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Commutators and diffeomorphisms of surfaces

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Abstract. For any compact oriented surface Σ we consider the group of diffeomorphisms of Σ which preserve a given area form. In this paper we show that the vector space of homogeneous quasi-morphisms on this group has infinite dimension. This result is proved by constructing explicitly and for each surface an infinite family of independent homogeneous quasi-morphisms. These constructions use simple arguments related to linking properties of the orbits of the diffeomorphisms.

1. *Introduction*

Given a group G, its *first derived subgroup* is the subgroup G' *generated* by commutators $[a, b] = aba^{-1}b^{-1}$ of elements a, b of G. Each element g of G' can be written as a product $\prod_{i=1}^{k}[a_i, b_i]$. The smallest integer k for which such an expression exists is the *commutator length* of g and is denoted by $comm(g)$. This function $comm : G' \to \mathbb{N}$ contains a lot of useful information on the structure of the group G (see, for instance, [**7**]). In this paper, we study the case where G is the group of area-preserving diffeomorphisms of compact surfaces.

Before stating our results, we recall some fundamental theorems in this area, obtained by M. Herman, J. Mather and W. Thurston in the 1970s (we refer to [**4**] for references and a general survey). Let M be a connected manifold and denote by $G = \mathrm{Diff}_0^r(M)$ the group of C^r-diffeomorphisms of M which have compact support and are isotopic to the identity through an isotopy with compact support. In this direction, the main result is that G is a simple group (at least if $r \neq \dim M + 1$). In particular, $G = G'$ and every element of G can be written as a product of commutators.

When M is the circle $\mathbb{S}^1$, Herman showed that every orientation-preserving C^∞-diffeomorphism of the circle is a product of *two* commutators (see [**15**]). Is the function *comm* bounded for higher-dimensional manifolds, like, for example, for the 2-sphere?

Since we are unable to answer these questions, we study the case of area-preserving diffeomorphisms of surfaces (which somehow correspond to the dimension $\frac{3}{2}\ldots$).

Let Σ be a closed oriented surface and *area* be an area form on Σ (normalized so that the total area is 1). Let $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$ denote the group of C^∞-diffeomorphisms of Σ which are isotopic to the identity. Calabi constructed homomorphisms

$$\mathfrak{Calabi}_\Sigma : \mathrm{Diff}_0^\infty(\Sigma, \mathit{area}) \to H_1(\Sigma, \mathbb{R})$$

when the genus of Σ is at least 2 and

$$\mathfrak{Calabi}_{\mathbb{T}^2} : \mathrm{Diff}_0^\infty(\mathbb{T}^2, \mathit{area}) \to \mathbb{R}^2/\mathbb{Z}^2$$

when Σ is the 2-torus $\mathbb{T}^2$ (see [**9**]). Banyaga showed that the kernels of these homomorphisms are simple groups and coincide with the first derived subgroups of $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$ and $\mathrm{Diff}_0^\infty(\mathbb{T}^2, \mathit{area})$. As for $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$, Banyaga showed that it is a simple group. Note that Banyaga's results cover many more higher-dimensional cases but we only mention here those which are relevant for this paper [**4**].

THEOREM 1.1. *Commutator length is an unbounded function on* $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$. *Commutator length is an unbounded function on the kernels of Calabi's homomorphisms on* $\mathrm{Diff}_0^\infty(\mathbb{T}^2, \mathit{area})$ *and on* $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$ *when* $\mathit{genus}(\Sigma) \geq 2$.

This theorem is not completely new. The case of the 2-torus has been established by Barge and Ghys in [**6**] (in a more general symplectic situation). The case of the 2-sphere is due to Entov and Polterovich (and their results also include many higher-dimensional symplectic situations: see [**10**]). Their method uses quantum homology and one of the motivations for this paper was to find a more elementary construction. To the best of our knowledge, the case of $\mathit{genus}(\Sigma) \geq 2$ is new.

The general strategy used to show that the function *comm* is unbounded is to construct *quasi-morphisms*. A function $\phi : G \to \mathbb{R}$ is a quasi-morphism if there exists a constant $D_\phi > 0$ (called the defect of ϕ) such that

$$|\phi(ab) - \phi(a) - \phi(b)| \leq D_\phi$$

for all a, b in G. Of course, any bounded function is a quasi-morphism so that we shall consider two quasi-morphisms as *equivalent* if their difference is bounded. We say that a quasi-morphism is *homogeneous* if $\phi(a^p) = p\phi(a)$ for every a in G and p in $\mathbb{Z}$. It is very easy to check that, for every quasi-morphism ϕ, the limit $\Phi(a) = \lim_{p\to+\infty}(1/p)\phi(a^p)$ exists and defines the unique homogeneous quasi-morphism equivalent to ϕ.

