Finite quotients of symplectic groups vs mapping class groups

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Abstract

We show that the Schur multiplier of $Sp(2g,\mathbb{Z}/D\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$, when D is divisible by 4 and $g \geq 4$. Further, for every prime p, we construct finite quotients of the mapping class group of genus $g \geq 3$ whose essential second homology has p-torsion, in contrast with the case of symplectic groups. Eventually, we prove that mapping class groups have Serre's property A_2 for trivial modules, whereas symplectic groups do not.

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1 Introduction and statements

Let $g \geq 1$ be an integer, and denote by M_g the mapping class group of a closed oriented surface of genus g and by $Sp(2g,\mathbb{Z})$ the symplectic group with integer coefficients. The choice of a basis in homology provides a surjective homomorphism $M_g \to Sp(2g,\mathbb{Z})$ and a natural question in this context is to compare the properties of these two groups. The leit-motive of this article is the (non)-residual finiteness property. For symplectic groups, Deligne's non-residual finiteness theorem from [9] states that the *universal central extension* $\widetilde{Sp(2g,\mathbb{Z})}$ is not residually finite since the image of its center under any homomorphism into a finite group has order at most two when $g \geq 3$. Our first motivation was to understand this result and give a sharp statement, namely to decide whether the image of the center might be of order two. Since these symplectic groups have the congruence subgroup property, this boils down to understanding the second homology of symplectic groups with coefficients the finite cyclic groups. In the sequel, for simplicity and unless otherwise explicitly stated, all (co)homology groups will be understood to be with trivial integer coefficients. An old theorem of Stein (see [49], Thm. 2.13 and Prop. 3.3.a) is that $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = 0$, when D is not divisible by 4. The case $D \equiv 0 \pmod 4$ remained open since then; this is explicitly mentioned for instance in ([42], Remarks after Thm. 3.8). Our first result settles this case:

Theorem 1.1. *The second homology group of finite principal congruence quotients of $Sp(2g,\mathbb{Z})$, $g \geq 4$ is*

$$H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{if } D \equiv 0 \pmod 4.$$

This computation shows that Deligne's statement of the non-residual finiteness theorem is sharp, because the image of the center of the universal central extension $\widetilde{Sp(2g,\mathbb{Z})}$ into the universal central extension of $Sp(2g,\mathbb{Z}/D\mathbb{Z})$ is of order two, when $D \equiv 0 \pmod 4$ and $g \geq 3$.

In comparison, recall that Beyl (see [4]) has showed that $H_2(SL(2,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0 (\mathrm{mod}\ 4)$ and Dennis and Stein proved using K-theoretic methods that for $n \geq 3$ we have $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0(\mathrm{mod}\ 4)$, while $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = 0$, for $D \not\equiv 0(\mathrm{mod}\ 4)$ (see [10], Cor. 10.2 and [38], section 12).

To better compare the behavior of the universal central extension of symplectic groups and mapping class groups we introduce the essential (second) homology group:

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Definition 1.1. *Let G be a discrete group and F a finite group. We define the essential second homology group $EH_2(F,G)$ of F relative to G as the union:*

$$EH_2(F,G) = \bigcup_{p:G\to F} p_*(H_2(G)) \subset H_2(F) \tag{1}$$

over all surjective homomorphisms $p : G \to F$, where $p_* : H_2(G) \to H_2(F)$ denotes the map induced in homology and all homology groups are considered with (trivial) integral coefficients.

Notice that by definition $EH_2(F,G)$ is a torsion group. In the case G is either a mapping class group or a symplectic group we have two very different situations. First, we have:

Theorem 1.2. *For any prime p there exist finite quotients F of M_g, $g \geq 3$, such that $EH_2(F, M_g)$ has p-torsion.*

On the other hand, a consequence of [9, 5, 3] is the existence of a uniform bound (independent on g) for the order of the torsion group $EH_2(F, Sp(2g, \mathfrak{A}))$, where $\mathfrak{A}$ is the ring of S-integers of a number field which is not totally imaginary and $g \geq 3$.

We prove Theorem 1.2 by exhibiting explicit finite quotients of the universal central extension of a mapping class group that arise from the so-called quantum representations. We refine here the approach in [14] where the first author proved that central extensions of M_g by $\mathbb{Z}$ are residually finite. In the meantime, it was proved in [15, 34] by more sophisticated tools that the set of quotients of mapping class groups contains arbitrarily large rank finite groups of Lie type. Notice however that the family of quotients obtained in Theorem 1.2 are of different nature than those obtained in [15, 36], although their source is the same (see Proposition 4.1 for details).

When G is a discrete group we denote by $\widehat{G}$ its *profinite completion*. Recall, following ([47], I.2.6) that G has property A_n for the finite $\widehat{G}$-module M if the homomorphism $H^k(\widehat{G}, M) \to H^k(G, M)$ is an isomorphism for $k \leq n$ and injective for $k = n + 1$. Deligne's theorem cited above actually is equivalent to the fact that $\mathrm{Sp}(2g, \mathbb{Z})$ has not property A_2 for the trivial $\mathrm{Sp}(2g, \mathbb{Z})$-modules (see also [20]). In contrast we prove:

Theorem 1.3. *For $g \geq 4$ the mapping class group M_g has property A_2 for trivial $\widehat{M_g}$-modules.*

The plan of this article is the following. In Section 3 we prove Theorem 1.1. Although it is easy to show that the groups $H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$ are cyclic, their non-triviality is more involved. We give three different proofs of the non-triviality, each one of them having its advantages and disadvantages in terms of bounds for detections or sophistication. The first proof relies on deep results of Putman in [42], and shows that we can detect this $\mathbb{Z}/2\mathbb{Z}$ factor on $H_2(Sp(2g, \mathbb{Z}/8\mathbb{Z}))$ for $g \geq 4$, providing even an explicit extension that detects this homology class. The second proof uses mapping class groups and Weil representations, it amounts to show that a suitable projective unitary representation does not lift to a linear representation of the mapping class group. This proof relies on deep results of Gervais [17]. The third proof is K-theoretical in nature and uses a generalization of Sharpe's exact sequence relating K-theory to symplectic K-theory due to Barge and Lannes [2]. Indeed, by the stability results, this $\mathbb{Z}/2\mathbb{Z}$ should correspond to a class in $KSp_2(\mathbb{Z}/4\mathbb{Z})$. This group admits a natural homomorphism to a Witt group of symmetric non-degenerate bilinear forms on free $\mathbb{Z}/4\mathbb{Z}$-modules, which we use to detect the non-triviality of that class. Finally in Section 4 we discuss the case of the mapping class groups and prove Theorems 1.2 using the quantum representations that arise from the $SU(2)/SO(3)$-TQFT's. These representations are the non-abelian counterpart of the Weil representations of symplectic groups.

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2 Residual finiteness of universal central extensions

In this section we collect results about universal central extensions of perfect groups, for the sake of completeness of our arguments. Every perfect group Γ has a universal central extension $\widetilde{\Gamma}$, and the kernel of the canonical projection map $\widetilde{\Gamma} \to \Gamma$ coincides with the center $Z(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$ and is canonically isomorphic to the second integral homology group $H_2(\Gamma)$. We will recall now how the residual finiteness problem for the universal central $\widetilde{\Gamma}$ of a perfect and residually finite groups Γ translates into an homological problem about $H_2(\Gamma)$. We start with a classical result for maps between universal central extensions of perfect groups.

Lemma 2.1. *Let Γ and F be perfect groups, $\widetilde{\Gamma}$ and $\widetilde{F}$ their universal central extensions and $p : \Gamma \to F$ be a group homomorphism. Then there exists a unique homomorphism $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ lifting p such that the following diagram is commutative:*

$$\begin{array}{ccccccc} 1 \to & H_2(\Gamma) & \to & \widetilde{\Gamma} & \to & \Gamma & \to 1 \\ & p_* \downarrow & & \widetilde{p} \downarrow & & \downarrow p & \\ 1 \to & H_2(F) & \to & \widetilde{F} & \to & F & \to 1 \end{array}$$

For a proof we refer the interested reader to ([31], chap VIII) or ([6], chap IV, Ex. 1, 7). If Γ is a perfect residually finite group, to prove that its universal central extension $\widetilde{\Gamma}$ is also residually finite we only have to find enough finite quotients of $\widetilde{\Gamma}$ to detect the elements in its center $H_2(\Gamma)$. The following lemma analyses the situation.

Lemma 2.2. *Let Γ be a perfect group and denote by $\widetilde{\Gamma}$ its universal central extension.*

1. *Let H be a finite index normal subgroup $H \subset \Gamma$ such that the image of $H_2(H)$ into $H_2(\Gamma)$ contains the subgroup $dH_2(\Gamma)$, for some $d \in \mathbb{Z}$. Let $F = \Gamma/H$ be the corresponding finite quotient of Γ and $p : \Gamma \to F$ the quotient map. Then $d \cdot p_*(H_2(\Gamma)) = 0$, where $p_* : H_2(\Gamma) \to H_2(F)$ is the homomorphism induced by p. In particular, if $p_* : H_2(\Gamma) \to H_2(F)$ is surjective, then $d \cdot H_2(F) = 0$.*

2. *Assume that F is a finite quotient of Γ satisfying $d \cdot p_*(H_2(\Gamma)) = 0$. Let $\widetilde{F}$ denote the universal central extension of F. Then the homomorphism $p : \Gamma \to F$ has a unique lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ and the kernel of $\widetilde{p}$ contains $d \cdot H_2(\Gamma)$.*

Observe that in point 2. of Lemma 2.2 the group F being finite, $H_2(F)$ is also finite, hence one can take $d = |H_2(F)|$.

Proof. The image of H into F is trivial and thus the image of $H_2(H)$ into $H_2(F)$ is trivial. This implies that $p_*(d \cdot H_2(\Gamma)) = 0$, which proves the first part of the lemma.

