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Classification of simplicial triangulations of topological manifolds

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0. Introduction

This paper undertakes the study of the existence and uniqueness of simplicial triangulations of a given topological manifold. Unlike a combinatorial triangulation of a given topological manifold M (a simplicial triangution of M in which the star of each vertex is piecewise linearly homeomorphic to a simplex), a simplicial triangulation of M is not necessarily piecewise-linearly homogeneous. Thus the usual "triangulating" techniques, often adapted from "smoothing" techniques, do not directly apply.

In 1969, R. Kirby and L. Siebenmann [19] showed that in each dimension greater than four there exist closed topological manifolds which admit no piecewise linear manifold structures and hence cannot be triangulated as combinatorial manifolds. Then in 1974, R. D. Edwards [10] showed that the double suspension of the Mazur homology 3-sphere is homeomorphic to S^5, thus exhibiting that a simplicial triangulation of a topological manifold need not be combinatorial. It is still unknown whether every topological manifold can be triangulated as a simplicial complex.

In this paper we relate the question of whether a given topological

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n-manifold M^n, with $n \geqq 7$ ($n \geqq 6$ if ∂M compact or $n \geqq 5$ if M closed), can be triangulated as a simplicial complex to a lifting problem, classify such triangulations up to a natural equivalence relationship in terms of homotopy classes of lifts, and then reduce the lifting problem to the existence of a certain PL homology 3-sphere.

To be more specific, let BTOP denote the stable classifying space for topological block bundles [31].

THEOREM 1. *There is a space* BTRI *and a natural map* BTRI $\to$ BTOP *such that if M is a topological n-manifold, $n \geqq 7$ ($n \geqq 6$ if ∂M compact or $n \geqq 5$ if M closed), and $\tau: M \to$* BTOP *classifies the stable topological tangent bundle of M, then there is a one-to-one correspondence between the set of concordance classes of simplicial triangulations of M and the set of vertical homotopy classes of lifts of τ to* BTRI.

Here, two simplicial triangulations are *concordant* if there exists a simplicial triangulation K of $M \times [0, 1]$ which restricts to triangulations on $M \times \{i\}$, $i = 0, 1$, compatible with the given ones.

THEOREM 2. *The fiber* TOP/TRI *of* BTRI $\to$ BTOP *is a* $K(\ker(\alpha: \theta_3^H \to Z_2), 4)$.

Here, θ_3^H denotes the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds; and $\alpha: \theta_3^H \to Z_2$ is the Kervaire-Milnor-Rochlin epimorphism $\alpha(H^3) = \sigma(H^3)/8(\text{mod}\, 2)$, where $\sigma(H^3)$ is the index of any parallelizable PL 4-manifold that H^3 bounds.

COROLLARY 3. *Let M be a topological n-manifold with N a codimension zero submanifold of ∂M such that a neighborhood of N in M is simplicially triangulated. If $n \geqq 7$ ($n \geqq 6$ if* cl$(\partial M - N)$ *is compact or $n \geqq 5$ if M is closed), then there exists a well-defined obstruction $\nabla \in H^5(M, N; \ker(\alpha: \theta_3^H \to Z_2))$ such that $\nabla = 0$ if and only if there exists a simplicial triangulation of M compatible with the given one near N. Furthermore the concordance classes of such triangulations on M are in one-to-one correspondence with the elements of $H^4(M, N; \ker(\alpha: \theta_3^H \to Z_2))$.*

THEOREM 4. *Every topological n-manifold, $n \geqq 7$ ($n \geqq 6$ if ∂M compact or $n \geqq 5$ if $\partial M = \varnothing$), can be triangulated as a simplicial complex if and only if there exists a* PL *homology 3-sphere H^3 such that*

(i) $\alpha(H^3) = 1$, *and*

(ii) $H^3 \# H^3$ *bounds an acyclic* PL *4-manifold.*

Let M be a given topological n-manifold, $n \geqq 7$ ($n \geqq 6$ if ∂M compact or $n \geqq 5$ if M closed) and let $\Delta \in H^4(M; Z_2)$ be the Kirby-Siebenmann obstruction to triangulating M as a combinatorial manifold [19]. Also, let $\beta: H^4(M; Z_2) \to H^5(M; \ker(\alpha))$ be the Bockstein associated with the short exact coefficient sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \theta_3^H \xrightarrow{\alpha} Z_2 \longrightarrow 0 .$$

THEOREM 5. *The topological manifold M can be triangulated as a simplicial complex if and only if* $\beta(\Delta) = 0$.

In this paper we also investigate the question of whether a given topological manifold can be triangulated as a simplicial homotopy manifold. Some sample theorems are:

THEOREM 6. *Suppose that every* PL *homotopy* 3*-sphere bounds a contractible* PL 4*-manifold. Then there is a natural one-to-one correspondence between the set of concordance classes of simplicial homotopy triangulations of a topological n-manifold M,* $n \geqq 7$ ($n \geqq 6$ *if* ∂M *compact or* $n \geqq 5$ *if M closed), and concordance classes of* PL *manifold structures on M.*

THEOREM 7. *Every topological n-manifold M,* $n \geqq 7$ ($n \geqq 6$ *if* ∂M *compact or* $n \geqq 5$ *if* $\partial M = \emptyset$), *can be triangulated as a simplicial homotopy manifold if and only if there exists a* PL *homotopy* 3*-sphere* H^3 *such that*

(i) $\alpha(H^3) = 1$, *and*

(ii) $H^3 \# H^3$ *bounds a contractible* PL 4*-manifold.*

Theorem 4 was first conjectured to be true by L. Siebenmann after he had shown in [34] that every oriented topological 5-manifold could be simplicially triangulated if and only if there exists a PL homology 3-sphere of Rochlin invariant one whose double suspension is homeomorphic to S^5. For closed topological n-manifolds M, $6 \leqq n \leqq 8$, with the integral Bockstein of the Kirby-Siebenmann obstruction to putting a PL manifold structure on M being zero, M. Scharlemann [33] proved that M could be simplicially triangulated provided there exists a PL homology 3-sphere of Rochlin invariant one whose double suspension is homeomorphic to S^5.

Further evidence for the existence of a classification theorem for simplicial triangulations of topological manifolds was given in [15] where we showed, without assuming any suspension theorems, that every polyhedral homology n-manifold P, $n \geqq 6$ ($n \geqq 5$ if $\partial P = \emptyset$ or if ∂P is a topological manifold), is canonically simple homotopy equivalent to a topological n-manifold. Then in 1974 R.D. Edwards [12] showed that the double suspension of every PL homology n-sphere, $n \geqq 4$, is homeomorphic to S^{n+2}. This allowed us in 1975 to prove Theorems 1, 6, and 7 and weaker versions of Theorems 2, 3, and 4 (cf. [16], [17]). However, in 1976, R.D. Edwards showed that the triple suspension of every PL homology 3-sphere is homeomorphic to S^6, and in 1977 J. Cannon [5] showed that the double suspension of *every* homology n-sphere is homeomorphic to S^{n+2}. This allowed us to simplify our proof of Theorem 1 and eliminate one of our obstructions to triangulating topological manifolds. In this paper we give the simplified proof. Also, T. Matumoto [24] has independently given a different proof of Theorem 4.

This paper is divided into 8 sections. In Section 1 we observe that every simplicially triangulated topological manifold is a polyhedral homology manifold and show that a closed polyhedral homology n-manifold, $n \geqq 5$, is locally Euclidean if and only if its $(n-1)$-dimensional links are 1-connected. We then show that every homology n-manifold, $n \geqq 5$, can be resolved via a PL contractible map to a triangulated topological n-manifold. We conclude Section 1 with the definition of a TRI manifold. In Section 2 we develop a theory of topological triangulated cone bundles and observing that this theory is stably equivalent to N. Martin and C. Maunder's theory of homology cobordism bundles [22], we produce the classifying space BTRI and a natural map of BTRI $\rightarrow$ BTOP.

Section 3 is devoted to embedding theorems of spheres and disks into triangulated topological manifolds so that we can use surgery techniques in Section 4 to prove our product structure theorem, which implies that if the product of a closed topological n-manifold M ($n \geqq 5$) with Euclidean space has a simplicial triangulation, then so does M. In Section 5 we use the product structure theorem to prove the classification theorem and Section 6 is devoted to the calculation of the homotopy groups of the fiber of BTRI $\rightarrow$ BTOP. Then in Section 7 we give necessary and sufficient conditions for topological manifolds to be triangulated as simplicial complexes in terms of the existence of certain PL homology 3-spheres.

Finally, in Section 8, we observe that the work of Sections 1–7 carries over to the question of when a topological manifold can be triangulated as a simplicial homotopy manifold.

We wish to thank R.D. Edwards for a trick which allowed us to prove Theorem 1.6 when $k = 4$. Also, we thank John Hollingsworth for constant inspiration, encouragement, and useful conversations throughout the duration of this work.

1. Recognizing topological manifolds among homology manifolds

Recall that a locally finite polyhedron M is a *homology* n-manifold if there exists a triangulation K of M such that for any $x \in M$ and any subdivision K' of K with x as a vertex, $H_*(lk(x, K'))$ is isomorphic to either $H_*(S^{n-1})$ or to $H_*(\text{point})$. Here, as in the rest of the paper, all homology, unless otherwise stated, is integral homology, and $lk(\sigma, K')$, for a simplex σ of K', denotes the link of σ in K'. The boundary of M, denoted ∂M, is the set of points x such that $H_*(lk(x)) = H_*(\text{point})$ and is a homology manifold without boundary. We refer the reader to [22] for basic properties about homology manifolds.

If X is a compact space, let $c(X)$ denote the cone on X, let $c'(X)$ denote the open cone on X, and let $\Sigma^r X$ denote the r-fold suspension of X.

LEMMA 1.1. *If X is a compact space, then*

a) *$c'(X) \times R^r$ is an open topological $(n + r)$-manifold if and only if $\Sigma^{r+1}X$ is homeomorphic to S^{n+r}.*

b) *$c'(X) \times R^r$ is a topological $(n + r)$-manifold with boundary if and only if $\Sigma^{r+1}X$ is homeomorphic to $[-1, 1]^{n+r}$.*

Proof. We only prove 1.1(a), as 1.1(b) follows from a similar argument. If $\Sigma^{r+1}X \approx S^{n+r}$ (where $\approx$ denotes homeomorphic to), then since $\Sigma^{r+1}X \approx S^{r-1} * \Sigma X$, and $\Sigma X \times R^r \approx [S^{r-1} * (\Sigma X)] - S^{r-1}$, where $*$ denotes join, it follows that both $\Sigma X \times R^r$ and $c'(x) \times R^r$ are open $(n + r)$-manifolds.

Conversely, suppose that $c'(X) \times R^r$ is an open $(n + r)$-manifold. Note that $c(X) \times [-1, 1]^r \approx c(\Sigma^r X)$, so that $c'(\Sigma^r X)$ is an open $(n + r)$-manifold. But then by gluing two copies of $c(\Sigma^r X)$ together along $\Sigma^r X$ we have by the generalized Schoenflies theorem [3] that $\Sigma(\Sigma^r X) = \Sigma^{r+1}X \approx S^{n+r}$. □

PROPOSITION 1.2. *If M is a simplicially triangulated topological n-manifold, then M is a homology n-manifold.*

Proof. Let σ be an i-simplex of M. Then the open star of σ in M which equals

$$\partial\sigma * c(lk(\sigma, M)) - \partial\sigma * lk(\sigma, M) \approx c'(lk(\sigma, M)) \times R^i$$

is a topological n-manifold. Hence, by (1.1), $\Sigma^{i+1} lk(\sigma, M)$ is an n-sphere or ball. Thus $lk(\sigma, M)$ has the homology of an $(n - i - 1)$ sphere or ball. □

Note that the proof of (1.1) and (1.2) implies

PROPOSITION 1.3. *A homology n-manifold M is a topological n-manifold if and only if for each i-simplex σ of M, $\Sigma^{i+1} lk(\sigma, M) \approx S^n$ if $\sigma \not\subset \partial M$ and $\Sigma^{i+1} lk(\sigma, M) \approx [-1, 1]^n$ if $\sigma \subset \partial M$.*

We now investigate to what extent the converse of (1.2) is true. This is accomplished by the following four theorems.

THEOREM 1.4. *Let H^k be a compact homology k-manifold having the integral homology of a k-sphere such that $H^k \times R^{n-k}$ is an open topological n-manifold. If $n - k \geqq 2$, then $\Sigma^{n-k}H^k \approx S^n$. If H^k is also 1-connected, $n - k \geqq 1$ and $k \geqq 4$, then $\Sigma^{n-k}H^k \approx S^n$.*

THEOREM 1.5. *A homology k-manifold M with a collared boundary, $k \geqq 5$ ($\geqq 6$ if $\partial M \neq \emptyset$), is a topological k-manifold if and only if for each vertex v of M both $lk(v, M)$ and $lk(v, \partial M)$ are 1-connected.*

THEOREM 1.6. *Let H^k be a compact homology k-manifold having the integral homology of S^k. If $k \geqq 4$, then H^k bounds a contractible homology $(k + 1)$-manifold V such that $V - \partial V$ is a topological $(k + 1)$-manifold.*

THEOREM 1.7. *Let M be a homology k-manifold with collared boundary, $k \geqq 5$ ($\geqq 6$ if $\partial M \neq \emptyset$). Then there exist a homology k-manifold N which is also a topological k-manifold and a* PL *contractible map $f: N \to M$ with $f^{-1}(\partial M) = \partial N$. Furthermore, if P is a collared codimension zero subpolyhedron of ∂M which is also a topological manifold, then f can be chosen to be a homeomorphism over P.*

Remark. Note that (1.7) strengthens the main result of [15].

To prove these theorems we will use the following theorem of J. Cannon [5] and R. D. Edwards [11].

THEOREM 1.8 (Double Suspension Theorem). *The double suspension of every* PL *homology n-sphere is homeomorphic to S^{n+2}.*

Proofs of (1.4)-(1.7). Let $(1.4)_r$, $(1.5)_r$, $(1.6)_r$ and $(1.7)_r$ denote the statements of (1.4), (1.5), (1.6), and (1.7), respectively, for $k \leqq r$ with k restricted by the statements of the respective theorems. We first show the validity of $(1.4)_r$ for $r = 4$, and then show that

$$(1.4)_r \Longrightarrow (1.5)_{r+1} \Longrightarrow (1.6)_r \Longrightarrow (1.7)_{r+1} \Longrightarrow (1.4)_{r+1}$$

for $r \geqq 4$, thus establishing the validity of the four theorems.

