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The Cohn localization of the free group ring

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In [**1**] P. Cohn suggested the construction of a localization of a ring with respect to a class of square matrices. Let us briefly recall the definitions.

Let Λ be a ring and Σ be a set of square matrices over Λ. A ring homomorphism $f:\Lambda\to S$ (S being a ring) is said to be Σ-*inverting* if every matrix in Σ is mapped by f to an invertible matrix in S. A ring homomorphism $\alpha:\Lambda\to\Lambda_\Sigma$ is the *universal* Σ-*inverting homomorphism* if it is Σ-inverting and any Σ-inverting ring homomorphism $\beta:\Lambda\to S$ factors uniquely by α, i.e. there is a unique ring homomorphism $\gamma:\Lambda_\Sigma\to S$ such that the accompanying triangle

$$\begin{tikzcd} & \Lambda \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \Lambda_\Sigma \arrow[rr, "\gamma"'] & & S \end{tikzcd}$$

commutes. It is easy to see that the universal Σ-inverting homomorphism $\alpha:\Lambda\to\Lambda_\Sigma$ exists and is unique up to isomorphism. The ring Λ_Σ is also called Cohn's localization of Λ with respect to Σ.

This localization construction turned out to be extremely useful in the algebraic topology of manifolds. It was shown in [**8**, **9**] that Γ-groups (homology surgery obstruction groups of Cappell and Shaneson) can be viewed as L-groups (Wall surgery obstruction groups) of the localization (in the sense of Cohn) of the group ring.

The aim of the present paper is to compute explicitly the Cohn localization of the free group ring Λ with respect to the class Σ of square matrices which become invertible after applying the augmentation $\epsilon:\Lambda\to k$. We show in this paper that Λ_Σ is isomorphic to a ring of 'rational functions' in non-commuting variables. These non-commutative rational functions are represented by their Taylor power series having some 'periodicity' properties. They were discovered in the theory of formal languages; we refer to [**6**] for a more comprehensive treatment and applications. In the subsequent sections we will describe these rational functions as solutions of some systems of linear equations [**6**] and will find a relation between rational functions and link modules (homology modules associated to links of codimension two).

In view of [**8**, **9**] and of the theory of Cappell and Shaneson [**3**], our computation of Λ_Σ is relevant to $\Gamma_*(\Lambda \to k)$ which is essentially isomorphic to the boundary link concordance group.

Our construction and proofs are purely algebraic. Nevertheless some (algebraic) notions and results which appeared first in link theory [**4**], naturally come into play.

1. *Rational functions*

In this section we will describe a non-commutative generalization of the notion of rational function. Our approach is a little different from [**6**]; the results of the next section show that we obtain the same notion.

1·1. Fix an integer $\mu > 0$ and a principal ideal domain k. Let F_μ denote the free group on μ generators $t_1, \ldots, t_\mu$ and let $\Lambda = k[F_\mu]$ be the group ring.

Consider also the ring $\Gamma = k\langle\langle x_1, \ldots, x_\mu \rangle\rangle$ of formal power series in non-commuting variables $x_1, \ldots, x_\mu$. The ring Λ is embedded in Γ via the Magnus embedding

$$t_i \mapsto 1 + x_i,$$
$$t_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \ldots.$$

It is convenient to use the following conventions on multiple indices. A *multi-index* α is a sequence of integers $\alpha = (i_1, \ldots, i_s)$ with $i_j \in \{1, \ldots, \mu\}$ for $j = 1, \ldots, s$. An empty sequence is also allowed. For a sequence $b = (b_1, \ldots, b_\mu)$ of symbols (letters) and for a multi-index $\alpha = (i_1, \ldots, i_s)$ define the monomials

$$b^\alpha = b_{i_1} b_{i_2} \ldots b_{i_s}, \quad b_\alpha = b_{i_s} b_{i_{s-1}} \ldots b_{i_1}$$

with the convention that

$$b^\phi = 1 = b_\phi.$$

Now, each element $\gamma \in \Gamma$ has a unique representation of the form

$$\gamma = \sum_\alpha a(\alpha)\, x^\alpha \quad (a(\alpha) \in k),$$

where α runs over all multi-indices.