Further, by Lemma 2.1 there exists an unique lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$. If $d \cdot p_*(H_2(\Gamma)) = 0$ then Lemma 2.1 yields $d \cdot \widetilde{p}(c) = d \cdot p_*(c) = 0$, for any $c \in H_2(\Gamma)$. This settles the second part of the lemma. □

Remark 2.1. It might be possible that the $d' \cdot p_*(H_2(\Gamma)) = 0$, for some proper divisor d' of d, so the first part of Lemma 2.2 can only give an upper bound of the orders of the image of the second cohomology. In order to find lower bounds we need additional information concerning the finite quotients F.

Lemma 2.3. *Let Γ be a perfect group, $\widetilde{\Gamma}$ its universal central extension, $p : \Gamma \to F$ be a surjective homomorphism onto a finite group F and $\widehat{p} : \widetilde{\Gamma} \to G$ be some lift of p to a central extension G of F by some finite abelian group C. Assume that the image of the center $Z(\widetilde{\Gamma}) = H_2(\Gamma)$ of $\widetilde{\Gamma}$ in G by $\widehat{p}$ contains an element of order q. Then there exists an element of $p_*(H_2(\Gamma)) \subset H_2(F)$ of order q.*

Proof. By Lemma 2.1 there exists a lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ of p into the universal central extension $\widetilde{F}$ of F. Then, by universality there exists a unique homomorphism $s : \widetilde{F} \to G$ of central extensions of F lifting the identity map of F. The homomorphisms $\widehat{p}$ and $s \circ \widetilde{p} : \widetilde{\Gamma} \to G$ are then both lifts of p. Using the centrality of C in G it follows that the map $\widetilde{\Gamma} \to C$ given by $x \mapsto \widehat{p}(x)^{-1} \cdot (s \circ \widetilde{p}(x))$ is a group homomorphism, and hence is trivial since $\widetilde{\Gamma}$ is perfect and C abelian. We conclude that $\widehat{p} = s \circ \widetilde{p}$.

Recall that the restriction of $\widetilde{p}$ to $H_2(\Gamma)$ coincides with the homomorphism $p_* : H_2(\Gamma) \to H_2(F)$ and that $H_2(F)$ is finite since F is so. Then, if $z \in H_2(\Gamma)$ is such that $\widehat{p}(z)$ has order q in C, the element $p_*(z) \in p_*(H_2(\Gamma)) \subset \widetilde{F}$ is sent by s onto an element of order q and therefore $p_*(z)$ has order a multiple of q, say aq. Then $(p_*(z))^a = p_*(z^a) \in p_*(H_2(\Gamma)) \subset \widetilde{F}$ has order q. □

3 Proof of Theorem 1.1

3.1 Preliminaries

According to ([40, Thm. 5]) and since symplectic groups are perfect for $g \geq 3$ (see e.g. [42], Thm. 5.1), it suffices to compute $H_2(Sp(2g, \mathbb{Z}/D\mathbb{Z}))$ when D is a power of a prime. Then, from Stein's computations for $D \not\equiv 0 \pmod 4$ (see [49, 51]), Theorem 1.1 is equivalent to the statement:

$$H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{ for all } g \geq 3, k \geq 2.$$

We will freely use in the sequel two classical results due to Stein. *Stein's isomorphism theorem* (see [49], Thm. 2.13 and Prop. 3.3.(a)) states that there is an isomorphism:

$$H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) \simeq H_2(Sp(2g, \mathbb{Z}/2^{k+1}\mathbb{Z})), \text{ for all } g \geq 3, k \geq 2.$$

Further, *Stein's stability theorem* (see [49]) states that the stabilization homomorphism $Sp(2g, \mathbb{Z}/2^k\mathbb{Z}) \hookrightarrow Sp(2g+2, \mathbb{Z}/2^k\mathbb{Z})$ induces an isomorphism:

$$H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) \simeq H_2(Sp(2g+2, \mathbb{Z}/2^k\mathbb{Z})), \text{ for all } g \geq 4, k \geq 1.$$

Therefore, to prove Theorem 1.1 it suffices to show that:

$$H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{ for some } g \geq 4, k \geq 2.$$

We provide hereafter three different proofs of this statement, each having its own advantage. For the first and the second proofs, the starting point is the intermediary result $H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$, for $g \geq 4$. This will be derived from Deligne's theorem ([9]). Then it will be enough to find a non-trivial extension of $Sp(2g, \mathbb{Z}/2^k\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ for some $g \geq 4, k \geq 2$.

In section 3.3 we show then that Putman's computations from ([42], Thm. F) provide us with a *non-trivial* central extension of $Sp(2g, \mathbb{Z}/8\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. The second proof seems more elementary and it provides a non-trivial central extension of $Sp(2g, \mathbb{Z}/4n\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, for all integers $n \geq 1$. Moreover, it does not use Stein's isomorphism theorem and relies instead on the study of the Weil representations of symplectic groups, or equivalently abelian quantum representations of mapping class groups. Since these representations come from theta functions this strategy is deeply connected to Putman's approach. In fact the proof of Theorem F in ([42]) is based on his Lemma 5.5 whose proof requires the transformation formulas for the classical theta nulls. The third proof, based on an extension of Sharpe's sequence in symplectic K-theory due to Barge and Lannes (see [2]) uses more sophisticated techniques but works already for $Sp(2g, \mathbb{Z}/4\mathbb{Z})$. Moreover, this last proof does not rely on Deligne's theorem.

3.2 An alternative for the order of $H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$

As our first step we prove, as a consequence of Deligne's theorem:

Proposition 3.1. *We have $H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z})) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$, when $g \geq 4$.*

Notice that although our proof hereafter works only for $g \geq 4$, the claim holds when $g = 3$ as well, by Stein's stability theorem.

Proof of Proposition 3.1. Let $p : Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2^k\mathbb{Z})$ be the reduction mod 2^k and $p_* : H_2(Sp(2g, \mathbb{Z})) \to H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$ the induced homomorphism. The first ingredient in the proof is the following result which seems well-known, and that we isolate for later reference:

Lemma 3.1. *The homomorphism $p_* : H_2(Sp(2g, \mathbb{Z})) \to H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$ is surjective, if $g \geq 4$.*

Now, it is a classical result that $H_1(Sp(2g, \mathbb{Z})) = 0$, for $g \geq 3$ and $H_2(Sp(2g, \mathbb{Z})) = \mathbb{Z}$, for $g \geq 4$ (see e.g. [42], Thm. 5.1). Note however that for $g = 3$, we have that $H_2(Sp(6, \mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ according to [50]. This implies that $H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$ is cyclic when $g \geq 4$ (this was also shown by Stein in [49]) and we only have to bound the order of this cohomology group.

Lemma 2.1 provides a lift between the universal central extensions $\widetilde{p} : \widetilde{Sp(2g, \mathbb{Z})} \to \widetilde{Sp(2g, \mathbb{Z}/2^k\mathbb{Z})}$ of the mod 2^k reduction map, such that the restriction of $\widetilde{p}$ to the center $H_2(Sp(2g, \mathbb{Z}))$ of $\widetilde{Sp(2g, \mathbb{Z})}$ is the homomorphism $p_* : H_2(Sp(2g, \mathbb{Z})) \to H_2(Sp(2g, \mathbb{Z}/2^k\mathbb{Z}))$. From Deligne's theorem [9] every finite index subgroup of the universal central extension $\widetilde{Sp(2g,\mathbb{Z})}$, for $g \geq 4$, contains $2\mathbb{Z}$, where $\mathbb{Z}$ is the central kernel $\ker(\widetilde{Sp(2g,\mathbb{Z})} \to Sp(2g,\mathbb{Z}))$. If c is a generator of the center $\mathbb{Z}$ we have $2p_*(c) = \widetilde{p}(2c) = 0$. According to Lemma 3.1 p_* is surjective and thus $H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z}))$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$, as claimed. □

3.3 First proof: an explicit extension detecting $H_2(Sp(2g,\mathbb{Z}/8\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$

According to Proposition 3.1 and Stein's stability theorem, in order to prove Theorem 1.1 it is enough to provide a non-trivial central extension of $Sp(2g,\mathbb{Z}/2^k\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, for some $g \geq 3$ and $k \geq 2$. Denote by $Sp(2g,2)$ the kernel of the mod 2 reduction map $Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/2\mathbb{Z})$. As ingredients of our proof we use the following results of Putman (see [42], Lemma 5.5 and Thm. F), which we state here in a unified way:

Proposition 3.2. *The pull-back $\widetilde{Sp(2g,2)}$ of the universal central extension $\widetilde{Sp(2g,\mathbb{Z})}$ under the inclusion homomorphism $Sp(2g,2) \to Sp(2g,\mathbb{Z})$ is a central extension of $Sp(2g,2)$ by $\mathbb{Z}$ whose extension class in $H^2(Sp(2g,2))$ is even.*

Let $\widetilde{G} \subset \widetilde{Sp(2g,2)}$ be a central extension of $Sp(2g,2)$ by $\mathbb{Z}$ whose extension class is half the extension class of $\widetilde{Sp(2g,2)}$. We have then a commutative diagram:

$$\begin{array}{ccccccccc}
1 \to & \mathbb{Z} & \to & \widetilde{G} & \to & Sp(2g,2) & \to 1 \\
& \downarrow \times 2 & & \downarrow & & \downarrow\cong & \\
1 \to & \mathbb{Z} & \to & \widetilde{Sp(2g,2)} & \to & Sp(2g,2) & \to 1 \\
& \downarrow\cong & & \downarrow & & \downarrow & \\
1 \to & \mathbb{Z} & \to & \widetilde{Sp(2g,\mathbb{Z})} & \to & Sp(2g,\mathbb{Z}) & \to 1.
\end{array} \tag{2}$$

In this section we will denote by $i : \mathbb{Z} \to \widetilde{Sp(2g,\mathbb{Z})}$ the inclusion of the center and by $p : \widetilde{Sp(2g,\mathbb{Z})} \to Sp(2g,\mathbb{Z})$ the projection killing $i(\mathbb{Z})$. Now $\widetilde{G}$ is a subgroup of index 2 of $\widetilde{Sp(2g,2)}$ and hence a normal subgroup of the form $\ker f$, where $f : \widetilde{Sp(2g,2)} \to \mathbb{Z}/2\mathbb{Z}$ is some group homomorphism. In particular, f factors through the abelianization homomorphism $F : \widetilde{Sp(2g,2)} \to H_1(\widetilde{Sp(2g,2)})$. If we denote by $\widetilde{K}$ the kernel of F, then $\widetilde{K} \subset \widetilde{G}$.