Step 1. $(1.4)_r$ holds for $r = 4$. For $k \leqq 3$, H^k is actually a PL manifold, so the result follows from (1.8). So assume that $k = 4$ and $n \geqq 5$. By (1.1) we must show that $c'(H^4) \times R^{n-5}$ is an open topological n-manifold. As H^4 is a homology 4-manifold, the only non PL sphere links are the links $L_i = lk(v_i, H^4)$ of a finite number of vertices $v_1, \cdots, v_m$ of H^4. Construct a boundary connected sum of star neighborhoods of the v_i in H^4 along suitable arcs and denote the resulting acyclic homology 4-manifold by M^4. Then ∂M^4 is PL homeomorphic to $L_1 \# \cdots \# L_m$ where $\#$ denotes connected sum. Thus $H^4 = P^4 \cup_{\partial M^4} M^4$, where P^4 is an acyclic PL 4-manifold with $\partial P^4 = \partial M^4$.

Let $X = c'(\partial M^4) \times R^{n-5}$, $Y = c'(P^4) \times R^{n-5}$, Q^4 denote the double of P^4, $Z = c'(Q^4) \times R^{n-5}$, and $W = c'(M^4) \times R^{n-5}$. If we can show that there exists a proper embedding $h: Y \to Z$, a collar C of X in W, a collar D of $h(X)$ in $\mathrm{cl}(Z - h(Y))$, and that Z is an open topological n-manifold, the proof will be complete. For then H can be extended to a homeomorphism $Y \cup C \to h(Y) \cup D$ where $Y \cup C$ is an open neighborhood of (cone point) $\times R^{n-5}$ in $c'(H^4) \times R^{n-5}$, and $h(Y) \cup D$ is an open topological n-manifold being an open subset of Z.

First suppose that $n \geqq 6$. Since Q^4 is a PL homology 4-sphere and $n \geqq 6$, (1.8) implies $\Sigma^{n-4} Q^4 \approx S^n$, so that Z is a topological n-manifold by (1.1). Now by the codimension one approximation theorem of Ancel-Cannon [1] (the version of Bryant-Edwards-Seebeck [4] suffices in this case), $Y \subset Z$ can be properly re-embedded via $h: Y \to Z$ so that $Z - h(Y)$ is 1-ULC at $h(X)$. But, by (1.8) and (1.1), X is a topological n-manifold, so that the codimension one taming theorem of R. Daverman [8] implies that $h(X)$ has a collar D in $\mathrm{cl}(Z - h(Y))$.

Let N^4 denote the double of M^4. Now N^4 is homeomorphic to $\Sigma L_1 \# \cdots \# \Sigma L_n$, so that since $\Sigma^2(\Sigma L_i) \approx S^n$ by (1.8), we have that $\Sigma^2 N^4 \approx S^n$ by [13]. Thus, (1.1) implies that $c'(N^4) \times R$ is a topological manifold. Next note that $W - X$ is 1-ULC at $X \subset c'(Q^4) \times R^{n-5}$ since M^4 is 1-connected and X is a topological manifold. Therefore, the codimension one taming theorem of R. Daverman [8] implies that X has a collar C in W, thus completing the case when $n \geqq 6$.

For $n = 5$, the proof is the same as above, except that (1.8) does not apply. To remedy this we note that since H^4 is 1-connected, Q^4 is a homotopy 4-sphere, and thus $\Sigma Q^4 \approx S^5$ by [34].

Step 2. For $r \geqq 4$, $(1.4)_r \Rightarrow (1.5)_{r+1}$. The necessity part of (1.5) follows from (1.1) and the fact that if $\Sigma X \approx S^n$, then X is 1-connected.

Conversely suppose $\partial M = \emptyset$, for if $\partial M \neq \emptyset$, apply the unbounded case to ∂M and $M - \partial M$, and then use the fact that ∂M is collared in M. So let P_i denote the statement that if σ is a $(k-i)$-simplex of M, then $\Sigma^{k-i+1} lk(\sigma, M) \approx S^k$. The statement P_i for $i = 0, 1, 2$ is well known to be true and P_3 is true by (1.8). So assume P_i to be true for $i < s$, where $4 \leqq s \leqq k \leqq r + 1$, and let σ be a $(k - s)$-simplex of M. Now $lk(\sigma, M) \times R^{k-s+1}$ is a topological $(k + 1)$-manifold, for if $(x, t) \in lk(\sigma, M) \times R^{k-s+1}$, then

$$X = lk\big((x, t), lk(\sigma, M) \times R^{k-s+1}\big) \approx lk\big(x, lk(\sigma, M)\big) * lk(t, R^{k-s+1})$$
$$\approx lk(x * \sigma, M) * S^{k-s} \approx \Sigma^{k-s+1} lk(x * \sigma, M) .$$

But by the induction hypothesis $\Sigma X \approx \Sigma^{k-s+2} lk(x * \sigma, M) \approx S^k$. Thus by (1.1)

$c'(X)$ is an open topological manifold neighborhood of (x, t) in $lk(\sigma, M) \times R^{k-s+1}$. Finally, $(1.4)_r$ implies that $\Sigma^{k-s+1} lk(\sigma, M) \approx S^k$, hence P_s is true and the induction is complete. An application of (1.3) completes the proof.

Step 3. For $r \geqq 4$, $(1.5)_{r+1} \Rightarrow (1.6)_r$.

Case 1: $r = 4$. Since H^4 is a compact homology 4-manifold, the only non PL sphere links are the links $L_i = lk(v_i, H^4)$ of a finite number of vertices $v_1, \cdots, v_m$ of H^4. Construct a boundary connected sum of star neighborhoods of the v_i in H^4 along suitable arcs and denote the resulting homology 4-manifold by M^4. Then ∂M^4 is PL homeomorphic to $L_1 \# \cdots \# L_m$ and $H^4 = P^4 \cup_{\partial M^4} M^4$, where P^4 is an acyclic PL 4-manifold with $P^4 = \partial M^4$.

Remove the interior of a PL 4-disk from the interior of P^4 and call the resulting PL 4-manifold W. Then W is a PL homology cobordism from ∂P^4 to S^3. Let $\mathcal{H}(W, \partial P^4)$ be a handlebody decomposition for W relative to ∂P^4. We can assume that $\mathcal{H}(W, \partial P^4)$ contains no 0- nor 4-handles. Let $C_*(W, \partial P^4)$ be the (relative) chain complex based on these handles. As $H_*(W, \partial P^4) = 0$, we have the split short exact sequence

$$0 \longrightarrow C_3(W, \partial P^4) \longrightarrow C_2(W, \partial P^4) \longrightarrow C_1(W, \partial P^4) \longrightarrow 0$$

so that we have a short exact sequence

$$0 \longrightarrow C_3(W, \partial P^4) \longrightarrow C_3 \oplus C_1 \longrightarrow C_1(W, \partial P^4) \longrightarrow 0$$

where $C_3(W, \partial P^4)$ maps isomorphically to C_3 and C_1 maps isomorphically to $C_1(W, \partial P^4)$. By sliding the 2-handles of $\mathcal{H}(W, \partial P^4)$ we can geometrically realize C_3 and C_1 as being freely generated by 2-handles and then by reordering these handles we can assume that the 2-handles generating C_3 are attached after the 2-handles generating C_1. Let W_1 be the union of ∂P^4 with the 1-handles of $\mathcal{H}(W, \partial P^4)$ and the 2-handles generating C_1, and let W_2 be the union of S^3 with the 3-handles of $\mathcal{H}(W, \partial P^4)$ and the 2-handles generating C_3. Then W_1 and W_2 are homology cobordisms between ∂P^4 and H^3, and H^3 and S^3, respectively, where H^3 is a PL homology 3-sphere, and $W = W_1 \cup_{H^3} W_2$. Furthermore, van Kampen's theorem implies that both $X_1 = M^4 \cup_{\partial M^4} W_1 \cup_{H^3} c(H^3)$ and $X_2 = c(S^3) \cup W_2 \cup_{H^3} c(H^3)$ are 1-connected.

Finally, let V be the contractible homology 5-manifold

$$V = c(X_1) \cup_{c(H^3)} c(X_2)\ .$$

Then $\partial V = H^4$ and $V - \partial V$ is a topological 5-manifold by $(1.5)_5$.

Case 2: $r \geqq 5$. By $(1.5)_{r+1}$ we need to show that H^k bounds a contractible homology $(k+1)$-manifold V such that the links of vertices of $V - \partial V$ are 1-connected. But for $k \geqq 5$ this is just a trick of C. Maunder

(Proposition 2.1 of [29]). Let P be a copy of H^k halfway between a and H^k in $a * H^k$, and let $x_1, \cdots, x_t$ be a set of generators for $\pi_1(P)$. Homology manifolds have enough general position properties so that each x_i can be represented by a disjoint embedded S^1, and since $H_1(P) = 0$, each S^1 can be spanned by a disjoint acyclic 2-dimension subpolyhedron D in P. Let N be a regular neighborhood of one such D and let Q be $a * N$ projected out to H^k. Now replace Q by $c(\partial Q)$. Do this for each x_i and call the resulting manifold V. Now $lk(a, V)$ is clearly 1-connected, and $\pi_1(\partial Q) = 0$ by general position. Thus all the links of vertices of $V - \partial V$ are 1-connected.

Step 4. For $r \geqq 4$, $(1.6)_r \Rightarrow (1.7)_{r+1}$. Assume $\partial M = \emptyset$. Let $v_1, v_2, \cdots$ be the vertices of a triangulation of M and let V_i, $i = 1, 2, \cdots$, be the contractible homology manifolds that $lk(v_i, M'')$ bound given by $(1.6)_r$, where M'' is a second barycentric subdivision of M. Let

$$N = \bigcup_i [(M - \operatorname{int}(v_i * lk(v_i, M''))) \cup V_i] .$$

Now N is a homology $(r + 1)$-manifold with links of vertices 1-connected, so that by the established $(1.5)_{r+1}$, N is a topological manifold. Define $f: N \to M$ by shrinking the complement of an inner collar of ∂V_i in each V_i to a point.

If $\partial M \neq \emptyset$, triangulate M so that ∂M and P are subcomplexes, resolve M as above noting that we need not touch P and then similarly extend this to a resolution of M.

Step 5. For $r \geqq 5$, $(1.7)_r \Rightarrow (1.4)_r$. By $(1.7)_r$ there exists a topological k-manifold N^k which is triangulated as a simplicial complex and a PL contractible map $f: N^k \to H^k$. Thus $f \times \mathrm{id}: N^k \times T^{n-k} \to H^k \times T^{n-k}$ ($T^{n-k} =$ $(n-k)$-fold Cartesian product of S^1), is a PL contractible map. By hypothesis $H^k \times T^{n-k}$ is a topological manifold, so the CE approximation theorem of M. Cohen [6] implies that $f \times \mathrm{id}$ is homotopic to a homeomorphism $h: N^k \times T^{n-k} \to H^k \times T^{n-k}$. We can assume that N^k is in fact a PL manifold, since the obstruction to putting a PL manifold structure on N^k lies in $H^4(N; Z_2) = 0$ [19]. Any homeomorphism $h': N^k \times R^{n-k} \to H^k \times R^{n-k}$ covering h satisfies $\|p_2(x) - p_2 h'(x)\| <$ constant for all $x \in N^k \times R^{n-k}$, where p_2 denotes projection to R^{n-k}. Let $r: R^{n-k} \to \operatorname{int} D^{n-k} = \{x \in R^{n-k} \mid \|x\| < 1\}$ be a ray preserving homeomorphism. Then $(\mathrm{id}|_H k \times r) \circ h' \circ (\mathrm{id}|_N k \times r^{-1}): N^k \times \operatorname{int} D^k \to H^k \times \operatorname{int} D^k$ extends to a homeomorphism $g: N^k * S^{n-k-1} \to H^k * S^{n-k-1}$. Here we regard $X * S^{n-k-1}$ as the quotient of $X \times D^{n-k}$ under the identification of $X \times \partial D^{n-k}$ to $\partial D^{n-k} = S^{n-k-1}$ by projection. Thus $\Sigma^{n-k} H^k \approx \Sigma^{n-k} N^k$ and if $n - k \geqq 2$, (1.8) implies that $\Sigma^{n-k} N^k \approx S^n$. If $n - k = 1$ and H^k, hence N^k, is 1-connected, then N^k is a PL homotopy k-sphere, so that $\Sigma H^k \approx \Sigma N^k \approx S^{k+1}$ [34]. $\square$

COROLLARY 1.9. *Let H^k be a compact homology k-manifold having the integral homology of a k-sphere. If $n - k \geqq 2$, then $\Sigma^{n-k} H^k \approx S^n$. If N^k is also 1-connected, $n - k \geqq 1$, $k \geqq 1$ and $k \geqq 4$, then $\Sigma^{n-k} H^k \approx S^n$.*

Proof. By (1.5), $H^k \times R^{n-k}$ is a topological manifold, so that the result follows from (1.4). $\square$

Motivated by (1.5), we call a (locally finite) polyhedron M a TRI *n-manifold* if M is a homology n-manifold with collared boundary with $lk(v, M)$ and $lk(v, \partial M)$, 1-connected for all $v \in M$. Thus if $n \geqq 6$ ($n \geqq 5$ if $\partial M = \varnothing$) a TRI n-manifold M is a topological n-manifold.

2. TRI cone bundles and their classifying space

We now describe a triangulated topological cone bundle theory, relate it to the topological block bundle theory of Rourke-Sanderson [31], and show that it is equivalent to the homology cobordism bundle theory of Martin-Maunder [22].