The augmentation $\epsilon: \Gamma \to k$ maps γ into

$$\epsilon(\gamma) = a(\phi) \in k.$$

It is a ring homomorphism; its restriction to Λ is the usual augmentation

$$\epsilon: \Lambda \to k, \quad \epsilon(t_i) = 1.$$

1·2. Let us define derivations

$$\partial_i: \Gamma \to \Gamma \quad (i = 1, \ldots, \mu).$$

If

$$\gamma = \sum_\alpha a(\alpha)\, x^\alpha$$

define $\partial_i \gamma$ as

$$\sum_\alpha a(i\alpha)\, x^\alpha,$$

where $i\alpha$ denotes $(i, i_1, \ldots, i_s)$ for $\alpha = (i_1, \ldots, i_s)$. Thus ∂_i acts as a cancellation of x_i from the left on monomials containing x_i on the left-most position, and sends to zero all other monomials. Each $\gamma \in \Gamma$ has a representation

$$\gamma = \epsilon(\gamma) + \sum_{i=1}^{\mu} x_i\, \partial_i(\gamma).$$

∂_i is a k-linear map; it is related to the product in Γ by the formula

$$\partial_i(\gamma_1 \gamma_2) = \partial_i(\gamma_1) . \gamma_2 + \epsilon(\gamma_1) . \partial_i(\gamma_2),$$

where $\gamma_1, \gamma_2 \in \Gamma$. From this it follows that $\partial_i(\Lambda) \supset \Lambda$ and the restriction of ∂_i on Λ coincides with the Fox derivative [2] with respect to t_i. (Note that ∂_i is also known as an $(\epsilon, 1)$-derivation or a transduction with respect to x_i.)

1·3. Using the conventions of 1·1 on multi-indices we shall define the *higher derivatives*

$$\partial_\alpha : \Gamma \to \Gamma$$

where $\alpha = (i_1, \ldots, i_s)$ is a multi-index and

$$\partial_\alpha = \partial_{i_s} \partial_{i_{s-1}} \ldots \partial_{i_1}.$$

1·4. For $\gamma \in \Lambda$ denote by V_γ the k-submodule of Γ generated (over k) by all derivatives $\partial_\alpha \gamma$, where α runs over all multi-indices (including $\alpha = \phi$).

Definition. An element $\gamma \in G$ will be said to be a *rational function* if V_γ is finitely generated over k. The set of all rational functions will be denoted by $\mathscr{R}$.

1·5. In case $\mu = 1$ the above definition is equivalent to the statement that $\gamma \in k[[x]]$ is the Taylor power series of a rational function of the form

$$\gamma = \frac{p(x)}{q(x)},$$

where $p(x), q(x)$ are polynomials with coefficients in k and $q(0) \in k^*$; cf. 2·4 below.

We will describe now some properties of rational functions in the sense of the definition 1·4.

PROPOSITION 1·6. (1) *Any element $\gamma \in \Lambda$ is a rational function.*

(2) *The sum (product) of two rational functions is a rational function.*

(3) *If $\gamma \in \Lambda$ is a rational function and $\epsilon(\gamma)$ is invertible in k then γ^{-1} is also a rational function.*

Proof. Let us prove (2) first. For $\gamma_1, \gamma_2 \in R$ we have

$$\gamma_1 + \gamma_2 \in V_{\gamma_1} + V_{\gamma_2}, \quad \gamma_1 . \gamma_2 \in V_{\gamma_1} V_{\gamma_2} + V_{\gamma_2}$$

and both modules $V_{\gamma_1} + V_{\gamma_2}$ and $V_{\gamma_1} V_{\gamma_2} + V_{\gamma_2}$ are finitely generated over k and invariant under $\partial_1, \ldots, \partial_\mu$.

To prove (3), assume that γ is a rational function with $\epsilon(\gamma) = 1$. Let us define W to be the set of all power series γ_1 represented in the form

$$\gamma_1 = \gamma_2 . \gamma^{-1} + c\gamma^{-1},$$

where $\gamma_2 \in V_\gamma$ and $c \in k$. W is finitely generated over k and is invariant under $\partial_1, \ldots, \partial_\mu$ because of the following formulae:

$$\partial_j(\gamma^{-1}) = -\partial_j(\gamma) . \gamma^{-1}:$$

$$\partial_j(\gamma_2 . \gamma^{-1}) = \partial_j(\gamma_2) . \gamma^{-1} - \epsilon(\gamma_2) \partial_j(\gamma) \gamma^{-1}.$$

Since $\gamma^{-1} \in W$ we get that γ^{-1} is a rational function.

