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Computation of characteristic classes of a manifold from a triangulation of it

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Abstract. This paper is devoted to the well-known problem of computing the Stiefel–Whitney classes and the Pontryagin classes of a manifold from a given triangulation of the manifold. In 1940 Whitney found local combinatorial formulae for the Stiefel–Whitney classes. The first combinatorial formula for the first rational Pontryagin class was found by Gabrielov, Gel'fand, and Losik in 1975. Since then, different authors have constructed several different formulae for the rational characteristic classes of a triangulated manifold, but none of these formulae provides an algorithm that computes the characteristic cycle solely from a triangulation of the manifold. In this paper a new local combinatorial formula recently found by the author for the first Pontryagin class is described; it provides the desired algorithm. This result uses a solution of the following problem: construct a function f on the set of isomorphism classes of three-dimensional PL-spheres such that for any combinatorial manifold the chain obtained by taking each simplex of codimension four with coefficient equal to the value of the function on the link of the simplex is a cycle.

Contents

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§ 1. Introduction

This paper focuses mainly on the Stiefel–Whitney and Pontryagin classes of manifolds. As is well known, the definitions of these characteristic classes makes essential use of the smooth structure. Nevertheless, the combinatorial invariance of the Stiefel–Whitney classes was proved soon after they were defined. In the 1950s Rokhlin and Schwartz [1] and Thom [2] proved independently that the rational Pontryagin classes are combinatorial invariants. However, their proof is not constructive, that is, it does not provide a way to compute the Pontryagin classes directly from a given triangulation of a manifold. Hence, the important problem of finding combinatorial formulae for characteristic classes, that is, the description of ways to construct simplicial cycles whose homology classes are Poincaré dual to given characteristic classes of the manifold from a triangulation of the manifold only, remained unsolved.

The problem of computing the Stiefel–Whitney numbers and the Pontryagin numbers of a manifold from a given triangulation of it is closely related to the previous problem. Here we should immediately single out the famous Hirzebruch L-genus: for an oriented $4k$-dimensional manifold this is a linear combination of Pontryagin numbers that gives the signature of the manifold. Thus, the L-genus admits a combinatorial description, because the cohomology ring of a triangulated manifold can be computed by a combinatorial procedure. A more effective way to compute the signature of a triangulated manifold was proposed by Ranicki and Sullivan [3]. For a given $4k$-dimensional triangulated manifold they constructed a symmetric bilinear form on the direct sum of the groups of simplicial $2k$-chains and $(2k+1)$-chains such that the signature of this form is equal to the signature of the manifold.

A problem of finding *local* formulae for characteristic classes is of special interest. A formula is said to be *local* if the coefficient of each simplex in the cycle constructed by the formula depends only on the structure of the manifold in a neighbourhood of this simplex. One can often obtain the following stronger locality condition for the rational Pontryagin classes: the coefficient of each simplex depends only on the combinatorial structure of its link.

In 1940 Whitney [4] obtained an explicit combinatorial formula for the Stiefel–Whitney classes. This formula was very simple. To obtain a cycle whose Poincaré dual represents the nth Stiefel–Whitney class of an m-dimensional combinatorial manifold K, one must take the sum modulo 2 of all $(m-n)$-dimensional simplices of the first barycentric subdivision K' of K (see § 3).

All known combinatorial formulae for the rational Pontryagin classes are much more complicated. Until recently, two main approaches to finding combinatorial formulae for the Pontryagin classes were known. The first was proposed by Gabrielov, Gel'fand, and Losik ([5], [6]) and developed by MacPherson [7], Gabrielov [8], and Gel'fand and MacPherson [9]. Diverse formulae obtained in these papers use the definitions of Pontryagin classes in terms of the curvature of the connection, the singularities of several generic sections of the tangent bundle, and the Gauss map. We describe this approach in §§ 4 and 6. The second approach was proposed by Cheeger [10] (see § 5). It is based on the Atiyah–Patodi–Singer construction (see [11]) of the η-invariant of a $(4k-1)$-dimensional Riemannian manifold. Recently, the author [12] proposed a new approach to finding combinatorial formulae for the Pontryagin classes. This approach is based on the use of the apparatus of bistellar moves. The local combinatorial formula thereby obtained for the first Pontryagin class is described in §§ 7–13.

Let us now discuss in more detail what the words *combinatorial formula* mean. This term often has in fact two different meanings. First, a combinatorial formula must construct from each combinatorial manifold a simplicial cycle determined solely by the combinatorial structure of the given manifold. Sometimes a combinatorial formula can be applied only if the combinatorial manifold satisfies some additional conditions. The cycle thus obtained can be a simplicial cycle defined either on the initial triangulation or on some subdivision of it. Second, it is often assumed that a combinatorial formula should provide an algorithm computing the required simplicial cycle from the given combinatorial manifold. We say that formulae of this kind are *algorithmically computable*.

At present, the following combinatorial formulae for the Pontryagin classes are known.

1) The MacPherson modification [7] of the Gabrielov–Gel'fand–Losik formula [5] for the first rational Pontryagin class (see § 4). This formula can be applied to a combinatorial manifold satisfying some additional conditions which distinguish the class of the so-called Brouwer manifolds (for the definition, see § 4). The cycle obtained is a simplicial cycle on the initial triangulation, and the coefficient of each simplex is determined solely by the combinatorial structure of the link of this simplex. The formula is not algorithmically computable, since the calculation requires operations with some complicated configuration spaces. At present, no algorithmic description for these spaces in combinatorial terms is known.

2) The Cheeger formula [10] for all Hirzebruch polynomials in the real Pontryagin classes (see § 5). This formula can be applied to any pseudomanifold with negligible boundary (for the definition, see § 5) and, in particular, to any combinatorial manifold. The cycle obtained is a simplicial cycle on the initial triangulation, and the coefficient of each simplex is determined solely by the combinatorial structure of the link of this simplex. The calculation by this formula can be reduced to the calculation of the spectrum of the Laplace operator on a pseudomanifold with locally flat metric. This spectrum can be computed only approximately, and therefore the Cheeger formula is not algorithmically computable. Moreover, it is not known whether or not the cycle constructed by the formula is rational.

3) The Gel'fand–MacPherson formula [9] for all rational normal Pontryagin classes (see § 6). This formula is not purely combinatorial in the first sense, namely, the cycle obtained depends not only on the combinatorial structure of a given triangulated manifold. This formula can be applied to a triangulated manifold only if this manifold is endowed either with a smooth structure or with its discrete analogue, a so-called *fixing cycle*. The formula gives a simplicial cycle on the first barycentric subdivision of the initial triangulation. The coefficient of each simplex depends both on the combinatorial structure of a neighbourhood of this

simplex and on the restriction of the smooth structure or the fixing cycle to this neighbourhood. For a given smooth structure or fixing cycle, the desired cycle can be computed by a purely combinatorial procedure.

4) The author's formula [12] for the first rational Pontryagin class. This formula can be applied to an arbitrary combinatorial manifold without any additional structures. This is the only known formula which is simultaneously local, combinatorial in the first sense, and algorithmically computable. It gives a cycle which is simplicial on the initial triangulation. The coefficient of each simplex depends only on the combinatorial structure of its link and can be found from the combinatorial type of the link by a finite combinatorial procedure.

Comparisons of the above formulae can be based on the following consideration. When constructing a combinatorial formula, one customarily uses some definition of characteristic classes of a smooth manifold. Two approaches were developed here. The first approach, which we refer to as algebro-topological, uses tools in algebraic topology, including the smooth structure. For instance, in the original paper by Pontryagin [13] a characteristic cycle is the cycle of singularities of k vector fields on the manifold. The second approach, which we call differential-geometric, uses the differential-geometric connection on the manifold and its curvature. (In the literature this approach is often called the Chern–Weil approach.) The differential-geometric definition of characteristic classes of real Riemannian manifolds was given by Pontryagin ([14], [15]), and that of the characteristic classes of complex Hermitian manifolds was given by Chern [16].

As noted by Buchstaber, the author's results [12] realize a new approach to the construction of characteristic classes of manifolds. Under this approach, an n-dimensional characteristic class of an m-dimensional triangulated manifold K is given by a *universal formula* of the form

$$f_\sharp(K) = \sum_{\sigma^{m-n}\in K} f(\operatorname{link}\sigma^{m-n})\sigma^{m-n},$$

where f is a chosen function on the isomorphism classes of oriented $(n-1)$-dimensional PL spheres. The universality property means that the function f is independent of the combinatorial manifold K, and the chain $f_\sharp(K)$ is a cycle for any K. The precise definitions are given in § 7. The basic result is that for any rational characteristic class there is a formula of the desired form. This result improves that of Levitt and Rourke [17] (see § 8). The function f is referred to as a *local formula* if the chain $f_\sharp(K)$ is a cycle for any combinatorial manifold K. For $n=4$ each rational local formula gives the first Pontryagin class up to multiplication by a rational constant. On the other hand, for $n=4$ all local formulae can be described explicitly by using bistellar moves (see §§ 9–11). Thus, we obtain an explicit description for all local formulae for the first rational Pontryagin class (Theorem 11.1). In § 12 we distinguish a single canonical local formula f_0 for the first Pontryagin class and describe a combinatorial procedure for computing the coefficient $f_0(L)$ from a given oriented three-dimensional PL sphere L.

As is well known, there are combinatorial manifolds whose rational Pontryagin classes cannot be represented by integral cocycles. However, for the kth Pontryagin class there is a universal constant N_k such that the cohomology class $N_k p_k(K)$ can be represented by an integral cocycle for any combinatorial manifold K. Let us now consider another problem. Suppose that for any combinatorial manifold K we have realized the homology class dual to a given rational Pontryagin class by a cycle of the form $f_\sharp(K)$, where f is function independent of the manifold K. What are the denominators of the coefficients of the cycles $f_\sharp(K)$, that is, of the values $f(L)$? It turns out that the denominators of the values $f(L)$ increase unboundedly as the number of vertices of the PL-sphere increases, even for the first Pontryagin class. Moreover, arbitrarily large powers of every prime appear in the denominators of values of the form $f(L)$. More exact bounds for the growth of the denominators of the values $f(L)$ are given in § 14.

In § 15 we prove that for any rational characteristic class there is a local formula f such that the problem of computing the number $f(L)$ from a given PL sphere L is algorithmically soluble. The complete proofs of the results in §§ 14 and 15 were given by the author in [12].

Of course, the problem of finding combinatorial formulae makes sense only for combinatorially invariant characteristic classes. In particular, the problem of finding combinatorial formulae for integral Pontryagin classes is not well posed. However, each integral Pontryagin class becomes combinatorially invariant after multiplying by some fixed positive integer, and therefore the corresponding problem of finding a combinatorial formula can be posed, although no such combinatorial formulae have yet been found.

Unless otherwise stated, all manifolds and triangulations in this paper are assumed to be piecewise-linear. By a cobordism we always mean an oriented piecewise-linear cobordism. A simplicial complex is called a PL sphere if some subdivision of it is isomorphic to some subdivision of the boundary of a simplex. A simplicial complex is called an *m-dimensional combinatorial manifold* if the link of each vertex of the complex is an $(m-1)$-dimensional PL sphere. We note that any PL triangulation of a PL manifold is a combinatorial manifold. All manifolds are assumed to be closed. By an isomorphism of oriented simplicial complexes we mean an orientation-preserving simplicial isomorphism. We denote the cone over a simplicial complex K by CK, the join of simplicial complexes K and L by $K * L$, and the link and the star of a simplex σ by $\operatorname{link}\sigma$ and $\operatorname{star}\sigma$, respectively.

