Dokl. Akad. Nauk SSSR
Tom 178 (1968), No. 2

Soviet Math. Dokl.
Vol. 9 (1968), No. 1

MASLOV-ARNOL'D CHARACTERISTIC CLASSES

UDC 513.833

D. B. FUKS

This note deals with the characteristic classes of Lagrange manifolds introduced by V. P. Maslov [1] and V. I. Arnol'd [2]. * We give an expression of these classes in terms of the classes of Stiefel-Whitney and of A. Borel.

1. Notation. Definitions. By $E = E^{2n}$ we denote the space of real coordinates $(p_1, \cdots, p_n, q_1, \cdots, q_n)$. The mapping $J\colon E \to E$ is defined by the formula $J(p_1, \cdots, p_n, q_1, \cdots, q_n) = (-q_1, \cdots, -q_n, p_1, \cdots, p_n)$. Obviously, $J^2 = -1$. An n-dimensional hyperplane $\Pi \subset E$ is called *Lagrangian* if the plane $J(\Pi)$ is orthogonal to it. The collection of (unoriented) Lagrange planes in the space E, passing through the origin, is denoted by Λ_n. The manifold Λ_n is diffeomorphic to the factor-space $U(n)/O(n)$ of the group of unitary matrices relative to the group of orthogonal matrices.

Let X be a finite CW-complex. We shall say that an *n-dimensional Lagrange bundle with base X* is given if we are given: (1) the $O(n)$-bundle ξ with base X; (2) the equivalence of the complexification $c\xi$ of the bundle ξ and of the trivial $U(n)$-bundle with base X. We shall always denote a Lagrange bundle by one and the same letter as the $O(n)$-bundle which determines it. We can see that the classes of equivalent n-dimensional Lagrange bundles with base X are in one-to-one correspondence with the homotopy classes of the mappings of X into Λ_n (here, to the identity mapping $\Lambda_n \to \Lambda_n$ there corresponds the bundle induced by the principal fiber $U(n) \to U(n)/O(n) = A_n$).

We note an important example. Let $M^n \subset E$ be a smooth submanifold of dimension n. The manifold M^n is called *Lagrangian* if all of its tangent planes are Lagrangian. The complexification of any plane $\Pi \subset E$ is canonically isomorphic to E and, therefore, the tangent bundle to a Lagrange manifold can be considered to be a Lagrange bundle. The mapping $M^n \to \Lambda_n$ corresponding to this bundle associates to the point $x \in M^n$ the plane $\Pi \subset E$ passing through the origin of space E and parallel to the plane tangent to the manifold M^n at the point x.

2. Subsets of manifold M^n. Let $l \le k \le n$; P_k is the linear hull of the vectors $p_1, p_2, \cdots, p_k$. By $\Lambda_n^{k,l}$ we denote a subset of manifold Λ_n, consisting of all Lagrange planes whose intersection with P_k is not less than l-dimensional. The sets $\Lambda_n^{k,l}$ are not submanifolds of Λ_n, but may be represented as images of manifolds in the following way. By $\widetilde{\Lambda}_n^{k,l}$ we denote the space of pairs (Π, Π'), where $\Pi \in \Lambda_n$, Π' is an l-dimensional plane in space E such that $\Pi' \subset \Pi \cap P_k$. The manifold $\widetilde{\Lambda}_n^{k,l}$ is the space of a smooth fiber with base $G_{k,l}$ (the Grassmann manifold) and layer Λ_{n-l}. A projection in this fiber takes (Π, Π') into Π'. The mapping $i = i_n^{k,l}\colon \widetilde{\Lambda}_n^{k,l} \to \Lambda_n$, where $i(\Pi, \Pi') = \Pi$, has as its own image $\Lambda_n^{k,l}$. On $i^{-1}(\Lambda_n^{k,l} \setminus \Lambda_n^{k,l+1})$ it is a diffeomorphism on $\Lambda_n^{k,l} \Lambda_n^{k,l+1}$ and the singularities of this mapping are concentrated on the set $i^{-1}(\Lambda_n^{k,l+1})$ which, as is easily seen, has the codimension $n - k + 1$. In what follows we shall mainly consider the sets $A_k = \Lambda_n^{n-k+1,1}$, $B_k = \Lambda_n^{n-k+1,2}$

* These articles introduced only the first (one-dimensional) classes from the classes being considered. The definition of the remaining classes was given by V. I. Arnol'd in his seminar.

and $\Lambda_n^k = \Lambda_n^{n,k}$. Note that *the dimension of manifold* Λ_n *equals* $n(n+1)/2$, *while the codimensions of* A_k, B_k *and* Λ_n^k *in* Λ_n *equal* k, $2k+1$ *and* $k(k+1)/2$, *respectively*