Lemma 3.2. *The image $K = p(\widetilde{K})$ under the projection $p : \widetilde{Sp(2g,\mathbb{Z})} \to Sp(2g,\mathbb{Z})$ is the kernel of the abelianization homomorphism $Sp(2g,2) \to H_1(Sp(2g,2))$. In particular K is the Igusa subgroup $Sp(2g,4,8)$ of $Sp(2g,4)$ consisting of those symplectic matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with the property that the diagonal entries of $AB^\top$ and $CD^\top$ are multiples of 8.*

Proof of Lemma 3.2. Let $\phi : Sp(2g,2) \to H_1(Sp(2g,2))/F(i(\mathbb{Z}))$ be the map defined by:

$$\phi(x) = F(\widetilde{x}),$$

where $\widetilde{x}$ is an arbitrary lift of x to $\widetilde{Sp(2g,2)}$. Then ϕ is a well-defined homomorphism and moreover, $p(\widetilde{K}) = \ker \phi$. The 5-term exact sequence associated to the central extension $\widetilde{Sp(2g,2)}$ is:

$$H_2(Sp(2g,2)) \to (H_1(\mathbb{Z}))_{Sp(2g,2)} \to H_1(\widetilde{Sp(2g,2)}) \to H_1(Sp(2g,2)) \to 0.$$

The image of $(H_1(\mathbb{Z}))_{Sp(2g,2)} \cong \mathbb{Z}$ into $H_1(\widetilde{Sp(2g,2)})$ in the sequence above is, by construction, the subgroup $F(i(\mathbb{Z}))$. Therefore p induces an isomorphism between $H_1(\widetilde{Sp(2g,2)})/F(\mathbb{Z})$ and $H_1(Sp(2g,2))$. This proves the first claim.

The second claim follows from Sato's computation of $H_1(Sp(2g,2))$ (see [46], Proposition 2.1) where he identifies the commutator subgroup $[Sp(2g,2), Sp(2g,2)]$ with $Sp(2g,4,8)$. □

Lemma 3.3. *The subgroups $\widetilde{K}$ and K are normal subgroups of $\widetilde{Sp(2g,\mathbb{Z})}$ and $Sp(2g,\mathbb{Z})$, respectively.*

Proof of Lemma 3.3. For any group G the subgroup $\ker(G \to H_1(G))$ is characteristic. Now the group $\widetilde{Sp(2g,\mathbb{Z})}$ acts by conjugation on its normal subgroup $\widetilde{Sp(2g,2)}$ and since $\widetilde{K}$ is a characteristic subgroup of $\widetilde{Sp(2g,2)}$ it is therefore preserved by the conjugacy action of $\widetilde{Sp(2g,\mathbb{Z})}$ and hence a normal subgroup. The proof of the other statement is similar. The fact that $Sp(2g,4,8)$ is a normal subgroup of $Sp(2g,\mathbb{Z})$ was proved by Igusa (see [27], Lemma 1.(i)). □

Lemma 3.4. *If $\widetilde{H} \subset \widetilde{G}$ is a finite index normal subgroup of $\widetilde{Sp(2g,\mathbb{Z})}$, then $\widetilde{H} \cap i(\mathbb{Z}) = 2 \cdot i(\mathbb{Z})$.*

Proof of Lemma 3.4. We have $\widetilde{H} \cap i(\mathbb{Z}) \subset \widetilde{G} \cap i(\mathbb{Z}) = 2 \cdot i(\mathbb{Z})$ so that $\widetilde{H} \cap i(\mathbb{Z}) = m \cdot i(\mathbb{Z}))$, with $m \geq 2$ or $m = 0$. Then $m \neq 0$ since $\widetilde{H}$ was supposed to be of finite index in $\widetilde{Sp(2g,\mathbb{Z})}$. If $m > 2$ then the projection homomorphism $\widetilde{Sp(2g,\mathbb{Z})} \to \widetilde{Sp(2g,\mathbb{Z})}/\widetilde{H}$ would send the center $i(\mathbb{Z})$ into $\mathbb{Z}/m\mathbb{Z}$, contradicting Deligne's theorem ([9]). Thus $m = 2$. □

Lemma 3.5. *The subgroup $\widetilde{K}$ is of finite index in $\widetilde{Sp(2g,\mathbb{Z})}$.*

Proof of Lemma 3.5. By the definition of $\widetilde{K}$ the statement is equivalent to prove that $H_1(\widetilde{Sp(2g,2)})$ is a finite abelian group. From the 5-term exact sequence above, and since the group $H_1(Sp(2g,2))$ is finite, it is enough to show that the image of $H_1(\mathbb{Z})_{Sp(2g,2)}$ is finite in $H_1(\widetilde{Sp(2g,2)})$ and this is precisely the image of the center of $\widetilde{Sp(2g,\mathbb{Z})}$. If this image is infinite in $H_1(\widetilde{Sp(2g,2)})$, then we could find finite abelian quotients of $\widetilde{Sp(2g,2)}$ for which the center is sent into $\mathbb{Z}/m\mathbb{Z}$, for arbitrary large m. Using induction we obtain finite representations of $\widetilde{Sp(2g,\mathbb{Z})}$ with the same property, contradicting Deligne's theorem. □

From this we already have that the following central extension between finite groups

$$1 \to \quad \mathbb{Z}/2\mathbb{Z} \quad \to \quad \widetilde{Sp(2g,\mathbb{Z})}/\widetilde{K} \quad \to \quad Sp(2g,2)/Sp(2g,4,8) \quad \to 1$$

is non-trivial. Therefore, there are finite quotients of $\widetilde{Sp(2g,\mathbb{Z})}$ in which the image of the center is not trivial.

The final proposition will provide now a whole familly of examples and will achieve our computation of $H_2(Sp(2g,\mathbb{Z}/2^k))$ for $g \geq 3$ and $k \geq 3$.

Proposition 3.3. *If $H \subset Sp(2g,4,8)$ is a principal congruence subgroup of $Sp(2g,\mathbb{Z})$ and $g \geq 3$ then $H_2(Sp(2g,\mathbb{Z})/H) = \mathbb{Z}/2\mathbb{Z}$. In particular for all $k \geq 3$, taking for H the level 8 principal congruence subgroup, we have $H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) \simeq H_2(Sp(2g,\mathbb{Z}/8\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $\widetilde{H} \subset \widetilde{K}$ be the pull-back of the central extension $\widetilde{Sp(2g,\mathbb{Z})}$ under the inclusion homomorphism $H \subset Sp(2g,\mathbb{Z})$, which is a normal finite index subgroup of $\widetilde{Sp(2g,\mathbb{Z})}$. Then the image of the center $i(\mathbb{Z})$ by the projection homomorphism $\widetilde{Sp(2g,\mathbb{Z})} \to \widetilde{Sp(2g,\mathbb{Z})}/\widetilde{H}$ is of order two. Therefore we have a central extension:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{Sp(2g,\mathbb{Z})}/\widetilde{H} \to Sp(2g,\mathbb{Z})/H \to 0.$$

If the extension $\widetilde{Sp(2g,\mathbb{Z})}/\widetilde{H}$ were trivial then we would obtain a surjective homomorphism $\widetilde{Sp(2g,\mathbb{Z})} \to \mathbb{Z}/2\mathbb{Z}$. But $H_1(\widetilde{Sp(2g,\mathbb{Z})}) = 0$ by universality. Therefore $H^2(Sp(2g,\mathbb{Z})/H; \mathbb{Z}/2\mathbb{Z}) \neq 0$. By Deligne's theorem $H_2(Sp(2g,\mathbb{Z})/H) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$ and hence $H_2(Sp(2g,\mathbb{Z})/H) = \mathbb{Z}/2\mathbb{Z}$. □

3.4 Second proof: detecting the non-trivial class via Weil representations

The projective representation that we use is related to the theory of theta functions on symplectic groups. Although the Weil representations of symplectic groups over finite fields of characteristic different from 2 is a classical subject present in many textbooks, the slightly more general Weil representations associated to finite rings of the form $\mathbb{Z}/\ell\mathbb{Z}$ received less consideration until recently. They first appeared in print as representations associated to finite abelian groups in [29] for genus $g = 1$ and were extended to locally compact abelian groups in ([54], Chapter I) andalso independently in the work of Igusa and Shimura on theta functions (see [26, 48, 25]) and in physics literature (see e.g. [23]). They were rediscovered as monodromies of generalized theta functions arising in the $U(1)$ Chern-Simons theory in [12, 19, 13] and then in finite-time frequency analysis (see [28] and references from there). In [12, 13, 19] these are projective representations of the symplectic group factorizing through the finite congruence quotients $Sp(2g, \mathbb{Z}/2\ell\mathbb{Z})$, which are only defined for even $\ell \geq 2$. However, for odd ℓ the monodromy of theta functions lead to representations of the theta subgroup of $Sp(2g, \mathbb{Z})$. These also factor through the image of the theta group into the finite congruence quotients $Sp(2g, \mathbb{Z}/2\ell\mathbb{Z})$. Notice however that the original Weil construction works as well for $\mathbb{Z}/\ell\mathbb{Z}$ with odd ℓ (see e.g. [22, 28]).

It is well-known (see [54] sections 43, 44 or [43], Prop. 5.8) that these projective Weil representations lift to linear representations of the integral metaplectic group, which is the pull-back of the symplectic group in a double cover of $Sp(2g, \mathbb{R})$. The usual way to resolve the projective ambiguities is to use the Maslov cocycle (see e.g. [53]). Moreover, it is known that the Weil representations over finite fields of odd characteristic and over $\mathbb{C}$ actually are linear representations. In fact the vanishing of the second power of the augmentation ideal of the Witt ring of such fields (see e.g. [52, 32]) implies that the corresponding metaplectic extension splits. This contrasts with the fact that Weil representations over $\mathbb{R}$ (or any local field different from $\mathbb{C}$) are true representations of the real metaplectic group and cannot be linearized (see e.g. [32]). The Weil representations over local fields of characteristic 2 is subtler as they are rather representations of a double cover of the so-called pseudo-symplectic group (see [54] and [21] for recent work).