Recall that a PL *cell complex* K on a polyhedron X is a locally finite covering of X by compact subpolyhedra, together with subpolyhedra $\partial\alpha$ of each element α of K, such that

(i) for each $\alpha \in K$, $\partial\alpha$ is a union of elements of K,

(ii) if α and β are distinct elements of K, then $(\alpha - \partial\alpha) \cap (\beta - \partial\beta) = \varnothing$, and

(iii) for each $\alpha \in K$, there is a PL homeomorphism $\alpha \cong c(\partial\alpha)$ rel $\partial\alpha$, where $\partial\alpha$ is a PL sphere.

If in (iii), $\partial\alpha$ is only assumed to be a polyhedral homology manifold having the integral homology of a sphere, then K is called an *H-cell complex*.

If K is a PL (H) cell complex on X, then $K \times I$ is the PL (H) cell complex on $X \times I$ with a typical cell being the cone on $(\partial\alpha \times I) \cup \alpha \times \{0, 1\}$, $\alpha \in K$.

If K is a PL (H) cell complex on a polyhedron X we denote this by $|K| = X$.

Let K be a PL cell complex on a polyhedron X. A TRI *q-cone bundle* ξ/X consists of a polyhedron $E(\xi)$ containing $X = |K|$ as a subpolyhedron such that

(i) for each p-dimensional cell $\sigma \in K$, there is a subpolyhedron $\beta_\sigma(\xi) \subset E(\xi)$ containing σ such that $\beta_\sigma(\xi)$ is a TRI $(p + q)$-manifold and such that $(\beta_\sigma(\xi), \sigma)$ is (topologically) homeomorphic to the cone on the standard sphere pair. β_σ is called the *block* of ξ over σ.

(ii) $E(\xi)$ is the union of the blocks $\beta_\sigma(\xi)$, $\sigma \in K$.

(iii) The interiors of the blocks are disjoint, and

(iv) $\beta_\sigma(\xi) \cap \beta_\tau(\xi)$ is the union of the blocks over the cells contained in $\sigma \cap \tau$.

Remark 2.1. Note that every TRI q-cone bundle is in fact a topological disk block bundle.

If ξ/K is a TRI cone bundle and L is a PL cell subcomplex of K, the *restriction* $\xi|L$ of ξ/K to L is defined by setting $\beta_\sigma(\xi|L) = \beta_\sigma(\xi)$ for each $\sigma \in L$.

Two TRI cone bundles are *isomorphic* if there is a PL homeomorphism $h: E(\xi_0) \to E(\xi_1)$ such that h is the identity on $|K|$ and $h(\beta_\sigma(\xi_0)) = \beta_\sigma(\xi_1)$ for each $\sigma \in K$.

Two cone bundles ξ_0, ξ_1/K are *concordant* if there is a TRI cone bundle $\mathfrak{N}/K \times I$ such that $\mathfrak{N}|K \times \{i\}$ is isomorphic with ξ_i, $i = 0, 1$.

We now refer the reader to [22] for the definition of and basic results concerning homology cobordism bundles. In summary, a homology cobordism q-bundle over an H cell complex K is a block bundle in which the block over a cell $\sigma \in K$ is a homology manifold homology cobordant to $\sigma \times [-1, 1]^q$.

THEOREM 2.2. *Let K be a* PL *cell complex. If $q \geqq 6$, then there is a natural one-to-one correspondence between the set* $\mathrm{TRI}_q(K)$ *of concordance classes of* TRI *q-cone bundles over K and elements of the set $\mathcal{H}_q(K)$ of concordance classes of homology cobordism q-bundles over K.*

Proof. As every TRI q-cone bundle is itself a homology cobordism bundle, there is a natural inclusion map $i: \mathrm{TRI}_q(K) \to \mathcal{H}_q(K)$. We construct an inverse $j: \mathcal{H}_q(K) \to \mathrm{TRI}_q(K)$ for $q \geqq 6$ as follows. Let $\bar{\xi}^q/K$ be a homology cobordism S^{q-1}-bundle associated with ξ^q/K. Now $\bar{\xi}^q/K$ is concordant as a homology cobordism bundle to a homology cobordism bundle ξ'/K which is a block spherical fibration (Proposition 3.1 of [27]). By (1.7), each block of ξ'/K is resolvable via a PL contractible map to a TRI manifold. Thus, by inductively resolving the blocks of ξ'/K, the mapping cylinders of these resolutions exhibits a concordance of ξ'/K to a homology cobordism bundle $\mathfrak{N}^q/K$ which is a spherical block fibration in which all blocks are TRI manifolds. Thus, the generalized Poincaré conjecture and the h-cobordism theorem imply that $\mathfrak{N}^q/K$ is in fact a topological block S^{q-1}-bundle. Then by coning the blocks of all the constructed bundles and concordance we have that ξ^q/K is concordant as a homology manifold to a TRI q-cone bundle $\mathfrak{N}^q/K$. Let $j: \mathcal{H}_q(K) \to \mathrm{TRI}_q(K)$ map the concordance class of ξ^q/K to the concordance class of $\mathfrak{N}^q/K$. It is now easy to verify that j is indeed an inverse for the inclusion map i: $\mathrm{TRI}_q(K) \to \mathcal{H}_q(K)$. □

Note that every TRI q-cone bundle can be considered as a TRI $(q+1)$-cone bundle by taking the product of the total space with $[-1, 1]$ and then observing that the correspondence of (2.2) commutes with this stabilization.

In [22], N. Martin and C. Maunder construct a classifying space $BH(q)$ for homology cobordism q-bundles over H-cell complexes. Hence (2.2) shows that $BH(q)$ can also be considered as a classifying space for TRI q-cone bundles if $q \geqq 6$. To emphasize this, we let $\mathrm{BTRI}(q)$ denote this classifying space for TRI q-cone bundles over PL cell complexes. So we have

LEMMA 2.3. *Let K be a* PL *cell complex. If $q \geqq 6$, then there is a one-to-one correspondence between the elements of* $\mathrm{TRI}_q(K)$ *and elements of* $[|K|, \mathrm{BTRI}(q)]$.

Let $\widetilde{\mathrm{BTOP}}(q)$ denote the classifying space for topological block bundles over PL cell complexes [31]. By (2.1) we have

THEOREM 2.4. *For $q \geqq 6$, there exists a natural map* $t: \mathrm{BTRI}(q) \to \widetilde{\mathrm{BTOP}}(q)$ *which induces a map* $t: \mathrm{BTRI} = \lim_{q\to\infty} \mathrm{BTRI}(q) \to \lim_{q\to\infty} \widetilde{\mathrm{BTOP}}(q) = \mathrm{BTOP}$.

Remark 2.5. Theorem 2.2 is in fact true for $q \geqq 3$, so that (2.3) and (2.4) are true for $q \geqq 3$, but for our triangulation work we only need a map of stable classifying spaces. Section 7 of [15] provides a proof if one uses (1.7) in place of Theorem 5.2 of [15].

We will discuss the fiber of $t: \mathrm{BTRI} \to \mathrm{BTOP}$ in Section 6.

3. Some embedding theorems

In this section we will show how to represent certain maps of disks into TRI manifolds by PL embedded disks up to an s-cobordism through a homology manifold. These results will be used in Section 4 to prove our product structure theorem.

Two closed homology m-manifolds are said to be *s-cobordant* if there is a homology $(m+1)$-manifold W such that ∂W is the disjoint union of M and N, and the inclusions $M \subset W$ and $N \subset W$ are simple homotopy equivalences. If M and N are homology manifolds with boundary, they are said to be *s-cobordant* if there is a homology $(m+1)$-manifold W such that $\partial W = M \cup W_0 \cup N$, where W_0 is an s-cobordism from ∂W to ∂N, and the inclusions $M \subset W$ and $N \subset W$ are simple homotopy equivalences. If $\partial M = \partial N$, then M and N are *relatively* s-cobordant if M and N are s-cobordant via an

s-cobordism W with $\partial W = N \cup \partial N \times I \cup M$. We denote an s-cobordism between M and N by $(W; M, N)$.

We first show how to embed disks below the middle dimension.

THEOREM 3.1. *Let M be a compact homology m-manifold with non-empty boundary. Suppose $f: (D^k, S^{k-1}) \to (M, \partial M)$ is a proper map such that $f|S^{k-1}$ is a* PL *embedding which extends to a* PL *embedding of $S^{k-1} \times D^{m-k}$ into ∂M. If $2k \leqq m - 1$, then there exist a relative s-cobordism $(W; M, M')$ and a proper map $F: (D^k, S^{k-1}) \times (I; 0, 1) \to (W; M, M')$ with*

$$F|(D^k, S^{k-1}) \times 0 = f,\ F|S^{k-1} \times I = f|S^{k-1} \times \mathrm{id}\ ,$$

and $F|(D^k, S^{k-1}) \times 1$ a proper PL *embedding which extends to a* PL *embedding of $(D^k, S^{k-1}) \times D^{m-k}$.*

Proof. The idea of this proof is due to Matsui [23]. We can assume that ∂M is PL collared in M, as this can be accomplished via a relative s-cobordism. In [26] Maunder develops general position theorems for maps of polyhedra into a homology manifold M. The basic philosophy is that given a map $f: P \to M$ of a k-dimensional polyhedron P into M, we may not be able to homotope f into general position, but there is a PL acyclic map $\alpha: K \to P$ where K is also a k-dimensional polyhedron, and a map $\beta: K \to M$ which is in general position, such that $f\alpha$ is arbitrarily close to β and hence homotopic to β. In our situation then there exists an acyclic polyhedron K of dimension k with $S^{k-1} \subset K$, a PL acyclic map $\alpha: K \to D^k$ with $\alpha|S^{k-1}$ the identity, and a PL embedding $\beta: K \to M$ with $\beta|S^{k-1} = f|S^{k-1}$. Furthermore, β is homotopic to $f\alpha$. Using a collar C of ∂M in M we can assume that $\beta(K) \cap C = f(S^{k-1}) \times I$. Let N be a regular neighborhood of $\beta(K)$ in M pushed off ∂M using the collar C. If $M_1 = \mathrm{cl}(M - N) \cup c(\partial N)$, then there is a PL acyclic map $g: M \to M_1$ with $g|C$ the identity. Now g induces an isomorphism on π_1 by van Kampen's theorem, since $\beta \sim f\alpha \sim 0$ implies $\pi_1(N) \to \pi_1(M)$ is the zero map. Thus, by [14], g is a simple homotopy equivalence. Also, there is a PL embedding $f_1: (D^k, S^{k-1}) \to (M_1, \partial M_1)$ with $f_1|S^{k-1} = f$, induced by $K \xrightarrow{\beta} M \xrightarrow{g} M_1$. The mapping cylinder of g provides a relative s-cobordism $(W_1; M, M_1)$. Also, the map

$$f \cup (f|S^{k-1} \times \mathrm{id}_I) \cup f_1 \colon D^k \cup (S^{k-1} \times I) \cup D^k \longrightarrow M \cup (\partial M \times I) \cup M_1$$

extends to a proper map $F_1: (D^k, S^{k-1}) \times (I; 0, 1) \to (W_1; M, M_1)$ which has all the desired properties except that f_1 may not extend to an embedding of $(D^k, S^{k-1}) \times D^{m-k}$. To remedy this situation note that $f_1(D^k, S^{k-1})$ has a normal homology cobordism bundle $\gamma/f_1(D^k)$ which is stably trivial rel $\gamma|f_1(S^{k-1})$. But since $\pi_i(BH, BH(r)) = 0$ for $i \leqq n$ and $r \geqq 3$ [25], γ is concordant rel $\gamma | f_1(S^{k-1})$ to the trivial bundle $D^k \times D^{m-k}$. Let $\mathfrak{N}/f_1(D^k) \times I$ be this concordance and let N' be a regular neighborhood of $f_1(D^k) \times I$ in $E(\mathfrak{N})$. Then $(M_1 \times I) \cup N'$, where the union is taken along a regular neighborhood of $f_1(D^k) \times 1$ in $M_1 \times 1$, is a relative s-cobordism $(W'; M_1, M')$ such that there is an embedding $f': (D^k, S^{k-1}) \to (M', \partial M')$ which extends to a PL embedding of $(D^k, S^{k-1}) \times D^{m-k}$. Now extend

$$f_1 \cup (f_1 | S^{k-1} \times \mathrm{id}_I) \cup f': D^k \cup (S^{k-1} \times I) \cup D^k \longrightarrow M_1 \cup (\partial M_1 \times I) \cup M$$

to a proper map $F': (D^k, S^{k-1}) \times (I; 0, 1) \to (W', M_1, M')$ and adjoin W_1 and W' along M_1, and F_1 and F' along f_1 to obtain the required s-cobordism. □

As a consequence of (3.1) we have

COROLLARY 3.2. *Let M be a closed homology m-manifold. Suppose $f: S^k \to M$ is a map and $2k \leqq m - 1$. Then there exist an s-cobordism $(W; M, M')$ and a proper map $F: S^k \times (I; 0, 1) \to (W; M, M')$ with $F | S^k \times 0 = f$, and $F | S^k \times 1$ extends to a* PL *embedding of $S^{k-1} \times D^{m-k}$.*

Our goal in the rest of this section is to treat the middle dimensional embedding problem for TRI manifolds. However, to do this we must first introduce the notion of a PL resolution of a homology manifold.

Let M be a compact homology n-manifold and let N be a codimension zero compact PL submanifold of M. A PL *resolution* of M rel N is a pair (P, f) where P is a PL n-manifold and $f: (P, \partial P) \to (M, \partial M)$ is a surjective PL map of pairs such that $f^{-1}(\partial M) = \partial P$, $f | f^{-1}(N)$ is a PL homeomorphism, and f is acyclic, i.e., $\tilde{H}_*(f^{-1}(x)) = 0$ for all $x \in M$. The pair (P, f) is called a *simple resolution of* M rel N if we further require that f be a simple homotopy equivalence.

M. Cohen [6] and D. Sullivan [38] have developed an obstruction theory for PL resolving M rel N. They show that there is a well-defined element $\sigma(M, N) \in H^4(M, N; \theta_3^H)$ whose vanishing is a necessary and sufficient condition for the existence of a PL resolution of M rel N.

PROPOSITION 3.3. *Let M be a compact homology m-manifold and let N be a codimension zero submanifold of ∂M. If M has a* PL *resolution* rel N, *then M has a simple resolution* rel N.

Proof. For simplicity we assume $N = \partial M = \emptyset$, as the general case is handled similarly.