Let K be an m-dimensional combinatorial manifold. By a *co-orientation* of a simplex $\sigma^n \in K$ we mean an orientation of the link of σ^n. Any m-dimensional simplex is assumed to be positively co-oriented. Let G be an Abelian group and let $\widehat{G}$ be the orientation sheaf of the manifold $|K|$ with fibre isomorphic to G. Let $\widehat{C}_*(K;G)$ be the chain complex of co-oriented simplicial chains of K with coefficients in G and let $\widehat{\partial}$ be the boundary operator of this complex. (The incidence coefficient of two co-oriented simplices $\tau^{k-1} \subset \sigma^k$ is equal to $+1$ if the orientation of $\operatorname{link}\sigma^k$ is induced by that of $\operatorname{link}\tau^{k-1}$, and is equal to -1 otherwise.) The homology of the complex $\widehat{C}_*(K;G)$ is equal to $H_*(|K|;\widehat{G})$. The homology classes Poincaré dual to the rational Pontryagin classes of $|K|$ belong to the group $H_*(|K|;\widehat{\mathbb{Q}})$. Therefore, the simplicial cycles representing these homology classes must belong to the group $\widehat{C}_*(K;\mathbb{Q})$.

§ 2. Formula for the Euler characteristic

In this section we consider a local formula for the Euler characteristic of a finite simplicial complex. This example can be regarded as an illustration of the definition of a local formula which will be given in § 7.

Let L be a finite simplicial complex. We denote the number of k-dimensional simplices of L by $f_k(L)$ and write

$$F(L) = 1 - \frac{f_0(L)}{2} + \frac{f_1(L)}{3} - \dots + \frac{(-1)^k f_{k-1}(L)}{k+1} + \cdots .$$

Proposition 2.1. *For any finite simplicial complex K,*

$$\chi(K) = \sum_{v \in K} F(\operatorname{link} v),$$

where the summation ranges over all vertices v of K.

Proof. Each k-dimensional simplex σ^k of K has exactly $k+1$ vertices. Hence, the number $f_k(K)$ is exactly $k+1$ times less than the number of pairs (v, σ^k) with v a vertex of K, σ^k a simplex of K, and $v \in \sigma^k$. On the other hand, each vertex $v \in K$ belongs to exactly $f_{k-1}(\operatorname{link} v)$ simplices $\sigma^k \in K$. Therefore,

$$f_k(K) = \frac{1}{k+1} \sum_{v \in K} f_{k-1}(\operatorname{link} v).$$

Thus,

$$\chi(K) = \sum_{k=0}^{\infty} (-1)^k f_k(K) = \sum_{k=0}^{\infty} \sum_{v \in K} \frac{(-1)^k f_{k-1}(\operatorname{link} v)}{k+1}.$$

If K is a combinatorial manifold, then a characteristic cycle representing the Euler class of the manifold can be computed by the local formula

$$E(K) = \sum_{v \in K} F(\operatorname{link} v) v.$$

We note that the coefficients $F(\operatorname{link} v)$ in the representation of the integral Euler cycle by means of the local formula have arbitrarily large denominators which disappear when passing to the non-local formula.

§ 3. Formula for the Stiefel–Whitney classes

Let K be an m-dimensional combinatorial manifold. The Stiefel–Whitney class w_n of K is an element of the cohomology with coefficients in $\mathbb{Z}_2$ if n is even and with coefficients in $\widehat{\mathbb{Z}}$ if n is odd, where $\widehat{\mathbb{Z}}$ stands for the orientation sheaf on the manifold $|K|$ with fibre $\mathbb{Z}$. We denote by W_n the homology class Poincaré dual to w_n. Then $W_n \in H_{m-n}(|K|; \mathbb{Z}_2)$ if n is even and $W_n \in H_{m-n}(|K|; \mathbb{Z})$ if n is odd. We denote by K' the first barycentric subdivision of K. If n is even, we denote by C_n the sum modulo 2 of all $(m-n)$-dimensional simplices of K'. Now let n be odd.

Let $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{m-n}$ be distinct non-empty simplices of K, and denote by $\tau(\sigma_0, \sigma_1, \dots, \sigma_{m-n})$ the simplex of K' whose vertices are the barycentres $b(\sigma_i)$ of the simplices σ_i. The orientation of the simplex $\tau(\sigma_0, \sigma_1, \dots, \sigma_{m-n})$ is given by the sequence of vertices $b(\sigma_0), b(\sigma_1), \dots, b(\sigma_{m-n})$. We introduce an integral $(m-n)$-dimensional chain C_n by the formula

$$C_n = \sum_{\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{m-n}} (-1)^{\dim \sigma_0 + \dim \sigma_1 + \cdots + \dim \sigma_{m-n}} \tau(\sigma_0, \sigma_1, \dots, \sigma_{m-n}).$$

Theorem 3.1. *The chain C_n is a cycle (modulo 2 if n is even and integral if n is odd) and represents the homology class W_n Poincaré dual to the nth Stiefel–Whitney class of K.*

This theorem was conjectured by Stiefel [18] and first proved by Whitney [4], but the complete proof was not published. For smooth manifolds the complete proof was first published by Halperin and Toledo [19]. To prove the theorem, they explicitly construct continuous tangent vector fields $F_1, F_2, \dots, F_m$ on $|K|$ which are smooth on each simplex and such that on each p-dimensional simplex $\sigma \in K$ the fields $F_1, F_2, \dots, F_p$ are linearly independent and the index of F_{p+1} (mod $F_1, F_2, \dots, F_p$) at the barycentre of σ is equal to ± 1. In fact, their proof works for an arbitrary combinatorial manifold. A sketch of a proof of Theorem 3.1 using another technique was published by Cheeger [20].

§ 4. Gabrielov–Gel'fand–Losik formula

The first explicit formula for the first rational Pontryagin class of a triangulated manifold was obtained by Gabrielov, Gel'fand, and Losik [5]. We describe here some ideas at the basis of this formula. Let M be a smooth manifold of dimension m and let K be a smooth triangulation of M. Let ∇ be a smooth connection in the tangent bundle of M. If a local trivialization of the tangent bundle is given, then the connection ∇ is given by a 1-form ω with values in $\mathfrak{gl}(m, \mathbb{R})$. Let $\Omega = d\omega - \omega \wedge \omega$ be the curvature form. Then

$$P^0(\nabla) = \operatorname{tr}(\Omega \wedge \Omega) = \operatorname{tr}\bigl(d\omega \wedge d\omega - 2\omega \wedge \omega \wedge d\omega\bigr)$$

is a well-defined 4-form on M. As is well known, the value of the first Pontryagin class of M on a piecewise-smooth 4-dimensional cycle Z is given by the formula

$$\bigl\langle p_1(M), Z \bigr\rangle = -\frac{1}{8\pi^2} \int_Z P^0(\nabla).$$

Let $\tau \in K$ be an $(m-1)$-dimensional co-oriented simplex. We denote by $\sigma_0(\tau)$ and $\sigma_1(\tau)$ the two m-dimensional simplices containing τ. Let $\rho \in K$ be an $(m-2)$-dimensional co-oriented simplex. We denote by $\sigma_1(\rho), \sigma_2(\rho), \dots, \sigma_{k(\rho)}(\rho)$ the m-dimensional simplices containing ρ and indexed in the direction of a positive circuit of the link of the simplex ρ.

Proposition 4.1. *Let a smooth connection ∇_σ on the tangent bundle of a manifold K be given on each m-dimensional simplex $\sigma \in K$. Let Z be a piecewise-smooth 4-dimensional cycle transversal to the triangulation K. Then*

$$\begin{aligned}\langle p_1(M), Z\rangle = -\frac{1}{8\pi^2}\Bigg(&\sum_{\dim\sigma=m}\int_{Z\cap|\sigma|} P^0(\nabla_\sigma)\\ &+\sum_{\dim\tau=m-1}\int_{Z\cap|\tau|} P^1(\nabla_{\sigma_0(\tau)},\nabla_{\sigma_1(\tau)})\\ &+\sum_{\dim\rho=m-2}\sum_{j=2}^{k(\rho)-1}\int_{Z\cap|\rho|} P^2(\nabla_{\sigma_1(\rho)},\nabla_{\sigma_j(\rho)},\nabla_{\sigma_{j+1}(\rho)})\Bigg),\end{aligned} \tag{$*$}$$

where

$$P^1(\nabla_0,\nabla_1) = \operatorname{tr}\Big((\omega_1-\omega_0)\wedge d(\omega_1+\omega_0)+\frac{2}{3}\omega_0\wedge\omega_0\wedge\omega_0-\frac{2}{3}\omega_1\wedge\omega_1\wedge\omega_1\Big),$$
$$P^2(\nabla_0,\nabla_1,\nabla_2) = -\operatorname{tr}\big(\omega_0\wedge\omega_1+\omega_1\wedge\omega_2+\omega_2\wedge\omega_0\big).$$

Here the ω_i stand for $\mathfrak{gl}(m,\mathbb{R})$-valued 1-forms corresponding to the connections ∇_i under some choice of a local trivialization.

The collection of the forms $P^0(\nabla)$, $P^1(\nabla_0,\nabla_1)$, $P^2(\nabla_0,\nabla_1,\nabla_2)$ is called a *difference cocycle*, since

$$dP^1(\nabla_0,\nabla_1) = P^0(\nabla_1)-P^0(\nabla_0),$$
$$-dP^2(\nabla_0,\nabla_1,\nabla_2) = P^1(\nabla_0,\nabla_1)+P^1(\nabla_1,\nabla_2)+P^1(\nabla_2,\nabla_0).$$

Let us choose the connections ∇_σ in such a way that:

(1) the connections ∇_σ are flat;
(2) the tangent bundle $T|\tau|$ is invariant under ∇_σ for any simplex $\tau\subset\sigma$;
(3) the restrictions of ∇_{σ_1} and ∇_{σ_2} to $T|\sigma_1\cap\sigma_2|$ coincide for any two m-dimensional simplices σ_1 and σ_2.

In this case the first two summands in the formula (*) vanish. Thus, computation of the integral over the 4-dimensional cycle Z reduces to computation of the integrals of certain 2-forms over the 2-dimensional chains $Z\cap|\rho|$. The next steps reduce the computation of these integrals first to the integration of certain 1-forms over the intersections of Z with simplices of codimension 3 and then to the determination of the index of intersection of Z with some $(m-4)$-dimensional cycle which turns out to be the desired cycle whose homology class is dual to the class $p_1(M)$. However, these steps are much more complicated and require an investigation of the topology of the configuration spaces $\Sigma(\sigma)$. We do not present here a full construction of the formula in [5]; however, we define the spaces $\Sigma(\sigma)$ and indicate the properties of them used in the construction of the formula.

Definition 4.1. By a *flattening* of a combinatorial manifold K at a simplex σ we mean a homeomorphism of $|C \operatorname{link} \sigma|$ onto a neighbourhood of the origin in $\mathbb{R}^{\operatorname{codim} \sigma}$ which takes the vertex of the cone to the origin and is linear on each simplex. A combinatorial manifold K is called a *Brouwer manifold* if a flattening of K at σ exists for any non-empty simplex $\sigma \in K$.

A combinatorial manifold need not be a Brouwer manifold [21]; however, any compact combinatorial manifold admits a barycentric subdivision which is a Brouwer manifold [22]. A Brouwer manifold need not be smoothable: for a combinatorial manifold to be smoothable it is necessary that there be a concordant choice of flattenings at the simplices.