3. **Orientability. Cohomology classes.** *The manifold* Λ_n *is orientable if and only if* n *is odd.* In order that the manifold $\tilde{\Lambda}_n^{k,l}$ be orientable it is necessary and sufficient that the layer Λ_{n-l} be orientable, while the base $G_{k,l}$ and its fiber be simultaneously orientable or unorientable. The layer Λ_{n-l} is orientable if and only if $n-l$ is odd, and the base $G_{k,l}$, if and only if k is even (excluding the cases $l=k$ and $l=0$). It is always trivial to restrict the fiber over the circle which is the generator $\pi_1(G_{k,l})$. *Hence, in particular, it follows that if* n *is odd, the manifold* $\Lambda_n^{k,l}$ *is orientable if and only if* k *and* l *are both even*, i.e., if and only if the layer and the base of the fiber are orientable.

The mapping $i_n^{k,l}$ defines the homology class of manifold Λ_n (the image of the fundamental class $[\tilde{\Lambda}_n^{k,l}]$); this class is integral if n is odd and k and l are even and has coefficients in group Z_2 for any n, k and l. If k is even and n and l are odd, the manifold $\tilde{\Lambda}_n^{k,l}$ is unorientable and its one-dimensional Stiefel class is equal to the image of the generating group $H^1(\Lambda_n; Z_2)$ under the homomorphism $(i_n^{n,l})^*$. Therefore, in this case the mapping $(i_n^{k,l})_*$ defines the homology class with coefficients in Z_T, the unique nontrivial local system of groups isomorphic to Z with base Λ_n. Finally, if $n = k$ is odd and l also is odd, then although the manifold $\tilde{\Delta}_n^{k,l} \setminus i^{-1}(\Lambda_n^{k,l+1})$ is orientable (the codimension of the set of singularities of mapping i equals 1). This allows us to treat $\Lambda_n^{n,l}$ as an integral cycle when l is odd. If l is even, then $\Lambda_n^{n,l}$ is a cycle with coefficients in Z_T. By applying Poincaré duality to the homology classes defined by cycles A_k, B_k, Λ_n^k, for any odd n we get the cohomology classes, respectively,

$$a_k \in H^k(\Lambda_n; Z_2), \qquad k = 1, 2, \ldots, n;$$

$$b_k \in \begin{cases} H^{2k+1}(\Lambda_n; Z), & k = 2, 4, \ldots, n-1; \\ H^{2k+1}(\Lambda_n; Z_T) & k = 1, 3, \ldots n-2; \end{cases}$$

$$\lambda_n^k \in \begin{cases} H^{(k(k+1))/2}(\Lambda_n, Z), & k = 1, 3, \ldots, n; \\ H^{(k(k+1))/2}(\Lambda_n; Z_T), & k = 2, 4, \ldots, n-1, \end{cases}$$

moreover, $\rho_2 \lambda_n^1 = a_1$ (ρ_2 is the reduction modulo 2).

The very same classes can be defined also in the cohomogies of Λ_n for even n. This can be done either geometrically (only, then, the "coorientability" of $\Lambda_n^{k,l}$ and not its orientability will be essential) or by considering the natural imbeddings $\Lambda_{n-1} \subset \Lambda_n \subset \Lambda_{n+1}$. We do not dwell on this in detail and in what follows we consider n to be arbitrary in the formulations of all the theorems and to be odd in the proofs.

Since the space Λ_n is a classifying space for Lagrange bundles, in the cohomologies of the base X of any Lagrange bundle ξ there are defined the characteristic classes $a_k(\xi) \in H^k(X; Z_2)$, $b_k(\xi) = H^{2k+1}(X; Z \text{ or } Z_T)$ and $\lambda_n^k(\xi) \in H^{k(k+1)/2}(X; Z \text{ or } Z_T)$. Here Z_T is the local system of groups isomorphic to Z with base X, defined by the bundle ξ.

The classes $a_k(\xi)$ *coincide with the Stiefel classes of bundle* ξ (and thus do not depend on the choice of the trivialization of the complexification of bundle ξ). By definition, *the classes* $\lambda_n^k(\xi)$ *are the Maslov-Arnol'd classes of the Lagrange bundle* ξ. Information on the classes $b_k(\xi)$ is contained in Theorem 4 presented below.

4. **Formulation of the results.**

Theorem 1. $\rho_2 \lambda_n^k = a_1 \cdots a_k = W^1 \ldots W^n$ $(k = 1, 2, \cdots, n)$.

