of incompressible tori such that T separates Y into two connected components Y_1 and Y_2. Let $\psi \in H^1(Y;\mathbb{Z})$. Then the following hold:*

1. *If $||\phi||_{T,Y} > 0$, then (Y, ψ) fibers over S^1 if and only if $(Y_1, \psi|_{Y_1})$ and $(Y_2, \psi|_{Y_2})$ fiber over S^1,*
2. *$||\psi||_{T,Y} = ||\psi|_{Y_1}||_{T,Y_1} + ||\psi|_{Y_2}||_{T,Y_2}$.*

Proof of Lemma 7.2. First note that an irreducible 3–manifold with boundary a union of tori has compressible boundary if and only if it is the solid torus. Since the lemma holds trivially in the case that $N = S^1 \times D^2$ we will from now on assume that N has incompressible boundary. This implies in particular that $||\phi||_T > 0$. The first statement is now an immediate consequence of Lemma 7.3 and the observation that $\Phi|_N = \phi$.

Now assume that (N, ϕ) satisfies Condition $(*)$. In the following we write $N_i = N, i = 1, 2$ and we think of W as $W = N_1 \cup_{\partial N_1 = \partial N_2} N_2$. Let $\alpha : \pi_1(N) \to G$ be a homomorphism to a finite group G. We write $n = |G|$, $V = \mathbb{Z}[G]$ and we slightly abuse notation by denoting by α the representation $\pi_1(W) \to \text{Aut}(V)$ given by left multiplication. We have to show that $\Delta^\alpha_{W,\Phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and that

$$\deg(\Delta^\alpha_{W,\Phi}) - \deg(\Delta^\alpha_{W,\Phi,0}) - \deg(\Delta^\alpha_{W,\Phi,2}) = n \,\|\Phi\|_T$$

(here we used Lemma 2.8 to rephrase the last condition). For any submanifold $X \subset W$ we denote the restriction of Φ and α to $\pi_1(X)$ by Φ and α as well. Evidently the restriction of Φ to $N = N_i, i = 1, 2$ just agrees with ϕ.

In order to prove the claims on $\Delta^\alpha_{W,\Phi}$ we will in the following express $\Delta^\alpha_{W,\Phi}$ in terms of $\Delta^\alpha_{N_i,\phi_i}$, $i = 1, 2$. The following statement combines the assumption that (N, ϕ) satisfies Condition $(*)$ with Lemmas 2.8 and 2.9.

Fact 1. For $i = 1, 2$ we have

$$\deg(\Delta^\alpha_{N_i,\Phi}) - \deg(\Delta^\alpha_{N_i,\Phi,0}) - \deg(\Delta^\alpha_{N_i,\Phi,2}) = n \,\|\Phi\|_{T,N_i}.$$

Furthermore for all j we have that $\Delta^\alpha_{N_i,\Phi,j}$ is monic.

We now turn to the twisted Alexander polynomials of the boundary tori of ∂N. The following is an immediate consequence of Theorem 6.5.

Fact 2. If $\Delta_{N,\phi} \neq 0$ (in particular if (N, ϕ) satisfies Condition $(*)$), then for any boundary component $T \subset \partial N$ the restriction of ϕ (and hence of Φ) to $\pi_1(T)$ is nontrivial.

This fact and a straightforward computation now gives us the following fact (cf. e.g. [KL99]).

Fact 3. Let $T \subset \partial N$ be any boundary component. Then

1. $\Delta^\alpha_{T,\Phi,i}$ is monic for any i,
2. $H_i(T; V[t^{\pm 1}]) = 0$ for $i \geq 2$, in particular $\Delta^\alpha_{T,\Phi,i} = 1$ for $i \geq 2$,
3. $\Delta^\alpha_{T,\Phi,0} = \Delta^\alpha_{T,\Phi,1}$.