Let K be a Brouwer manifold and let $\sigma \in K$ be a simplex of codimension k. The space of flattenings of K at the simplex σ is a topological space with an action of the group $GL(k, \mathbb{R})$. The orbit space of this action is denoted by $\Sigma(\sigma)$ and is called a *configuration space*. One can readily prove that $\Sigma(\sigma)$ is contractible if $\operatorname{codim} \sigma = 2$. As is also known, $\Sigma(\sigma)$ is connected [23] and simply connected [24] if $\operatorname{codim} \sigma = 3$. The formula in [5] can be applied only to triangulations K satisfying the so-called condition (A): the space $\Sigma(\sigma)$ is connected for every $\sigma \in K$ such that $\operatorname{codim} \sigma = 4$. It is not known which triangulations satisfy this condition.

Let q be the number of vertices of the complex given by $\operatorname{link} \sigma$. Then corresponding to each flattening $\psi\colon |C \operatorname{link} \sigma| \to \mathbb{R}^k$ is a q-tuple of non-zero vectors in $\mathbb{R}^k$. The space $\Sigma(\sigma)$ has a natural stratification: it can be represented as the union of the disjoint strata $\Sigma_c(\sigma)$, $c = 0, 1, 2, \dots$, where the stratum $\Sigma_0(\sigma)$ consists of the equivalence classes of flattenings for which the corresponding configuration of q vectors in $\mathbb{R}^k$ is in general position, that is, does not contain k linearly dependent vectors, and the stratum $\Sigma_c(\sigma)$, $c > 0$, consists of all equivalence classes of flattenings for which the corresponding configuration of vectors has a c-fold degeneracy.

The formula obtained in [5] can be applied to any Brouwer manifold satisfying condition (A). To calculate with the help of this formula, one must choose the following additional structures on the manifold K: a point $y_\tau \in \Sigma_0(\tau)$ for each simplex τ of codimension 4, a point $y_\sigma \in \Sigma_0(\sigma)$ for each simplex σ of codimension 3, and a curve $z_{\sigma,\tau}(t)$ in $\Sigma_0(\sigma) \cup \Sigma_1(\sigma)$ such that $z_{\sigma,\tau}(0) = y_\sigma$ and $z_{\sigma,\tau}(1)$ is the image of y_τ under the natural map $\Sigma(\tau) \to \Sigma(\sigma)$ for each pair $\tau \subset \sigma$ of simplices such that $\operatorname{codim} \tau = 4$ and $\operatorname{codim} \sigma = 3$. Moreover, one must choose an additional combinatorial structure, the so-called *hypersimplicial system*. After choosing all these structures, one can by a straightforward combinatorial computation obtain a rational simplicial cycle whose homology class is Poincaré dual to the first Pontryagin class of the manifold K. We note that the stratifications of the spaces $\Sigma(\rho)$ are rather complicated, and there is no combinatorial way to choose the points y_σ and y_τ and the curves $z_{\sigma,\tau}$. Hence, the Gabrielov–Gel'fand–Losik formula is not algorithmically computable. The only case in which the additional structures can be chosen in some combinatorial way is the case in which a global smoothing of the manifold K is given.

A generalization of the Gabrielov–Gel'fand–Losik formula [5] for the higher Pontryagin classes was obtained by Gabrielov [8]. To compute the kth Pontryagin class of K, it is assumed that the manifold K satisfies the following condition (A_{4k})

generalizing the condition (A):

$$\widetilde{H}_q(\Sigma_\sigma;\mathbb{Q})=0 \quad \text{for every} \quad 0\leqslant q\leqslant 4k-\operatorname{codim}\sigma$$

for any simplex $\sigma\in K$. This condition is very restrictive, and there is no way to find out whether or not a given Brouwer manifold satisfies it. All the more so, there is no combinatorial way to make calculations in accordance with the formula. The specific feature of the Gabrielov approach is that, instead of the definition of the Pontryagin classes in terms of the curvature of the connection, he uses the definition of the Pontryagin classes in terms of degeneracies in systems of sections of the tangent bundle (see [13], [25]).

The formula obtained in [5] is not local. In [6] an averaging procedure over the choice of additional structures is described, and it enables one to construct a rational simplicial cycle for which the coefficient of each simplex is determined solely by the combinatorial structure of the link of the simplex and the homology class of the cycle is dual to the class $p_1(|K|)$ for any Brouwer manifold K satisfying condition (A). However, to make calculations with the help of the formula, one must know much more about the spaces $\Sigma(\rho)$: one must not only choose some points y_σ and y_τ and the curves $z_{\sigma,\tau}$ but also describe all such triples of points and curves satisfying some special conditions.

In [7] MacPherson modified the formula in [6], enabling him to get rid of the condition (A). Thus, the resulting formula can be applied to an arbitrary Brouwer manifold, and the coefficient of a simplex in the cycle obtained depends only on the combinatorial structure of the link of the simplex. However, this formula still contains a step related to the description of the stratification of the spaces $\Sigma(\rho)$, and this step cannot be carried out combinatorially. In [7] MacPherson gave a new proof that the resulting formula defines a cycle whose homology class is Poincaré dual to the first Pontryagin class. The idea of the proof is to construct a homology analogue of the Gauss map. If M^m is a smooth manifold, then for any embedding $M^m\hookrightarrow\mathbb{R}^{n-1}$ one has the Gauss map $g\colon M^m\to G_{n-1,m}$, where $G_{n-1,m}$ is the Grassmannian manifold of all m-dimensional subspaces of $\mathbb{R}^{n-1}$. Taking the composition of the map g and the natural embedding $G_{n-1,m}\hookrightarrow G_{n,m+1}$, we obtain a map $g_1\colon M^m\to G_{n,m+1}$. We have $g^*\gamma^m_{n-1}\cong TM^m$ and $g_1^*\gamma^{m+1}_n\cong TM^m\oplus\varepsilon^1$, where TM^m is the tangent bundle of M^m, γ^m_n is the tautological vector bundle over $G_{n,m}$, and ε^1 is a trivial line bundle. Now let K be an m-dimensional Brouwer manifold with n vertices. Then the standard embedding $|K|\hookrightarrow\Delta^{n-1}\subset\mathbb{R}^{n-1}$ is well defined. MacPherson explicitly constructs a *homology Gauss* 4-*map*, that is, a chain map $f\colon C_i(K^*;\mathbb{Q})\to C_i(G_{n,m+1};\mathbb{Q})$, $i\leqslant 4$, such that $f^*p_1(\gamma^{m+1}_n)=p_1(|K|)$, where K^* stands for the cellular decomposition of $|K|$ dual to the triangulation K.

§ 5. Cheeger formula

Cheeger's approach [10] is based on the construction of the Hodge theory for pseudomanifolds with a locally flat metric that satisfy certain local topological conditions. A simplicial complex of dimension m is called a pseudomanifold if each simplex of it is contained in some m-dimensional simplex and each $(m-1)$-dimensional simplex is contained in exactly two m-dimensional simplices. Let K be a compact m-dimensional pseudomanifold and let Σ^{m-2} be the

$(m-2)$-skeleton of K. We introduce a metric on $|K|$ such that its restriction to each simplex coincides with the Euclidean metric on the regular simplex with edge 1. Then $|K| \setminus |\Sigma^{m-2}|$ is a non-compact smooth Riemannian manifold. We denote by $H^*_{(2)}(|K| \setminus |\Sigma^{m-2}|;\mathbb{R})$ the L_2-cohomology spaces of this manifold. By definition, we set $H^*_{(2)}(K;\mathbb{R}) = H^*_{(2)}(|K| \setminus |\Sigma^{m-2}|;\mathbb{R})$. One can readily see that the L_2-cohomology spaces thus defined for a pseudomanifold K are invariant under passage to subdivisions, and hence are piecewise-linear invariant.

Definition 5.1. A pseudomanifold K is called a *pseudomanifold with negligible boundary* if $H^k_{(2)}(\operatorname{link}\sigma;\mathbb{R}) = 0$ for any non-empty simplex $\sigma \in K$ such that $\dim \operatorname{link}\sigma = 2k$.

This condition holds if the closures of the operators d and d^* in the spaces of L_2-forms on the manifold $|K| \setminus |\Sigma^{m-2}|$ are conjugate. In particular, any combinatorial manifold is a pseudomanifold with negligible boundary. The L_2-cohomology spaces of compact pseudomanifolds with negligible boundary are always finite-dimensional.

Cheeger defines an analogue of the Atiyah–Patodi–Singer functional $\eta(L)$ for any pseudomanifold L of dimension $(4k-1)$ with negligible boundary. Let ϕ_j be co-closed $(2k-1)$-dimensional eigenforms of the Laplace operator on $|L| \setminus |\Sigma^{4k-3}|$ and let μ_j be the corresponding eigenvalues. The forms ϕ_j can be normalized by the condition $d\phi_j = \pm\frac{\sqrt{\mu_j}}{2k-1} * \phi_j$. Let us consider the function

$$\eta(s) = \int_{|L|} \frac{2}{\sqrt{\pi}} \sum_j \mu_j^{-(s+\frac{1}{2})} \phi_j \wedge d\phi_j, \qquad \operatorname{Re} s < -\frac{1}{2}\,.$$

The function η can be meromorphically continued to the entire complex plane, and the point 0 is not a pole of this continuation. In this case $\eta(L) = \eta(0)$. We note that the value $\eta(L)$ is real and depends only on the combinatorial type of the pseudomanifold L.

Theorem 5.1 (Cheeger [10]). *Let K be an oriented m-dimensional pseudomanifold with negligible boundary. Then the chain*

$$c_{m-4k}(K) = \sum_{\sigma^{m-4k} \in K} \eta(\operatorname{link}\sigma^{m-4k})\sigma^{m-4k}$$

is a cycle. The homology class of the cycle $c_{m-4k}(K)$ is invariant under passage to subdivisions of K, and hence this class is a piecewise-linear invariant. If K is a combinatorial manifold, then the homology class of the cycle $c_{m-4k}(K)$ is Poincaré dual to the $4k$-dimensional Hirzebruch L-class of the manifold $|K|$.

The Pontryagin classes can be represented as polynomials in the Hirzebruch L-classes, that is, the Hirzebruch L-classes are a generating system in the Pontryagin ring $\mathbb{Q}[p_1, p_2, \dots]$. Theorem 5.1 can be regarded as a definition of the homology L-classes for pseudomanifolds with negligible boundary. The cycle $c_{m-4k}(K)$ is determined solely by the combinatorial structure of the pseudomanifold K, and the Cheeger formula is combinatorial in this sense. However, no combinatorial way to compute the value $\eta(L)$ from a given pseudomanifold L is known. Moreover, it is not known whether or not the values $\eta(L)$ are rational. Thus, Theorem 5.1 gives real cycles only.

§ 6. Gel'fand–MacPherson formula

In [9] Gel'fand and MacPherson obtained combinatorial formulae for all rational normal Pontryagin classes. The normal Pontryagin classes of a manifold M are the classes $\tilde{p}_k(M) \in H^{4k}(M;\mathbb{Q})$ such that $(1 + p_1(M) + p_2(M) + \cdots) \smile (1 + \tilde{p}_1(M) + \tilde{p}_2(M) + \cdots) = 1$. The approach used in [9] is a discretization of the following approach to the definition of the Pontryagin classes. Let M be an m-dimensional smooth manifold with m odd, and let E be the total space of the bundle $\eta = TM \oplus \varepsilon^1$, where ε^1 stands for a trivial line bundle over M. We denote by $\pi\colon \mathcal{Y} \to M$ the Grassmannian bundle of $(m-1)$-dimensional planes in η and by ξ the tautological 2-dimensional vector bundle over $\mathcal{Y}$.

Proposition 6.1. *The normal Pontryagin classes of M satisfy the equality*

$$\tilde{p}_k(M) \frown [M] = (-1)^k \pi_*(e(\xi)^{m-1+2k} \frown [\mathcal{Y}]),$$

where $e(\xi)$ is the Euler class of the vector bundle ξ with coefficients in the orientation sheaf of ξ.