By hypothesis M has a PL resolution so that $\sigma(M) = 0$. Now by [6] or [20] there exist homology manifolds M_i, $i = 0, 1, \cdots, m$ and PL acyclic maps $g_i: M_{i+1} \to M_i$ so that $f = g_{m-1} \circ \cdots \circ g_0: M_m \to M_0 = M$ is a PL resolution of M. We further observe by the construction of the g_i in [6] or [20] and by van Kampen's theorem, that the elements in the kernel of the $(g_i)_*: \pi_1(M_{i+1}) \to \pi_1(M_i)$ lie in simply connected sets. But then each g_i, and hence f, induces an isomorphism on π_1. Therefore, by [14], f is a simple homotopy equivalence. □

THEOREM 3.4. *Let M be a compact* TRI *$2k$-manifold, $k \geqq 3$, with non-empty boundary. Suppose that $f_i: (D^k, S^{k-1}) \to (M, \partial M)$, $1 \leqq i \leqq r$, are proper maps which are homotopic to disjoint topologically locally flat embeddings with trivial normal bundles. Then there exist an s-cobordism $(W; M, M')$ and proper maps $F_i: (D^k, S^{k-1}) \times (I; 0, 1) \to (W; M, M')$, $1 \leqq i \leqq r$, with $F_i | (D^k, S^{k-1}) \times 0 = f_i$ and $F_i | (D^k, S^{k-1}) \times 1$ disjoint proper* PL *embeddings which extend to proper* PL *embeddings of $(D^k \times D^k, S^{k-1} \times D^k)$.*

Proof. As in the proof of (3.1) we can assume that ∂M is PL collared in M. Let $f'_i: (D^k \times D^k, S^{k-1} \times D^k) \to (M, \partial M)$ be disjoint topological embeddings with $f'_i | (D^k \times 0, S^{k-1} \times 0)$ properly homotopic to f_i, $1 \leqq i \leqq r$. Let

$$(S_i, T_i) = f'_i(D^k \times D^k, S^{k-1} \times D^k) , \quad (Q_i, R_i) = \big(\mathrm{cl}(M - S_i), \mathrm{cl}(M - T_i)\big) ,$$

and $g_i: (M, \partial M) \to (S_i/Q_i, T_i/R_i)$ be the natural collapse maps. Since the range of each g_i can be considered as the Thom space of trivial PL bundles over (D^k, S^{k-1}), the homology transversality theorem (Theorem 3.7 of [15]) implies that we can homotope each g_i rel Q_i to maps, which we still call g_i, so that $(P_i, \partial P_i) = g_i^{-1}(D^k \times 0, S^{k-1} \times 0)$ is a proper homology k-submanifold of M with a trivial normal homology cobordism bundle, γ_i/P_i, inside S_i. Let $\mathfrak{N}_i/P_i \times I$ be the concordance with $\mathfrak{N}_i | P_i \times 0 = \gamma_i/P_i$ and $\mathfrak{N}_i | P_i \times 1 = P_i \times D^k$. Let N_i be a relative regular neighborhood of $P_i \times I$ in the homology manifold $E(\mathfrak{N}_i)$. Then the homology manifolds N_i furnish s-cobordisms between a regular neighborhood of P_i in M and, by uniqueness of relative regular neighborhoods, $P_i \times D^k$. Thus by attaching the N_i to $M \times I$ over a regular neighborhood of P_i in $M \times 1$, we obtain an s-cobordism from $(M, \partial M)$ to a homology manifold which we also call $(M, \partial M)$, and with $(P_i, \partial P_i)$ disjointly embedded in $(M, \partial M)$ with neighborhoods of the form $(P_i \times D^k, \partial P_i \times D^k)$. We also extend the maps g_i rel Q_i over the s-cobordism to obtain maps

$$g_i: (M, \partial M) \longrightarrow (S_i/Q_i, T_i/R_i) \quad \text{with} \quad (P_i, \partial P_i) = g_i^{-1}(D^k \times 0, S^{k-1} \times 0) .$$

As P_i has a trivial normal bundle in M, it is stably parallelizable (as a homology manifold). Thus, by the classification theorem for PL resolutions (Theorem 5.1 of [9]) and by (3.3), there exist PL k-manifolds P'_i and PL acyclic maps $h_i: P'_i \to P_i$ with $h_i | \partial P'_i$ and h_i simple homotopy equivalences. By adjoining the (simplicial mapping cylinder of h_i) $\times D^k$ to M over $P_i \times D^k$

we have that $(M, \partial M)$ is s-cobordant to a homology manifold, which we again call $(M, \partial M)$, with $(P_i', \partial P_i')$ PL embedded in $(M, \partial M)$ with neighborhoods of the form $(P_i' \times D^k, \partial P_i' \times D^k)$. We also extend the maps g_i rel Q_i over the s-cobordism to obtain maps $g_i: (M, \partial M) \to (S_i/Q_i, T_i/R_i)$ with $(P_i', \partial P_i') = g_i^{-1}(D^k \times 0, S^{k-1} \times 0)$. Our goal is now to do surgery on the P_i' away from Q_i, using (3.1), so that the P_i' are disks.

We first do surgery on the $\partial P_i'$ away from the R_i so that the $\partial P_i'$ are spheres. So inductively assume that $g_i: \partial P_i' \to \partial D^k \times 0$ is $(j-1)$-connected for $2j \leqq k-1$. Let $\alpha: S^j \to \partial P_i'$ be an embedding representing a non-zero generator of $\pi_j(\partial P_i')$ which extends to an embedding of $S^j \times D^{k-j-1}$. Since we inductively assume that the R_i have been kept fixed, α extends to a map $\alpha: (D^{j+1}, S^j) \to (\partial M, \partial P_i)$ with $g_i\alpha(D^{j+1}) \subset f_i(D^k \times 0)$. By (3.1) there is an s-cobordism W_i between ∂M and a homology manifold $\bar{M}$, with W_i fixing $\partial P_i' \times D^k$ and an embedding $\beta: (D^{j+1}, S^j) \to (\bar{M}, \partial P_i)$ which extends to an embedding of $(D^{j+1}, S^j) \times D^{k-j-1}$ and with β homotopic to α rel S^j. Also, we extend the maps g_i over the s-cobordism to yield maps $g_i: (\tilde{M}, \partial\tilde{M} = \bar{M}) \to (S_i/Q_i, T_i/R_i)$ with $\partial P_i' = g_i^{-1}(S^{k-1} \times 0)$, where $\tilde{M}$ is M union the above s-cobordism along ∂M. By PL surgery we can use the embeddings β to kill the homotopy element α and deform the maps g_i so that the transversal preimage of $S^{k-1} \times 0$ is identified with the resulting surgery. So if $k \geqq 6$, we continue in this fashion until each $g_i^{-1}(S^{k-1} \times 0)$ is S^{k-1}. For $k = 5$, using the above procedure we can assume that the $\partial P_i'$ are 1-connected. Since the index of each $\partial P_i'$ is zero we can ambiently take connected sums of $\partial P_i'$ with connected sums of $S^2 \times S^2$ so that we can assume that $\partial P_i'$ is PL homeomorphic to a connected sum of copies of $S^2 \times S^2$ (cf. [40]). Now by adding 3-handles, using (3.1) as above, we can assume that the $\partial P_i'$ are 4-spheres. For $k = 4$ we can assume that the $\partial P_i'$ are connected. Since 3-dimensional cobordism is zero, we have that $\partial P_i'$ is cobordant through a spin 4-manifold W to S^3. But W can be realized by adding only 2 and 3 handles to $\partial P_i'$, so by using (3.1) as above we can embed this normal cobordism. Thus we can assume that the $\partial P_i'$ are 3-spheres.

So after performing the above surgeries we have the following situation. There are an s-cobordism $\bar{W}$ between $(M, \partial M)$ and a homology manifold $(\bar{M}, \partial\bar{M})$, proper maps $F_i: (D^k, S^{k-1}) \times (I; 0, 1) \to (W; M, \bar{M})$ with $F_i|(D^k, S^{k-1}) \times 0 = f_i$ and $F_i|S^{k-1} \times 1$ disjoint PL embeddings which extend to embeddings of $S^{k-1} \times D^k$ into $\partial\bar{M}$, and maps $g_i: (\bar{M}, \partial\bar{M}) \to (S_i/Q_i, T_i/R_i)$ with $g_i|\partial\bar{M}$ transverse to $S^{k-1} \times 0$ and $g_i^{-1}(S^{k-1} \times 0) = F_i(S^{k-1} \times 1)$. Now use the relative homology transversality theorem (Theorem 3.7 of [15]) to deform the g_i, rel ∂M, so that $(P_i', \partial P_i' = S^{k-1}) = g_i^{-1}(D^k \times 0, S^{k-1} \times 0)$ are homology k-manifolds with trivial homology cobordism bundles γ_i/P'_i with $\gamma_i|P'_i = S^{k-1} \times D^k$. Now proceed, as we did above for $\partial P'_i$, to make P'_i a disk. The result now follows. □

The proof of (3.4) also applies to show

THEOREM 3.5. *Let M be a compact* TRI $(2k+1)$*-manifold, $k \geqq 3$, with non-empty boundary. Suppose that $f_i: (D^{k+1}, S^k) \to (M, \partial M)$, $1 \leqq i \leqq r$, are proper maps which are homotopic to disjoint topologically locally flat embeddings with trivial normal bundles. Then there exist an s-cobordism $(W; M, M')$ and proper maps $F_i: (D^{k+1}, S^k) \times (I; 0, 1) \to (W; M, M')$, $1 \leqq i \leqq r$, with $F_i|(D^{k+1}, S^k) \times 0 = f_i$ and $F_i|(D^{k+1}, D^k) \times 1$ disjoint proper* PL *embeddings which extend to proper* PL *embeddings of $(D^{k+1} \times D^k, S^k \times D^k)$.*

4. A product structure theorem

In this section we prove a product structure theorem from which our main theorem classifying TRI manifold structures on a given topological manifold routinely follows. We begin with some definitions.

A TRI *manifold structure* Σ on a topological manifold M is a maximal family of PL related embeddings of compact TRI manifolds. If Σ is a TRI manifold structure on M, there exists a locally finite complex K which is a TRI manifold and a homeomorphism $h: |K| \to M$ which is PL related to every element of Σ, and such a homeomorphism determines a TRI manifold structure on M.

Let N be a codimension zero topological submanifold of ∂M. Then a TRI *manifold structure on M near N* is a TRI manifold structure Σ_0 on a neighborhood U of N in M such that Σ_0 restricts to a TRI manifold structure on N. Let Σ be a TRI manifold structure on $M \times I$ $(I = [0, 1])$. By restriction it gives TRI manifold structures $\Sigma_i \times \{i\}$ on $M \times \{i\}$ for $i = 0, 1$. It is said to give a TRI *concordance* rel Σ' if Σ equals $\Sigma' \times I$ on a neighborhood of $N \times I$ in $M \times I$, where Σ' is a given TRI manifold structure on M near N. When the TRI manifold structure on M is clear from the context, we will sometimes refer to M as a TRI manifold.

THEOREM 4.1. *Let M^n be a compact connected topological n-manifold and let θ be a* TRI *manifold structure on $M \times R$. Let Σ_0 be a* TRI *manifold structure on M near ∂M such that $\Sigma_0 \times R$ agrees with θ near $\partial M \times R$. If $n \geqq 5$, then there is a topological s-cobordism $(W; M, M')$* rel *a neighborhood of ∂M in M, a* TRI *manifold structure Σ' on M' coinciding with Σ_0 near $\partial M = \partial M'$, and a* TRI *manifold structure Σ on $W \times (-1, 1)$ with*

$$\Sigma | M \times (-1, 1) = \theta | M \times (-1, 1) \quad \textit{and} \quad \Sigma | M' \times (-1, 1) = \Sigma' \times (-1, 1) .$$

Moreover, any such Σ *determines a* TRI *manifold structure* Γ *on* M *coinciding with* Σ_0 *near* ∂M, *unique up to concordance* rel Σ_0, *such that* $\Gamma \times (-1, 1)$ *is concordant* rel $\Sigma_0 \times (-1, 1)$ *to* $\theta | M \times (-1, 1)$.

Proof. Our proof is modelled on W. Browder's *Structures on* $M \times R$ [2]. However, in our case M may not be simply connected so we will use $\Lambda = Z(\pi_1(M))$ coefficients in homology, noting that the kernels of degree one maps satisfy Poincaré duality (see [41]) and hence the algebra of Section 2 of [2] holds in our case.

We will assume that $\partial M = \emptyset$, as our proof can easily be modified to keep ∂M fixed throughout the construction.

Let θ be the given TRI structure on $M \times R$ and let $\bar{\theta}$ be the induced TRI manifold structure on $M \times (2, 3)$. Let $\pi: M \times (2, 3) \to (2, 3)$ be projection and subdivide $(M \times (2, 3))_{\bar{\theta}}$ so that π has a simplicial approximation which we also call π. Let $x \in (2, 3)$ be a point in the interior of a 1-simplex. Then $K = \pi^{-1}(x)$ is a polyhedron with a neighborhood PL homeomorphic to $K \times R$. As $K \times R$ is a TRI manifold, K is a homology n-manifold. By choosing an appropriate component of K we may assume that K is connected, closed, and divides $M \times (2, 3)$ into two homology manifolds A and B with boundary $K = A \cap B$. Let $g: K \to M$ be the inclusion $K \to M \times (2, 3)$ followed by the projection $p: M \times (2, 3) \to M$. Then $g: K \to M$ is of degree 1. We proceed to kill the kernel of $g_*: \pi_*(K) \to \pi_*(M)$ by doing ambient surgery on K inside $M \times (2, 3)$ up to s-cobordisms through homology manifolds, and then use (1.7) to do the surgery up to s-cobordisms through TRI manifolds.