Let $\ell \geq 2$ be an integer and denote by $\langle , \rangle$ the standard bilinear form on $(\mathbb{Z}/\ell\mathbb{Z})^g \times (\mathbb{Z}/\ell\mathbb{Z})^g \to \mathbb{Z}/\ell\mathbb{Z}$. The Weil representation we consider is a representation in the unitary group of the complex vector space $\mathbb{C}^{(\mathbb{Z}/\ell\mathbb{Z})^g}$ endowed with its standard Hermitian form. Notice that the standard basis of this vector space is canonically labeled by elements in $(\mathbb{Z}/\ell\mathbb{Z})^g$.

It is well-known (see e.g. [27]) that $Sp(2g, \mathbb{Z})$ is generated by the matrices having one of the following forms: $\begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix}$ where $B = B^\top$ has integer entries, $\begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix}$ where $A \in GL(g, \mathbb{Z})$ and $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$.

We can now define the Weil representations on these generating matrices as follows:

$$\rho_{g,\ell} \begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix} = \mathrm{diag} \left(\exp \left(\frac{\pi\sqrt{-1}}{\ell} \langle m, Bm \rangle \right) \right)_{m \in (\mathbb{Z}/\ell\mathbb{Z})^g}, \tag{3}$$

where diag stands for diagonal matrix with given entries;

$$\rho_{g,\ell} \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} = (\delta_{A^\top m, n})_{m,n \in (\mathbb{Z}/\ell\mathbb{Z})^g}, \tag{4}$$

where δ stands for the Kronecker symbol;

$$\rho_{g,\ell} \begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} = \ell^{-g/2} \exp \left(-\frac{2\pi\sqrt{-1}\langle m, n \rangle}{\ell} \right)_{m,n \in (\mathbb{Z}/\ell\mathbb{Z})^g}. \tag{5}$$

It is proved in [13, 19], that for even ℓ these formulas define a unitary representation $\rho_{g,\ell}$ of $Sp(2g, \mathbb{Z})$ in $U(\mathbb{C}^{(\mathbb{Z}/\ell\mathbb{Z})^g})/R_8$. Here $U(\mathbb{C}^N) = U(N)$ denotes the unitary group of dimension N and $R_8 \subset U(1) \subset U(\mathbb{C}^N)$ is the subgroup of scalar matrices whose entries are roots of unity of order 8. For odd ℓ the same formulas define representations of the theta subgroup $Sp(2g, 1, 2)$ (see [27, 26, 13]). Notice that by construction $\rho_{g,\ell}$ factors through $Sp(2g, \mathbb{Z}/2\ell\mathbb{Z})$ for even ℓ and through the image of the theta subgroup in $Sp(2g, \mathbb{Z}/\ell\mathbb{Z})$ for odd ℓ.

Proposition 3.4. *The projective Weil representation $\rho_{g,\ell}$ of $Sp(2g, \mathbb{Z})$, for $g \geq 3$ and even ℓ does not lift to linear representations of $Sp(2g, \mathbb{Z})$, namely it determines a generator of $H^2(Sp(2g, \mathbb{Z}/2\ell\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Remark 3.1. For odd ℓ it was already known that the projective Weil representations $\rho_{g,\ell}$ lift to linear representations (see [1], Appendix AIII).

3.4.1 Outline of the proof

We use again, as in the first proof, Proposition 3.1. The projective Weil representation $\rho_{g,\ell}$ determines a central extension of $Sp(2g, \mathbb{Z}/2\ell\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, since it factors through the metaplectic central extension, by [54]. We will prove that this central extension is non-trivial thereby proving the claim. The pull-back of this central extension by the homomorphism $Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/2\ell\mathbb{Z})$ is a central extension of $Sp(2g, \mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ and it is enough to prove that this last extension is non-trivial. It turns out to be easier to describe the pull-back of this central extension over the mapping class group M_g of the genus g closed orientable surface. Denote by $\widetilde{M_g}$ the pull-back of the central extension above under the homomorphism $M_g \to Sp(2g,\mathbb{Z})$. By the stability results of Harer for $g \geq 5$, and the low dimensional computations in [41] and [30] for $g \geq 4$, the natural homomorphism $M_g \to Sp(2g,\mathbb{Z})$, obtained by choosing a symplectic basis in the surface homology induces isomorphisms $H_2(M_g;\mathbb{Z}) \to H_2(Sp(2g,\mathbb{Z});\mathbb{Z})$ and $H^2(Sp(2g,\mathbb{Z});\mathbb{Z}) \to H^2(M_g;\mathbb{Z})$ for $g \geq 4$. In particular in this range the class of the central extension $\widetilde{M_g}$ is an element of $H^2(M_g;\mathbb{Z}/2\mathbb{Z})$. Therefore, we can reformulate Proposition 3.4 at least for $g \geq 4$ in an equivalent form but involving the mapping class group instead; the case $g =$ beeing recovererd from this by Stein's stability theorem:

Proposition 3.5. *If $g \geq 4$ then the class of the central extension $\widetilde{M_g}$ is a generator of $H^2(M_g;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.*

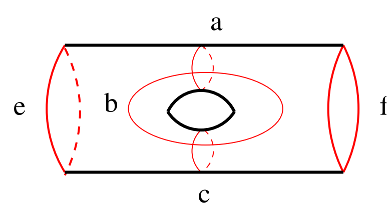
3.4.2 A presentation of $\widetilde{M_g}$

The method we use is due to Gervais (see [17]) and was already used in [16] for computing central extensions arising in quantum Teichmüller space. We start with a number of notations and definitions. Recall that $\Sigma_{g,r}$ denotes the orientable surface of genus g with r boundary components. If γ is a curve on a surface then D_γ denotes the right Dehn twist along the curve γ.

Definition 3.1. *A chain relation C on the surface $\Sigma_{g,r}$ is given by an embedding $\Sigma_{1,2} \subset \Sigma_{g,r}$ and the standard chain relation on this 2-holed torus, namely*

$$(D_a D_b D_c)^4 = D_e D_d,$$

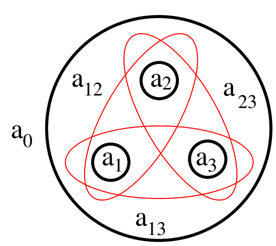
where a,b,c,d,e,f are the following curves of the embedded 2-holed torus:



Definition 3.2. *A lantern relation L on the surface $\Sigma_{g,r}$ is given by an embedding $\Sigma_{0,4} \subset \Sigma_{g,r}$ and the standard lantern relation on this 4-holed sphere, namely*

$$D_{a_{12}} D_{a_{13}} D_{a_{23}} D_{a_0}^{-1} D_{a_1}^{-1} D_{a_2}^{-1} D_{a_3}^{-1} = 1, \quad (6)$$

where $a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ are the following curves of the embedded 4-holed sphere:



The key step in proving Proposition 3.5 and hence Proposition 3.4 is to find an explicit presentation for the central extension $\widetilde{M_g}$. By definition, if we choose arbitrary lifts $\widetilde{D}_a \in \widetilde{M_g}$ for each of Dehn twists $D_a \in M_g$, then $\widetilde{M_g}$ is generated by the elements $\widetilde{D}_a$ plus a central element z of order at most 2. Specifically, as a consequence of Gervais' presentation [17] of the universal central extension of the mapping class group, the group $\widetilde{M_g}$ in the next proposition determines canonically a non-trivial central extension of M_g by $\mathbb{Z}/2\mathbb{Z}$.

Proposition 3.6. *Suppose that $g \geq 3$. Then the group $\widetilde{M_g}$ has the following presentation.*

1. *Generators:*

(a) *With each non-separating simple closed curve a in Σ_g is associated a generator $\widetilde{D}_a$;*

(b) *One (central) element z.*

2. *Relations:*

(a) *Centrality:*

$$z\widetilde{D}_a = \widetilde{D}_a z, \tag{7}$$

for any non-separating simple closed curve a on Σ_g;

(b) *Braid type 0-relations:*

$$\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a, \tag{8}$$

for each pair of disjoint non-separating simple closed curves a and b;

(c) *Braid type 1-relations:*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b, \tag{9}$$

for each pair of non-separating simple closed curves a and b which intersect transversely at one point;

(d) *One lantern relation on a 4-holed sphere subsurface with non-separating boundary curves:*

$$\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = \widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}, \tag{10}$$

(e) *One chain relation on a 2-holed torus subsurface with non-separating boundary curves:*

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z\widetilde{D}_e\widetilde{D}_d. \tag{11}$$

(f) *Scalar equation:*

$$z^2 = 1. \tag{12}$$

Moreover $z \neq 1$.

3.4.3 Proof of Proposition 3.6

By definition $\widetilde{M}_g$ fits into a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{M}_g & \longrightarrow & M_g & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow \rho_{g,\ell} & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{\rho_{g,\ell}(M_g)} & \longrightarrow & \rho_{g,\ell}(M_g) & \longrightarrow & 1,
\end{array}$$

where $\widetilde{\rho_{g,\ell}(M_g)} \subset U(\mathbb{C}^{(\mathbb{Z}/\ell\mathbb{Z})^g})$. This presents $\widetilde{M}_g$ as a pull-back and therefore the relations claimed in Proposition 3.6 will be satisfied if and only if they are satisfied when we project them into M_g and $\widetilde{\rho_{g,\ell}(M_g)} \subset U(\mathbb{C}^{(\mathbb{Z}/\ell\mathbb{Z})^g})$. If this is the case then $\widetilde{M}_g$ will be a quotient of the group obtained from the universal central extension by reducing mod 2 the center and that surjects onto M_g. But, as the mapping class group is Hopfian there are only two such groups: first, $M_g \times \mathbb{Z}/2\mathbb{Z}$ with the obvious projection on M_g and second, the mod 2 reduction of the universal central extension. Then relation (e) shows that we are in the latter case.

The projection on M_g is obtained by killing the center z, and by construction the projected relations are satisfied in M_g and we only need to check them in the unitary group.