Now (1) follows from (2) and (3). Each element γ of Λ is a polynomial in $t_i = 1+x_i$ and $t_i^{-1} = (1+x_i)^{-1}$. By (2) it is enough to check that t_i and t_i^{-1} belong to R. For t_i it is evident and for t_i^{-1} it follows from (3).

2. *Systems of linear equations*

PROPOSITION 2·1. *Consider a system of equations of the form*

$$\sum_{j=1}^{n} \lambda_{ij}\gamma_j = \delta_i \quad (i = 1, \ldots, n),$$

with coefficients λ_{ij}, δ_i rational functions. Assume that $\det(\epsilon(\lambda_{ij}))$ *is invertible in k. Then*

(1) *there exists a unique solution $(\gamma_1, \ldots, \gamma_n)$ with each component γ_i a rational function and*

(2) *any rational function is a component of solution of a system of the above type with coefficients λ_{ij}, δ_i belonging to Λ.*

Proof. (1) We may assume additionally that

$$\epsilon(\lambda_{ij}) = \delta_{ij},$$

the Kronecker symbol. Then the above system can be written in the matrix form as

$$(I+A)a = b,$$

where
$$a = (\gamma_1, \ldots, \gamma_n), \quad b = (\delta_1, \ldots, \delta_n),$$

and I is the unit matrix while each element of A belongs to the augmentation ideal of Γ. Thus

$$a = (I - A + A^2 - A^3 + \ldots)b$$

and the power series converges in Γ. This proves that there is a unique solution of the above system in Γ.

Let us use the induction on n to show that all components γ_i of the solution are rational.

If $n = 1$ then the system has the form

$$\lambda . \gamma = \delta,$$

with $\epsilon(\lambda) = 1$. So $\gamma = \lambda^{-1}\delta$ and the result follows from Proposition 1·6.

Now consider a general system as above with $\epsilon(\lambda_{ij}) = \delta_{ij}$. From the first equation one finds

$$\gamma_1 = \lambda_{11}^{-1}\delta_1 - \lambda_{11}^{-1}\lambda_{12}\gamma_2 - \ldots - \lambda_{11}^{-1}\lambda_{1n}\gamma_n.$$

Substituting the expression in other equations of the system, we get a new system

$$\sum_{j=2}^{n} \mu_{ij}\gamma_j = \sigma_i \quad (i = 2, 3, \ldots, n)$$

(of $n-1$ equations in $n-1$ variables) with

$$\mu_{ij} = \lambda_{ij} - \lambda_{i1}\lambda_{11}^{-1}\lambda_{1j}, \quad \sigma_i = \delta_i - \lambda_{i1}\lambda_{11}^{-1}\delta_1.$$

Firstly, one sees that the coefficients μ_{ij}, σ_i are rational. Secondly,

$$\epsilon(\mu_{ij}) = \epsilon(\lambda_{ij}) - \epsilon(\lambda_{i1}).\epsilon(\lambda_{11}^{-1}).\epsilon(\lambda_{1j}) = \delta_{ij},$$

and our statement (1) now follows by induction.

To prove (2), assume that γ is a rational function. Let $e_1, \ldots, e_n$ be a system of generators (over k) of V_γ. One may assume that $\gamma = e_1$. Because V_γ is invariant under $\partial_1, \ldots, \partial_\mu$, we have

$$\partial_j e_i = \sum_{k=1}^{n} a_{ij}^k e_k \quad (i = 1, \ldots, n; j = 1, \ldots, \mu).$$

Thus,

$$e_i = \epsilon(e_i) + \sum_{j,k} x_j a_{ij}^k e_k,$$

which can be rewritten in the form

$$\sum_{j=1}^{n} \lambda_{ij} e_j = \sigma_i \quad (i = 1, \ldots, n)$$

with

$$\lambda_{ij} = \delta_{ij} - \sum_{s=1}^{\mu} x_s a_{is}^j, \quad \sigma_k = \epsilon(e_i).$$

It is clear that λ_{ij}, $\sigma_i \in \Lambda$ and $\epsilon(\lambda_{ij}) = \delta_{ij}$. This completes the proof.