A general approach to the construction of the characteristic classes using the direct image of the Euler class of the Grassmannization of the initial vector bundle was given by Buchstaber in [26] (for a detailed exposition, see [27]).

A discrete analogue of the above construction uses *oriented matroids*. (For an introduction to the theory of oriented matroids, see [28] and [29].) Let K be an oriented m-dimensional simplicial manifold with m odd. The use of oriented matroids enables one to construct a simplicial complex Y which is a combinatorial analogue of the space $\mathcal{Y}$. An analogue of the map π is a simplicial map $\widehat{\pi}\colon Y \to K'$, where K' is the first barycentric subdivision of K. There is a canonical topological circle bundle over Y, and the rational simplicial cocycle Ω representing the Euler class of this circle bundle can be computed combinatorially. Moreover, the simplicial complex Y and the cocycle Ω can be computed from the triangulation K *locally*, that is, the structure of the pre-image $\widehat{\pi}^{-1}(|L|)$ and the restriction $\Omega\,|_{\widehat{\pi}^{-1}(|L|)}$ depend only on the combinatorial structure of the manifold K in a neighbourhood of the subcomplex L.

Theorem 6.1 (Gel'fand–MacPherson [9]). *Let $\phi \in C_{3m-2}(Y;\mathbb{Z})$ be a simplicial cycle such that $\widehat{\pi}_*(\Omega^{m-1} \frown \phi) = [K']$, and let $\zeta_k \in C_{m-4k}(K';\mathbb{Q})$ be the simplicial cycle given by the formula*

$$\zeta_k = (-1)^k \widehat{\pi}_*\biggl(\biggl(\frac{1}{2}\Omega\biggr)^{m-1+2k} \frown \phi\biggr).$$

Then ζ_k represents the homology class Poincaré dual to the class $\tilde{p}_k(|K|)$.

This formula can readily be generalized to the case of an even-dimensional manifold and also to the case of a non-orientable manifold.

Any cycle ϕ satisfying the condition of Theorem 6.1 is said to be *fixing* and is an analogue of the fundamental cycle of the manifold $\mathcal{Y}$. A fixing cycle cannot be chosen canonically, because in contrast to $\mathcal{Y}$ the space Y is not a manifold. Gel'fand and MacPherson regard a fixing cycle as a structure on K that is a combinatorial analogue of a smooth structure. A cycle ϕ can be (locally) recovered from a given global smoothing of the manifold $|K|$, but no way is known to construct a fixing cycle for an arbitrary combinatorial manifold K. Thus, the Gel'fand–MacPherson formula can be applied only to a simplicial manifold with a given fixing cycle or global smoothing, which is a disadvantage of this formula as compared with the formulae in [7] and [10]. On the other hand, the merit of the Gel'fand–MacPherson formula is that the cycle ζ_k can be computed by a finite combinatorial procedure if a fixing cycle (or a smoothing) is given. The cycle ζ_k is locally determined by a triangulation K and a fixing cycle (or a smoothing), that is, the coefficient of each simplex of K' in the cycle ζ_k depends only on the combinatorial structure of the manifold K' in a neighbourhood of this simplex and on the restriction of the fixing cycle (or the smoothing) to the pre-image of this neighbourhood.

§ 7. Local formulae

In the rest of the paper we describe a local combinatorial formula obtained by the author in [12] for the first Pontryagin class. This is the first formula which, for an arbitrary combinatorial manifold without any additional structures, enables one to give a purely combinatorial computation of a cycle whose homology class is dual to the first Pontryagin class of the manifold. The resulting cycle is simplicial on the initial triangulation of the manifold rather than on some subdivision of it. Moreover, the coefficient of each simplex is determined solely by the combinatorial structure of the link of this simplex. Thus, the resulting cycle is of the form

$$f_\sharp(K) = \sum_{\sigma^{m-4}\in K} f(\operatorname{link}\sigma^{m-4})\sigma^{m-4},$$

where $m = \dim K$ and f is a function on the isomorphism classes of oriented 3-dimensional PL spheres. If $m > 4$, then the simplices $\sigma^{m-4} \in K$ have no distinguished orientation, and therefore the above sum makes sense only if the value $f(L)$ changes sign when the orientation of the PL-sphere L is changed. In § 2 we constructed a function F that gives a local formula for the Euler class of an oriented manifold. Here the value $F(L)$ does not depend on the orientation of the PL-sphere L. This difference corresponds to the fact that the Euler class of an oriented manifold changes sign when the orientation of the manifold is changed, whereas the Pontryagin classes do not depend on the orientation of the manifold. The derivation of a formula for the first Pontryagin class is broken up into two steps. First, using the apparatus of bistellar moves, we can describe all functions f such that the chain $f_\sharp(K)$ is a cycle for any combinatorial manifold K (see §§ 9–11, 13). Second, it turns out that any such function f gives a cycle whose homology class is dual to the first Pontryagin class multiplied by some constant (§ 8, Theorem 8.3).

We now give the precise definitions. Let $\mathcal{T}_n$ be the set of all isomorphism classes of the oriented $(n-1)$-dimensional PL spheres. (We assume that $\mathcal{T}_0 = \{\varnothing\}$ and $\mathcal{T}_{-n} = \varnothing$ for $n > 0$.) As a rule, we do not distinguish between a PL sphere and its isomorphism class. For any PL sphere $L \in \mathcal{T}_n$ we denote by $-L$ the PL sphere obtained from L by changing the orientation. A PL sphere $L \in \mathcal{T}_n$ is said to be *symmetric* if it is isomorphic to the PL sphere $-L$. Let G be an Abelian group. We denote by $\mathcal{T}^n(G)$ the Abelian group of all functions $f\colon \mathcal{T}_n \to G$

changing sign when the orientation of the PL sphere is changed (assuming that $\mathcal{T}^0(G) = G$ and $\mathcal{T}^{-n}(G) = 0$ for $n > 0$).

Let K be an m-dimensional combinatorial manifold. To each function $f \in \mathcal{T}^n(G)$ we assign a co-oriented chain $f_\sharp(K) \in \widehat{C}_{m-n}(K;G)$ by the formula

$$f_\sharp(K) = \sum_{\sigma^{m-n} \in K} f(\operatorname{link} \sigma^{m-n})\sigma^{m-n}$$

(the summand $f(\operatorname{link} \sigma^{m-n})\sigma^{m-n}$ does not depend on the choice of the co-orientation of the simplex σ^{m-n}.)

Definition 7.1. A function $f \in \mathcal{T}^n(G)$ is called a *local formula* if the co-oriented chain $f_\sharp(K)$ is a cycle for any combinatorial manifold K.

We define the differential $\delta\colon \mathcal{T}^n(G) \to \mathcal{T}^{n+1}(G)$ by the formula

$$(\delta f)(L) = \sum f(\operatorname{link} v),$$

where the summation ranges over all vertices v of the PL sphere L and the orientation of $\operatorname{link} v$ is induced by the orientation of L (as the orientation of the boundary of the star of the vertex v). It is easy to see that $\delta^2 = 0$. Thus, $\mathcal{T}^*(G)$ is equipped with the structure of a cochain complex.

The following two propositions can be proved immediately.

Proposition 7.1. *A function f is a local formula if and only if f is a cocycle in the cochain complex $\mathcal{T}^*(G)$. If f is a coboundary in the cochain complex $\mathcal{T}^*(G)$, then the co-oriented chain $f_\sharp(K)$ is a boundary for any combinatorial manifold K.*

Proposition 7.2. *Let $f \in \mathcal{T}^n(G)$ be a local formula and let K be a combinatorial manifold with boundary. Then $\widehat{\partial} f_\sharp(K) = i(f_\sharp(\partial K))$, where $i\colon \widehat{C}_*(\partial K;G) \to \widehat{C}_*(K;G)$ is the natural embedding and $f_\sharp(K)$ is the chain in which every $(\dim K - n)$-dimensional simplex $\sigma \in K$ appears with coefficient $f(\operatorname{link}\sigma)$ if σ is not contained in ∂K and coefficient $f(\operatorname{link}\sigma \cup_{\partial \operatorname{link}\sigma} C\partial \operatorname{link}\sigma)$ if σ is contained in ∂K.*

Corollary 7.1. *If K_1 and K_2 are two triangulations of a PL manifold M^m and $f \in \mathcal{T}^n(G)$ is a local formula, then the cycles $f_\sharp(K_1)$ and $f_\sharp(K_2)$ are homologous.*

Thus, each cohomology class $\psi \in H^n(\mathcal{T}^*(G))$ determines the homology class of the form $\psi_\sharp(M^m) \in H_{m-n}(M^m;\widehat{G})$ for any manifold M^m, and hence, by Poincaré duality, it determines a cohomology class $\psi^\sharp(M^m) \in H^n(M^m;G)$. If $m = n$ and the manifold M^n is oriented, then the class ψ determines an element $\psi^\star(M^n)$ of the group G by the formula $\psi^\star(M^n) = \langle \psi^\sharp(M^n), [M^n]\rangle$. The following corollary to Proposition 7.2 is needed in § 8.

Corollary 7.2. *Suppose that $\psi \in H^n(\mathcal{T}^*(G))$. Then $\psi^\star(M_1^n) = \psi^\star(M_2^n)$ for any cobordant oriented manifolds M_1^n and M_2^n.*

§ 8. Cohomology of the complex $\mathcal{T}^*(\mathbb{Q})$

By characteristic classes we mean elements of the cohomology group $H^*(\mathrm{BPL};G)$, where BPL is the classifying space of stable piecewise-linear bundles. If $p \in H^*(\mathrm{BPL};G)$, then we denote the corresponding characteristic class of a manifold M^m by $p(M^m)$. In the case of $G = \mathbb{Q}$ we have $H^*(\mathrm{BPL};\mathbb{Q}) = H^*(\mathrm{BO};\mathbb{Q})$, and the characteristic classes are exactly the polynomials in the Pontryagin classes.

Definition 8.1. A local formula $f \in \mathcal{T}^*(G)$ is called a *local formula for a characteristic class* $p \in H^*(\mathrm{BPL}; G)$ if for any combinatorial manifold K the cycle $f_\sharp(K)$ represents the homology class Poincaré dual to the cohomology class $p(|K|)$ for any combinatorial manifold K.

Theorem 8.1. *Each rational local formula is a local formula for some rational characteristic class.*

Proof. It follows from Corollary 7.2 that there is a well-defined homomorphism $\star\colon H^n(\mathcal{T}^*(G)) \to \mathrm{Hom}(\Omega_n, G)$ taking each cohomology class ψ to the homomorphism $\psi^\star$, where Ω_* stands for the oriented piecewise-linear cobordism ring. There is a canonical isomorphism $\mathrm{Hom}(\Omega_n, \mathbb{Q}) \cong H^n(\mathrm{BPL}; \mathbb{Q})$. Hence, the homomorphism $\star$ determines a homomorphism $\sharp\colon H^n(\mathcal{T}^*(\mathbb{Q})) \to H^n(\mathrm{BPL}; \mathbb{Q})$. Thus, corresponding to any local formula $f \in \mathcal{T}^n(\mathbb{Q})$ is the rational characteristic class $p = \sharp(\psi)$, where ψ is the cohomology class represented by the cocycle f. However, we have not proved yet that f is a local formula for the characteristic class p. Indeed, from the definition of the homomorphism $\sharp$ it follows only that $\psi^\sharp(M^n) = p(M^n)$ for any n-dimensional manifold M^n. Therefore, to prove that f is a local formula for p, it remains to prove the following proposition.