Suppose Φ is a homogeneous quasi-morphism with defect D_Φ and g an element of G of the form $\prod_{i=1}^k [a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1}$. We have

$$|\Phi(g) - (\Phi(a_1) + \Phi(b_1) + \Phi(a_1^{-1}) + \Phi(b_1^{-1}) + \cdots)| \leq (4k-1)D_\Phi$$

so that

$$4k - 1 \geq \frac{|\Phi(g)|}{D_\Phi}.$$

Taking the minimum of k over all expressions of g as a product of commutators, we get

$$\mathit{comm}(g) \geq \frac{1}{4}\left(\frac{|\Phi(g)|}{D_\Phi} + 1\right).$$

This inequality for g^p gives

$$comm(g^p) \geq p\frac{|\Phi(g)|}{4D_\Phi}.$$

Therefore, *if one can construct a non-trivial homogeneous quasi-morphism, the commutator length is unbounded*. This is the method that we shall follow.

Somehow, this is the only possible method. Indeed, for an element g of the first derived group of a group G, let us define the *stable commutator length* $\|g\|$ of g by $\|g\| = \lim_{p\to+\infty}(1/p)comm(g^p)$ (the limit exists by subadditivity). In [**7**], Bavard shows that $\|g\|$ is the lower bound of $|\Phi(g)|/2\sup_{a,b}|\Phi([a,b])|$ when Φ describes all non-trivial homogeneous quasi-morphisms. So, one could say that stable commutator length and homogeneous quasi-morphisms are in duality. Note also that homogeneous quasi-morphisms are *class functions*, i.e. they are constant on conjugacy classes of the group.

The following theorem is stronger than Theorem 1.1.

THEOREM 1.2. *For every closed oriented surface Σ, there exist homogeneous quasi-morphisms $\Phi : \mathrm{Diff}_0^\infty(\Sigma, area) \to \mathbb{R}$ which are non-trivial, even when restricted to the kernel of Calabi's homomorphism (for $\Sigma \neq \mathbb{S}^2$). Moreover, the vector space of these homogeneous quasi-morphisms is infinite dimensional.*

We believe that these explicit quasi-morphisms have a dynamical interest, besides their algebraic interest. For instance, they are invariant under a topological conjugacy which is area preserving.

The general spirit of our constructions is to measure some kind of generalized 'rotation number'. Recall that the classical Poincaré rotation number is a map $\rho : \mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{R}/\mathbb{Z}$ defined on the group of orientation-preserving homeomorphisms of the circle. Its lift to universal covers $\tilde{\rho} : \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1) \to \mathbb{R}$ (with $\tilde{\rho}(\mathrm{id}) = 0$) is called the translation number. It turns out that $\tilde{\rho}$ is the *unique* homogeneous quasi-morphism defined on $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ (up to a multiplicative constant) (see [**14**]). Therefore, one can consider the constructions in the present paper as (further) attempts to generalize rotation numbers to higher-dimensional situations in the same spirit as Schwartzman's asymptotic cycles and Arnold's asymptotic Hopf invariant [**1, 2, 20**].

This paper essentially contains elementary constructions of homogeneous quasi-morphisms on diffeomorphisms groups. In §§2, 3 and 4, we construct *one* non-trivial quasi-morphism in the cases of the torus, surfaces of genus at least 2, and on the sphere. Sections 5 and 6 generalize these constructions to produce infinitely many independent quasi-morphisms. These sections are largely independent and can be read in almost any order. In this introduction, we restricted ourselves to surfaces with no boundary but the case of area-preserving diffeomorphisms of the disc will often serve as a first case of study.

2. *The cases of the disc and the torus*

This section contains no new result and serves as an introduction to our techniques and as a reminder of [**6**].

