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CLASSIFICATION OF SIMPLICIAL TRIANGULATIONS OF TOPOLOGICAL MANIFOLDS

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In this note we announce theorems which classify simplicial (not necessarily combinatorial) triangulations of a given topological n-manifold M, $n \geqslant 7$ ($\geqslant 6$ if $\partial M = \varnothing$), in terms of homotopy classes of lifts of the classifying map $\tau: M \longrightarrow BTOP$ for the stable topological tangent bundle of M to a classifying space $BTRI_n$ which we introduce below. The (homotopic) fiber of the natural map $j: BTRI_n \longrightarrow BTOP$ is described in terms of certain groups of PL homology 3-spheres. We also give necessary and sufficient conditions for a closed topological n-manifold M, $n \geqslant 6$, to possess a simplicial triangulation.

The proofs of these results incorporate recent geometric results of F. Ancel and J. Cannon [1], J. Cannon [2], R. D. Edwards [4], and D. Galewski and R. Stern [5].

In [8], R. Kirby and L. Siebenmann show that in each dimension greater than four there exist closed topological manifolds which admit no piecewise linear manifold structure and hence cannot be triangulated as a combinatorial manifold. Also, R. D. Edwards [3] has recently shown that the double suspension of the Mazer homology 3-sphere is homeomorphic to S^5, thus showing that a simplicial triangulation of a topological manifold *need not* be combinatorial. But it is still unknown whether or not every topological manifold can be triangulated as a simplical complex.

Our classification theorems for simplicial triangulations on a given topological manifold take the following forms:

Let $BTOP$ denote the classifying space for stable topological block bundles.

THEOREM 1. *There is a space $BTRI_n$ and a natural map $BTRI_n \longrightarrow BTOP$ such that if M is a topological n-manifold, $n \geqslant 7$ ($\geqslant 6$ if $\partial M = \varnothing$) and $\tau: M \longrightarrow BTOP$ classifies the stable topological tangent bundle of M, then there is a one-to-one correspondence between the set of concordance classes of simplicial triangulations of M and the set of vertical homotopy classes of lifts of τ to $BTRI_n$.*

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The obvious relative versions of Theorem 1 also hold true.

THEOREM 2. *The fiber TOP/TRI_n of $BTRI_n \longrightarrow BTOP$ has only two non-zero homotopy groups, namely π_3 and π_4, and the following sequence is exact.*

$$0 \longrightarrow \pi_4 \longrightarrow \ker(\alpha\colon \theta_3^H \longrightarrow Z_2) \longrightarrow \theta_3^{TRI_n} \longrightarrow \pi_3 \longrightarrow 0.$$

Here θ_3^H denotes the group of PL homology 3-spheres, modulo those which bound acyclic PL 4-manifolds, under the operation of connected sum; $\alpha\colon \theta_3^H \longrightarrow Z_2$ is the Kervaire-Milnor-Rochlin map $\alpha(H^3) = I(H^3)/8 \bmod 2$, where $I(H^3)$ is the index of a parallelizable PL 4-manifold that H^3 bounds; and $\theta_3^{TRI_n}$ is the group of PL homology 3-spheres modulo those which bound acyclic homology 4-manifolds W with $W \times R^{n-4}$ a topological manifold, under the operation of connected sum. Note that if $\Sigma^{n-3}\, H^3$ is homeomorphic to S^n, then H^3 represents the zero element of $\theta_3^{TRI_n}$.

THEOREM 3. (i) $\pi_3(TOP/TRI_n) \subseteq Z_2$,

(ii) $\pi_3(TOP/TRI_n) = 0$ *if and only if there exists a PL homology 3-sphere H^3 with $\alpha(H^3) = 1$ and the $(n-3)$-suspension of H^3, $\Sigma^{n-3}\, H^3$, is homeomorphic to S^n.*

(iii) $\pi_4(TOP/TRI_n) = 0$ *if and only if every PL homology 3-sphere H^3 with $\alpha(H^3) = 0$ and which bounds an acyclic homology 4-manifold W with $W \times R^{n-4}$ a topological manifold, bounds an acyclic PL 4-manifold.*

THEOREM 4. *There exists a PL homology 3-sphere H^3 such that*

(i) $\alpha(H^3) = 1$,

(ii) $H^3 \# H^3$ *bounds an acyclic PL 4-manifold, and*

(iii) $\Sigma^{n-3}\, H^3$ *is homeomorphic to S^n.*

If and only if every closed topological n-manifold, $n \geqslant 6$, can be triangulated as a simplicial complex.

REMARK. For $M = 5$ and M^n oriented, Siebenmann [10] has shown under conditions (i) and (iii) that M is simplicially triangulable. M. Scharlemann has pointed out that if M^5 is unoriented, then (i), (iii) and the fact that $H^3 \# H^3$ bounds a *contractible PL* 4-manifold implies the result. For $6 \leqslant n \leqslant 8$, Theorem 4 was proven by M. Scharlemann [9], where in place of (ii) he has the orientability condition that the integral Bockstein of the Kirby-Siebenmann obstruction to putting a PL structure on M is zero. T. Matumoto has claimed a version of Theorem 4 under the stronger hypothesis that (iii) be replaced by the condition that $\Sigma^{n-4} H^3$ is homeomorphic to S^{n-1}.

We also investigate the question of whether a given topological n-manifold, $n \geqslant 6$, can be triangulated as a simplicial homotopy manifold. For example;

PROPOSITION 5. *Suppose that every PL homotopy 3-sphere bounds a contractible PL 4-manifold. Then there is a one-to-one correspondence between the set of concordance classes of simplicial homotopy manifold triangulations of* *a topological n-manifold M, $n \geqslant 6$, and concordance classes of PL manifold structures on M.*

PROPOSITION 6. *Suppose there exists a bad counterexample to the* 3 *dimensional Poincaré conjecture; namely suppose there exists a PL homotopy* 3-*sphere* H^3, *with*

(i) $\alpha(H^3) = 1$, *and*

(ii) $H^3 \# H^3$ *bounds a contractible PL* 4-*manifold.*

Then every topological n-manifold, $n \geqslant 6$, can be triangulated as a simplicial homotopy manifold.

Details of these and related results will appear in [6] and [7].

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