2·2. One may consider systems of equations of the form

$$\sum_{i=1}^{n} \gamma_i \lambda_{ij} = \delta_j \quad (j = 1, \ldots, n)$$

with λ_{ij}, $\delta_j \in \mathscr{R}$, assuming $\det(\epsilon(\lambda_{ij})) \in k^*$. It is clear that statements (1) and (2) of Proposition 2·1 are true with respect to these systems.

2·3. Let Σ denote the set of all square matrices $u = (\lambda_{ij})$ over Λ with $\det(\epsilon(\lambda_{ij})) \in k^*$.

Using the terminology of Cohn [1], chapter 7, we may reformulate Proposition 2·1 and the remark of 2·2 as follows: the inclusion $\Lambda \to \Gamma$ is a Σ-inverting ring homomorphism and the set of rational functions $\mathscr{R}$ coincides with the Σ-rational closure of Λ in Γ.

2·4. The following expression

$$\gamma = [1 - x(1-y^2)^{-1} x - y(1-x^2)^{-1} y]^{-1}$$

gives an example of a rational function. In fact, any rational function can be represented by a *finite algebraic formula* (similar to that given above); this follows from the proof of Proposition 2·1.

2·5. Let $\delta_i : \Gamma \to \Gamma$ be the cancellation of x_i *from the right* (cf. 1·2). One may use δ_i instead of ∂_i and define rational functions as those γ for which $\{\delta_\alpha \gamma\}_\alpha$ has finite rank over k. It follows from the above remark (and also from 2·2) that this gives the same class of formal power series.

3. *Link modules*

3·1. A finitely generated left Λ-module M is of *type L* (or a *link module*) if $\mathrm{Tor}_q^\Lambda(k, M) = 0$ for all† q, where k is regarded as a right Λ-module with trivial action via the augmentation map. As was shown by Sato [7], this condition is equivalent to the following: the map

$$M^\mu = \underbrace{M \times \ldots \times M}_{\mu \text{ times}} \to M,$$

† Note that Tor_q^Λ vanishes identically for $q > 2$.

given by $(m_1, \ldots, m_\mu) \mapsto \Sigma_{i=1}^{\mu} (t_i - 1) m_i$ is a bijection. In other words, each $m \in M$ has a unique representation in the form

$$m = \sum_{i=1}^{\mu} (t_i - 1) m_i.$$

Because of this it is possible [4] to define the *derivations* $\partial_i : M \to M$ for $i = 1, \ldots, \mu$ by $\partial_i(m) = m_i$, where m_i is the element of M appearing in the above decomposition. Now we can think of M as also having a left module structure over the ring

$$D = k\langle \partial_1, \ldots, \partial_\mu \rangle$$

of polynomials in non-commuting variables $\partial_1, \ldots, \partial_\mu$. Any Λ-homomorphism $f : M_1 \to M_2$ between modules of type L is also a D-homomorphism; the converse is also true.

3·2. A relation between the D-module structure and the Λ-module structure is given by

$$\partial_i(\lambda . m) = \partial_i(\lambda) . m + \epsilon(\lambda) . \partial_i(m),$$

where $\lambda \in \Lambda$, $m \in M$ and $\partial_i(\lambda)$ is the Fox derivative with respect to t_i.

3·3. Following [4], let us define some other operations on a module M of type L. For $i = 1, \ldots, \mu$ define

$$\pi_i(m) = (t_i - 1)\, \partial_i(m) \quad (m \in M).$$

Then

$$m = \pi_1(m) + \ldots + \pi_\mu(m),$$

$$\pi_i \circ \pi_i = \pi_i, \quad \pi_i \circ \pi_j = 0 \quad \text{for } i \neq j.$$

3·4. Let M be a Λ-module of type L. A *lattice* in M is a k-submodule $A \subset M$ which

(*a*) is invariant under ∂_i, π_i, for $i = 1, \ldots, \mu$,

(*b*) generates M over Λ, and

(*c*) is finitely generated over k.

It is proved in [4] that any module of type L admits a lattice and any such lattice determines the whole module: two modules of type L are isomorphic if and only if they admit lattices which are isomorphic as D-modules, cf. [4], lemmas 1·5 and 2·6. (Note: because of property (*a*) above, each lattice is a D-submodule of M.)