Lemma 3.6. *One can choose the lifts of Dehn twists in $\widetilde{M}_g$ so that all braid type relations are satisfied and the lift of the lantern relation is trivial, namely*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c\widetilde{D}_d = \widetilde{D}_u\widetilde{D}_v\widetilde{D}_w,$$

for the non-separating curves on an embedded $\Sigma_{0,4} \subset \Sigma_g$.

Proof. This is standard, see [16]. □

We say that the lifts of the Dehn twists are *normalized* if all braid type relations and one lantern relation are lifted in a trivial way.

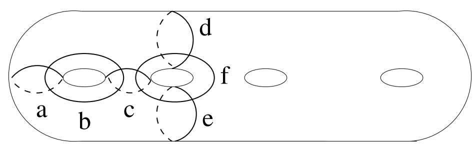
Now the proposition follows from:

Lemma 3.7. *If all lifts of the Dehn twist generators are normalized then* $(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z\widetilde{D}_e\widetilde{D}_d$*, where* $z \neq 1$*,* $z^2 = 1$*.*

Proof. We denote by T_γ the action of D_γ in homology. Moreover we denote by R_γ the matrix in $U(\mathbb{C}^{(\mathbb{Z}/\ell\mathbb{Z})^g})$ corresponding to the prescribed lift $\rho_{g,\ell}(T_\gamma)$ of the projective representation. The level ℓ is fixed through this section and we drop the subscript ℓ from now on.

Our strategy is as follows. We show that the braid relations are satisfied by the matrices R_γ. It remains to compute the defect of the chain relation in the matrices R_γ.

Consider an embedding of $\Sigma_{1,2} \subset \Sigma_g$ such that all curves from the chain relation are non-separating, and thus like in the figure below:



The subgroup generated by D_a, D_b, D_c, D_d, D_e and D_f act on the homology of the surface Σ_g by preserving the symplectic subspace generated by the homology classes of a, e, b, f and being identity on its orthogonal complement. The Weil representation behaves well with respect to the direct sum of symplectic matrices and this enables us to focus our attention on the action of this subgroup on the 4-dimensional symplectic subspace generated by a, e, b, f and to use $\rho_{2,\ell}$. In this basis the symplectic matrices associated to the above Dehn twists are:

$$T_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_b = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix},$$

$$T_d = T_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \; T_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that $T_b = J^{-1}T_aJ$, where J is the matrix of the standard symplectic structure.

Set $q = \exp\left(\frac{\pi i}{\ell}\right)$, which is a 2ℓ-th root of unity. We will change slightly the basis $\{\theta_m, m \in (\mathbb{Z}/\ell\mathbb{Z})^g\}$ of our representation vector space in order to exchange the two obvious parabolic subgroups of $Sp(2g, \mathbb{Z})$. Specifically we fix the basis given by $-S\theta_m$, with $m \in (\mathbb{Z}/\ell\mathbb{Z})^g$. We have then:

$$R_a = \operatorname{diag}(q^{\langle L_a x, x\rangle})_{x\in(\mathbb{Z}/\ell\mathbb{Z})^2}, \text{ where } L_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R_c = \operatorname{diag}(q^{\langle L_c x, x\rangle})_{x\in(\mathbb{Z}/\ell\mathbb{Z})^2}, \text{ where } L_c = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$R_e = R_d = \operatorname{diag}(q^{\langle L_e x, x\rangle})_{x\in(\mathbb{Z}/\ell\mathbb{Z})^2}, \text{ where } L_e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We set now:

$$R_b = S^3 R_a S \text{ and } R_f = S^3 R_e S,$$

where $S = \rho_{g,\ell}\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$.

Lemma 3.8. *The matrices* R_a, R_b, R_c, R_f, R_e *are normalized lifts, namely the braid relations are satisfied.*

We postpone the proof of this lemma a few lines. Let us denote by $G(u,v)$ the Gauss sum:

$$G(u,v) = \sum_{x\in\mathbb{Z}/v\mathbb{Z}} \exp\left(\frac{2\pi\sqrt{-1}ux^2}{v}\right).$$

Denote by $\omega = \frac{1}{2}G(1,2\ell)$. The lift of the chain relation is of the form:

$$(R_aR_bR_c)^4 = \mu R_eR_d,$$

with $\mu \in U(1)$. Our aim now is to compute the value of μ. Set $X = R_aR_bR_c$, $Y = X^2$ and $Z = X^4$. We have then:

$$X_{m,n} = \ell^{-1}\omega\delta_{n_2,m_2}q^{-(n_1-m_1)^2+m_1^2+(n_1+n_2)^2}.$$

This implies $Y_{m,n} = 0$ if $\delta_{m_2,n_2} = 0$. If $m_2 = n_2$ then:

$$Y_{m,n} = \ell^{-2}\omega^2 \sum_{r_1\in\mathbb{Z}/\ell\mathbb{Z}} q^{-(m_1-r_1)^2+m_1^2+(r_1+n_2)^2-(n_1-r_1)^2+r_1^2+(n_1+n_2)^2} =$$

$$= \ell^{-2}\omega^2 \sum_{r_1\in\mathbb{Z}/\ell\mathbb{Z}} q^{m_2^2+n_2^2+2n_1n_2+2r_1(m_1+m_2+n_1)}.$$

Therefore $Y_{m,n} = 0$, unless $m_1+m_2+n_1 = 0$. Assume that $m_1+m_2+n_1 = 0$. Then:

$$Y_{m,n} = \ell^{-1}\omega^2 q^{m_2^2+n_2^2+2n_1n_2} = \ell^{-1}\omega^2 q^{-2m_1m_2}.$$

It follows that: $Z_{m,n} = \sum_{r\in(\mathbb{Z}/\ell\mathbb{Z})^2} Y_{m,r}Y_{r,n}$ vanishes, except when $m_2 = r_2 = n_2$ and $r_1 = -(m_1+m_2)$, $n_1 = -(r_1+r_2) = m_1$. Thus Z is a diagonal matrix. If $m = n$ then:

$$Z_{m,n} = \ell^{-2}\omega^4 Y_{m,r}Y_{r,n} = \ell^{-2}\omega^4 q^{-2m_1m_2-2r_1r_2} = \ell^{-2}\omega^4 q^{m_2^2}.$$

We have therefore obtained:

$$(R_aR_bR_c)^4 = \ell^{-2}\omega^4 T_e^2$$

and thus $\mu = \ell^{-2}\omega^4 = \left(\frac{G(1,2\ell)}{2\ell}\right)^4$. This proves that whenever ℓ is even we have $\mu = -1$. Since this computes the action of the central element z, it follows that $z \neq 1$. □

Proof of Lemma 3.8. We know that R_b is S^3R_aS, where S is the S-matrix, up to an eight root of unity. The normalization of this root of unity is given by the braid relation:

$$R_aR_bR_a = R_bR_aR_b$$

We have therefore:

$$(R_b)_{m,n} = \ell^{-2} \sum_{x\in(\mathbb{Z}/\ell\mathbb{Z})^2} q^{\langle L_ax,x\rangle+2\langle n-m,x\rangle}$$

This entry vanishes except when $m_2 = n_2$. Assume that $n_2 = m_2$. Then:

$$(R_b)_{m,n} = \ell^{-1} \sum_{x_1\in\mathbb{Z}/\ell\mathbb{Z}} q^{x_1^2+2(n_1-m_1)x_1} = \ell^{-1}q^{(n_1-m_1)^2} \sum_{x_1\in\mathbb{Z}/\ell\mathbb{Z}} q^{(x_1+n_1-m_1)^2} = \ell^{-1}q^{(n_1-m_1)^2}\omega$$

where $\omega = \frac{1}{2}\sum_{x\in\mathbb{Z}/2\ell\mathbb{Z}} q^{x^2}$ is a Gauss sum. We have first:

$$(R_aR_bR_a)_{m,n} = \ell^{-1}\omega\delta_{m_2,n_2}q^{-(n_1-m_1)^2+m_1^2+n_1^2} = \ell^{-1}\omega\delta_{m_2,n_2}q^{2n_1m_1}$$

Further

$$(R_bR_a)_{m,n} = \ell^{-1}\omega\delta_{m_2,n_2}q^{-(n_1-m_1)^2+n_1^2}$$

so that:

$$(R_bR_aR_b)_{m,n} = \ell^{-2}\omega^2 \sum_{r\in(Z/\ell\mathbb{Z})^2} \delta_{m_2,r_2}\delta_{n_2,r_2}q^{-(n_1-r_1)^2+r_1^2-(r_1-n_1)^2} =$$

$$= \ell^{-1}\omega^2\delta_{m_2,n_2}q^{2m_1n_1} \sum_{r_1\in Z/\ell\mathbb{Z}} q^{-(r_1-m_1+n_1)^2} = \ell^{-1}\omega\delta_{m_2,n_2}q^{2n_1m_1}$$

Similar computations hold for the other pairs of non-commuting matrices in the set R_b, R_c, R_f, R_e. This ends the proof of Lemma 3.8 □

3.5 Third proof: K-theory computation of $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$

We give below one more proof using slightly more sophisticated tools which were developed by Barge and Lannes in [2] and which allow us to dispose of Deligne's theorem. Recall that according to Stein's stability theorem ([49]) it is enough to prove that $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))=\mathbb{Z}/2\mathbb{Z}$, for g large. It is well-known that the second homology of the linear and symplectic groups can be interpreted in terms of the K-theory group K_2. Denote by $K_1(\mathfrak{A}), K_2(\mathfrak{A})$ and $KSp_1(\mathfrak{A})$, $KSp_2(\mathfrak{A})$ the groups of algebraic K-theory of the stable linear groups and symplectic groups over the commutative ring $\mathfrak{A}$, respectively, see [24] for definitions. Our claim is equivalent to the fact that $KSp_2(\mathbb{Z}/4\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$. The key ingredient in this proof is the exact sequence from ([2], Thm. 5.4.1) which is a generalization of Sharpe's exact sequence (see [24], Thm.5.6.7) in K-theory:

$$K_2(\mathbb{Z}/4\mathbb{Z})\to KSp_2(\mathbb{Z}/4\mathbb{Z})\to V(\mathbb{Z}/4\mathbb{Z})\to K_1(\mathbb{Z}/4\mathbb{Z})\to 1, \quad (13)$$

where the las map is known as the *discriminant*. We first show:

Lemma 3.9. *The homomorphism $K_2(\mathbb{Z}/4\mathbb{Z})\to KSp_2(\mathbb{Z}/4\mathbb{Z})$ is trivial.*

Proof of Lemma 3.9. Recall from [2] that this homomorphism is induced by the hyperbolization inclusion $GL(g,\mathbb{Z}/4\mathbb{Z})\to Sp(2g,\mathbb{Z}/4\mathbb{Z})$, which sends the matrix A to $A\oplus(A^{-1})^\top$. Therefore it would suffice to show that the pull-back of the universal central extension over $Sp(2g,\mathbb{Z}/4\mathbb{Z})$ by the hyperbolization morphism $SL(g,\mathbb{Z}/4\mathbb{Z})\to Sp(2g,\mathbb{Z}/4\mathbb{Z})$ is trivial.