We first resolve K inside of $M \times (2, 3)$ so that K is a TRI manifold. As $n \geqq 5$, (1.5) shows that we only need to alter the $(n - 1)$-dimensional links of K so that they are 1-connected. Let v be a vertex of K; then by (1.6), $lk(v, K)$ bounds a contractible manifold M_v with $W_v - \partial M_v$ a TRI manifold. So replace $D(v, K)$ by W_v and replace $D(v, M \times (2, 3)) \approx S^0 * D(v, K)$ by $S^0 * W_v$. Do this for all vertices of K and let K' denote the repaired K and let N denote the repaired $M \times (2, 3)$. Then $K' \subset N$ are both TRI manifolds and there is a proper PL contractible map $f: N \to M \times (2, 3)$ with $f | K': K' \to K$ a PL contractible map and with f the identity off a neighborhood of K' in N. Now K' is connected, closed, and divides N into two TRI manifolds A' and B', with $K' = A' \cap B'$. Note that we can assume that K' is PL bicollared in N, for, if necessary, we can split N open along K' and fill in with $K' \times I$. Let $g': K' \to N$ be the composition $K' \overset{i}{\subset} N \overset{f}{\to} M \times (2, 3) \overset{p}{\to} M$, where p is projection. We then have the homotopy commutative diagram

$$\begin{array}{ccccc} N & \overset{i}{\supset} & K' & \xrightarrow{g'} & M \\ \uparrow f & & \uparrow f|K' & \nearrow g & \\ M(2,3) & \supset & K & & \end{array}$$

Next we kill the kernel of $g'_*: \pi_1(K') \to \pi_1(M)$ by doing surgery on K' inside N up to s-cobordism through homology manifolds.

Choose a set of normal generators $g_1, \cdots, g_r$ of $\ker(g'_*: \pi_1(K') \to \pi'(M))$. As K is a TRI manifold, its top dimensional links are 1-connected, so that K is an ND(3) manifold in the sense of Stallings [35]. Thus by Theorem 5.4.11 of [35], these generators can be represented by disjoint PL embeddings $g_i: S^1 \to K'$. As N is a topological manifold, the g_i extend to topological embeddings $g_i: D^2 \to N$ meeting K' topologically transversally. Thus, for each $1 \leqq i \leqq r$, $g_i(D_2) \cap K'$ is a collection of disjoint topologically embedded circles. Again, as K' is ND(3), every map of a circle can be arbitrarily closely approximated by a PL embedding. Thus, by the homotopy extension property, topological general position [39], and the fact that K' is bicollared, we can assume that the $g_i: D^2 \to N$, $1 \leqq i \leqq r$, are disjoint topological embeddings with $g_i(D^2) \cap K'$ a collection of disjoint PL embedded circles. Choose an innermost one of $g_1(D^2) \cap K'$ which bounds a topologically embedded disk δ, with $\delta \subset A'$ or $\delta \subset B'$, say $\delta \subset A'$. Let P be a regular neighborhood of simplicial neighborhood of δ in A' with $g_1(D^2) \cap g_i(D^2) \cap P = \delta$, $2 \leqq i \leqq r$. By (3.1), P is s-cobordant rel ∂P to P' with δ PL embedded in P' with trivial normal bundle. Thus by attaching this s-cobordism to $N \times I$ we have K' embedded in a homology manifold N' with N' properly simple homotopy equivalent to $M \times (2, 3)$, equal to $M \times (2, 3)$ off a neighborhood of K' and divided into two homology manifolds A_0 and B_0 with boundary $K' = A_0 \cap B_0$. There exist PL embeddings $g_i: S^1 \to K'$ representing generators of $\ker(g_*: \pi_1(K') \to \pi_1(N'))$ which extend to disjoint topological embeddings $g_i: D^2 \to N'$ with $g_i(D^2) \cap K'$ a collection of disjoint PL embedded circles. Also an innermost circle of $g_1(D^2) \cap K'$ bounds a PL embedded disk ζ with a PL trivial normal bundle neighborhood $\bar{P}$. Now let $A_1 = \mathrm{cl}(A_0 - \bar{P})$, $B_1 = B_0 \cup \bar{P}$ and $K_1 = A_1 \cap B_1$ and say that K_1 was obtained from K' by adding the handle $\bar{P}$. Do this for all subdisks which bound innermost circles of $K_1 \cap g_1(D^2)$ until we finally have exchanged $g_1(D^2)$ and killed the generator g_1 of $\ker(g_*: \pi_1(K') \to \pi_1(N))$. Now do this for all the g_i, $1 \leqq i \leqq r$, until we have the following situation. There are (new) homology manifolds K' and N with $K' \subset N$ a PL bicollared homology manifold; N is properly simple homotopy equivalent via a (new) map f to $M \times (2, 3)$, N is equal to $M \times (2, 3)$

off a neighborhood of K', and K' divides N into two homology manifolds A' and B' with boundary $K' = A' \cap B'$. Let $a: K' \to A'$, $b: K' \to B'$ and $i: K' \to N$ be inclusions and note that they induce isomorphisms on π_1 and that i is degree 1.

We now proceed to kill $\ker(i_*: \pi_*(K') \to \pi_*(N))$ (cf. [2; Prop. 4.1]). So let us inductively assume that $\pi_j(N, K') = 0$ for $j < k$ and $k < n/2$. Note that as in [2], $\ker i_* = \ker a_* \oplus \ker b_*$. Now $\pi_k(a) = \ker(i_*: H_{k-1}(K'; \Lambda) \to H_{K-1}(A'; \Lambda))$ is finitely generated so let $x \in \pi_k(a)$ be a generator. Then there is a mapping $h: (D^k, S^{k-1}) \to (A', K')$ representing x. Now apply (3.1) and (3.2) so that we can assume that h is a PL embedding which extends to a PL embedding of $(D^k, S^{k-1}) \times D^{n-k}$. Next exchange a regular neighborhood of $h(D^k)$ from A' to B', to obtain new K', A', B', i, a, and b. We note that as in [2] we have reduced the number of generators of $\pi_k(a)$ and kept $\pi_k(b)$ the same. Now do this for all generators of $\pi_k(a)$ and $\pi_k(b)$ until they are zero and hence $\pi_k(i) = 0$. Thus by induction we can assume that $\pi_j(N, K') = 0$ for $2j < n$.

We now have the following situation. There is a (new) homology manifold N with the proper simple homotopy type $M \times (2, 3)$, a (new) connected homology manifold $K' \subset N$ with N equal to $M \times (2, 3)$ off a neighborhood of K', K' divides N into two (new) homology manifolds A' and B' with boundary $K' = A' \cap B'$, $B' \cup M \times [0, 2]$ is connected and $\pi_j(i) = 0$ for $2j < n$, where $i: K' \subset N$. We now consider two cases, $n = 2k + 1$ and $n = 2k$.

Case 1: $n = 2k + 1$. Let $g_1, \cdots, g_r$ represent a basis of

$$\pi_{k+1}(a) \approx \operatorname{Ker}(a_*: H_k(K'; \Lambda) \longrightarrow H_k(A'; \Lambda))$$

where $g_i: (D^{k+1}, S^k) \to (A', K')$. By resolving K' in N as in the beginning of the proof we can assume K' is a bicollared TRI manifold in N. Now by topological surgery (§ 4 of [41]) we can apply (3.4) to A' so that we can assume that the g_i are disjoint PL embeddings with trivial normal bundle neighborhoods in a new N which is properly simple homotopy equivalent to $M \times (2, 3)$ and which is equal to $M \times (2, 3)$ off a neighborhood of K' (but now K', A', B' are again just homology manifolds). Exchange these neighborhoods from A' to B' and let $\bar{K}$, $\bar{A}$, and $\bar{B}$ be the new K', A', and B', respectively. We now apply this process again to the free generators of $\pi_{k+1}(b) = \operatorname{Ker}(b_*: H_k(\bar{K}; \Lambda) \to H_k(\bar{B}; \Lambda))$ and conclude as in (4.2) of [2] that $i: \bar{K} \to N$ is a homotopy equivalence and by resolving $\bar{K}$ in the new N we can again assume that $\bar{K}$ is TRI bicollared in N.

Let W_1 be the region between $\bar{K}$ and $M \times 0$. Then W_1 is a topological h-cobordism between $M \times 0$ and $\bar{K}$ with Whitehead torsion $\tau(W_1)$. We can easily construct a TRI h-cobordism, W_2, between K and a TRI n-manifold M' with Whitehead torsion $\tau(W_2) = -\tau(W_1)$. Thus $W = W_1 \cup_{\bar{K}} W_2$ is a topological s-cobordism between $M \times 0$ and M' with $M \times (-1, 0] \cup_{M \times 0} W$ a TRI $(n + 1)$-manifold. W will be the desired cobordism between M and M'.

Case 2. $n = 2k$. We now proceed as in Case 1. By using topological surgery we invoke (3.5) so that if $g_1, \cdots, g_r$ represents a basis of $\pi_{k+1}(a) \approx \ker\big(a_*: H_k(K'; \Lambda) \to H_k(A'; \Lambda)\big)$, we can assume that they are represented by disjoint PL embeddings $g_i: (D^{k+1}, S^k) \to (A', K')$ with trivial PL normal bundles. Abstractly attach disjoint handles $D_i^{k+1} \times D^k$ to A' by identifying $\partial D_i^{k+1} \times D^k$ with $g_i(S^k) \times D^k$ and let $W_1 = M \times [0, 2] \cup B' \cup \bigcup_i (D_i^{k+1} \times D^k)$. Thus W_1 is an h-cobordism between $M \times 0$ and say, M'' with Whitehead torsion $\tau(W_1)$ and with $M \times (-1, 0] \cup_{M \times 0} W_1$ a TRI $(n + 1)$-manifold. As in Case 1 let W_2 be a TRI h-cobordism from M'' to M' with $\tau(W_2) = -\tau(W_1)$. The desired topological s-cobordism W from $M \times 0$ to M' is $W_1 \cup W_2$.

Now let $(W; M, M')$ be the topological s-cobordism produced by the above construction with $\bar{\Sigma}'$ the TRI manifold structure on M' and $\bar{\Sigma}$ the TRI manifold structure on $W' = W \cup \big(M \times (-1, 0]\big)$ with $\bar{\Sigma} | M \times (-1, 1) = \theta | M \times (-1, 1)$ and $\Sigma | M' = \Sigma'$, where $M \times [0, 1)$ is the collar of M in W. There is a PL ambient isotopy $h_t: 2D^2 \to 2D^2$ of the 2-disk $2D^2 = \{x \in r^2 \,|\, \|x\| \leqq 2\}$ which fixes $\partial(2D^2)$ and interchanges the axes of $D^2 = \{x \in R^2 \,|\, \|x\| \leqq 1\}$. This isotopy extends to an ambient topological isotopy of $W' \times R$ which leaves $W' \times R$ fixed on a neighborhood of $M' \times R$ and interchanges the axes in the fibers of $M \times D^2$ in $W' \times R$. Thus there is a TRI manifold structure Σ on $W \times (-1, 1)$, induced by $\bar{\Sigma} \times R$ and h_1, with $\Sigma | M \times (-1, 1) = \theta | M \times (-1, 1)$ and with $\Sigma | M' \times (-1, 1) = \Sigma' \times (-1, 1)$.

By the topological s-cobordism theorem there is a homeomorphism $h: (W; M, M') \to M \times \big([0, 1]; 0, 1\big)$ with $h | M$ the identity. Then $h | M': M' \to M$ determines a structure Γ on M and $h \times \text{id}: \big(W \times (-1, 1)\big) \to (M \times I) \times (-1, 1)$ exhibits the desired concordance between $\Gamma \times (-1, 1)$ and $\theta | M \times (-1, 1)$.

To see the uniqueness of Γ up to concordance, suppose $(\bar{W}, M, \bar{M})$ is another topological s-cobordism given by the above construction. Then by the topological s-cobordism theorem there is a homeomorphism $\bar{h}: (\bar{W}; M, \bar{M}) \to M \times \big([-1, 0]; 0, -1\big)$ with $\bar{h} | M$ the identity. Thus $\bar{h} | \bar{M}: \bar{M} \to M$ determines a structure $\bar{\Gamma}$ on M. Note that $(h \cup \bar{h}) \times \text{id}: (W \cup_M \bar{W}) \times (-1, 1) \to M \times [-1, 1] \times (-1, 1)$ determines a TRI structure $\bar{\Sigma}$ on $M \times [-1, 1] \times (-1, 1)$ with

$$\bar{\Sigma} | (M \times -1) \times (-1, 1) = \bar{\Gamma} \times (-1, 1)$$

and

$$\Sigma | (M \times 1) \times (-1, 1) = \Gamma \times (-1, 1) .$$

Now applying relative existence to $(M \times [-1, 1] \times (-1, 1))_{\bar{\Sigma}}$, we obtain a concordance between Γ and $\bar{\Gamma}$. □

Remark 4.2. It is not difficult to see that the above construction can be used to produce a TRI s-cobordism between $M \times R$ and a TRI manifold L, properly containing W (W as in (4.1)), which is a product outside any given neighborhood of $M \times 0$ and with $M' \subset L$ (M' as in (4.1)) a simple homotopy equivalence. Moreover, we can assume $M' \subset L$ has a neighborhood homeomorphic to $M' \times I$.