We begin with some general remarks which hold true for any compact connected oriented manifold M. According to Moser's lemma, two volume forms on M with the same total volume can be transformed into each other by some diffeomorphism isotopic to the identity. Therefore, in Theorems 1.1 and 1.2, the choice of the area form *area* on a given Σ is not relevant and all the groups $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$ are isomorphic. Moreover, a strong version of Moser's lemma asserts that, for any volume form *vol*, the group of volume-preserving diffeomorphisms $\mathrm{Diff}^\infty(M, \mathit{vol})$ is a deformation retract of $\mathrm{Diff}^\infty(M)$ (see, for instance, [**4**]). It follows that, for instance, a volume-preserving diffeomorphism which is isotopic to the identity is also isotopic to the identity through volume-preserving diffeomorphisms.

2.1. *The disc.* Let us denote by $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, \mathit{area})$ the group of C^∞-area-preserving diffeomorphisms of the 2-disc $\mathbb{D}^2$ which are the identity in a neighborhood of the boundary. This group is contractible (see [**11**]). Of course, it is convenient to choose the area form as the usual (normalized) area on the disc.

We shall construct the so-called *Ruelle's quasi-morphism* for diffeomorphisms of the disc (introduced in [**19**] and described as a quasi-morphism in [**6, 12**]). Let g be an element of $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, \mathit{area})$ and choose an isotopy $(g_t)_{t\in[0,1]}$ between $g_0 = \mathrm{id}$ and $g_1 = g$. For each point x in the disc, consider the differential $dg_t(x)$. Using the natural trivialization of the tangent bundle of the disc, we can consider this differential as a 2×2 matrix, element of $\mathrm{SL}(2, \mathbb{R})$. The first column $v_t(x)$ of $dg_t(x)$ is a non-zero vector in $\mathbb{R}^2$. Denote by $Ang_g(x) \in \mathbb{R}$ the variation of the angle (or of the argument) of this curve $v_t(x)$ of non-zero vectors when t runs from 0 to 1 (as a unit for angles, we prefer to use the full turn instead of the radian). This number could *a priori* depend on the choice of the isotopy g_t but the contractibility of $\mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, \mathit{area})$ shows that it does not, so that it depends only on g and x (hence the notation $Ang_g(x)$). Let us define

$$r(g) = \int_{\mathbb{D}^2} Ang_g(x)\, d\, area(x).$$

Consider now two elements g and h of $\mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, \mathit{area})$ and choose two isotopies g_t and h_t as before. Considering the concatenation of these isotopies, one sees that

$$|Ang_{gh}(x) - Ang_h(x) - Ang_g(h(x))| < 1/2.$$

Indeed, if A_t is a curve in $\mathrm{SL}(2, \mathbb{R})$ and if u, v are two non-zero vectors in $\mathbb{R}^2$, the variations of the arguments of $A_t(u)$ and $A_t(v)$ differ by, at most, one half-turn. It follows that r is a quasi-morphism. After 'homogenization', we get *Ruelle's homogeneous quasi-morphism*

$$\mathfrak{Ruelle}_{\mathbb{D}^2}(g) = \lim_{p\to+\infty} \frac{1}{p} r(g^p).$$

We still have to check that $\mathfrak{Ruelle}_{\mathbb{D}^2}$ is non-trivial. Let us consider a function $\omega : [0, 1] \to \mathbb{R}$ which is equal to zero in a neighborhood of 1 and constant in a neighborhood of 0. Define an area-preserving diffeomorphism F_ω of the disc by

$$F_\omega(x) = Rot_{\omega(\|x\|)}(x)$$

where Rot_θ denotes the rotation around the origin by an angle θ. The computation of Ruelle's number in this case is easy. We get

$$\mathfrak{Ruelle}_{\mathbb{D}^2}(F_\omega) = \frac{1}{\pi} \int_0^1 \omega(t) 2\pi t\, dt = 2 \int_0^1 t\omega(t)\, dt$$

so that the quasi-morphism $\mathfrak{Ruelle}_{\mathbb{D}^2}$ is indeed non-trivial.

Note that $\mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$ is not a simple group: Calabi defined a *homomorphism* $\mathfrak{Calabi}_{\mathbb{D}^2} : \mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area) \to \mathbb{R}$. We postpone the definition of $\mathfrak{Calabi}_{\mathbb{D}^2}$ to §5.2 and we shall see that $\mathfrak{Ruelle}_{\mathbb{D}^2}$ is also non-trivial on the kernel of $\mathfrak{Calabi}_{\mathbb{D}^2}$ (which is a simple group).

2.2. *The torus.* We are going to construct a Ruelle quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ using the same idea as for the disc. Before going to the construction, we quickly describe the construction of Calabi's *homomorphism* $\mathfrak{Calabi}_{\mathbb{T}^2}$.