PROPOSITION 3·5. *The following conditions are equivalent:*

(*a*) *$\gamma \in \Gamma$ is a rational function;*

(*b*) *there exists a left Λ-submodule $M \subset \Gamma/\Lambda$ which is of type L and contains the image of γ under $\Gamma \to \Gamma/\Lambda$.*

Proof. (*a*) $\Rightarrow$ (*b*) Let $W \subset \Gamma/\Lambda$ be the image of V_γ under $\Gamma \to \Gamma/\Lambda$. W is finitely generated over k and $\partial_i(W) \subset W$, for $i = 1, \ldots, \mu$. Consider the left Λ-module $M \subset \Gamma/\Lambda$ generated by W over Λ. Then $\partial_i(m) \in M$ for $m \in M$ and we see that each $m \in M$ has a unique representation of the form

$$m = \sum_{i=1}^{\mu} x_i m_i$$

with $m_i = \partial_i(m)$. By the Sato theorem [7], M is of type L.

(*b*) $\Rightarrow$ (*a*) Suppose that $\bar{\gamma}$, the image of γ under the projection $\Gamma \to \Gamma/\Lambda$, belongs to a submodule $M \subset \Gamma/\Lambda$ of type L. From the proof of lemma 1·5 in [4] it follows that there is a lattice $W \subset M$ containing $\bar{\gamma}$. Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_n \in W$ generate W over k. One may assume that $\bar{\gamma}_1 = \bar{\gamma}$. Because $\partial_i(W) \subset W$ for $i = 1, \ldots, \mu$, there is a presentation

$$\partial_i(\bar{\gamma}_j) = \sum_{r=1}^{n} a_{ij}^r \bar{\gamma}_r \quad (a_{ij}^r \in k).$$

Let $\gamma_i \in \Gamma$ be a representative of $\bar{\gamma}_i$ for $i = 1, \ldots, \mu$; we may choose $\gamma_1 = \gamma$. Then we have

$$\partial_i \gamma_j = \sum_{r=1}^{n} a_{ij}^r \gamma_r + \sigma_{ij}$$

with $\sigma_{ij} \in \Lambda$. Because $\gamma_j = \epsilon(\gamma_j) + \sum_{i=1}^{\mu} x_i \partial_i(\gamma_j)$, we obtain that $(\gamma_1, \ldots, \gamma_n)$ is a solution of the system

$$\sum_{j=1}^{n} \lambda_{ij} \gamma_j = b_i$$

with
$$\lambda_{ij} = \delta_{ij} - \sum_{s=1}^{\mu} x_s a_{si}^j, \quad b_i = \epsilon(\gamma_i) + \sum_{s=1}^{\mu} x_s \sigma_{si}.$$

We observe that $\epsilon(\lambda_{ij}) = \delta_{ij}$ and by Proposition 2·1 we see that $\gamma = \gamma_1$ is a rational function. This completes the proof.

COROLLARY 3·6. *If M is a left Λ-module of type L then the image of any Λ-homomorphism $M \to \Gamma/\Lambda$ belongs to $\mathscr{R}/\Lambda$.*

Proof. The result follows from Proposition 3·5 because the image of $M \to \Gamma/\Lambda$ is a module of type L.

COROLLARY 3·7. *$\mathscr{R}/\Lambda$ is the union of all left Λ-submodules $M \subset \Gamma/\Lambda$ of type L.*

4. Σ-*local modules*

4·1. As in 2·3, let Σ denote the set of all square matrices $u = (\lambda_{ij})$ over Λ with $\det(\epsilon(\lambda_{ij})) \in k^*$.

A right Λ-module X is called Σ-*local* if for any $n \times n$ matrix $u = (\lambda_{ij}) \in \Sigma$, the map

$$X^n \to X^n, \quad (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n),$$

where
$$y_j = \sum_{i=1}^{n} x_i \lambda_{ij} \quad (j = 1, \ldots, n),$$

is a bijection.

Examples of Σ-local Λ-modules are provided by Γ and $\mathscr{R}$, cf. 2·3. Λ_Σ is also an example.

PROPOSITION 4·2. *A right Λ-module X is Σ-local if and only if*

$$\operatorname{Tor}_q^{\Lambda}(X, M) = 0$$

for every left Λ-module M of type L without k-torsion and for every q.