Suppose that we have a central extension by $\mathbb{Z}/2\mathbb{Z}$ over $Sp(2g,\mathbb{Z}/4\mathbb{Z})$. It defines therefore a class in $H^2(Sp(2g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$. We want to show that the image of the hyperbolization homomorphism

$$h: H^2(Sp(2g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})\to H^2(SL(g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$$

is trivial, when $g\geq 3$.

Since these groups are perfect we have the following isomorphisms coming from the universal coefficient theorem:

$$H^2(Sp(2g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})\cong \mathrm{Hom}(H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})),\mathbb{Z}/2\mathbb{Z}),$$
$$H^2(SL(g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})\cong \mathrm{Hom}(H_2(SL(g,\mathbb{Z}/4\mathbb{Z})),\mathbb{Z}/2\mathbb{Z}).$$

As recalled in Lemma 3.1 that the obvious homomorphism $H_2(Sp(2g,\mathbb{Z}))\to H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$ is surjective. This implies that the dual map $H^2(Sp(2g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})\to H^2(Sp(2g,\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$ is injective.

In the case of SL we can use the same arguments to prove surjectivity. Anyway it is known (see [44]) that $H_2(SL(g,\mathbb{Z}))\to H_2(SL(g,\mathbb{Z}/4\mathbb{Z}))$ is an isomorphism and that both groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, when $g\geq 3$. This shows in particular that $H^2(SL(g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z})\to H^2(SL(g,\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$ is injective. Now, we have a commutative diagram:

$$\begin{array}{ccc} H^2(Sp(2g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z}) & \hookrightarrow & H^2(Sp(2g,\mathbb{Z});\mathbb{Z}/2\mathbb{Z}) \\ h\downarrow & & \downarrow H \\ H^2(SL(g,\mathbb{Z}/4\mathbb{Z});\mathbb{Z}/2\mathbb{Z}) & \hookrightarrow & H^2(SL(g,\mathbb{Z});\mathbb{Z}/2\mathbb{Z}). \end{array} \quad (14)$$

The vertical arrow H on the right side is the hyperbolization homomorphism induced by the hyperbolization inclusion $SL(g,\mathbb{Z})\to Sp(2g,\mathbb{Z})$. We know that $H^2(Sp(2g,\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$ is generated by the mod 2 Maslov class. But the restriction of the Maslov cocycle to the subgroup $SL(g,\mathbb{Z})$ is trivial because $SL(g,\mathbb{Z})$ fixes the standard direct sum decomposition of $\mathbb{Z}^{2g}$ in two Lagrangian subspaces. This proves that H is zero. Since the horizontal arrows are injective it follows that h is the zero homomorphism.

An alternative argument is as follows. The hyperbolization homomorphism $H: K_2(\mathbb{Z}/4\mathbb{Z})\to KSp_2(\mathbb{Z}/4\mathbb{Z})$ sends the Dennis-Stein symbol $\{r,s\}$ to the Dennis-Stein symplectic symbol $[r^2,s]$, see e.g. ([52], section 6). According to [49] the group $K_2(\mathbb{Z}/4\mathbb{Z})$ is generated by $\{-1,-1\}$ and thus its image by H is generated by $[1,-1]=0$. □

It is known that:

$$K_1(\mathbb{Z}/4\mathbb{Z})=(\mathbb{Z}/4\mathbb{Z})^*=\mathbb{Z}/2\mathbb{Z}, \quad (15)$$

and the problem is to compute the discriminant map $V(\mathbb{Z}/4\mathbb{Z})\to\mathbb{Z}/2\mathbb{Z}$.

For an arbitrary ring R, the group $V(R)$ for a ring R is defined as follows (see [2, Section 4.5.1]). Consider the set of triples $(L; q_0, q_1)$, where L is a free R-module of finite rank and q_0 and q_1 are non-degenerate symmetric bilinear forms. Two such triples $(L; q_0, q_1)$ and $(L'; q_0', q_1')$ are equivalent, if there exists an R-linear isomorphism $a : L \to L'$ such that $a^* \circ q_0' \circ a = q_0$ and $a^* \circ q_1' \circ a = q_1$. Under direct sum this triples form a monoid. The group $V(R)$ is by definition the quotient of the associated Grothendieck-Witt group by the subgroup generated by Chasles' relations, that is the subgroup generated by the elements of the form:

$$[L; q_0, q_1] + [L; q_1, q_2] - [L; q_0, q_2].$$

There is a canonical map from $V(R)$ to the Grothendieck-Witt group of symmetric non-degenerate bilinear forms over free modules that sends $[L; q_0, q_1]$ to $q_1 - q_0$. Since $\mathbb{Z}/4$ is a local ring, we know that $SK_1(\mathbb{Z}/4\mathbb{Z}) = 1$ and hence by [2, Corollary 4.5.1.5] we have a pull-back square of abelian groups:

$$\begin{array}{ccc} V(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & I(\mathbb{Z}/4\mathbb{Z}) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/4\mathbb{Z})^* & \longrightarrow & (\mathbb{Z}/4\mathbb{Z})^*/((\mathbb{Z}/4\mathbb{Z})^*)^2, \end{array}$$

where $I(\mathbb{Z})$ is a the augmentation ideal of the Grothendieck-Witt ring of $\mathbb{Z}/4\mathbb{Z}$. The structure of the group of units in $\mathbb{Z}/4\mathbb{Z}$ is well-known, and the bottom arrow in the square is then an isomorphism $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, so $V(\mathbb{Z}/4\mathbb{Z}) \simeq I(\mathbb{Z}/4\mathbb{Z})$ and the kernel of $V(Z/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^* \simeq K_1(\mathbb{Z}/4\mathbb{Z})$ is the kernel of the discriminant homomorphism $I(\mathbb{Z}/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^*/((\mathbb{Z}/4\mathbb{Z})^*)^2$. To compute $V(\mathbb{Z}/4\mathbb{Z})$ it is therefore enough to compute the Witt ring $W(\mathbb{Z}/4\mathbb{Z})$. Recall that this is the quotient of the monoid of symmetric non-degenerate bilinear forms over finitely generated projective modules modulo the sub-monoid of *split* forms. A bilinear form is split if the underlying free module contains a direct summand N such that $N = N^\perp$. Since, by a classical result of Kaplansky, finitely generated projective modules over $\mathbb{Z}/4\mathbb{Z}$ are free by [39, Lemma 6.3] any split form can be written in matrix form as:

$$\begin{pmatrix} 0 & I \\ I & A \end{pmatrix},$$

for some symmetric matrix A. Isotropic submodules form an inductive system, and therefore any isotropic submodule is contained in a maximal one and these have all the same rank, in case of a split form this rank is necessarily half of the rank of the underlying free module, which is therefore even. The main difficulty in the following computation is due to the fact that as 2 is not a unit in $\mathbb{Z}/4\mathbb{Z}$, so that the classical Witt cancellation lemma is not true. As usual in this context, for any invertible element u of $\mathbb{Z}/4\mathbb{Z}$ we denote by $\langle u \rangle$ the non-degenerate symmetric bilinear form on $\mathbb{Z}/4\mathbb{Z}$ of determinant u.

Proposition 3.7. *The Witt ring $W(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, and it is generated by the class of $\langle -1 \rangle$.*

The computation of $W(\mathbb{Z}/4\mathbb{Z})$ was also obtained by Gurevich and Hadani in [21].

The discriminant of $\omega = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is 1 and in the proof of Proposition 3.7 below we show that its class is non-trivial in $W(\mathbb{Z}/4\mathbb{Z})$ and hence it represents a non-trivial element in the kernel of the discriminant map $I(\mathbb{Z}/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^*/(\mathbb{Z}/4\mathbb{Z})^{*2}$. From the Cartesian diagram above we get that it also represents a non-trivial element in the kernel of leftmost vertical homomorphism $V(\mathbb{Z}/4\mathbb{Z}) \to K_1(\mathbb{Z}/4\mathbb{Z})$. In particular $KSp_2(\mathbb{Z}/4\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$.

Proof of Proposition 3.7. Thus given a free $\mathbb{Z}/4\mathbb{Z}$-module L, any non-degenerate symmetric bilinear form on L is an orthogonal sum of copies of $\langle 1 \rangle$, of $\langle -1 \rangle$ and of a bilinear form β on a free summand N such that for all $x \in N$ we have $\beta(x, x) \in \{0, 2\}$. Fix a basis $e_1, \cdots, e_n$ of N. Let B denote the matrix of β in this basis. Expanding the determinant of β along the first column we see that there must be an index $i \geq 2$ such that $\beta(e_1, e_i) = \pm 1$, for otherwise the determinant would not be invertible. Without loss of generality we may assume that $i = 2$ and that $\beta(e_1, e_2) = 1$. Replacing if necessary e_j for $j \geq 3$ by $e_j - \frac{\beta(e_1, e_j)}{\beta(e_1, e_2)} e_2$, we may assume that B is of the form:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & a & c \\ 0 & {}^t c & A \end{pmatrix}$$

where A and c are a square matrix and a row matrix respectively, of the appropriate sizes, and $a \in \{0, 2\}$. But then β restricted to the submodule generated by e_1, e_2 defines a non-singular symmetric bilinear form and therefore $N = \langle e_1, e_2\rangle \oplus \langle e_1, e_2\rangle^{\perp}$, where on the first summand the bilinear form is either split or ω. By induction we have that any symmetric bilinear form is an orthogonal sum of copies of $\langle 1\rangle, \langle -1\rangle$, of

$$\omega = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and split spaces.