Let us equip the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the 'Lebesgue measure' $area = dx\,dy$. The group $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ contains the translations $\mathbb{T}^2 \subset \mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ as a subgroup. This inclusion is a homotopy equivalence (see, for instance, [**11**]).

Let g be an element of $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ and $(g_t)_{t\in[0,1]}$ an isotopy between $g_0 = \mathrm{id}$ and $g_1 = g$ as before. Lifting to the universal cover $\mathbb{R}^2$, we get an isotopy $(\tilde{g}_t)_{t\in[0,1]}$ of $\mathbb{R}^2$ from $\tilde{g}_0 = \mathrm{id}$ to some lift $\tilde{g}_0 = \tilde{g}$ of g. Consider the function

$$x \in \mathbb{R}^2 \mapsto \tilde{g}_1(x) - x \in \mathbb{R}^2.$$

This is a $\mathbb{Z}^2$-periodic function on $\mathbb{R}^2$ which can, therefore, be integrated over a fundamental domain. Changing the isotopy would change the lift $\tilde{g}_1$ by some integral translation so that the integral of $\tilde{g}_1(x) - x$ on some fundamental domain is well defined modulo $\mathbb{Z}^2$. This defines a map

$$\mathfrak{Calabi}_{\mathbb{T}^2} : \mathrm{Diff}_0^\infty(\mathbb{T}^2, area) \to \mathbb{R}^2/\mathbb{Z}^2$$

which is *Calabi's homomorphism.* The fact that this is indeed a homomorphism is elementary. Note that the restriction of $\mathfrak{Calabi}_{\mathbb{T}^2}$ to the subgroup of translations $\mathbb{R}^2/\mathbb{Z}^2$ is the identity map, so that the kernel of $\mathfrak{Calabi}_{\mathbb{T}^2}$ is a contractible group.

Now we define Ruelle's quasi-morphism. Since the tangent bundle of the torus is trivialized, for each point x, we can again consider the curve $dg_t(x)$ as a curve in $\mathrm{SL}(2, \mathbb{R})$ and the variation of the argument of the first column. Unlike the group $\mathrm{Diff}^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$, the group $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ is not contractible but any loop is homotopic to a loop in the translation subgroup, hence to a loop with constant differential. It follows that $Ang_g(x)$ is again well defined and that one can again define a quasi-morphism $\mathfrak{Ruelle}_{\mathbb{T}^2}$ on $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ by the same formula.

If we choose a small disc D in the torus, it is clear that the group of area-preserving diffeomorphisms of this disc which are the identity near the boundary can be embedded as a subgroup of $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ (extending by the identity). Clearly this subgroup is contained in the kernel of $\mathfrak{Calabi}_{\mathbb{T}^2}$ and the restriction of Ruelle's quasi-morphism of the bigger group coincides with Ruelle's quasi-morphism of the subgroup (strictly speaking, we have to extend the definitions in the obvious manner when the area is not normalized since the small disc cannot have area 1). It follows that *Ruelle's quasi-morphism is non-trivial when restricted to the kernel of Calabi's homomorphism.*

3. *Surfaces of genus at least 2*

We now consider a closed oriented surface Σ of genus at least 2 equipped with an area form *area*. In this case, the group $\mathrm{Diff}_0^\infty(\Sigma, area)$ is contractible [**11**].

We first recall the construction of Calabi's homomorphism

$$\mathfrak{Calabi}_{\Sigma} : \mathrm{Diff}_0^{\infty}(\Sigma, area) \to H_1(\Sigma, \mathbb{R}).$$

Let α be a *closed* 1-form on Σ. For each element g in $\mathrm{Diff}_0^{\infty}(\Sigma, area)$, choose an isotopy $(g_t)_{t\in[0,1]}$ as above. For each point x in Σ, consider the integral of α on the arc $t \in [0, 1] \mapsto g_t(x) \in \Sigma$. Since the group $\mathrm{Diff}_0^{\infty}(\Sigma, area)$ is contractible, the homotopy class of this arc, relative to its end points, is independent of the isotopy, so that the integral only depends on g and x. Denote this number by $I_\alpha(g, x)$. Clearly, one has

$$I_\alpha(gh, x) = I_\alpha(h, x) + I_\alpha(g, h(x)).$$

Let us define

$$\mathfrak{Calabi}_{\alpha,\Sigma}(g) = \int_{\Sigma} I_\alpha(g, x)\, d\ area(x)$$