We now consider the following Mayer–Vietoris sequence:

$$\begin{array}{ccccccc} 0 & \to & H_2(N_1; V[t^{\pm 1}]) \oplus H_2(N_2; V[t^{\pm 1}]) & \to & H_2(W; V[t^{\pm 1}]) & \to & \\ H_1(\partial N; V[t^{\pm 1}]) & \to & H_1(N_1; V[t^{\pm 1}]) \oplus H_1(N_2; V[t^{\pm 1}]) & \to & H_1(W; V[t^{\pm 1}]) & \to & \\ H_0(\partial N; V[t^{\pm 1}]) & \to & H_0(N_1; V[t^{\pm 1}]) \oplus H_0(N_2; V[t^{\pm 1}]) & \to & H_0(W; V[t^{\pm 1}]) & \to & 0. \end{array}$$

Recall that we assume that (N, ϕ) (and hence $(N_i, \phi), i = 1, 2$) satisfy Condition $(*)$. By Lemmas 2.8 and 2.9 and Facts 1 and 3 it follows that all homology modules in the above long exact sequence but possibly $H_1(W; V[t^{\pm 1}])$ and $H_2(W; V[t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion. But then evidently $H_1(W; V[t^{\pm 1}])$ and $H_2(W; V[t^{\pm 1}])$ also have to be

$\mathbb{Z}[t^{\pm 1}]$–torsion. Furthermore it follows from Fact 3, [Tu01, Theorem 3.4] and [Tu01, Theorem 4.7] that

$$\frac{\Delta^\alpha_{W,\Phi,1}}{\Delta^\alpha_{W,\Phi,0}\Delta^\alpha_{W,\Phi,2}} = \frac{\Delta^\alpha_{N_1,\Phi,1}}{\Delta^\alpha_{N_1,\Phi,0}\Delta^\alpha_{N_1,\Phi,2}} \cdot \frac{\Delta^\alpha_{N_2,\Phi,1}}{\Delta^\alpha_{N_2,\Phi,0}\Delta^\alpha_{N_2,\Phi,0}}. \tag{5}$$

Note that $\Delta^\alpha_{W,\Phi,0}$ and $\Delta^\alpha_{W,\Phi,2}$ are monic by Lemma 2.8, it now follows from Fact 1 and Equality (5) that $\Delta^\alpha_{W,\Phi,1}$ is monic as desired.

Finally we can appeal to Lemma 7.3 to conclude that $||\Phi||_{T,W} = ||\Phi||_{T,N_1} + ||\Phi||_{T,N_2}$. It therefore follows from Fact 1 and Equation (5) that

$$\deg(\Delta^\alpha_{W,\Phi}) - \deg(\Delta^\alpha_{W,\Phi,0}) - \deg(\Delta^\alpha_{W,\Phi,2}) = n\,\|\Phi\|_{T,W}$$

as required. □

Let N be a 3–manifold with empty or toroidal boundary. We write $\pi = \pi_1(N)$. Let $\tilde{\pi} \subset \pi$ be a finite index subgroup and $\phi \in H^1(N;\mathbb{Z})$ nontrivial. We now say that the pair $(\tilde{\pi}, \phi)$ has Property (M) if the twisted Alexander polynomial $\Delta^{\pi/\tilde{\pi}}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and if

$$\deg(\Delta^{\pi/\tilde{\pi}}_{N,\phi}) = [\pi : \tilde{\pi}]\,\|\phi\|_T + (1 + b_3(N))\mathrm{div}\phi_{\tilde{\pi}}$$

holds.

The first statement of the following lemma is well–known, the second one can be easily verified and the third is an immediate consequence of the second statement.

Lemma 7.4. *Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be nontrivial. Let $k \neq 0 \in \mathbb{Z}$. Then the following hold:*

1. *(N, ϕ) fibers over S^1 if and only if $(N, k\phi)$ fibers over S^1,*
2. *Let $\tilde{\pi} \subset \pi$ be a finite index subgroup. Then $(\tilde{\pi}, \phi)$ has Property (M) if and only if $(\tilde{\pi}, k\phi)$ has Property (M),*
3. *(N, ϕ) satisfies Condition $(*)$ if and only if $(N, k\phi)$ satisfies Condition $(*)$.*

We will also need the following lemma.

Lemma 7.5. *Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be non–trivial. Suppose that all finite index* normal *subgroups of $\pi_1(N)$ have Property (M), then in fact all finite index subgroups of $\pi_1(N)$ have Property (M).*

Proof. We write $\pi := \pi_1(N)$. Let $\phi \in H^1(N;\mathbb{Z})$ be non–trivial. By Lemma 7.4 (2) we can without loss of generality assume that ϕ is primitive. Let $\tilde{\pi} \subset \pi$ be a finite index subgroup. We denote by $\hat{\pi} \subset \pi$ the core of $\tilde{\pi}$, i.e. $\hat{\pi} = \cap_{g\in\pi} g\tilde{\pi}g^{-1}$. Note that $\hat{\pi}$ is normal in π and contained in $\tilde{\pi}$.