COROLLARY 4.3. *Let M^n be a compact topological n-manifold, θ a* TRI *manifold structure on $M \times R$, N a codimension zero submanifold of ∂M, and Σ_0 a* TRI *manifold structure on M near N such that $\Sigma_0 \times R$ agrees with θ near $N \times R$. If $n \geqq 6$, then there is a topological s-cobordism* $(W; M, M')$ rel *a neighborhood of N in M, a* TRI *manifold structure Σ' on M' coinciding with Σ_0 near N, and a* TRI *manifold structure Σ on $W \times (-1, 1)$ with $\Sigma | M \times (-1, 1) = \theta | M \times (-1, 1)$ and $\Sigma | M' \times (-1, 1) = \Sigma' \times (-1, 1)$. Moreover, any such Σ determines a* TRI *manifold structure Γ on M coinciding with Σ_0 near N, unique up to concordance* rel Σ_0, *such that $\Gamma \times (-1, 1)$ is concordant* rel $\Sigma_0 \times (-1, 1)$ *to $\theta | M \times (-1, 1)$.*

Proof. Let Σ_0 be the TRI manifold structure on a neighborhood U of N which restricts to the TRI manifold structure on N, and P be a regular neighborhood of N in U. Now P is a homology manifold with $\operatorname{int} P$ a TRI manifold and with $\bar{P} = \operatorname{cl}(\partial P \cap \operatorname{int} M)$ PL bicollared. We can now use (1.7) to ambiently repair the top dimensional links of P in M so that we can assume that Σ_0 induces a TRI manifold structure on P. Now let $\bar{N} = \operatorname{cl}(\partial M - P)$, let $(\bar{W}; \bar{N}, N')$ be a TOP s-cobordism rel a neighborhood of ∂N in $\bar{N}$ given by applying (4.1) to $\bar{N}$, and let $\bar{M}$ be the result of attaching $\bar{W}$ to M along $\bar{N}$. Noting that $\bar{M}$ has a TRI manifold structure near $\partial \bar{M}$, we apply (4.1) to $\bar{M}$ to get a TOP s-cobordism $(W; \bar{M}, M')$ rel a neighborhood of $\partial \bar{M}$ in $\bar{M}$. By construction we can view W as a TOP s-cobordism between M and M' rel a neighborhood of N in M with a TRI manifold structure N' on M' coinciding with Σ_0 near N, and a TRI manifold structure Σ on $W \times (-1, 1)$ with

$$\Sigma | M \times (-1, 1) = \theta | M \times (-1, 1) \quad \text{and} \quad \Sigma | M' \times (-1, 1) = \Sigma' \times (-1, 1) .$$

Finally apply the s-cobordism theorem to W as in the end of the proof (4.1) to obtain the desired TRI structure Γ on M and note that the uniqueness of Γ up to concordance follows from relative existence as in 4.1. □

COROLLARY 4.4 (Product Structure Theorem). *Let M^q be a connected* *topological q-manifold and let θ be a* TRI *manifold structure on* $M \times R^n$. *Let* N *be codimension zero submanifold of* ∂M *and* Σ_0 *a* TRI *manifold structure on* M *near* N *such that* $\Sigma_0 \times R^n$ *agrees with* θ *near* $N \times R^n$. *If* $q \geqq 7$ ($q \geqq 6$ *if* $\mathrm{cl}(\partial M - N)$ *is compact or* $q \geqq 5$ *if* M *is closed*), *then there exists a* TRI *manifold structure* Γ *on* M *coinciding with* Σ_0 *near* N, *unique up to concordance* $\mathrm{rel}\,\Sigma_0$, *with* $\Gamma \times (-1, 1)^n$ *concordant* $\mathrm{rel}\,\Sigma_0 \times (-1, 1)^n$ *to* $\theta | M \times (-1, 1)^n$.

Proof. The theorem need only be proved for $n = 1$ for we can inductively reduce the $M \times R^n$ case to $M \times R$. We also assume for simplicity that $\partial M = \varnothing$, for the case when $\partial M \neq \varnothing$ follows similarly.

Filter M by codimension zero compact topological submanifolds $M_0 \subset M_0^+ \subset M_1 \subset M_1^+ \subset \cdots$ such that each $S_i = \mathrm{cl}(M_i^+ - M_i)$ is a compact codimension zero topological submanifold of M_i^+, $(S_i, \partial_- S_i)$ is homeomorphic to $(\partial_- S_i \times [0, 1], 0)$ where $\partial_- S_i = S_i \cap M_i$, and each $B_i = \mathrm{cl}(M_i - M_{i-1}^+)$ is a compact codimension zero submanifold of M_i. Such filtrations exist (Theorem 5.9.2 in Essay III of [19]) provided $n \geqq 6$. Call such a filtration a TOP filtration on M.

Next we construct a TOP filtration $M_0' \subset M_0^{+\prime} \subset M_1' \subset M_1^{+\prime} \subset \cdots$ on M and a TRI manifold structure θ' on $M \times R$ concordant to θ so that θ' restricts to a TRI manifold structure on each $\partial S_i' \times (-2, 2)$ where $S_i' = \mathrm{cl}(M_i^{+\prime} - M_i')$.

For each i, let $A_i = S_i \times [-3, 3] \subset M \times R$. Then ∂A_i is a closed topological manifold homeomorphic to $\partial_- S_i \times S^1$, and θ restricts to a TRI manifold structure on the interior of a small closed neighborhood C_i on ∂A_i in $M \times R$ homeomorphic to $\partial A_i \times [-1, 1]$, where ∂A_i corresponds to $\partial A_i \times 0$, $D_i = C_i \cap M \times 0 \subset M \times R$ is the disjoint union of D_i^- and D_i^+ where D_i^- and D_i^+ are small closed bicollared neighborhoods in M of $\partial_- S_i$ and $\partial_+ S_i$, respectively, $\partial_+ S_i = S_i \cap M_{i+1}^+$, and $C_i \cap (M \times (-2, 2)) = D_i \times (-2, 2) \subset M \times R$. By Remark 4.2 applied to $\mathrm{Int}\, C_i$, there exists a TOP s-cobordism $(W_i; C_i, C_i')$ with the following properties:

1) $X_i = (M \times R \times [0, 1]) \cup_{C_i \times 1} W_i$ has a TRI manifold structure Σ_i with $\Sigma_i | M \times R \times [0, 1] = \theta \times [0, 1]$ and Σ_i restricts to a TRI manifold structure on C_i', and

2) $\delta W_i = \partial W_i - \mathrm{Int}(C_i \cup C_i')$ is a TOP s-cobordism between ∂C_i and $\partial C_i'$.

Let $X = \bigcup_i X_i$ and Σ be the TRI structure on X induced from the Σ_i. By applying the TOP s-cobordism theorem first to δW_i and then to W_i, we get homeomorphisms $g_i \colon (W_i', C_i, C_i') \to C_i \times ([1, 2], 1, 2)$ with $g_i | C_i$ the identity. Therefore $g = (\bigcup_i g_i) \cup \mathrm{id} | M \times R \times [0, 1]$ is a homeomorphism from X to $Y = M \times R \times [0, 1] \cup (\cup_i C_i \times [1, 2])$ with $g | M \times R \times 0$ the identity.

By utilizing collar structures, it is easy to see that there exists a homeomorphism $h: Y \to M \times R \times [0, 1]$ such that

$$h | M \times R \times 0 = \mathrm{id}\,, \quad h | \mathrm{cl}\Big[Y - \Big(C_i \times \Big[\frac{1}{2}, 2\Big]\Big)\Big] = \mathrm{id}\,,$$

$$h\Big(C_i \times \Big[\frac{1}{2}, 2\Big]\Big) = C_i \times \Big[\frac{1}{2}, 1\Big] \quad \text{and,} \quad h(D_i \times t \times 2) = D_i \times t \times 1$$

for $-2 \leqq t \leqq 2$. Therefore $h \circ g(\Sigma)$ determines a concordance of $M \times (-2, 2)$ from $\theta|_{M \times (-2,2)}$ to a TRI manifold structure, say θ', and there is a TOP filtration $M_0' \subset M_0^{+\prime} \subset M_1' \subset M_1^{+\prime} \subset \cdots$ of M defined by $S_i' = \mathrm{cl}(M_i^{+\prime} - M_i') = h(D_i^- \times 2)$. Also from the above construction it easily follows that θ' restricts to a TRI manifold structure on $\partial S_i' \times (-2, 2)$.

To complete the proof we apply 4.2 inductively to the new filtration to obtain a TRI manifold Γ on M. The uniquencess of Γ, up to concordance, follows from relative existence as in 4.1. □

5. The classification theorem

Recall from Section 2 that a TRI cone bundle η/K with base a (finite dimensional) polyhedron $X = |K|$ is a topological disk block bundle over X and hence has an underlying microbundle which we denote by $\mathrm{mic}(\eta)/X$ [30]. Let ξ/X be a microbundle. A TRI *reduction of* ξ/X *to a* TRI *cone bundle*, also called a TRI *reduction* of ξ, is a TRI cone bundle ξ'/X such that $\mathrm{mic}(\xi')$ is identical to ξ/X as microbundles, i.e., $\mathrm{mic}(\xi')$ is *micro-identical* to $\mathrm{mic}(\xi)$. A *stable* TRI *reduction* of ξ is a TRI reduction η of $\xi \oplus \varepsilon^s$, $s > 0$, where $\varepsilon^s: X \to X \times R^s \to X$ is the standard trivial microbundle.

A TRI *concordance* between TRI reductions η_0 and η_1 of ξ is a TRI reduction $\eta/X \times I$ of $\xi \times I$ such that the reduction $\eta | X \times i = \eta_i \times i$ for $i = 0, 1$. If L is a subcomplex of K and U is a neighborhood of $C = |L|$ in X such that $\eta | U \times I = \eta_0 | U \times I$, then η is called a TRI *concordance* rel C. A *stable concordance* rel C of stable TRI reductions η_0 and η_1 of ξ is just a TRI concordance rel C between stabilizations η_0 and η_1.

If ξ_0 is a stable TRI reduction of $\xi | C$ which extends to a stable TRI reduction of $\xi | V$ for some neighborhood V of C in X we write

$$\mathrm{TOP/TRI}(\xi \,\mathrm{rel}\, C, \xi_0)$$

for the set of stable concordance classes rel C of stable TRI reductions ξ' of ξ that coincide near C with ξ_0 in the sense that $\xi' | U$ is stably the same as $\xi_0 | U$ for some neighborhood U of C.

By (2.4) there is a natural map $t\colon \mathrm{BTRI} \to \mathrm{BTOP}$, where BTOP classifies stable topological microbundles. We can, and do, assume that $t\colon \mathrm{BTRI} \to \mathrm{BTOP}$ is a Hurewicz fibration. Let ξ/X be a microbundle over a polyhedron $X = |K|$ and let $\xi\colon X \to \mathrm{BTOP}$ classify ξ/X. Let ξ_0 be a TRI reduction of $\xi\,|\,C$, C a subpolyhedron of X, which extends to a TRI reduction of $\xi\,|\,U$, U some neighborhood of C in X, and let $\xi_0\colon U \to \mathrm{BTRI}$ classify $\xi_0\,|\,U$. We then have $t\xi_0 = \xi\,|\,U$. Define

$$\mathrm{Lift}(\xi \operatorname{rel} C,\, \xi_0)$$

to be the set of liftings of ξ to BTRI through $t\colon \mathrm{BTRI} \to \mathrm{BTOP}$ that equal ξ_0 near C, modulo the equivalence relation of vertical homotopy rel C.

By the homotopy lifting property we have

PROPOSITION 5.1. *There is a one-to-one correspondence between the elements of* $\mathrm{TOP/TRI}(\xi \operatorname{rel} C,\, \xi_0)$ *and* $\mathrm{Lift}(\xi \operatorname{rel} C,\, \xi_0)$.

Remark. As X is finite dimensional $\xi\colon \mathrm{BTOP} = \lim_{n\to\infty} \mathrm{BTOP}(n)$ factors through a classifying map $f\colon X \to \mathrm{BTOP}(n)$. Let γ^n denote the universal topological microbundle over $\mathrm{BTOP}(n)$. Technically, the homotopy lifting property only establishes a one-to-one correspondence between the elements of $\mathrm{TOP/TRI}\big(f^*(\gamma^n) \operatorname{rel} C,\, \xi_0\big)$ and $\mathrm{Lift}(\xi \operatorname{rel} C,\, \xi_0)$, so (5.1) is true only when a *specific* stable micro-isomorphism between $f^*(\gamma^n)$ and ξ is given. However, we choose to ignore this technical point as in all our uses of (5.1) such a choice will be obvious.

Let M be a topological m-manifold and let $\mathcal{S}_{\mathrm{TRI}}(M)$ denote the set of TRI concordance classes of TRI manifold structures on M. If Σ_0 is a TRI manifold structure on M near a codimension zero submanifold N of ∂M, let $\mathcal{S}_{\mathrm{TRI}}(M \operatorname{rel} N,\, \Sigma_0)$ denote the set of TRI concordance classes rel Σ_0 of TRI manifold structures on M.

We now fix a proper topological embedding of M in a Euclidean space R^q_+, $q \gg m$, that is a PL embedding near N, and fix a PL manifold neighborhood Q of M in R^q_+ that admits a deformation retraction $r\colon Q \to M$ with r a PL map to Σ_0 on the preimage of a neighborhood of N in M. Let $\tau(M)$ denote the topological tangent microbundle of M.

PROPOSITION 5.2. *If M has a* TRI *manifold structure Σ then $r^*\tau(M)$ has a* TRI *reduction.*

Proof. Homotope r (rel a neighborhood of N) to a PL map $r_1\colon Q \to M$ and note that by the microbundle homotopy theorem $r_1^*\tau(M)$ is micro-isomorphic to $r^*\tau(M)$. Since $\tau(M)$ is the microbundle $M \xrightarrow{\Delta} M \times M \xrightarrow{\pi} M$ with $\Delta(x) = (x, x)$ and $\pi(x, y) = x$, the induced microbundle $r'^*\tau(M)$ has a total space $Er_1^*\tau(M) = \{(y, r_1(y), x) \mid \in Q \times M \times Q\}$ with projection $(y, r_1(y), x) \to y$ and zero section $y \to (y, r_1(y), r_1(y))$. Now Q is a subpolyhedron of $E(r_1^*(M)) \approx Q \times M$. Assume $\partial M = \emptyset$. Triangulate Q and $Q \times M$ so that Q is a full subcomplex. Now $r_1^*(M)$ has a natural TRI cone bundle structure over the dual cell complex of Q by assigning to every dual cell of a simplex σ in Q the dual cell D_σ of σ in $Q \times M$. Note that

$$D_\sigma \approx S^{q-\dim\sigma-1} * D(\sigma, M) = \Sigma^{q-\dim\sigma} D(\sigma, M)$$

is topologically homeomorphic to an appropriate dimensional PL ball if q is sufficiently large. Thus $r_1^*\tau(M)$, hence $r^*\tau(M)$, has a TRI reduction.