Proposition 8.1. $\psi^\sharp(M^m) = p(M^m) = \sharp(\psi)(M^m)$ *for any manifold* M^m, *where* $m \geqslant n$.

Sketch of the proof. It can be shown that $\psi^\sharp(M^m)\big|_{N^n} = \psi^\sharp(N^n)$ for any submanifold $N^n \subset M^m$ with trivial normal bundle. If $m > 2n + 1$ and M^m is orientable, then the proposition follows from Thom's result [30] claiming that for any homology class $z \in H_n(M^m; \mathbb{Z})$ there is a non-zero integer q such that the homology class qz can be realized by a submanifold with trivial normal bundle. If $n < m \leqslant 2n + 1$, then one must replace M^m by $M^m \times S^n$. If M^m is non-orientable, then one must pass to the two-sheeted orienting cover of M^m.

The first result on the existence of local formulae for characteristic classes was obtained by Levitt and Rourke [17].

Theorem 8.2 (Levitt–Rourke [17]). *Let* $p \in H^n(BPL; \mathbb{Q})$ *be a rational characteristic class and let* $m \geqslant n$ *be an integer. Then there is a function* $f \in \mathcal{T}^n(\mathbb{Q})$ *such that the homology class represented by the cycle* $f_\sharp(K)$ *is Poincaré dual to the cohomology class* $p(|K|)$ *for any oriented* m*-dimensional combinatorial manifold* K.

Remark 8.1. A similar result was obtained by King [31] in the case of a smooth manifold with a smooth triangulation (for $m = n$ and real coefficients).

In [12] the author proved the following theorem.

Theorem 8.3. *Let* p *be an arbitrary rational characteristic class. Then a local formula for* p *exists and is unique up to adding an arbitrary coboundary in the complex* $\mathcal{T}^*(\mathbb{Q})$. *Thus, the homomorphism*

$$\sharp\colon H^*(\mathcal{T}^*(\mathbb{Q})) \to H^*(\mathrm{BPL}; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots], \qquad \deg p_i = 4i,$$

is an isomorphism.

The part of Theorem 8.3 claiming the existence of a local formula for an arbitrary rational characteristic class is an improvement of Theorem 8.2 and follows immediately from Theorems 8.2 and 8.1.

Levitt and Rourke also obtained results on the possibility of local computation of characteristic classes with arbitrary coefficients for combinatorial manifolds with a given *local ordering of vertices*, that is, a partial ordering on the vertices whose restriction to the set of vertices of the star of each vertex in each simplex is a total ordering. Let $\mathcal{D}_m$ be the set of all oriented PL triangulations of an m-dimensional disc with a total ordering of the vertices up to an isomorphism preserving the ordering of the vertices.

Theorem 8.4 (Levitt–Rourke [17]). *Let $p \in H^n(BPL; G)$ be a characteristic class and let $m \geqslant n$ be an integer. Then there is a function $g\colon \mathcal{D}_m \to G$ such that the homology class represented by the cycle*

$$\sum_{\sigma^{m-n} \in K} g(\operatorname{star} \sigma^{m-n})\sigma^{m-n}$$

is Poincaré dual to the cohomology class $p(|K|)$ for any oriented m-dimensional combinatorial manifold K with local ordering of vertices.

Remark 8.2. In Theorem 8.4 we mean *piecewise linear* characteristic classes. If $G = \mathbb{Z}$, then the integral Pontryagin classes are not PL characteristic classes but some multiples of them are.

§ 9. Bistellar moves

Let K be a combinatorial manifold. Suppose that the simplicial complex K contains a simplex $\sigma_1 \in K$ such that $\operatorname{link} \sigma_1 = \partial\sigma_2$ is the boundary of a simplex and the simplex σ_2 does not belong to K. Then $\sigma_1 * \partial\sigma_2$ is a full subcomplex of K. By the *bistellar move* associated with the simplex σ_1 we mean the transformation taking K into the simplicial complex

$$\beta(K) = (K \setminus (\sigma_1 * \partial\sigma_2)) \cup (\partial\sigma_1 * \sigma_2).$$

If $\dim \sigma = 0$, then we assume that $\partial\sigma = \varnothing$, and for any simplex σ we assume that $\sigma * \varnothing = \sigma$. Thus, stellar subdivisions of simplices of maximal dimension and their inverse transformations are special cases of bistellar moves. Any bistellar move results in a combinatorial manifold PL homeomorphic to the original manifold. All the types of bistellar moves for $\dim K = 2$ and $\dim K = 3$ are shown in Figs. 1 and 2, respectively.

By Pachner's theorem (see [32] and also [33]), if K_1 and K_2 are two PL triangulations of the same manifold, then K_1 can be transformed into K_2 by a finite sequence of bistellar moves (here triangulations are treated as purely combinatorial objects, that is, we do not distinguish between isomorphic triangulations). In particular, any two m-dimensional PL spheres can be transformed one into the other by a finite sequence of bistellar moves.

§ 10. The graphs Γ_n

We introduce a graph Γ_n for any positive integer n. The set of vertices of Γ_n is the set $\mathcal{T}_{n+1}$ of oriented n-dimensional PL spheres. Let $L_1, L_2 \in \mathcal{T}_{n+1}$. We say that two bistellar moves β_1 and β_2 transforming L_1 into L_2 and associated with simplices σ_1

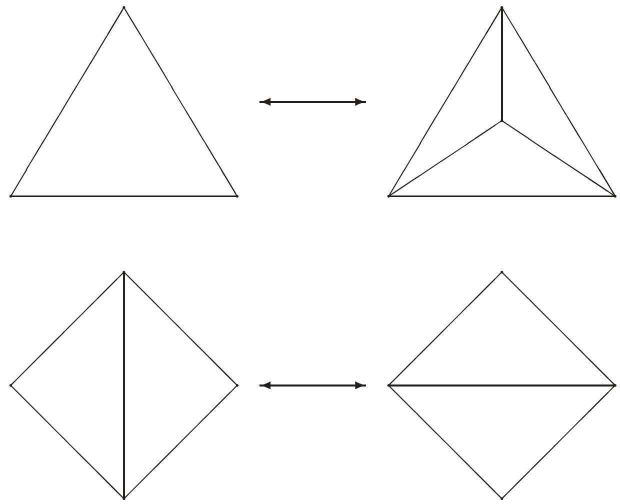

Figure 1. Bistellar moves for $\dim K = 2$

and σ_2, respectively, are *equivalent* if there is an automorphism of the PL sphere L_1 that takes the simplex σ_1 to σ_2. The edges joining two distinct vertices L_1 and L_2 of the graph Γ_n are in a one-to-one correspondence with the equivalence classes of bistellar moves taking L_1 to L_2. We now describe the edges of Γ_n with both ends coinciding with some vertex L. A bistellar move β taking the PL sphere L into itself is said to be *inessential* if it is equivalent to the inverse bistellar move β^{-1}. No edge of the graph Γ_n is assigned to equivalence classes of inessential bistellar moves. The other equivalence classes of bistellar moves taking L into itself can be partitioned into pairs of mutually inverse classes. The edges of Γ_n joining the vertex L to itself are in one-to-one correspondence with these pairs of equivalence classes. By Pachner's theorem, the graph Γ_n is connected. For any essential bistellar move β we denote the corresponding edge of Γ_n by e_β. Corresponding to the symbols e_β and $e_{\beta^{-1}}$ are the same edges but with opposite orientations.

Let $C_*(\Gamma_n;\mathbb{Z})$ be the cellular chain complex of the graph Γ_n. The group $\mathbb{Z}_2$ acts on Γ_n by changing the orientations of the PL spheres and on the group $\mathbb{Q}$ by changing the sign. Hence, one can define the equivariant cochains $C^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}_2}(C_*(\Gamma_n;\mathbb{Z}),\mathbb{Q})$ and the equivariant cohomology $H^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$ (we assume that the group $\mathbb{Z}_2$ acts trivially on the group $\mathbb{Z}$). We denote by d the differential of the complex $C^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$ and by $B^1_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) \subset C^1_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$ the subgroup of equivariant coboundaries. Since the graph Γ_n is connected, we have $H^0_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) = 0$. Therefore, the homomorphism

$$d\colon C^0_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) \to C^1_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$$

is a monomorphism.

Obviously, $C^0_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q}) = \mathcal{T}^n(\mathbb{Q})$. Therefore, the differential

$$\delta\colon C^0_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q}) \to C^0_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$$

is well defined.

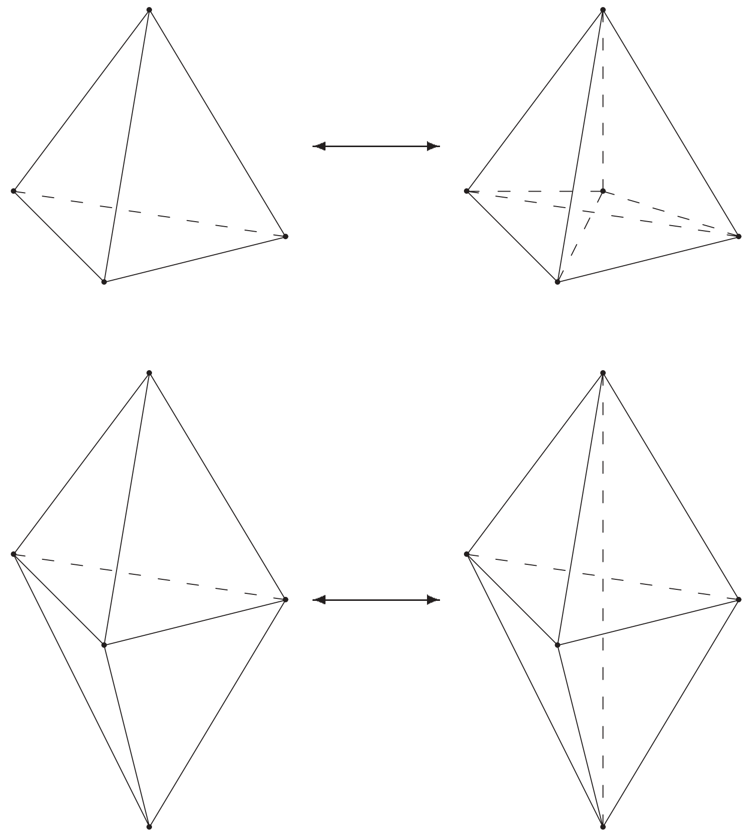

Figure 2. Bistellar moves for $\dim K = 3$

Let $L_1, L_2 \in \mathcal{T}_{n+1}$ and let β be a bistellar move taking L_1 to L_2. We can assume that L_1 and L_2 are simplicial complexes with the same set V of vertices (if $\dim \sigma_1 > 0$ and $\dim \sigma_2 > 0$, then this is really the case, because otherwise we can introduce a fictitious vertex v_0 for one of the complexes L_1 or L_2, where v_0 is not a simplex of the corresponding complex). For any vertex $v \in V$ the bistellar move β either preserves the complex given by $\operatorname{link} v$ or induces a bistellar move β_v transforming the complex $\operatorname{link}_{L_1} v$ to the complex $\operatorname{link}_{L_2} v$. Let $W \subset V$ be the subset of all vertices v such that the bistellar move β_v is not inessential. (We assume that $v_0 \notin W$.) Let us define the differential $\delta\colon C^1_{\mathbb{Z}_2}(\Gamma_{n-1}; \mathbb{Q}) \to C^1_{\mathbb{Z}_2}(\Gamma_n; \mathbb{Q})$ by the formula

$$(\delta h)(e_\beta) = \sum_{v \in W} h(e_{\beta_v}).$$

It is easy to show that $\delta^2 = 0$ and $\delta d = d\delta$.