By Proposition 3.1 the class ϕ is dual to a connected Thurston norm minimizing surface Σ. We write $A = \pi_1(\Sigma)$ and $B = \pi_1(N \setminus \nu\Sigma)$ as before.

We write $\hat{B} := B \cap \hat{\pi}$ and $\hat{A}^{\pm} := (\iota_{\pm})^{-1}(\hat{B})$. We now pick representatives $g_1, \dots, g_m$ for the equivalence classes of $B\backslash\pi/\tilde{\pi}$. For $i = 1, \dots, m$ we write $\tilde{B}_i := B \cap g_i\tilde{\pi}g_i^{-1}$ and $\tilde{A}_i^{\pm} := (\iota_{\pm})^{-1}(\tilde{B}_i)$.

Since $\hat{\pi} \subset \pi$ is normal and since we assume that normal finite index subgroups have Property (M) we can now apply Proposition 3.2 and Lemma 2.1 to conclude that

$$\iota_{\pm} : H_j(A; \mathbb{Z}[B/\hat{B}]) \to H_j(B; \mathbb{Z}[B/\hat{B}])$$

are isomorphisms for $j = 0, 1$. It now follows from Corollary 2.5 that the maps

$$\iota_{\pm} : A/\hat{A}^{\pm} \to B/\hat{B} \text{ and } \iota_{\pm} : A/[\hat{A}^{\pm}, \hat{A}^{\pm}] \to B/[\hat{B}, \hat{B}]$$

are isomorphisms. Recall that $\hat{\pi}$ is normal in π, it follows that $\hat{B} \subset B$ is normal and for any i we have $\hat{B} = B \cap g_i\hat{\pi}g_i^{-1} \subset B \cap g_i\tilde{\pi}g_i^{-1} = \tilde{B}_i$. We now deduce from Lemma 2.7 that

$$\iota_{\pm} : A/\tilde{A}_i^{\pm} \to B/\tilde{B}_i \text{ and } \iota_{\pm} : A/[\tilde{A}_i^{\pm}, \tilde{A}_i^{\pm}] \to B/[\tilde{B}_i, \tilde{B}_i]$$

are bijections for $i = 1, \dots, m$. It now follows from Lemma 2.4 that the maps

$$\iota_{\pm} : H_j(A; \mathbb{Z}[\pi/\tilde{\pi}]) \to H_j(B; \mathbb{Z}[\pi/\tilde{\pi}])$$

are isomorphisms. It now follows from Proposition 3.2 that $\tilde{\pi}$ also has Property (M). □

We will now use the previous lemma to prove the following lemma.

Lemma 7.6. *Let N be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(N; \mathbb{Z})$ be non–trivial. Let $p : N' \to N$ be a finite cover. We write $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$. Then the following hold:*

1. *ϕ' is nontrivial,*
2. *(N, ϕ) fibers over S^1 if and only if (N', ϕ') fibers over S^1,*
3. *if (N, ϕ) satisfies Condition $(*)$, then (N', ϕ') satisfies Condition $(*)$.*

Proof. The first statement is well–known. The second statement is a consequence of [He76, Theorem 10.5]. We now turn to the third statement. Assume that (N, ϕ) satisfies Condition $(*)$.

Let $\tilde{\pi}$ be a normal finite index subgroup of $\pi' = \pi_1(N')$. We have to show that $(\tilde{\pi}, \phi')$ has Property (M). Note that $\tilde{\pi}$ viewed as a subgroup of $\pi = \pi_1(N)$ is not necessarily normal. It nonetheless follows from the assumption that (N, ϕ) satisfies Condition $(*)$ and from Lemma 7.5 that the twisted Alexander polynomial $\Delta_{N,\phi}^{\pi/\tilde{\pi}} \in \mathbb{Z}[t^{\pm 1}]$ is monic and that

$$\deg(\Delta_{N,\phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}]\, \|\phi\|_T + (1 + b_3(N))\operatorname{div}\phi_{\tilde{\pi}}$$

holds. It now follows easily from Lemma 2.8, $b_3(N) = b_3(N')$, and the multiplicative property of the Thurston norm under finite covers (cf. [Ga83, Corollary 6.13]) that the twisted Alexander polynomial $\Delta^{\pi'/\tilde{\pi}}_{N',\phi'} \in \mathbb{Z}[t^{\pm 1}]$ is monic and that the following equality holds:

$$\deg(\Delta^{\pi'/\tilde{\pi}}_{N',\phi'}) = [\pi' : \tilde{\pi}]\, \|\phi'\|_T + (1 + b_3(N'))\operatorname{div}\phi'_{\tilde{\pi}}.$$

In particular $(\tilde{\pi}, \phi')$ has Property (M). □

We are now finally in a position to prove Theorem 1.2.