Note that if $\partial M \neq \emptyset$ we alter the above construction of a TRI cone bundle structure on $r_1^*\tau(M)$ slightly as follows. The construction is unaltered for simplices of M not in ∂M. Let $\sigma \in \partial M$. We replace $D(\sigma, Q)$ by $\tilde{D}(\sigma, Q)$ with $D(\sigma, Q) = \tilde{D}(\sigma, Q)$ as a subset of Q, and $\partial D(\sigma, Q) = \partial D(\sigma, Q) \cup \tilde{D}(\sigma, \partial Q)$. Now push the cone point of $D(\sigma, Q)$ into $Q - \partial Q$ using a collar of ∂Q to realize $\tilde{D}(\sigma, Q)$ as a cone on $\partial\tilde{D}(\sigma, M)$, so that these new cells form a PL cell complex on Q. A similar construction applied to the $\tilde{D}_\sigma$ yields the desired TRI cone bundle. □

With the data preceding (5.2) we now have that $r^*\tau(M)$ restricted to a neighborhood of N in Q has a TRI reduction that we call ξ_0. Also if we have a given TRI manifold structure Σ on M coinciding with Σ_0 near N the proof of (5.2) again shows that $r^*\tau(M)$ admits a TRI reduction ξ rel ξ_0. The construction $\Sigma \to \xi$ determines a well-defined map

$$\pi\colon \mathcal{S}_{\mathrm{TRI}}(M \operatorname{rel} N, \Sigma_0) \longrightarrow \mathrm{TOP/TRI}\left(r^*\tau(M) \operatorname{rel} N, \xi_0\right).$$

Suppose now that $r^*\tau(M)$ admits a stable TRI reduction rel ξ_0. Then $E(r^*\tau(M) \oplus \varepsilon^s)$ admits a TRI manifold structure near the zero section. However, there is an open embedding $h\colon E(r^*\tau(M) \oplus \Sigma^s) \to M \times R^t$, for some t, that sends the zero section $\Delta(M)$ of $\tau(M)$ to $M \times 0 \subset M \times R^t$ and that gives a PL embedding of a neighborhood of $\Delta(N)$ in $M \times R^t$ (cf. [19], [30]). We thus have a TRI manifold structure θ on $M \times R^t$ near $M \times 0$ that coincides with $\Sigma_0 \times R^t$ near $N \times 0$. Then (4.4) produces a concordance rel $\Sigma_0 \times (-1, 1)^t$ of $\theta \mid M \times (-1, 1)^t$ to a TRI manifold structure that is a product of the form $\Sigma \times (-1, 1)^t$. This construction $\xi \to \Sigma$ determines a well-defined map

$$\sigma\colon \mathrm{TOP/TRI}\left(r^*\tau(M) \operatorname{rel} N, \xi_0\right) \longrightarrow \mathcal{S}_{\mathrm{TRI}}(M \operatorname{rel} N, \xi_0).$$

THEOREM 5.3. *If $m \geqq 7$ ($m \geqq 6$ if $\mathrm{cl}(\partial M - N)$ is compact or $m \geqq 5$ if M is closed), then σ is bijective.*

Proof. *σ is onto.* To illustrate this one shows $\sigma\pi$ = identity, but this is just a chase of definitions which we leave to the reader.

σ is injective. Let ξ_1, ξ_2 be two stable TRI reductions of $r^*\tau(M)\oplus\Sigma^s$, $s \geqq 0$, agreeing near N in the total space with the reductions determined by $\Sigma_0 \times R^s$. Suppose $\sigma(\xi_1) = \sigma(\xi_2)$. We want to show that ξ_1 and ξ_2 are stably concordant rel N. As $\sigma(\xi_1) = \sigma(\xi_2)$, we get a TRI concordance Γ rel N from a TRI manifold structure on an open neighborhood U of M as a subset of $E(\xi_1)$ to a TRI manifold structure on U as a subset of $E(\xi_2)$. Let $Q' = U \cap Q$ and let ξ be a TRI cone bundle neighborhood of Q' in U as constructed in (5.2). This yields a stable TRI concordance rel N of $\xi_1 | Q'$ to $\xi_2 | Q'$. By pulling Q into Q' via a PL homotopy and applying the (relative) microbundle homotopy theorem we get the desired concordance from $\xi_1 | Q$ to $\xi_2 | Q$. □

Combining (5.1) and (5.3) we have

CLASSIFICATION THEOREM 5.4. *Let M be a topological m-manifold, Σ_0 a* TRI *manifold structure on M near a codimension zero submanifold N of ∂M, τ_0: $U \to$* BTRI *the map classifying the stable* TRI *cone bundle structure on $\tau(M) | U$, U a neighborhood of N in M, determined by Σ_0, and let $\tau: M \to$* BTOP *classify $\tau(M)$ such that $t\tau_0 = \tau$ near N.*

If $m \geqq 7$ ($m \geqq 6$ if $\mathrm{cl}(\partial M - N)$ is compact or $m \geqq 5$ if M is closed) then M admits a TRI *manifold structure Σ coinciding with Σ_0 near N if and only if τ has a lifting $M \to$* BTRI *equal to τ_0 near N. In fact there is a bijection*

$$\mathcal{S}_{\mathrm{TRI}}(M \operatorname{rel} N, \Sigma_0) \longrightarrow \mathrm{Lift}(\tau \operatorname{rel} N, \tau_0) .$$

6. The fiber of t: BTRI → BTOP

Consider the classifying spaces BTRI, BTOP (cf. §2), and BPL (the classifying space for stable PL microbundles). Then there is a homotopy commutative diagram

$$(6.1)\qquad \begin{array}{ccc} \mathrm{BPL} & \xrightarrow{\ i\ } & \mathrm{BTRI} \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle t} \\ & & \mathrm{BTOP} \end{array}$$

where i, j, and t are the forgetful maps. The (homotopy-theoretic) fiber of j is a $K(Z_2, 3)$ (cf. [19]) and the fiber of i is a $K(\theta_3^H, 3)$ (cf. [20]).

THEOREM 6.2. *The fiber* TOP/TRI *of t:* BTRI → BTOP *is a K*(kernel $(\alpha: \theta_3^H \to Z_2)$, 4).

Proof. Since the fibers of i and j are a $K(\theta_3^H, 3)$ and $K(Z_2, 3)$, respectively, the homotopy exact sequence of the triple (BTOP, BTRI, BPL) implies that all the homotopy groups of TOP/TRI are zero except possible in dimensions 3 and 4, in which case we have the exact sequence

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \pi_4(\mathrm{TOP/TRI}) & \xrightarrow{t} & \pi_3(\mathrm{TRI/PL}) & \xrightarrow{a} & \pi_3(\mathrm{TOP/PL}) & \longrightarrow & \pi_3(\mathrm{TOP/TRI}) & \longrightarrow & 0 \\
 & & \| & & \wr\| & & \wr\| & & \| & & \\
0 & \longrightarrow & \pi_4(\mathrm{TOP/TRI}) & \longrightarrow & \theta_3^H & \xrightarrow{\alpha} & Z_2 & \longrightarrow & \pi_3(\mathrm{TOP/TRI}) & \longrightarrow & 0\,.
\end{array}$$

The result follows once we identify a with the Kervaire-Milnor-Rochlin homomorphism $\alpha\colon \theta_3^H \to Z_2$. To do this we recall the construction of an isomorphism $e\colon \theta_3^H \to \pi_3(\mathrm{TRI/PL})$ from [20] and an isomorphism $d\colon \pi_3(\mathrm{TOP/PL}) \to Z_2$ from [19] and show that the following diagram commutes:

$$\begin{array}{ccc}
\pi_3(\mathrm{TRI/PL}) & \xrightarrow{t} & \pi_3(\mathrm{TOP/PL}) \\
\uparrow{\scriptstyle e} & & \downarrow{\scriptstyle d} \\
\theta_3^H & \xrightarrow{\alpha} & Z_2
\end{array}\quad .$$

The isomorphism e is constructed as follows (cf. [20]): Let H^3 be a PL homology 3-sphere representing an element of θ_3^H and remove a PL 3-disk from the boundary of $c(H^3)$. Now consider $c(H^4)$ as a space over $\Delta^3 \times I$ such that over $(\Delta^3 \times \{0, 1\}) \cup (\Delta^2 \times I)$ it is the trivial point bundle. But $c(H^3) \times S^q$, for $q \geqq 1$, is a topological manifold by the double suspension theorem [5] and $c(H^3) \times S^q$ is a space over $\Delta^3 \times I$ which is the product bundle over $(\Delta^3 \times \{0, 1\}) \cup (\Delta^2 \times I)$. By coning these blocks we get a representative for an element $e(H^3) \in \pi_3(\mathrm{TRI/PL})$. Now $dte(H^3)$ is precisely the obstruction to putting a PL manifold structure on $c(H^4) \times S^q$ extending the natural PL manifold structure on $H^3 \times S^q$. But by Theorem C of [34], $c(H^3) \times S^q$ has such a PL manifold structure if and only if $\alpha(H^3) = 0$. Thus $dte(H^3) = \alpha(H^3)$.

□

Theorems 5.4 and 6.1 now establish Corollary 3 of the introduction.

COROLLARY 6.3. $\pi_4(\mathrm{TOP/TRI}) = 0$ *if and only if* $\theta_3^H = Z_2$.

For a given topological m-manifold M, $m \geqq 7$ ($m \geqq 6$ if ∂M compact or $m \geqq 5$ if M closed), we can now observe that the one obstruction to putting a TRI manifold structure on M is a cohomology class $\nabla(M) \in H^5(M; \ker(\alpha\colon \theta_3^H \to Z_2))$, namely the one obstruction to lifting the classifying map $\tau\colon M \to \mathrm{BTOP}$ for $\tau(M)$ to BTRI. The obstruction to sectioning the fibration

$$K(\ker(\alpha), 4) \longrightarrow \mathrm{BTRI} \xrightarrow{t} \mathrm{BTOP}$$

is the *universal* triangulation obstruction $\nabla \in H^5(\mathrm{BTOP}; \ker(\alpha))$ and $\nabla(M) = \tau^*(\nabla)$ by naturality of obstructions. Let $\Delta \in H^4(\mathrm{BTOP}; Z_2)$ denote the universal obstruction to combinatorially triangulating topological n-manifolds, $n \geqq 6$ ($m \geqq 5$ if ∂M compact) i.e., the obstruction to a section of the fibration

$$K(Z_2, 3) \longrightarrow \text{BPL} \longrightarrow \text{BTOP} .$$

Also, let $\beta: H^4(\text{BTOP}; Z_2) \to H^5(\text{BTOP}; \ker(\alpha))$ be the Bockstein homomorphism associated with the short exact coefficient sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \theta_3^H \xrightarrow{\alpha} Z_2 \longrightarrow 0$$

As (6.1) commutes, standard obstruction theory (see for instance § 38.8 of [36]) implies

COROLLARY 6.4. $\beta(\Delta) = \nabla$.

Remark 6.5. It follows from Siebenmann [34] that all oriented topological 5-manifolds without boundary can be triangulated as simplicial complexes. But it is *not* known if there exists an unorientable closed 5-manifold M such that M is not combinatorially triangulable but is simplicially triangulable. In fact, if such an example exists, then, for instance, *all* closed topological n-manifolds $n \geqq 5$ can be simplicially triangulated! (See [18] for a discussion of this phenomenon.)

7. Necessary and sufficient conditions for the existence of simplicial triangulations of topological manifolds

THEOREM 7.1. *Every topological m-manifold M, $m \geqq 7$ ($m \geqq 6$ if ∂M compact or $m \geqq 5$ if $\partial M = \varnothing$), can be triangulated as a simplicial complex if and only if there exists a* PL *homology 3-sphere H^3 such that*

(i) *H^3 bounds a parallelizable* PL *4-manifold with index* 8, i.e., $\alpha(H^3) = 1$, *and*

(ii) *$H^3 \# H^3$ bounds an acyclic* PL *4-manifold.*

Proof. Necessity. Let M be a closed topological m-manifold, $m \geqq 5$, and let $\Delta(M) \in H^4(M; Z_2)$ denote the Kirby-Siebenmann obstruction to putting a PL manifold structure on M. There exists a closed topological 5-manifold with $\text{Sq}^1 \Delta(M) \neq 0$ (see [18] for an explicit construction), where $\text{Sq}^1: H^4(M; Z_2) \to H^5(M; Z_2)$ is the Steenrod squaring operation, i.e., Sq^1 is the Bockstein homomorphism associated with the short exact coefficient sequence $0 \to Z_2 \xrightarrow{2} Z_4 \to Z_2 \to 0$. Suppose that M has a simplicial triangulation and let θ denote the finitely generated subgroup of θ_3^H generated by the 3-dimensional links of M. Let $i: \theta \to \theta_3^H$ denote the inclusion homomorphism. Suppose that every element $[H^3]$ of θ with $\alpha(H^3) = 1$ does not have order 2 in θ_3^H. We construct a homomorphism $\gamma: \theta \to Z_4$ which commutes in the diagram

$$\begin{array}{ccccccccc} & & & & \theta & \longrightarrow & \theta_3^H & & \\ & & & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & Z_4 & \xrightarrow{\times 2} & Z_4 & \xrightarrow{r} & Z_2 & \longrightarrow & 0 \end{array}$$

where $r: Z_4 \to Z_2$ is reduction mod 2, as follows. There exist elements $H_1, \cdots, H_r$ of θ such that $\theta = [H_1] \oplus \cdots \oplus [H_r]$ where $[H_i]$ is the cyclic subgroup of θ generated by H_i. If $\alpha(H_i) = 0$, let $\gamma(H_i) = 0$. If $\alpha(H_i) = 1$ and $[H_i] \cong Z$, let γ be given by reduction mod 4. If $\alpha(H_i) = 1$ and $[H_i] \cong Z/p^{2^j}Z$ where, by assumption, $p^{2^j} \geqq 4$, then there is an element H of $[H_i]$ with $\alpha(H) = 1$ and of order p. Thus p must be even, and so $p^{2^j} = 4q$ for some q. Then let γ be given again by reduction mod 4.

Now let $\sigma(M) \in H^4(M; \theta_3^H)$ be the obstruction to PL resolving M to a PL manifold (cf. Section 3). Indeed, there is an element $\tilde{\sigma}(M) \in H^4(M; \theta)$ with $i^*\tilde{\sigma}(M) = \sigma(M)$. Thus

$$\mathrm{Sq}^1\alpha^*\sigma(M) = \mathrm{Sq}^1\alpha^* i^*\tilde{\sigma}(M) = \mathrm{Sq}^1 r^*\gamma^*\big(\tilde{\sigma}(M)\big) = 0\ ,$$

since $\mathrm{Sq}^1 r^* = 0$. But by (6.2), $\alpha^*\sigma(M) = \Delta(M)$, so that $\mathrm{Sq}^1\Delta(M) = 0$, a contradiction. Thus there must exist a PL homology 3-sphere H^3 with $\alpha(H^3) = 1$ and $H^3 \# H^3$ bounding an acyclic PL 4-manifold.