§ 11. Local formulae for the first Pontryagin class

This section is devoted to the explicit description of all local formulae for the first Pontryagin class. By Theorem 8.1, each local formula $f \in \mathcal{T}^4(\mathbb{Q})$ is a local formula for the first Pontryagin class multiplied by some rational constant. Thus, one must first find all functions $f\colon \mathcal{T}_4 \to \mathbb{Q}$ that are local formulae. Any function $f \in \mathcal{T}^4(\mathbb{Q})$ can be regarded as a $\mathbb{Z}_2$-equivariant zero-dimensional cellular cochain on the graph Γ_3.

Proposition 11.1. *A function $f \in \mathcal{T}^4(\mathbb{Q}) = C^0_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})$ is a local formula if and only if there is a cochain $h \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ such that $df = \delta h$.*

Proof. Suppose that there is a cochain $h \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ such that $df = \delta h$. Then $d\delta f = \delta d f = \delta^2 h = 0$. Since $d\colon C^0_{\mathbb{Z}_2}(\Gamma_4;\mathbb{Q}) \to C^1_{\mathbb{Z}_2}(\Gamma_4;\mathbb{Q})$ is a monomorphism, it follows that $\delta f = 0$. Thus, f is a local formula.

We now assume that f is a local formula.

Let $L_1, L_2 \in \mathcal{T}_n$ and let β be a bistellar move transforming the PL sphere L_1 to the PL sphere L_2. We define the sets V and W as in § 10. Let the bistellar move β replace the full subcomplex $\sigma_1 * \partial\sigma_2$ of the simplicial complex L_1 by the full subcomplex $\partial\sigma_1 * \sigma_2$ of the simplicial complex L_2. We consider the cone CL_1 with vertex u_1 and the cone CL_2 with vertex u_2. Then $L_\beta = CL_1 \cup CL_2 \cup (\sigma_1 * \sigma_2)$ is a simplicial complex on the set $V \cup \{u_1, u_2\}$ of vertices. Obviously, L_β is an n-dimensional PL sphere. We choose an orientation of L_β such that the induced orientation of the complex determined by $\operatorname{link} u_2 = L_2$ coincides with the given orientation of L_2. Then $L_\beta \in \mathcal{T}_{n+1}$.

Let us consider a cochain $h \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ such that $h(e_\beta) = f(L_\beta)$ for any edge e_β of Γ_2. We prove that $\delta h = df$. Let e_β be an arbitrary edge of Γ_3, where β is a bistellar move taking a three-dimensional PL sphere L_1 to a PL sphere L_2. Obviously, the links of all vertices $v \in V \setminus W$ in the complex L_β are symmetric. The link of any vertex $v \in W$ in the complex L_β is isomorphic to the complex $-L_{\beta_v}$. The links of the vertices u_1 and u_2 are isomorphic to the complexes $-L_1$ and L_2, respectively. Since f is a local formula, it follows that

$$
\begin{aligned}
0 = (\delta f)(L_\beta) &= -\sum_{v\in W} f(L_{\beta_v}) + f(L_2) - f(L_1) \\
&= -\sum_{v\in W} h(e_{\beta_v}) + f(\partial e_\beta) = -(\delta h)(e_\beta) + (df)(e_\beta).
\end{aligned}
$$

This proves Proposition 11.1.

We now describe the subgroup $A \subset C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ formed by the cochains h such that $\delta h \in B^1_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})$, that is, $\delta h = df$ for some cochain $f \in C^0_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})$. Since the differentials d and δ commute, it follows that $\delta\colon C^*_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q}) \to C^*_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})$ is a chain map. Therefore, the induced homomorphism $\delta^*\colon H^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q}) \to H^1_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})$ is well defined. We denote the kernel of δ^* by $\widetilde{A}$. Suppose that $h \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$; it is clear that $h \in A$ if and only if $[h] \in \widetilde{A}$.

Obviously, $H^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q}) \cong \operatorname{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2;\mathbb{Z}),\mathbb{Q})$. Therefore, to describe the subgroup $\widetilde{A}$, we must choose a generating system in the group $H_1(\Gamma_2;\mathbb{Z})$. Let us consider the cycles in Γ_2 shown in Figs. 3–8. These figures must be understood as follows: one considers an arbitrary two-dimensional PL sphere containing a subcomplex shown in a figure. To this PL sphere we then apply bistellar moves shown in the figure. It is assumed that the orientation of the PL sphere is given by clockwise circuits of the vertices of the triangles shown in the picture. If an angle is marked by an arc, then it is assumed that the indicated number gives the number of triangles inside the angle that abut on the vertex. Thus, we obtain six infinite series of cycles in Γ_2. We note that the cycles shown in Figs. 3–5 correspond to the commutation of a pair of bistellar moves.

Proposition 11.2. *The homology classes represented by the cycles in Figs.* 3–8 *generate the group* $H_1(\Gamma_2;\mathbb{Z})$.

In this paper we omit the proof of Proposition 11.2. It is based on the following two assertions, proved in [34] (see also [29]).

1) Any triangulation of the two-dimensional sphere can be realized as the boundary of a convex simplicial polytope.

2) Two combinatorially equivalent three-dimensional convex simplicial polytopes can be deformed into each other in the class of convex simplicial polytopes of the same combinatorial type.

We denote the set of all cycles shown in Figs. 3–8 by S. Let us define a function $c\colon S\to\mathbb{Q}$ by assigning a number shown in the corresponding figure to each cycle, where

$$\rho(p,q)=\frac{q-p}{(p+q+2)(p+q+3)(p+q+4)}\,,$$

$$\eta(p)=\frac{1}{(p+2)(p+3)}\,.$$

We thus obtain a function $c\colon S\to\mathbb{Q}$.

Remark 11.1. Similar numerical expressions arose in [35] in the solution of quite another problem, namely, in finding a formula for the Chern–Euler class of an S^1-bundle in terms of singularities of the restrictions of a Morse function on the total space to the fibres of the bundle.

Proposition 11.3. *The function c can be extended to a well-defined cohomology class $c\in H^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})=\operatorname{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2;\mathbb{Z}),\mathbb{Q})$. Then $\widetilde{A}$ is the one-dimensional vector space generated by the cohomology class c.*

A sketch of the proof of this proposition is given in § 13.

It follows from Proposition 11.1 that the homomorphism $d^{-1}\delta\colon A\to C^0_{\mathbb{Z}_2}(\Gamma_3;\mathbb{Q})=\mathcal{T}^4(\mathbb{Q})$ is well defined and the image of $d^{-1}\delta$ is the subgroup formed by all local formulae.

Theorem 11.1. *The map $d^{-1}\delta$ provides a bijection between the set of all $\mathbb{Z}_2$-equivariant cocycles representing the cohomology class c and the set of all local formulae for the first Pontryagin class.*

Proof. Let us consider the epimorphism $j\colon\widetilde{A}\to H^4(\mathcal{T}^*(\mathbb{Q}))$ induced by the homomorphism $d^{-1}\delta$. We have $\dim\widetilde{A}=1$ by Proposition 11.3 and $\dim H^4(\mathcal{T}^*(\mathbb{Q}))=1$ by Theorem 8.3. Thus, the epimorphism j is an isomorphism, and there is a rational number $\lambda\neq 0$ such that $j(\lambda c)=\phi$, where $\phi\in H^4(\mathcal{T}^*(\mathbb{Q}))$ is the cohomology class represented by local formulae for the first Pontryagin class.

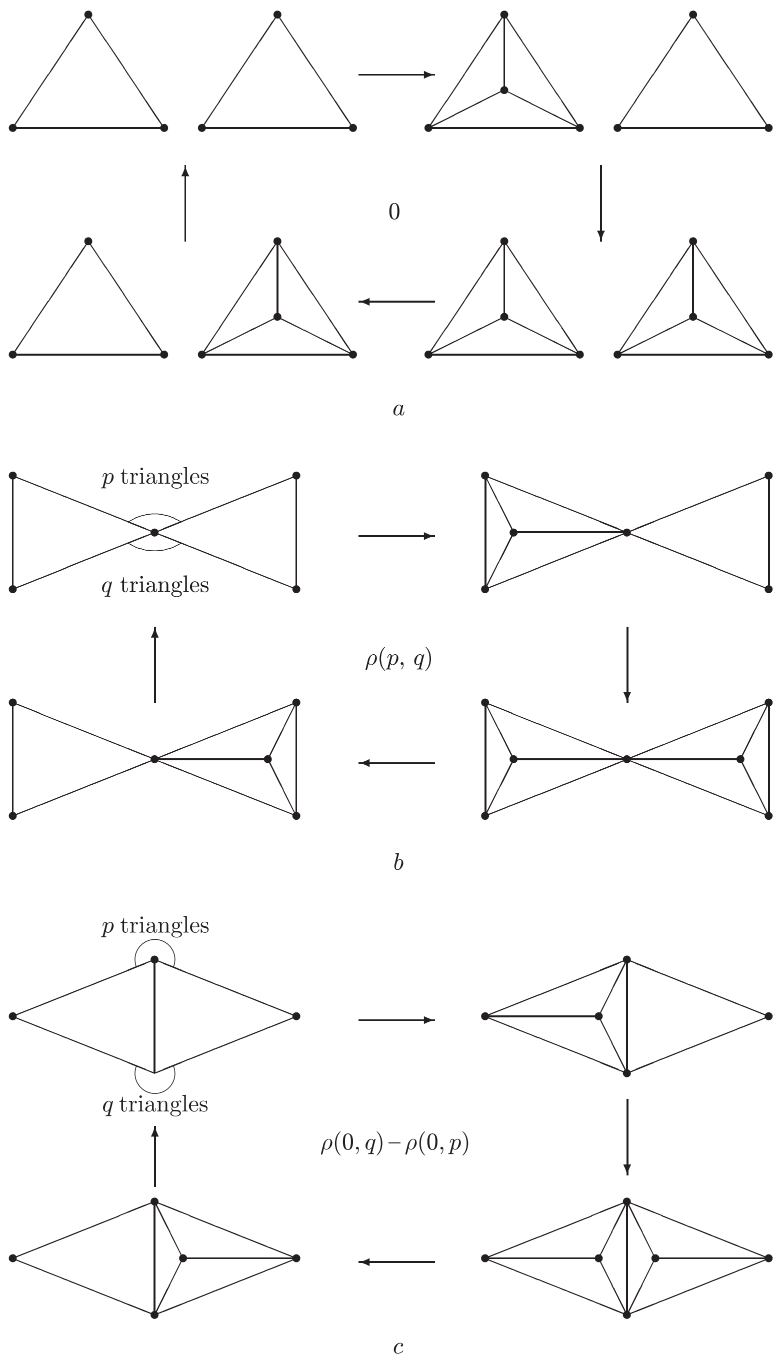
0
a
p triangles
q triangles
ρ(p, q)
b
p triangles
q triangles
ρ(0, q) − ρ(0, p)
c