Corollary. *The Maslov-Arnol'd classes, reduced modulo 2, do not depend on the choice of the* *trivialization of the complexification.*

Theorem 2. $\lambda_n^{2s+1} = \lambda_n^1 b_2 b_4 \cdots b_{2s}$ $(s = 1, 2, \cdots, (n-1)/2)$.

Theorem 3. $\lambda_n^{2s} = b_1 b_3 \cdots b_{2s-1}$ $(s = 1, 2, \cdots, (n-1)/2)$.

Corollary. *If any of the Maslov-Arnol'd classes equals zero, then all the subsequent ones also equal zero.*

Theorem 4. *The ring of weak integral cohomologies of manifold Λ_n for odd n is an exterior algebra with the generators $\lambda_n^1, b_2, b_4, \cdots, b_{n-1}$.*

The last theorem signifies that the classes λ_n^1 and b_k coincide with the generators of the ring of cohomologies of the homogeneous space $U(n)/O(n)$, found by A. Borel [3].

Since Λ_n is a classifying space for Lagrange bundles, the relations comprising Theorems 1–3 are fulfilled also for the characteristic classes $a_k(\xi)$, $b_k(\xi)$, $\lambda_n^k(\xi)$ of any Lagrange bundle ξ. The proofs of Theorems 1–3 are analogous. In §5 we prove only Theorem 1. §6 is devoted to the study of integral cohomologies of the space Λ_n. In particular, in it we prove Theorem 4.

5. **Intersections.** We prove the equality $\rho_2\lambda_n^k = a_k(\rho_2\lambda_n^{k-1})$. By definition, the class $D(\rho_2\lambda_n^{k-1})$ is the image of the fundamental class mod 2 of the manifold $\tilde{\Lambda}_n^{n,k-1}$ under the mapping $i_n^{n,k-1}$. We fix a small number $\epsilon > 0$. We denote by $\tilde{A}_k^{(\epsilon)}$ the set of pairs (Π, l), where $\Pi \subset E$ is a Lagrange plane and l is a straight line lying in the intersection of Π with the linear hull of the vectors $p_1\cos\epsilon + q_1\sin\epsilon, \cdots, p_{n-k+1}\cos\epsilon + q_{n-k+1}\sin\epsilon$. Obviously, $\tilde{A}_k^{(\epsilon)} \approx \tilde{\Lambda}_n^{n-k+1,1}$ and the mapping $i' : \tilde{A}_k^{(\epsilon)} \to \Lambda_n$, where $i'(\Pi, l) = \Pi$, is homotopic to $i_n^{n-k+1,1}$. Therefore, the image under the mapping i' of the fundamental class mod 2 of manifold $\tilde{A}_k^{(\epsilon)}$ is $D(a_k)$. It turns out that the mapping $j = i_n^{n,k-1} \times i'$: $\tilde{\Lambda}_n^{n,k-1} \times \tilde{A}_k^{(\epsilon)} \to \Lambda_n \times \Lambda_n$ is transversally-regular relative to the diagonal $\Delta(\Lambda_n) \subset \Lambda_n \times \Lambda_n$. The total preimage $j^{-1}(\Delta(\Lambda_n))$ is that manifold the image of whose fundamental class is dual in the sense of Poincaré to the product $a_k(\rho_2\lambda_n^{k-1})$. By $\approx{\Lambda}_n^k$ we denote the manifold whose points are the triples (Π, Π', l), where $\Pi \subset E$ is a Lagrange plane, $\Pi' \subset \Pi \cap P_n$ is a k-dimensional plane, $l \subset \Pi' \cap P_{n-k+1}$ is a straight line. Let us consider the mapping η: $\approx{\Lambda}_n^k \to \tilde{\Lambda}_n^{n,k-1} \times \tilde{A}_k^{(\epsilon)}$, where $\eta(\Pi, \Pi', l) = ((\phi_\epsilon\Pi, \Pi'/l), (\phi_\epsilon\Pi, \phi_\epsilon l))$. Here, Π'/l is the orthogonal complement of the straight line l in the plane Π', ϕ_ϵ: $E \to E$ is a mapping which rotates the plane $(l, J(l))$ by an angle ϵ (so that $(\phi_\epsilon\Pi, \phi_\epsilon l) \in \tilde{A}_k^{(\epsilon)}$) and is an identity mapping on the orthogonal complement of this plane. It is easy to see that this mapping is a diffeomorphism of $\approx{\Lambda}_n^k$ onto $j^{-1}(\Delta(\Lambda_n))$. The composition $\approx{\Lambda}_x^k \xrightarrow{\eta} \tilde{\Lambda}_n^{n,k-1} \times A^{(\epsilon)} \to \Delta(\Lambda_n) \subset \Lambda_n \times \Lambda_n$ is homotopic to the mapping which takes (Π, Π', l) into Π. The latter is the composition of the mapping $\approx{\Lambda}_n^k \to \Lambda_n^{n,k}((\Pi, \Pi', l) \to (\Pi, \Pi'))$ of degree 1 and of the mapping $i_n^{n,k}$. From this ensues the assertion we had to prove.