It's a classical fact (see [39, Chapter I]) that in $W(\mathbb{Z}/4\mathbb{Z})$ one has $\langle 1\rangle = -\langle -1\rangle$. Also $\langle -1\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$ is isometric to $\omega \oplus \langle -1\rangle \oplus \langle 1\rangle$. To see this notice that, if $e_1, \ldots, e_4$ denotes the preferred basis for the former bilinear form, then the matrix in the basis $e_1 + e_2, e_1 + e_3, e_1 - e_2 - e_3, e_4$ is precisely $\omega \oplus \langle -1\rangle \oplus \langle 1\rangle$. Also, in the Witt ring $\langle 1\rangle \oplus \langle -1\rangle = 0$, so $4\langle -1\rangle = \omega$, and a direct computation shows that $\omega = -\omega$, in particular ω has order at most 2. All these show that $W(\mathbb{Z}/4\mathbb{Z})$ is generated by $\langle -1\rangle$ and that this form is of order at most 8. It remains to show that ω is a non-trivial element to finish the proof.

Assume the contrary, namely that there is a split form σ such that $\omega \oplus \sigma$ is split. We denote by A the underlying module of ω and by $\{a, b\}$ its preferred basis. Similarly, we denote by S the underlying space of σ of dimension $2n$ and by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ a basis that exhibits it as a split form. By construction $e_1, \ldots, e_n$ generate a totally isotropic submodule E of rank n in $A \oplus S$, and since it is included into a maximal isotropic submodule, we can adjoin to it a new element v such that $v, e_1, \ldots, e_n$ is a totally isotropic submodule of $A \oplus S$, and hence has rank $n+1$. By definition there are unique elements $x, y \in \mathbb{Z}/4\mathbb{Z}$ and elements $\varepsilon \in E$ and $\phi \in F$ such that $v = xa + yb + \varepsilon + \phi$. Since v is isotropic we have:

$$2x^2 + 2xy + 2y^2 + \sigma(\varepsilon, \phi) + \sigma(\phi, \phi) \equiv 0 \pmod 4.$$

Since $E \oplus \mathbb{Z}/4\mathbb{Z}v$ is totally isotropic, then $\sigma(e_i, v) = \sigma(e_i, \phi) = 0$, for every $1 \le i \le n$. In particular, since ϕ belongs to the dual module to E with respect to σ, $\phi = 0$, so the above equation implies:

$$2x^2 + 2xy + 2y^2 \equiv 0 \pmod 4.$$

But now this can only happen when x and y are multiples of 2 in $\mathbb{Z}/4\mathbb{Z}$. Therefore reducing mod 2, we find that v mod 2 belongs to the $\mathbb{Z}/2\mathbb{Z}$ -vector space generated by the mod 2 reduction of the elements $e_1, \ldots, e_n$, and by Nakayama's lemma this contradicts the fact that the $\mathbb{Z}/4\mathbb{Z}$-module generated by $v, e_1, \ldots, e_n$ has rank $n+1$. □

4 Mapping class group quotients

4.1 Preliminaries on quantum representations

The results of this section are the counterpart of those obtained in section 3.4, by considering $SU(2)$ – instead of abelian – quantum representations. The first author proved in [14] that central extensions of the mapping class group M_g by $\mathbb{Z}$ are residually finite and we improve here this result by showing that the second essential homology of the mapping class group has arbitrarily large torsion. Recall that for $g \ge 4$ the mapping class group is perfect and $H_2(M_g) = \mathbb{Z}$ (see, for instance [41]), while for $g = 3$ this group is still perfect but $H_2(M_3) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (see [45]).

The quantum representation considered here is a projective representation ρ_k of M_g, depending on an integer parameter k, which lifts to a linear representation $\widetilde{\rho}_k : M_g(12) \to U(N(k, g))$ of some central extension $M_g(12)$ of M_g by $\mathbb{Z}$. Masbaum, Roberts ([34]) and Gervais ([17]) gave a precise description of this extension. For $g \ge 3$, $H^2(M_g, \mathbb{Z})$ is generated by the so-called signature class χ (see [30]).Then the cohomology class $c_{M_g(12)} \in H^2(M_g, \mathbb{Z})$ of the extension $M_g(12)$ equals 12χ. We denote more generally by $M_g(n)$ the central extension by $\mathbb{Z}$ whose class is $c_{M_g(n)} = n\chi$. Their explicit presentations show that $M_g(1)$ embeds as a finite index subgroup into $M_g(n)$, by sending the generator c of the center of $M_g(1)$ into the n-th power of the generator of $M_g(n)$. Notice that for $g \ge 4$ the signature extension $\chi = M_g(1)$ is the universal central extension of M_g, but this is not the case when $g = 3$.

Let c be the generator of the center of $M_g(1)$, which by construction is 12 times the generator of the center of $M_g(12)$. The quotient $M_g(1)_n$ of $M_g(1)$ obtained by imposing the new relation $c^n = 1$ is a non-trivial central extension of M_g by $\mathbb{Z}/n\mathbb{Z}$. We will say that a quantum representation $\widetilde{\rho_p}$ *detects* the center of $M_g(1)_n$ if it factors through $M_g(1)_n$ and is injective on its center.

Recall that the image of the generator c of the center of $M_g(1)$ by quantum representations $\widetilde{\rho_p}$ is torsion. For instance, the SO(3)-TQFT with parameter $A = -\zeta_p^{(p+1)/2}$, where ζ_p is a primitive p-th root of unity, (so that A is a primitive $2p$ root of unity with $A^2 = \zeta_p$ provides the quantum representation $\widetilde{\rho}_p$. By carefully choosing the parametrizing root of unity ζ_p for varying p we show:

Lemma 4.1. *For each prime power q^s there exists some quantum representation $\widetilde{\rho_p}$ which detects the center of $M_g(1)_{q^s}$.*

Proof. By [34] we know that $\widetilde{\rho}_p(c) = A^{-12-p(p+1)}$, where $A = -\zeta_p^{(p+1)/2}$ is a $2p$-root of unity.

1. If q is a prime number $q \geq 5$ we let $p = q^s$. Then $2p$ divides $p(p+1)$ and $\widetilde{\rho}_p(c) = A^{-12} = \zeta_p^{-6}$ is of order $p = q^s$. Thus the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{q^s}$.

2. If $q = 2$, we set $p = 2$. Then $\widetilde{\rho}_2(c) = \zeta_2$, and $\widetilde{\rho_2}$ detects the center of $M_g(1)_2$.

3. Set now $p = 12r$ for some integer $r > 1$ to be fixed later. Then $\widetilde{\rho}_p(c) = A^{-12-12r(12r+1)} = \zeta_{2r}^{-1-r(12r+1)} = \zeta_{2r}^{-1-r}$. This $2r$-th root of the unit has order l.c.m.$(1+r, 2r)/(1+r) = 2r/$g.c.d.$(1+r, 2r)$. An elementary computation shows that g.c.d.$(1+r, 2r) = 1$ or 2 depending on whether r is even or odd.

   - If $r = 2^s$, then $\zeta_{2\cdot 2^s}^{-1-2^s}$ is of order 2^{s+1} and the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{2^{s+1}}$.
   - If $r = 3^s$, then $\zeta_{2\cdot 3^s}^{-1-3^s}$ is of order 3^s and the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{3^s}$.

□

4.2 Proof of Theorem 1.2

We wish to prove that for any prime p there exist finite quotients F of M_g, $g \geq 3$, such that $EH_2(F, M_g)$ has p-torsion.

End of proof of Theorem 1.2. By a classical result of Malcev [33], finitely generated subgroups of linear groups over a commutative unital ring are residually finite. This applies to the images of quantum representations. Hence there are finite quotients $\widetilde{F}$ of these for which the image of the generator of the center is not trivial. By Lemma 4.1 we may find quantum representations for which the order of the image of the center can have arbitrary prime power order p. Hence, for any prime p there are finite quotients $\widetilde{F}$ of $M_g(1)$ in which the image of the center has an element of order p. We apply Lemma 2.3 to the quotient F of $\widetilde{F}$ by the image of the center to get finite quotients of the mapping class group with arbitrary primes in the essential homology $EH_2(F, M_g)$. □

Concrete finite quotients with arbitrary torsion in their essential homology from mapping class groups can be explicitly constructed as follows.

Let p be a prime different form 2 and 3. According to Gilmer and Masbaum ([18]) we have that $\widetilde{\rho}_p(\widetilde{M_g}(1)) \subset U(N(p,g)) \cap GL(\mathcal{O}_p)$ for prime p, where $\mathcal{O}_p$ is the following ring of cyclotomic integers $\mathcal{O}_p = \begin{cases} \mathbb{Z}[\zeta_p], & \text{if } p \equiv -1 (\bmod 4) \\ \mathbb{Z}[\zeta_{4p}], & \text{if } p \equiv 1 (\bmod 4). \end{cases}$

Let then consider the principal ideal $\mathfrak{m} = (1 - \zeta_p)$ which is a prime ideal of $\mathcal{O}_p$. It is known that prime ideals of $\mathcal{O}_p$ are maximal and then $\mathcal{O}_p/\mathfrak{m}^n$ is a finite ring for every n. Let then $\Gamma_{p,\mathfrak{m},n}$ be the image of $\widetilde{\rho}_p(M_g(1))$ into the finite group $GL(N(p,g), \mathcal{O}_p/\mathfrak{m}^n)$ and $F_{p,\mathfrak{m},n}$ be the quotient $\Gamma_{p,\mathfrak{m},n}/(\widetilde{\rho}_p(c))$ by the image of the center of $M_g(1)$. The image $\widetilde{\rho}_p(c)$ of the generator c into $\Gamma_{p,\mathfrak{m},n}$ is the scalar root of unity ζ_p^{-6}, which is a non-trivial element of order p in the ring $\mathcal{O}_p/\mathfrak{m}^n$ and hence an element of order p into $GL(N(p,g), \mathcal{O}_p/\mathfrak{m}^n)$. Notice that this is a rather exceptional situation, which does not occur for other prime ideals in unequal characteristic (see Proposition 4.1).