In the proof (cf. 4·4) we will need the following lemma.

LEMMA 4·3. *Let $u = (\lambda_{ij})$ be an $n \times n$ matrix in Σ. Thus u determines a homomorphism (of left Λ-modules)*

$$u : \Lambda^n \longrightarrow \Lambda^n, \quad u(a_1, \ldots, a_n) = (b_1, \ldots, b_n),$$

where

$$b_j = \sum_{i=1}^{n} a_i \lambda_{ij} \quad (j = 1, \ldots, n).$$

Then

(1) $\ker(u) = 0$;

(2) $\operatorname{coker}(u)$ *is a left Λ-module of type L without k-torsion*;

(3) *any module of type L without k-torsion is isomorphic to* $\operatorname{coker}(u)$ *for some $u \in \Sigma$.*

4·4. *Proof of Proposition* 4·2. Let X be a Σ-local module and M be a module of type L without k-torsion. According to statement (3) of Lemma 4·3, M has a free resolution

$$0 \longrightarrow \Lambda^n \xrightarrow{u} \Lambda^n \longrightarrow M \longrightarrow 0$$

with $u \in \Sigma$. Now $\operatorname{Tor}_q^{\Lambda}(X, M)$ is the homology of

$$0 \longrightarrow X^n \xrightarrow{u} X^n \longrightarrow 0$$

which vanishes since X is Σ-local.

The converse statement follows similarly.

4·5. *Proof of Lemma* 4·3. Statement (1) follows from

$$\ker[u : \Lambda^n \longrightarrow \Lambda^n] \subset \ker[u : \Gamma^n \longrightarrow \Gamma^n]$$

since the group on the right is trivial (cf. the beginning of the proof of Proposition 2·1).

To prove (2), consider $M = \operatorname{coker}(u)$, $u \in \Sigma$. Then

$$0 \longrightarrow \Lambda^n \xrightarrow{u} \Lambda^n \longrightarrow M \longrightarrow 0$$

is a resolution of M and $\operatorname{Tor}_*^{\Lambda}(k; M)$ may be computed as the homology of

$$0 \longrightarrow k \otimes_{\Lambda} \Lambda^n \xrightarrow{1 \otimes u} k \otimes_{k} \Lambda^n \longrightarrow 0,$$

which coincides with

$$0 \longrightarrow k^n \xrightarrow{\epsilon(u)} k^n \longrightarrow 0.$$

The last complex is acyclic because $\epsilon(u)$ is invertible over k.

By similar arguments $\operatorname{Tor}_*^{\Lambda}(\Gamma; M) = 0$ and this can be used to show that M has no k-torsion: from the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Gamma/\Lambda \longrightarrow 0$$

we get an isomorphism

$$M \approx \operatorname{Tor}_1^{\Lambda}(\Gamma/\Lambda; M)$$

and thus M is isomorphic to the kernel of

$$1 \otimes u : \Gamma/\Lambda \otimes_{\Lambda} \Lambda^n \longrightarrow \Gamma/\Lambda \otimes_{\Lambda} \Lambda^n.$$

It is clear that this kernel (and M as well) are free of k-torsion since the modules Γ/Λ and $\Gamma/\Lambda \otimes_{\Lambda} \Lambda^n = (\Gamma/\Lambda)^n$ are free of k-torsion. This proves (2).

Let us prove (3). Consider a module M of type L and $N \subset M$ be a lattice (cf. (3·4). N is finitely generated over k and has no k-torsion (assuming that M has no k-torsion). Thus N is free over k. Consider the homomorphism

$$u:\Lambda \otimes_k N \longrightarrow \Lambda \otimes_k N, \quad u(\lambda \otimes n) = \lambda \otimes n - \sum_{i=1}^{\mu} \lambda(t_i - 1) \otimes \partial_i(n),$$

where $\lambda \in \Lambda$, $n \in N$. It is clear that $u \in \Sigma$ and thus coker (u) is a module of type L. We want to show that coker $(u) \approx M$.