Figure 3

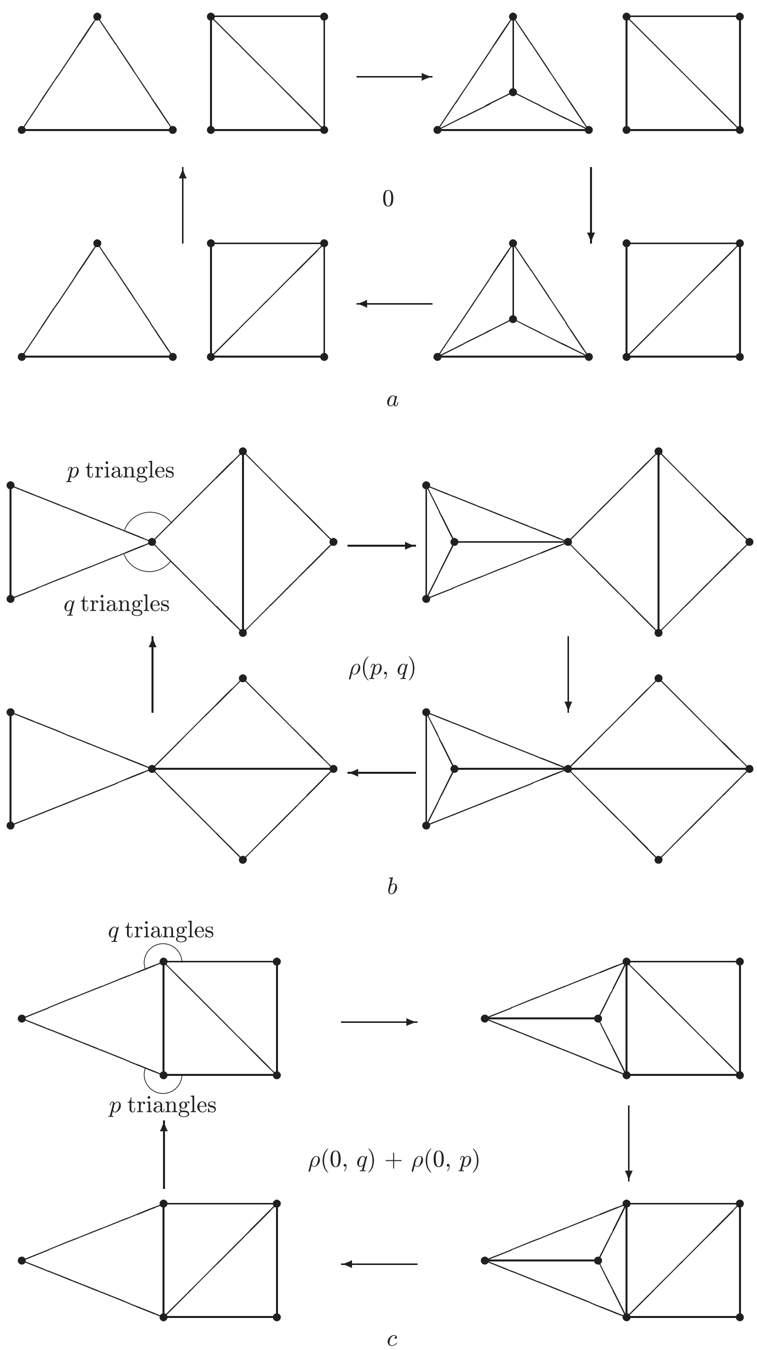
0
a
p triangles
q triangles
ρ(p, q)
b
q triangles
p triangles
ρ(0, q) + ρ(0, p)
c

Figure 4

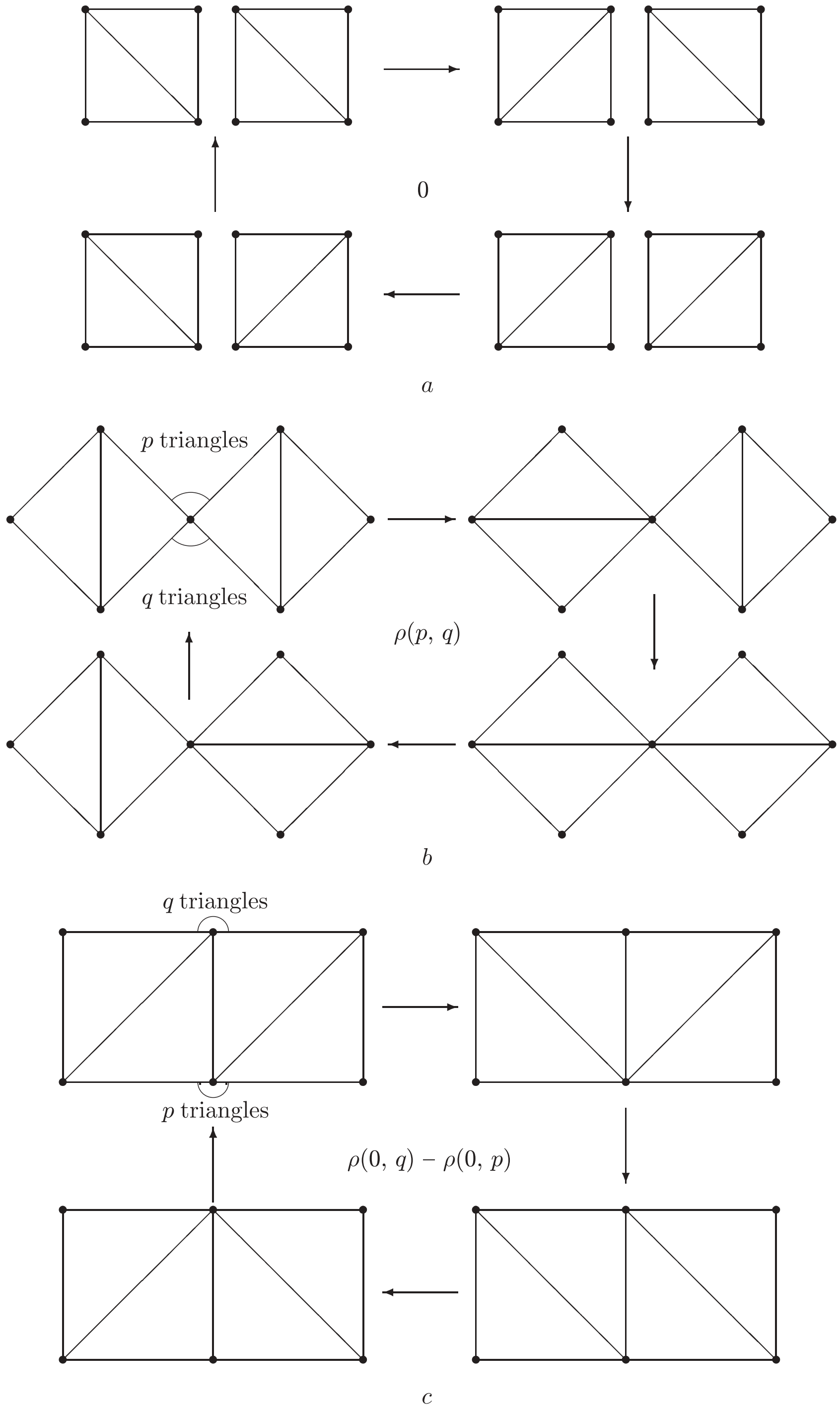
0
a
p triangles
q triangles
ρ(p, q)
b
q triangles
p triangles
ρ(0, q) − ρ(0, p)
c

Figure 5

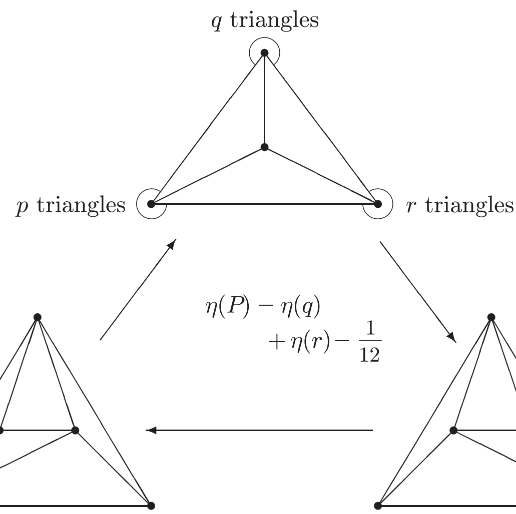

Figure 6

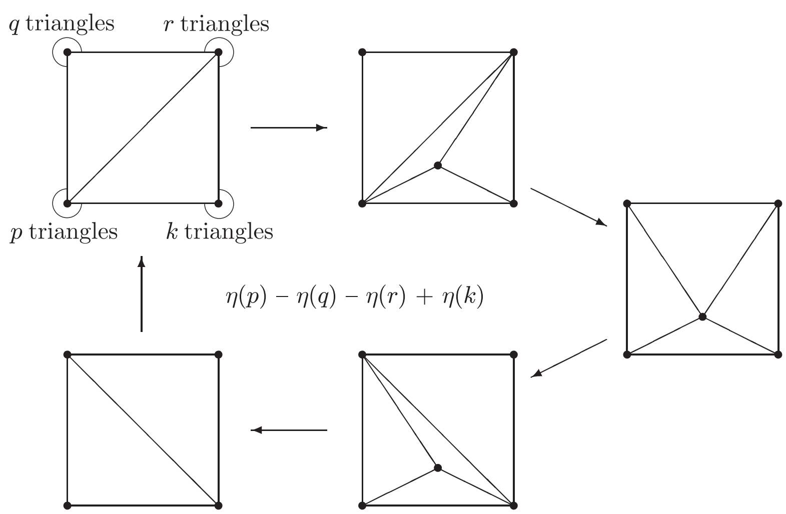

Figure 7

We prove that $d^{-1}\delta$ is a monomorphism. Since j is a monomorphism, we have $\ker(d^{-1}\delta) \subset B^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$. Moreover, $(d^{-1}\delta)\,\big|_{B^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})} = \delta d^{-1}$. Obviously, $\mathcal{T}^2(\mathbb{Q}) = 0$, and $H^3(\mathcal{T}^*(\mathbb{Q})) = 0$ by Theorem 8.3. Hence, $\delta\colon \mathcal{T}^3(\mathbb{Q}) \to \mathcal{T}^4(\mathbb{Q})$ is a monomorphism. Thus, $\delta d^{-1}\colon B^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q}) \to \mathcal{T}^4(\mathbb{Q})$ is a monomorphism.

Hence, $d^{-1}\delta$ is a monomorphism. Therefore, the homomorphism $d^{-1}\delta$ is a bijection between the set of all $\mathbb{Z}_2$-equivariant cocycles representing the cohomology class λc and the set of all local formulae for the first Pontryagin class. It remains to prove that $\lambda = 1$. To this end, it suffices to consider an oriented 4-dimensional combinatorial manifold K with a known first Pontryagin number and compute the

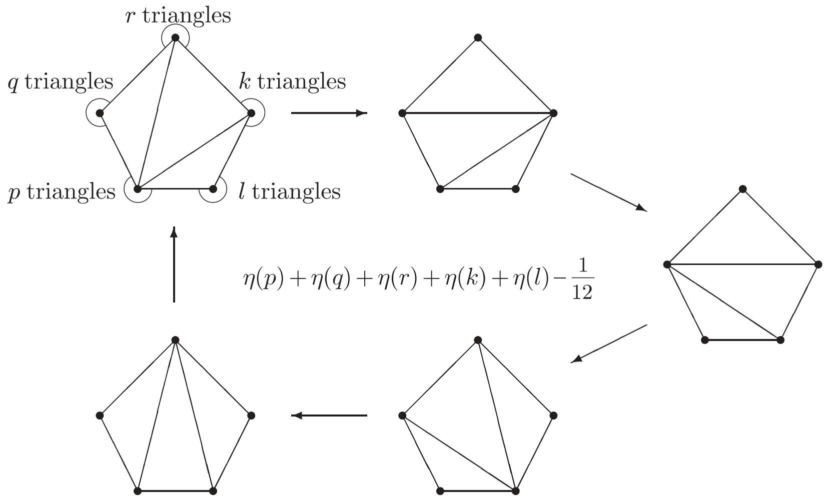

Figure 8

value $\psi^\star(|K|)$, where $\psi = j(c)$. As K we take a triangulation of $\mathbb{C}P^2$ with 9 vertices (constructed in [36]; see also [37]). One can see by a direct calculation that $\lambda = 1$.