Proof of Theorem 1.2. First note that the combination of Theorem 1.9 and Lemmas 7.1, 7.2, 7.6 and 7.4 shows that it suffices to show the following claim:

Claim. Assume we are given a pair (N, ϕ) where

(1) N is a closed irreducible 3–manifold such that the fundamental group of each JSJ component is residually p, and
(2) ϕ is primitive.

If (N, ϕ) satisfies Condition $(*)$, then (N, ϕ) fibers over S^1.

Let (N, ϕ) be a pair as in the claim which satisfies Condition $(*)$. If $||\phi||_T = 0$, then it follows from [FV08b, Proposition 4.6] that (N, ϕ) fibers over S^1.

We can and will therefore henceforth assume that $||\phi||_T > 0$. We denote the tori of the JSJ decomposition of N by $T_1, \dots, T_r$. We pick a connected Thurston norm minimizing surface Σ dual to ϕ and a tubular neighborhood $\nu\Sigma = \Sigma \times [-1, 1] \subset N$ as in Section 6.1. In particular we can and will throughout assume that $\Sigma \times t$ and the tori $T_1, \dots, T_r$ are in general position for any $t \in [-1, 1]$ and that for any $i \in \{1, \dots, r\}$ any component of $\Sigma \cap T_i$ represents a nontrivial element in $\pi_1(T_i)$. Furthermore as in Section 6 we assume that our choice of Σ minimizes the number $\sum_{i=1}^r b_0(\Sigma \cap T_i)$.

Let $A_1, \dots, A_m$ be the components of the intersection of the tori $T_1, \dots, T_r$ with $M := N \setminus \Sigma \times (-1, 1)$. Furthermore let $M_1, \dots, M_n$ be the components of M cut along $A_1 \cup \dots \cup A_m$. Recall that any M_i is a submanifold of a JSJ component of N.

For $i = 1, \dots, m$ write $C_i = A_i \cap \Sigma^-$. It follows from Lemma 6.3 that for $i = 1, \dots, m$ the surface A_i is an annulus which is a product on C_i, i.e. C_i consists of one component and $\pi_1(C_i) \to \pi_1(A_i)$ is an isomorphism.

In order to show that M is a product on Σ^- it suffices to show that $\pi_1(\Sigma_i^-) \to \pi_1(M_i)$ is an isomorphism for any $i \in \{1, \dots, n\}$. So let $i \in \{1, \dots, n\}$. Since (N, ϕ) satisfies Condition $(*)$ it follows from Proposition 1.7 that the maps $\pi_1(\Sigma^\pm) \to \pi_1(M)$ induce an isomorphism of prosolvable completions. By Theorem 6.4 (1) the surfaces $\Sigma_i^\pm$ are connected, and by Theorem 6.4 (3) the inclusion induced maps $\pi_1(\Sigma_i^\pm) \to \pi_1(M_i), i = 1, \dots, n$ also induce isomorphisms of prosolvable completions. By Theorem 6.4 (2) we have that the group $\pi_1(M_i)$ is a subgroup of the fundamental group of a JSJ component of N. By our assumption this implies that $\pi_1(M_i)$ is residually p, in particular residually finite solvable.

In the following we view M_i as a sutured manifold with sutures given by $\gamma_i = \partial N \cap M_i$. We can pick orientations such that $R_-(\gamma_i) = \Sigma_i^-$ and $R_+(\gamma_i) = \Sigma_i^+$.

Since $\Sigma \subset N$ is Thurston norm minimizing it follows that (M_i, γ_i) is a taut sutured manifold. We can therefore now apply Theorem 4.1 to conclude that (M_i, γ_i) is a product sutured manifold, i.e. $\pi_1(\Sigma_i^-) \to \pi_1(M_i)$ is an isomorphism. □

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