so that

$$\mathfrak{Calabi}_{\alpha,\Sigma}(gh) = \mathfrak{Calabi}_{\alpha,\Sigma}(g) + \mathfrak{Calabi}_{\alpha,\Sigma}(h)$$

and $\mathfrak{Calabi}_{\alpha,\Sigma}$ is a homomorphism. If α is an exact form, a differential of a function $u : \Sigma \to \mathbb{R}$, one has $I_\alpha(g, x) = u(g(x)) - u(x)$ and $\mathfrak{Calabi}_{\alpha,\Sigma}(g) = 0$. Since $\mathfrak{Calabi}_{\alpha,\Sigma}$ is clearly linear as a function of α, we see that, g being fixed, $\mathfrak{Calabi}_{\alpha,\Sigma}$ is a linear form on the de Rham cohomology group $H^1(\Sigma, \mathbb{R})$. Therefore, any element g of $\mathrm{Diff}_0^{\infty}(\Sigma, area)$ defines an element of the dual of $H^1(\Sigma, \mathbb{R})$, hence an element $\mathfrak{Calabi}_{\Sigma}(g)$ of the homology group $H_1(\Sigma, \mathbb{R})$. This map $\mathfrak{Calabi}_{\Sigma}$ is Calabi's homomorphism. According to Banyaga, the kernel of $\mathfrak{Calabi}_{\Sigma}$ is a simple group.

We now construct a homogeneous quasi-morphism $\mathfrak{Ruelle}_{\Sigma}$ in the spirit of Ruelle's quasi-morphism, which is non-trivial on the kernel of $\mathfrak{Calabi}_{\Sigma}$. Choose on Σ a metric with curvature -1 so that the universal cover of Σ is identified with the Poincaré disc with its usual hyperbolic metric. Let g be an element of $\mathrm{Diff}_0^{\infty}(\Sigma, area)$, x a point in Σ and $(g_t)_{t\in[0,1]}$ an isotopy between id and g. Since the differential of g_t in x is a linear map between two different vector spaces and the tangent bundle is not trivial, it is not possible to identify this differential to a 2×2 matrix, as we did for the disc and the torus.

Let us lift the isotopy g_t to an isotopy $\tilde{g}_t$ of the universal cover, i.e. of the Poincaré disc $\mathbb{D}^2$. If $\tilde{x}$ is a point of $\mathbb{D}^2$ and $\tilde{v}$ is a non-zero vector tangent to $\mathbb{D}^2$ at the point $\tilde{x}$, we can consider the geodesic starting from $\tilde{x}$ with initial velocity $\tilde{v}$ which 'converges' to some point at infinity $\pi(\tilde{x}, \tilde{v})$ in the boundary $\partial\mathbb{D}^2$, which is a circle. For each $(\tilde{x}, \tilde{v})$, we can, therefore, consider the curve $t \in [0, 1] \mapsto \pi(\tilde{g}_t(\tilde{x}), d\tilde{g}_t(\tilde{x})(\tilde{v})) \in \partial\mathbb{D}^2$ and count the 'number of full turns' of this curve around the circle. More precisely, the number of full turns of a curve $t \in [0, 1] \mapsto \gamma(t) \in \mathbb{R}/\mathbb{Z}$ is the integral part of $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ where $\tilde{\gamma}$ is any lift of γ to $\mathbb{R}$ (note that this integer does not depend on any choice of a parameterization on the circle). Since $\mathrm{Diff}_0^{\infty}(\Sigma, area)$ is contractible, this number does not depend on the choice of the isotopy as well: let us denote it by $r(g, \tilde{x}, \tilde{v})$. If $\tilde{v}_1$ and $\tilde{v}_2$ are two non-zero vectors tangent to the same point $\tilde{x}$, it is clear that $r(g, \tilde{x}, \tilde{v}_1)$ and $r(g, \tilde{x}, \tilde{v}_2)$ differ by, at most, 1. Therefore, we can define $r(g, \tilde{x}) = \inf_{\tilde{v}} r(g, \tilde{x}, \tilde{v})$.