Consider the following diagram

$$\begin{tikzcd} & M & & \\ \Lambda \otimes_k N \arrow[r, "u"] & \Lambda \otimes_k N \arrow[u, "f"] \arrow[r, "e"] & \operatorname{coker}(u) \arrow[ul, "g"'] \arrow[r] & 0 \\ & N \arrow[u, "\phi"] \arrow[ur, "\psi"'] & & \end{tikzcd}$$

where $$\phi(n) = 1 \otimes n \quad (n \in N),$$

$$f(\lambda \otimes n) = \lambda n \quad (\lambda \in \Lambda, n \in N),$$

$$\psi = e \circ \phi.$$

Since $f \circ u = 0$ we get a Λ-homomorphism $g:\operatorname{coker}(u) \to M$ with $g \circ e = f$. The map $\psi: N \to \operatorname{coker}(u)$ is a D-homomorphism:

$$\psi(n) = e(1 \otimes n) = e\left(\sum_{i=1}^{\mu} (t_i - 1) \otimes \partial_i(n)\right) = \left(\sum_{i=1}^{\mu} (t_i - 1) e(1 \otimes \partial_i(n)\right) = \sum_{i=1}^{\mu} (t_i - 1)\psi(\partial_i n)$$

and thus $$\partial_i \psi(n) = \psi(\partial_i n) \quad (i = 1, \ldots, \mu).$$

Moreover ψ is a monomorphism because $g \circ \psi = f \circ \phi$ coincides with the inclusion $N \to M$. Thus ψ provides a D-isomorphism between N (which is a lattice in M) and $\psi(N)$ (which is a lattice in coker (u)). That M and coker (u) are isomorphic now follows from lemma 2·6 of [**4**]. This completes the proof of Lemma 4·3.

THEOREM 4·6. *Let $\mathscr{R} \subset \Gamma$ be the ring of rational functions. Consider $\mathscr{R}$ as a left Λ-module. If X is a Σ-local right Λ-module then the map*

$$X = X \otimes_\Lambda \Lambda \longrightarrow X \otimes_\Lambda \mathscr{R}$$

is an isomorphism.

Proof. We have an exact sequence

$$\operatorname{Tor}_1^\Lambda (X; \mathscr{R}/\Lambda) \longrightarrow X \otimes_\Lambda \Lambda \longrightarrow X \otimes_\Lambda \mathscr{R} \longrightarrow X \otimes_\Lambda \mathscr{R}/\Lambda \longrightarrow 0.$$

Therefore 4·6 will follow if we prove that $\operatorname{Tor}_*^\Lambda (X; \mathscr{R}/\Lambda) = 0$. Consider all submodules $M_\alpha \subset \mathscr{R}/\Lambda$ of type L. By Corollary 3·7, $\mathscr{R}/\Lambda$ is the direct limit of $\{M_\alpha\}$. Now from Proposition 4·2 we get

$$\operatorname{Tor}_*^\Lambda(X; \mathscr{R}/\Lambda) = \lim_{\alpha} \operatorname{Tor}_*^\Lambda (X; M_\alpha) = 0$$

as required.

COROLLARY 4·7. *Each Σ-local right Λ-module X has an $\mathscr{R}$-module structure extending its Λ-module structure. If X and Y are two Σ-local Λ-modules and $f: X \to Y$ is a Λ-homomorphism then f is also an $\mathscr{R}$-homomorphism.*

5. *The main theorem*

THEOREM 5·1. *The ring of rational functions $\mathscr{R}$ is isomorphic to the ring Λ_Σ, the Cohn localization of Λ with respect to Σ.*

Proof. Let $\alpha: \Lambda \to \Lambda_\Sigma$ and $\beta: \Lambda \to \mathscr{R}$ denote the canonical inclusions. Since β is Σ-inverting (cf. Section 2), there is a unique ring homomorphism $\gamma: \Lambda_\Sigma \to \mathscr{R}$ with $\gamma \circ \alpha = \beta$. On the other hand, since Λ_Σ is a Σ-local right Λ-module it has a right $\mathscr{R}$-module structure (by Corollary 4·7). Let us denote the product in this structure of $\lambda \in A_\Sigma$ and $r \in \mathscr{R}$ by $(\lambda, r) \in \Lambda_\Sigma$. For each $a \in \Lambda_\Sigma$ the left multiplication by a,

$$f_a: \Lambda_\Sigma \longrightarrow \Lambda_\Sigma, \quad f_a(\lambda) = a\lambda \quad (\lambda \in \Lambda_\Sigma),$$

is a homomorphism of right Λ-modules. By Corollary 4·7, f_a is also an $\mathscr{R}$-homomorphism. Thus

$$a(\lambda, r) = (a\lambda, r) \tag{$*$}$$

for $a \in \Lambda_\Sigma$, $\lambda \in \Lambda_\Sigma$, $r \in \mathscr{R}$.