Sufficiency. If there exists a PL homology 3-sphere satisfying (i) and (ii), then we have a *split* exact sequence

$$0 \longrightarrow \pi_4(\mathrm{TOP}/\mathrm{TRI}) \longrightarrow \theta_3^H \underset{\longleftarrow}{\overset{\alpha}{\longrightarrow}} Z_2 \longrightarrow 0$$

so that the associated Bockstein $\beta: H^4(M; Z_2) \to H^5\big(M; \ker(\alpha)\big)$ is the zero homomorphism. The result now follows from (6.4) provided M is not an open 5-manifold. However, Siebenmann [34] along with the double suspension theorem [5] implies that all open 5-manifolds can be simplicially triangulated. □

Remark 7.2. Let M be a topological manifold satisfying the hypothesis of (7.1) and let H^3 be a PL homology 3-sphere satisfying (i) and (ii) of (7.1). Then one can actually show that M can be triangulated so that its 3-dimensional homology sphere links are PL homeomorphic to connected sums of H^3, $-H^3$ and S^3 as follows. If in Section 2 we *further* required in the definition of a TRI cone bundle that all the blocks be cones on TRI manifolds whose 3-dimensional homology sphere links are PL homeomorphic to connected sums of H^3, $-H^3$, and S^3, we can construct a classifying $\mathrm{BTRI}(H^3)$ for such bundles. There are natural forgetful maps $k: \mathrm{BPL} \to \mathrm{BTRI}(H^3)$ and $l: \mathrm{BTRI}(H^3) \to \mathrm{BTOP}$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{BTRI}(H^3) \\ & \overset{k}{\nearrow} & \downarrow l \\ \mathrm{BPL} & \xrightarrow{j} & \mathrm{BTOP} \end{array}$$

commutes. The fiber for k can be shown to be a $K(Z_2, 3)$, so that the fiber of l is contractible. Thus $M \times R^n$, for some n, has a TRI manifold structure such that each of its 3-dimensional homology sphere links is PL homeomorphic to connected sums of H_1^3, $-H^3$, or S^3. One then checks that (4.4) does not destroy this property.

Our proof of (7.1) and Remark (7.2) allows us to (seemingly) strengthen its statement. In particular, let

$$\mathrm{Sq}_k\colon H^4(\ ;Z_2) \longrightarrow H^5(\ ;Z_k)$$

denote the Bockstein homomorphism associated with the short exact coefficient sequence

$$0 \longrightarrow Z_k \xrightarrow{\times 2} Z_{2k} \xrightarrow{r} Z_2 \longrightarrow 0$$

where r is reduction mod 2. Note that $\mathrm{Sq}^1 = \mathrm{Sq}_2$.

COROLLARY 7.3. *Let M be a closed simplicially triangulated m-manifold, $m \geqq 5$, such that* $\mathrm{Sq}_{2k}\Delta(M) \neq 0$. *Then there exists a* PL *homology 3-sphere H^3 such that*

(i) $\alpha(H^3) = 1$, *and*

(ii) *the $2k$-fold connected sum of H^3 bounds a* PL *acyclic 4-manifold. Furthermore, if there exists a* PL *homology 3-sphere satisfying* (i) *and* (ii) *above, then every topological m-manifold M, $m \geqq 7$ ($m \geqq 6$ if ∂M compact or $m \geqq 5$ if $\partial M = \emptyset$) with* $\mathrm{Sq}_k\Delta(M) = 0$, *can be triangulated as a simplicial complex.*

COROLLARY 7.4. *If M is a topological m-manifold as in* (7.1) *and the integral Bockstein of $\Delta(M)$ is zero, then M can be triangulated as a simplicial complex.*

8. Triangulating topological manifolds as homotopy manifolds

In this final section we note that the work of Sections 1–7 can be modified to classify triangulations of topological manifolds as polyhedral homotopy manifolds (see Remark (7.2)). In fact this classification can be accomplished by much easier geometry!

A (polyhedral) *homotopy manifold* of dimension n is a polyhedron P such that there is a triangulation K of P in which given any k-simplex α in K, $lk(\alpha, K)$ is homotopy equivalent to either S^{n-k-1} or a point. The set of simplicies with the latter property form a subpolyhedron which is called the *boundary* of the manifold, denoted ∂P, and we insist further that P be a homotopy $(n-1)$-manifold without boundary. As usual, this definition is independent of any triangulation of P chosen.

L. Siebenmann [34] has shown that every homotopy m-manifold is a topological manifold if $m \geqq 5$ ($\geqq 6$ if $\partial M \neq \emptyset$) (see Theorem 1.5). M. Cohen [6] has a theory for resolving a homotopy manifold M via a PL contractible map to a PL manifold and the obstruction lies in $H^4(M; \theta_3^h)$, where θ_3^h is the Kervaire-Milnor group of homotopy 3-spheres. Following [22] and Section 2 one can define a theory of hTRI-cone bundles which are topological disk block bundles in which all the blocks are triangulated as homotopy manifolds, and one can construct the resulting stable classifying space BhTRI. There is a (homotopy) commutative diagram of forgetful maps

$$\begin{array}{ccc} \mathrm{BPL} & \xrightarrow{\;i\;} & \mathrm{B}h\mathrm{TRI} \\ \downarrow & \swarrow{\scriptstyle t'} & \downarrow{\scriptstyle j} \\ \mathrm{BTOP} & \longrightarrow & \mathrm{BTRI} \end{array}$$

with the fiber hTRI/PL of i being a $K(\theta_3^h, 3)$.

The proofs of the results of Sections 2–5 hold equally well for homotopy manifolds. In fact, the proofs can be *immensely* simplified by observing that every homotopy m-manifold satisfies the $ND(m)$ condition of Stallings [35], so has all the general position properties of a PL m-manifold. With the obvious definitions, we then have the following results.

CLASSIFICATION THEOREM 8.1. *Let M be a topological m-manifold, Σ_0 a homotopy manifold structure on M near a codimension zero submanifold N of M, τ_0: $U \to$* BhTRI *the map classifying the structure on $\tau(M)|U$, U a neighborhood of N in M, determined by Σ_0, and let τ: $M \to$* BTOP *classify $\tau(M)$ such that $t'\tau_0 = \tau$ near N.*

Suppose $m \geqq 7$ ($m \geqq 6$ if $\mathrm{cl}(\partial M - N)$ compact or $m \geqq 5$ if $\partial M = N$ and M compact). Then M admits a homotopy manifold structure Σ coinciding with Σ_0 near N if and only if τ has a lifting $M \to$ BhTRI *equal to τ_0 near N. In fact there is a bijection*

$$\mathcal{S}_{h\mathrm{TRI}}(M \,\mathrm{rel}\, N, \Sigma_0) \longrightarrow \mathrm{Lift}(\tau \,\mathrm{rel}\, N, \tau_0)\ .$$

THEOREM 8.2. *The homotopy groups of* TOP/hTRI *of t':* BhTRI $\to$ BTOP *are zero except possibly for π_3 and π_4. Furthermore there is an exact sequence*

$$0 \longrightarrow \pi_4 \longrightarrow \theta_3^h \xrightarrow{\;\alpha\;} Z_2 \longrightarrow \pi_3 \longrightarrow 0$$

where α is the Kervaire-Milnor-Rochlin map.

See [21] for a result related to (9.2).

COROLLARY 8.3. *Every topological m-manifold M, $m \geqq 7$ ($m \geqq 6$ if ∂M compact or $m \geqq 5$ if $\partial M = \emptyset$), has a homotopy manifold structure if and only if there exists a* PL *homotopy 3-sphere with $\alpha(H^3) = 1$ and $H^3 \# H^3$ bounding a contractible* PL *4-manifold.*

Note that if $\theta_3^h = 0$, then $i: \mathrm{BPL} \to \mathrm{B}h\mathrm{TRI}$ is a homotopy equivalence, so we have

COROLLARY 8.4. *Suppose that every* PL *homotopy 3-sphere bounds a contractible* PL *4-manifold,* i.e., $\theta_3^h = 0$. *Then there exists a one-to-one correspondence between the set of concordance classes of homotopy manifold structures of a given topological m-manifold, $n \geqq 7$ ($m \geqq 6$ if ∂M compact or $m \geqq 5$ if $\partial M = \emptyset$), and concordance classes of* PL *manifold structures on M.*

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BIBLIOGRAPHY

[1] F. ANCEL and J. CANNON, The locally flat approximation of cell-like embedding relations, Ann. of Math. **109** (1979), 61-86.

[2] W. BROWDER, Structures on $M \times R$, Proc. Camb. Phil. Soc. **61** (1965), 337-345.

[3] M. BROWN, A proof of the generalized Schoenflies theorem, Bull. A.M.S. **66** (1960), 74-76.

[4] J.L. BRYANT, R.D. EDWARDS, C.L. SEEBECK III, Approximating codimension one submanifolds with 1-LC embeddings, Notices A.M.S. **20** (1973), A-28, Abstract 73T-G17.

[5] J. CANNON, Shrinking cell-like decompositions of manifolds: Codimension three, Ann. of Math. **110** (1979), 83-112.

[6] M. COHEN, Homeomorphisms between homotopy manifolds and their resolutions, Inv. Math. **10** (1970), 239-250.

[7] M. COHEN and D. SULLIVAN, On the regular neighborhood of a two-sided submanifold, Topology **9** (1970), 141-147.

[8] R. DAVERMAN, Locally nice codimension one manifolds are locally flat, Bull. A.M.S. **79** (1973), 410-413.

[9] A. EDMONDS and R. STERN, Resolutions of homology manifolds: A classification theorem, J. London Math. Soc. (2), **11** (1975), 474-480.

[10] R.D. EDWARDS, The double suspension of a certain homology 3-sphere is S^5, Notices A.M.S. **22** (1975), A-334.

[11] ———, Approximating certain cell-like maps by homeomorphisms, preprint.

[12] ———, Suspensions of homology spheres, to appear in Ann. of Math.

[13] R.D. EDWARDS and M.G. SCHARLEMANN, A remark on suspensions of homology spheres, Notices A.M.S. **21** (1974), A-324.

[14] D. GALEWSKI and J. HOLLINGSWORTH, PL acyclic maps on manifolds, preprint.

[15] D. GALEWSKI and R. STERN, The relationship between homology and topological manifolds via homology transversality, Inv. Math. **39** (1977), 277-292.

[16] ———, Classification of simplicial triangulations of topological manifolds, Bull. A.M.S. **82** (1976), 916-918.

[17] D. GALEWSKI and R. STERN, Simplicial triangulations of topological manifolds, A. M. S. Proc. Symposia in Pure Math. **32** (1977), 7-12.

[18] ———, A universal 5-manifold with respect to simplicial triangulations, to appear in Proc. Georgia Topology Conf., 1977.

[19] R. KIRBY and L. SIEBENMANN, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Studies No. 88, Princeton University Press, 1977.

[20] N. MARTIN, On the difference between homology and piecewise linear block bundles, J. London Math. Soc. **6** (1973), 197-204.

[21] ———, Locally flat embeddings of homotopy manifolds, Quart. J. Math. Oxford (2), **27** (1976), 59-67.

[22] N. MARTIN and C. MAUNDER, Homology cobordism bundles, Topology **10** (1971), 93-110.

[23] A. MATSUI, Surgery on 1-connected homology manifolds, Tohoku Math. J. **25** (1974), 169-172.

[24] T. MATUMOTO, Triangulations of manifolds, A.M.S. Proc. Symposia in Pure Math. **32** (1977), 3-6.

[25] T. MATSUMOTO and Y. MATSUMOTO, The unstable difference between homology cobordism and piecewise linear block bundles, Tohoku Math. J. (2), **27** (1975), 000-000.

[26] C. R. F. MAUNDER, General position theorems for homology manifolds, J. London Math. Soc. (2) **4** (1972), 760-768.

[27] ———, An H-cobordism theorem for homology manifolds, Proc. London Math. Soc. (3) **25** (1972), 137-155.

[28] ———, Surgery on homology manifolds, I: The absolute case, Proc. London Math. Soc. (3) **32** (1976), 480-520.

[29] ———, Improving general position properties of homology manifolds, Proc. London Math. Soc. (3) **25** (1972), 689-700.

[30] J. MILNOR, "Microbundles I," Topology **3** suppl. 1 (1964), 53-80.

[31] C. P. ROURKE and B. J. SANDERSON, On topological neighborhoods, Composition Math. **22** (1970), 387-424.

[32] H. SATO, Constructing manifolds by homotopy equivalences I, Ann. Inst. Fourier, Grenoble, **22** (1972), 271-286.

[33] M. SCHARLEMAN, Simplicial triangulations of non-combinatorial manifolds of dimension less than 9, Trans. A. M. S. **219** (1976), 269-287.

[34] L. SIEBENMANN, Are non-triangulable manifolds triangulable?, pp. 77-84 in *Topology of Manifolds*, ed. by J. C. Cantrell and C. H. Edwards, Markham, Chicago, 1970.

[35] J. STALLINGS, *Lectures on Polyhedral Topology*, Tata Institute of Fundamental Research, Bombay, 1968.

[36] N. STEENROD, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, New Jersey, 1951.

[37] R. STERN, PL homology 3-spheres and triangulations of manifolds, Proc. LSU topology conference, Vol. 2, No. 2 (1977), 621-630.

[38] D. SULLIVAN, Singularities in spaces, Liverpool Symposium, Springer Lecture Notes **209** (1971), 196-206.

[39] R. SUMMERHILL, General position for compact subsets of Euclidean space, General Topology **3** (1973), 339-345.

[40] C. T. C. WALL, On simply-connected 4-manifolds, Proc. London Math. Soc. **39** (1964), 141-149.

[41] ———, *Surgery on Compact Manifolds*, London Math. Soc. Monograph #1, Academic Press, 1970.

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