Lemma 2.3 implies then that the image $p_*(H_2(M_g))$ within $H_2(F_{p,\mathfrak{m},n})$ contains an element of order p. This result also shows the contrast between the mapping class group representations and the Weil representations:

Corollary 4.1. *If $g \geq 3$, p is prime and $p \notin \{2,3\}$ (or more generally, p does not divide 12 and not necessarily prime), then $\widetilde{\rho}_p(\widetilde{M_g})$ is a non-trivial central extension of $\rho_p(M_g)$. Furthermore, under the same hypotheses on g and p, if $\mathfrak{m} = (1 - \zeta_p)$, then the extension $\Gamma_{p,\mathfrak{m},n}$ of the finite quotient $F_{p,\mathfrak{m},n}$ is non-trivial.*

Proof. If the extension were trivial then by universality $\widetilde{\rho_p}$ would kill the center of $\mathcal{M}_g(1)$, and this is not the case. The same argument yields the second claim. □

Remark 4.1. Although the group M_2 is not perfect, because $H_1(M_2) = \mathbb{Z}/10\mathbb{Z}$, it still makes sense to consider the central extension $\widetilde{M_2}$ arising from the TQFT. Then the computations above show that the results of Theorem 1.2 and Corollary 4.1 hold for $g = 2$ if p is a prime and $p \notin \{2, 3, 5\}$.

Remark 4.2. The finite quotients $F_{p,\mathfrak{m},n}$ associated to the ramified principal ideal $\mathfrak{m} = (1-\zeta_p)$ were previously considered by Masbaum in [35].

When $p \equiv -1 (\mathrm{mod}\, 4)$ the authors of [15, 36] found many finite quotients of M_g by using more sophisticated means. However, the results of [15, 36] and the present ones are of a rather different nature. Assume that $\mathfrak{n}$ is a prime ideal of $\mathcal{O}_p$ such that $\mathcal{O}_p/\mathfrak{n}$ is the finite field $\mathbb{F}_q$ with q elements. In fact the case of equal characteristics $\mathfrak{n} = \mathfrak{m} = (1 - \zeta_p)$ is the only case where non-trivial torsion can arise, according to:

Proposition 4.1. *If $\mathfrak{n}$ is a prime ideal of unequal characteristic (i.e. such that* g.c.d.$(p, q) = 1$*) and $p, q \geq 5$ then $EH_2(F_{p,\mathfrak{n},n}, M_g) = 0$. Moreover, for all but finitely many prime ideals $\mathfrak{n}$ of unequal characteristic both groups $\Gamma_{p,\mathfrak{n},1}$ and $F_{p,\mathfrak{n},1}$ coincide with $SL(N(p, g), \mathbb{F}_q)$ and hence $H_2(F_{p,\mathfrak{n},1}) = 0$.*

Proof. The image of a p-th root of unity scalar in $SL(N(p, g), \mathbb{F}_q)$ is trivial, as soon as g.c.d.$(p, q) = 1$. Thus $\Gamma_{p,\mathfrak{n},n} \to F_{p,\mathfrak{n},n}$ is an isomorphism and hence the image of $H_2(M_g)$ into $H_2(F_{p,\mathfrak{n},n})$ must be trivial. A priori this does not mean that $H_2(F_{p,\mathfrak{n},n}) = 0$. However, Masbaum and Reid proved in [36] that for all but finitely many prime ideals $\mathfrak{n}$ in $\mathcal{O}_p$ the image $\Gamma_{p,\mathfrak{n},1} \subset GL(N(p, g), \mathbb{F}_q)$ is the whole group $SL(N(p, g), \mathbb{F}_q)$. It follows that the projection homomorphism $\widetilde{M_g}(1) \to SL(N(p, g), \mathbb{F}_q)$ factors through $M_g \to SL(N(p, g), \mathbb{F}_q)$. But $H_2(SL(N, \mathbb{F}_q)) = 0$, for $N \geq 4, q \geq 5$, as $SL(N, \mathbb{F}_q)$ itself is the universal central extension of $PSL(N, \mathbb{F}_q)$. □

4.3 Property A_2 and proof of Theorem 1.3

We will now prove that for $g \geq 4$ the mapping class group M_g has Serre's property A_2 for trivial modules. Our first result reduces the case to study residual finiteness of central extensions of the mapping class group.

Proposition 4.2.
1. *A residually finite group G has property A_2 for all finite G-modules M if and only if any extension by a finite abelian group is residually finite.*
2. *Moreover for* (1.) *to hold true for all finite trivial G-modules it is enough to consider central extensions of G.*

Hence Theorem 1.3 is then a direct consequence of:

Proposition 4.3. *Let $g \geq 4$ be an integer. For any finitely generated abelian group A and any central extension*

$$1 \to A \to E \to M_g \to 1$$

the group E is residually finite.

It will convenient to rephrase property A_n as follows. A residually finite group G is said to have property (D_n) for a module M if for each $x \in H^j(G, M)$, $1 \leq j \leq n$, there exists a subgroup $H \subset G$ of finite index in G such that the image of x in $H^j(H, M)$ is zero. Following ([47], Ex.1) properties A_n and D_n are equivalent. It is easy to see that these properties are also equivalent when we let the modules M run over the class of finite trivial $\widehat{G}$-modules.

4.3.1 Proof of proposition 4.2 (1)

Assume that every extension of G by a finite abelian group is residually finite. Let $x \in H^2(G; A)$ be represented by the extension:

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1.$$

By the equivalent property D_2, it is enough to find a finite index subgroup $H \subset G$ such that x is zero in $H^2(H; A)$. Since E is residually finite, for each non-trivial element $a \in A$ choose a finite quotient F_a of E in which the image of a is not identity. Let B_a be the image of A in F_a, and $Q_a = E_a/B_a$. Denote by F_A, B_A

and Q_A the products of these (finitely many) groups over the set of non-trivial elements a in A. Then the diagonal map $E \to F_A$ fits into a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & E_A & \longrightarrow & F_A & \longrightarrow & Q_A & \longrightarrow & 1 \end{array}$$

Let K be the kernel $\ker(G \to Q_A)$. Then K is a finite index normal subgroup and the pull back of x to $H^2(K; A)$ is trivial.

Conversely, assume that extension G has property A_2 and let

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

be an extension of G by the finite abelian group A. Then, by ([47], Ch. I.2.6, Ex. 2), we have a natural short exact sequence of profinite completions:

$$1 \longrightarrow \widehat{A} \longrightarrow \widehat{E} \longrightarrow \widehat{G} \longrightarrow 1$$

that fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{E} & \longrightarrow & \widehat{G} & \longrightarrow & 1 \end{array}$$

Since A is finite $A \simeq \widehat{A}$, and since G is residually finite the rightmost downward arrow is an injection. By the five lemma the homomorphism $E \hookrightarrow \widehat{E}$ is also injective, and hence E is residually finite, as a subgroup of a profinite group.

4.3.2 Proof of proposition 4.2 (2)

One can easily step from $\mathbb{F}_p$ coefficients to any trivial G-module. Condition (D_n) for G and all trivial $\widehat{G}$-modules $\mathbb{F}_p$ implies (D_n) for G and all trivial $\widehat{G}$-modules M. This follows from decomposing the abelian group M in p-primary components and then use induction on the rank of the composition series of M and the 5-lemma.

4.3.3 Proof of Proposition 4.3

We will use below that a group is residually finite if and only if finite index subgroups are residually finite. Observe that since M_g and A are finitely generated, so is E. Recall that M_g is perfect for $g \geq 4$. The five term exact sequence in homology associated to the central extension E yields the exact sequence:

$$H_2(M_g; \mathbb{Z}) \to A \to H_1(E, \mathbb{Z}) \to 0.$$

Any element $f \in E$ that is not in A, projects non-trivially in the mapping class group and is therefore detected by a finite quotient of this latter group. If $f \in A$ but is not in the image of $H_2(M_g; \mathbb{Z})$, then it projects non-trivially into the finitely generated abelian group $H_1(E; \mathbb{Z})$, and is therefore detected by a finite abelian quotient of E. It remains to detect the elements in the image of $H_2(M_g; \mathbb{Z})$. Recall the following elementary result:

Lemma 4.2. *Let A be a finitely generated abelian group, B a subgroup of A. Then there exists a direct factor C of A that contains B as a subgroup of finite index.*

Apply this lemma to the image B of $H_2(M_g; \mathbb{Z})$ into A, let p_C be the projection onto the subgroup C and consider the push-out diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & M_g & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle p_C} & & \downarrow & & \| & & \\
1 & \longrightarrow & C & \longrightarrow & E_C & \longrightarrow & M_g & \longrightarrow & 1
\end{array}$$

Then it is sufficient to prove that E_C is residually finite in order to show that E is residually finite.

Now, the mapping class group M_g is perfect, and therefore we have a push-out diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H_2(M_g;\mathbb{Z}) & \longrightarrow & M_g(1) & \longrightarrow & M_g & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & C & \longrightarrow & E_C & \longrightarrow & M_g & \longrightarrow & 1,
\end{array}$$

where the first row is the universal central extension, and the arrow $H_2(M_g;\mathbb{Z}) \to C$ is the one appearing in the five term exact sequence of the bottom extension. Recall that for $g \geq 4$, $H_2(M_g;\mathbb{Z}) = \mathbb{Z}$. Two cases could occur:

1. Either $H_2(M_g;\mathbb{Z}) \to C$ is injective and in this case E_C contains the residually finite group $M_g(1)$ as a subgroup of finite index, and this is known to be residually finite (see [14]).

2. Or the image of $H_2(M_g;\mathbb{Z}) \to C$ is a cyclic group $\mathbb{Z}/k\mathbb{Z}$ and E_C contains as a finite index subgroup the reduction mod k of the universal central extension. But Lemma 2.3 shows that for any integer $n \geq 2$, the group $M_g(1)_n$ obtained by reducing mod n a generator of the center of $M_g(1)$ is residually finite.

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