6. Rational cohomologies of the space Λ_n for odd n can be easily found from the Cartan-Serre theorem. *The ring $H^*(\Lambda_n; Q)$ for odd n is an external algebra of the generators $\beta_k \in H^{4k-3}(\Lambda_n; Q)$, $k = 1, \cdots, (n-1)/2$.*

In the space Λ_n consider the filtration $* = \Lambda_n^n \subset \cdots \subset \Lambda_n^1 \subset \Lambda_n^0 = \Lambda_n$. In this filtration we construct the (integral) spectral sequence of the cohomology groups. The term $E_1^{p,q}$ of this spectral sequence is $H^{p+q}(\Lambda_n^{n-p+1}, \Lambda_n^{n-p})$; the term E_∞ is adjoint to $H^*(\Lambda_n)$. We can show that this spectral sequence is multiplicative. The difference $\Lambda_n^p \setminus \Delta_n^{p-1}$ is the space of a vector bundle with base $G_{n,p}$ and a layer of dimension $((n-p)(n-p+1))/2$ and, moreover, this bundle is oriented if p is odd and unoriented if p is even. Therefore, $E_1^p = \Sigma_q E_1^{p,q}$ is none other than the complete cohomology group of the Thom space of this bundle, i.e., $E_1^{p,q} = H^{p+q-(p(p+1))/2}(G_{n,n-p}; Z)$ when p is even and $E_1^{p,q} =$

$H^{p+q-(p(p+1))/2}(G_{n,n-p}; Z_T)$ when p is odd, where Z_T is the only nontrivial local system of groups isomorphic to Z possible over $G_{n,n-p}$. In particular, for odd $p \leq n$ the groups $E_1^{p,p(p-1)/2}$ are isomorphic to Z; the groups $E_1^{1,q}$ are all finite, except $E_1^{2,q}$ the only infinite ones are the groups $E_1^{2,3+4s}$, $s = 0, \cdots, (n-3)/2$, and moreover, each of them is isomorphic to the direct sum of group Z and a finite group.

A simple calculation of ranks shows that the analogous spectral sequence with coefficients in the rational numbers is trivial. Hence it ensues that the elements of the groups $E_1^{p,p(p-1)/2}$ are not images of any differentials (from dimension considerations the elements of this group are cycles of all differentials). Since $E_1^{r,(p(p+1)/2)-r} = 0$ when $r > p$, the group $E_\infty^{p,p(p-1)/2}$ is a subgroup of the group $H^{p(p-1)/2}(\Lambda_n)$. This subgroup is characterized by the fact that its elements go to zero under the homomorphism induced by the imbedding $\Lambda_n^{n-p-1} \subset \Lambda_n$. In particular, the class λ_n^p possesses this property. Furthermore, the value of this class on the cycle Λ_{n-p} equals 1; consequently it is not divisible by any integer and hence is the generator of this subgroup.

Thus, the classes $\lambda_n^p = \lambda_n^1 b_2 \cdots b_{p-1} \in H^*(\Lambda_n)$ are not divisible and are of infinite order. Hence Theorem 4 follows from the calculation, carried out above, of the rational cohomologies of the space Λ_n.

Note that the generators $\lambda_n^1, b_2, \cdots, b_{n-1}$ of the ring of weak cohomologies of Λ_n are associated in E_1 with the generators of the free parts of the groups $E_1^{1,0}, E_1^{2,3}, \cdots, E_1^{2,2n-3}$.

In conclusion we remark that all the results of this note can be obtained from only the spectral sequence of §6 without recourse to any geometry.

Moscow State University
Received 17/MAR/67

BIBLIOGRAPHY

[1] V. P. Maslov, *Theory of perturbations and asymptotic methods*, Moscow, 1965. (Russian)

[2] V. I. Arnol'd, Funkcional. Anal. i Priložen. 1 (1967), no. 1.

[3] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) 57 (1953), 115; Russian transl. Chapter IV in *Fiber spaces*, IL, Moscow, 1958, p. 163. MR 14, 490; MR 22 #12522.

Translated by:
N. H. Choksy