Let us now define a homomorphism $\delta: \mathscr{R} \to \Lambda_\Sigma$ by

$$\delta(r) = (1, r) \in \Lambda_\Sigma \quad (r \in \mathscr{R}),$$

where 1 is the unit element of Λ_Σ. Thus δ is a ring homomorphism: using $(*)$ we have

$$\delta(r)\,\delta(r') = (1, r)(1, r') = ((1, r)\,1, r') = ((1, r), r') = (1, rr') = \delta(rr').$$

It is clear that for $r \in \Lambda$ we get $\delta(r) = r$. Thus we have the following diagram of ring homomorphisms

$$\begin{tikzcd} & \Lambda \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \Lambda_\Sigma \arrow[rr, shift left, "\gamma"] & & \mathscr{R} \arrow[ll, shift left, "\delta"] \end{tikzcd}$$

with $$\gamma \circ \alpha = \beta, \quad \delta \circ \beta = \alpha.$$

Now one gets

$$(\delta \circ \gamma) \circ \alpha = \alpha \quad \text{and thus} \quad \delta \circ \gamma = 1_{\Lambda_\Sigma}$$

by the universal property of α. On the other hand

$$(\gamma \circ \delta) \circ \beta = \beta$$

and so $\gamma \circ \delta: \mathscr{R} \to \mathscr{R}$ is a ring homomorphism which acts as identity on Λ. Since any rational function $r \in \mathscr{R}$ is a component if solution for a linear system with coefficients in Λ (with matrix in Σ), cf. Proposition 2·1 (2), then applying $\gamma \circ \delta$ to this system we will get another solution of the same system. From uniqueness of the solution (cf. Proposition 2·1 (1)) it follows now that $\gamma \circ \delta = 1_{\mathscr{R}}$. Thus γ and δ are mutually inverse ring homomorphisms.

5·2. We will show now that the ring of rational functions is the Cohn localization with respect to other 'k-points' of Λ.

Let $\omega = (\omega_1, \ldots, \omega_\mu)$ be an ordered set of invertible elements $\omega_i \in k^*$. It defines a ring homomorphism $\epsilon_\omega : \Lambda \to k$ where $\epsilon_\omega(t_i) = \omega_i$. Denote by Σ_ω the set of all square matrices (λ_{ij}) over Λ with

$$\det(\epsilon_\omega(\lambda_{ij})) \in k^*.$$

THEOREM 5·3. *The ring homomorphism $\beta_\omega : \Lambda \to \mathcal{R}$, where $\beta_\omega(t_i) = \omega_i(x_i + 1)$ for $i = 1, \ldots, \mu$ is the universal Σ_ω-inverting ring homomorphism.*

Proof. For $\omega = (1, \ldots, 1)$ this was proved in Theorem 5·1. For general ω it follows from the following commutative diagram

$$\begin{tikzcd}
& \mathcal{R} & \\
\Lambda \arrow[ur, "\beta"] \arrow[dr, "\epsilon"'] & & \Lambda \arrow[ul, "\beta_\omega"'] \arrow[ll, "f"'] \arrow[dl, "\epsilon_\omega"] \\
& k &
\end{tikzcd}$$

where β is the Magnus embedding and f is the ring automorphism of Λ defined by $f(t_i) = \omega_i t_i$ for $i = 1, \ldots, \mu$.

5·4. *Remark.* We are thankful to the referee for pointing out Lewin's paper [5] to us. There he gave a representation of Cohn's *universal field of fractions* U of Λ in the Mal'cev–Neumann ring of formal series. U is a (non-commutative) field; it can be represented as Λ_Φ, where Φ is the class of all *full* square matrices, cf. [1]. Lewin showed in [5] that U is isomorphic to the rational closure of Λ in any Mal'cev–Neumann embedding of Λ.

There is an obvious ring homomorphism $f : \mathcal{R} \to U$ (because our Σ is embedded in Φ). It seems plausible that f is an inclusion; however, f is not surjective (for obvious reasons).

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