Remark 11.2. Actually, in the proof of this theorem we need only the part of Theorem 8.3 claiming that the map $\sharp\colon H^*(\mathcal{T}^*(\mathbb{Q})) \to H^*(\mathrm{BPL};\mathbb{Q})$ is an epimorphism (the equality $H^3(\mathcal{T}^*(\mathbb{Q})) = 0$ can readily be verified directly.)

§ 12. Canonical choice of a formula

Theorem 11.1 describes all rational local formulae for the first Pontryagin class. It is now useful to describe at least one canonical local formula f_0 for the first Pontryagin class, that is, find a canonical cocycle $\widehat{c}_0 \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ representing the cohomology class c. The problem of finding a canonical 1-dimensional cocycle representing a given cohomology class arose in [9] in a quite different situation. Here we use an approach similar to that used in [9].

By $\mathcal{T}_3^{(l)}$ we denote the set of all isomorphism classes of oriented 2-dimensional PL spheres that can be obtained from the boundary of a tetrahedron by using at most l bistellar moves. We denote by $\Gamma_2^{(l)}$ the full subgraph of Γ_2 spanned by the set $\mathcal{T}_3^{(l)}$ of vertices. Then $\Gamma_2^{(l)}$ is a finite connected graph admitting an explicit combinatorial construction. We shall use induction to define cocycles $\widehat{c}_0^{(l)} \in C^1_{\mathbb{Z}_2}(\Gamma_2^{(l)};\mathbb{Q})$ representing the cohomology classes $c|_{\Gamma_2^{(l)}}$ such that the restriction of $\widehat{c}_0^{(l)}$ to $\Gamma_2^{(l-1)}$ coincides with $\widehat{c}_0^{(l-1)}$. Suppose that the cocycle $\widehat{c}_0^{(l-1)}$ has been constructed. Among all cocycles $b \in C^1_{\mathbb{Z}_2}(\Gamma_2^{(l)};\mathbb{Q})$ such that $[b] = c|_{\Gamma_2^{(l)}}$ and $b|_{\Gamma_2^{(l-1)}} = \widehat{c}_0^{(l-1)}$ we choose a cocycle $\widehat{c}_0^{(l)}$ such that the sum of its squared values on the edges of $\Gamma_2^{(l)}$ is minimal. The problem of choosing such a cocycle is a minimization problem for a quadratic functional on a plane in a finite-dimensional vector space. Therefore, the desired cocycle exists, it is unique, it is rational, and its computation reduces to the solution of a system of linear equations with rational coefficients.

We denote by $\widehat{c}_0$ the cocycle on Γ_2 whose restrictions to the graphs $\Gamma_2^{(l)}$ coincide with the cocycles $\widehat{c}_0^{(l)}$, respectively. Then $f_0 = d^{-1}\delta\widehat{c}_0$ is a canonical local formula for the first Pontryagin class.

We now describe how to make specific calculations by using this formula, that is, how to find the value $f_0(L)$ for a given PL sphere $L \in \mathcal{T}_4$. To this end, the following steps must be carried out.

1) Choose a sequence of bistellar moves $\beta_1, \beta_2, \dots, \beta_l$ taking the boundary of a 4-dimensional simplex to the PL sphere L. Denote by L_j the PL sphere obtained from $\partial\Delta^4$ by the bistellar moves $\beta_1, \beta_2, \dots, \beta_{j-1}$ and denote by W_j the set of all vertices $v \in L_j$ such that the bistellar move β_j induces on the link of v a bistellar move β_{jv} that is not inessential. We note that all edges $e_{\beta_{jv}}$ belong to $\Gamma_2^{(l)}$.

2) Compute the graphs $\Gamma_2^{(j)}$ and the cocycles $\widehat{c}_0^{(j)}$, $j = 1, 2, \dots, l$, in succession.

3) Compute the value $f_0(L)$ by the formula

$$f_0(L) = \sum_{j=1}^{l} \sum_{v \in W_j} \widehat{c}_0^{(l)}(e_{\beta_{jv}}).$$

§ 13. Sketch of the proof of Proposition 11.3

Let c' be an arbitrary element of the group $\tilde{A}$. We prove first that there is a $\lambda \in \mathbb{Q}$ such that $c'([\alpha]) = \lambda c(\alpha)$ for any $\alpha \in S$.

Let α be the cycle shown in Fig. 3a, let L be the original two-dimensional PL sphere (that is, the PL sphere shown at the upper left corner of Fig. 3a), and let σ_1 and σ_2 be the triangles of the PL sphere that are shown in the figure. We consider a three-dimensional PL sphere $K \in \mathcal{T}_4$ containing a vertex u whose link is isomorphic to the PL sphere L and whose star is a full subcomplex of K. By $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ we denote the tetrahedra of K spanned by the vertex u and the triangles σ_1 and σ_2, respectively. The cycle α was obtained upon commuting the bistellar moves associated with the triangles σ_1 and σ_2. Let us now commute the bistellar moves associated with the tetrahedra $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$. We obtain a cycle $\gamma \in C_1(\Gamma_3; \mathbb{Z})$. The cycle γ induces for each vertex v of K a cycle γ_v in Γ_2 consisting of edges corresponding to the induced bistellar moves of the link of v. In this case

$$\sum_v c'([\gamma_v]) = \delta^*(c')([\gamma]) = 0,$$

because $c' \in \tilde{A} = \ker\delta^*$. The cycle γ_v is homologous to zero for any vertex $v \in K$ except for u, and the cycle γ_u coincides with α. Therefore, $c'([\alpha]) = 0$.

Now let α_1 and α_2 be two cycles as shown in Fig. 3b with the same pairs of numbers (p,q). As in the previous case, there is a cycle $\gamma \in C_1(\Gamma_3; \mathbb{Z})$ starting from some three-dimensional PL sphere K such that $\gamma_u = \alpha_1$ and $\gamma_v = -\alpha_2$ for some vertices u and v of the complex K and $\gamma_w = 0$ for the other vertices w of K. Hence, $c'([\alpha_1]) = c'([\alpha_2])$. Thus, the value of c' on a cycle shown in Fig. 3b depends only on the pair (p,q). We denote this value by $\rho'(p,q)$. In the same way we can prove that the function ρ' can be extended to a function $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ such that $\rho'(p,q) = -\rho'(q,p)$ and $c'([\alpha]) = \rho'(0,q) - \rho'(0,p)$ for any cycle α shown in Fig. 3c, but the extension is not unique. To fix one of these extensions, we assume that $\rho'(0,1) = \frac{7}{2}\rho'(1,2)$ (this choice is motivated by the relation $\rho(0,1) = \frac{7}{2}\rho(1,2)$.)

Let us consider an oriented triangulation L of the two-dimensional sphere and let L contain a vertex x on which $p+q+r+3$ triangles abut. Among the triangles abutting on the vertex x we choose three triangles σ_0, σ_1, and σ_2 such that upon going around the vertex x clockwise, we pass in succession through the triangle σ_0, through r other triangles, through σ_1, through p other triangles, through σ_2, and through the q remaining triangles. We denote by L_j the complex obtained from L by applying the bistellar move associated with the triangle σ_j, by α_j the cycle obtained upon commuting the bistellar moves associated with the triangles σ_{j+1} and σ_{j+2} of L, and by α'_j the cycle obtained upon commuting the bistellar moves associated with the triangles σ_{j+1} and σ_{j+2} of the complex L_j, where the sums of indices are taken modulo 3. One can readily see that

$$\alpha'_0+\alpha'_1+\alpha'_2=\alpha_0+\alpha_1+\alpha_2.$$

Hence,

$$\begin{aligned}\rho'(p,q+r+2)+\rho'(q,r+p+2)+\rho'(r,p+q+2)\\ =\rho'(p,q+r+1)+\rho'(q,r+p+1)+\rho'(r,p+q+1).\end{aligned}$$

One can readily derive from this equality that there is a constant $\lambda\in\mathbb{Q}$ such that $\rho'(p,q)=\lambda\rho(p,q)$ for every p and q. Hence, $c'([\alpha])=\lambda c(\alpha)$ for any cycle α shown in Fig. 3. It can be proved in a similar way that $c'([\alpha])=\lambda c(\alpha)$ for every $\alpha\in S$.

The assertion proved above immediately implies that $\dim\widetilde{A}\leqslant 1$. Since an epimorphism $\sharp\colon H^*(\mathcal{T}^*(\mathbb{Q}))\to H^*(\mathrm{BPL};\mathbb{Q})$ exists, it follows that $\dim H^4(\mathcal{T}^*(\mathbb{Q}))\geqslant 1$. On the other hand, the homomorphism $d^{-1}\delta\colon A\to\mathcal{T}^4(\mathbb{Q})$ induces an epimorphism $j\colon\widetilde{A}\to H^4(\mathcal{T}^*(\mathbb{Q}))$. Thus, $\dim\tilde{A}=\dim H^4(\mathcal{T}^*(\mathbb{Q}))=1$. Hence, the cohomology class c is well defined and generates the one-dimensional vector space $\widetilde{A}$.

§ 14. Denominators of the values of local formulae

Let $f\colon\mathcal{T}_n\to\mathbb{Q}$ be a local formula. We shall now clarify the growth of the denominators of the values $f(L)$ as the number of vertices of L increases. We denote by $\mathcal{T}_{n,l}$ the set of all oriented $(n-1)$-dimensional PL spheres having at most l vertices, and by $\mathrm{den}_l(f)$ the least common multiple of the denominators of all the values $f(L)$, $L\in\mathcal{T}_{n,l}$. The following two theorems give bounds for the growth of $\mathrm{den}_l(f)$ as a function of l.

Theorem 14.1. *Let $\psi\in H^n(\mathcal{T}^*(\mathbb{Q}))$ be an arbitrary cohomology class. Then there is a local formula f representing the class ψ and an integer constant $b\neq 0$ such that the number $\mathrm{den}_l(f)$ divides the product $b(l+1)!$ for any l.*

Theorem 14.2. *Let $f\in\mathcal{T}^4(\mathbb{Q})$ be an arbitrary local formula for the first Pontryagin class. Then the number $\mathrm{den}_l(f)$ is divisible by the least common multiple of the numbers $1,2,\dots,l-3$ for any even integer $l\geqslant 10$.*

Theorem 14.1 can be derived from results in [17] and Theorem 8.1. Theorem 14.2 can readily be proved by using the explicit description in § 11 of all local formulae for the first Pontryagin class. Theorem 14.2 readily implies the following assertion.

Corollary 14.1. $H^4(\mathcal{T}^*(G))=0$ *for any proper subgroup* $G\subset\mathbb{Q}$.

§ 15. Existence of algorithms to compute local formulae

Theorem 15.1. *Let $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ be an arbitrary cohomology class. Then there is a local formula f representing the cohomology class ψ and such that the value $f(L)$ is algorithmically computable from a given triangulation $L \in T_n$.*

Remark 15.1. Novikov (see [38], pp. 166–167) proved that the problem of determining whether or not a given simplicial complex L is a triangulation of an $(n-1)$-dimensional PL sphere is algorithmically undecidable for $n \geqslant 6$. The exact formulation of Theorem 15.1 is as follows: there is an algorithm whose input is an oriented simplicial complex L and whose output is the value $f(L)$ if $L \in T_n$, whereas no output is produced in finite time if $L \notin T_n$.

Remark 15.2. Obviously, if $n \geqslant 4$, then there is an uncomputable coboundary $f \in \mathcal{T}^n(\mathbb{Q})$. Therefore, there are uncomputable rational local formulae.

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