The concatenation of two isotopies yields

$$|r(gh, \tilde{x}, \tilde{v}) - r(h, \tilde{x}, \tilde{v}) - r(g, \tilde{h}_1(\tilde{x}), d\tilde{h}_1(\tilde{v}))| \leq 1$$

so that

$$|r(gh, \tilde{x}) - r(h, \tilde{x}) - r(g, \tilde{h}_1\tilde{x})| \leq 4.$$

Of course, the isotopy $\tilde{g}_t$ commutes with the isometric action of the fundamental group of Σ on the Poincaré disc. Therefore, g being fixed, the function $\tilde{x} \in \mathbb{D}^2 \mapsto r(g, \tilde{x}) \in \mathbb{Z}$ is invariant by this action and defines a function $r(g, x)$ on Σ. Therefore, if we set

$$r(g) = \int_\Sigma r(g, x)\, d\ area(x),$$

we get a quasi-morphism on $\mathrm{Diff}_0^\infty(\Sigma, area)$. By homogenization, we finally set

$$\mathfrak{Ruelle}_\Sigma(g) = \lim_{p\to+\infty} \frac{1}{p} r(g^p).$$

We still have to show that the homogeneous quasi-morphism $\mathfrak{Ruelle}_\Sigma$ is non-trivial on the kernel of Calabi's homomorphism $\mathfrak{Calabi}_\Sigma$. Choose an area-preserving embedding of a small disc D in Σ. The group $\mathrm{Diff}_0^\infty(D, \partial D, area)$ embeds in $\mathrm{Diff}_0^\infty(\Sigma, area)$ (extending by the identity outside of D). We claim that the restriction of $\mathfrak{Ruelle}_\Sigma$ to this subgroup coincides with Ruelle's invariant on the disc $\mathfrak{Ruelle}_{\mathbb{D}^2}$. Indeed, we have to consider an isotopy g_t in $\mathrm{Diff}_0^\infty(D, \partial D, area)$ and a curve of the form $t \in [0, 1] \mapsto dg_t(x)(v)$ in the tangent bundle of D. The definition of Ruelle's quasi-morphism $\mathfrak{Ruelle}_{\mathbb{D}^2}$ uses the triviality of this tangent bundle and the definition of $\mathfrak{Ruelle}_\Sigma$ uses the projection to infinity π. Since $g_t(x)$ stays in the disc D, the two numbers 'variation of the argument of $dg_t(v)$' and 'number of full turns of $\pi(g_t(x), dg_t(x)(v))$' differ by a bounded amount, independently of g, x, v, only depending on the embedding of D in Σ. Hence, $\mathfrak{Ruelle}_\Sigma$ and the restriction of $\mathfrak{Ruelle}_\Sigma$ to $\mathrm{Diff}_0^\infty(D, \partial D, area)$ differ by some bounded quantity. Since these two maps are homogeneous quasi-morphisms, they must be identical. Note also that $\mathrm{Diff}_0^\infty(D, \partial D, area)$ is clearly contained in the kernel of Calabi's homomorphism so that we established that $\mathfrak{Ruelle}_\Sigma$ *is indeed non-trivial on the kernel of Calabi's homomorphism* $\mathfrak{Calabi}_\Sigma$.

Observe that the definition of $\mathfrak{Ruelle}_\Sigma$ involves the choice of a metric with curvature -1 on Σ but one can easily show that it is independent of this choice. Indeed, let m_1, m_2 be two negatively curved metrics on Σ and $\tilde{m}_1, \tilde{m}_2$ their lifts to the universal cover $\tilde{\Sigma}$. Any $\tilde{m}_1$-geodesic is a $\tilde{m}_2$-quasigeodesic and stays at a bounded distance from a unique $\tilde{m}_1$-geodesic. This gives a natural identification of the boundaries at infinity of $\tilde{\Sigma}$ for $\tilde{m}_1$ and $\tilde{m}_2$ and shows that the number of full turns of $\pi(\tilde{g}_t(\tilde{x}), d\tilde{g}_t(\tilde{x})(\tilde{v}))$ is the same if one counts it with the projection π associated to $\tilde{m}_1$ or $\tilde{m}_2$.

4. *The 2-sphere*

The case of the 2-sphere is the most interesting. The group $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ is a deformation retract of $\mathrm{Diff}_0^\infty(\mathbb{S}^2)$ which coincides with the group of orientation-preserving diffeomorphisms of the sphere. Both groups retract on the group SO(3) of isometries (Smale's theorem).

Before we begin our construction of a quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$, we shall describe a quasi-morphism on the discrete group $\mathrm{PSL}(2, \mathbb{Z})$. For more information, see [**6**].

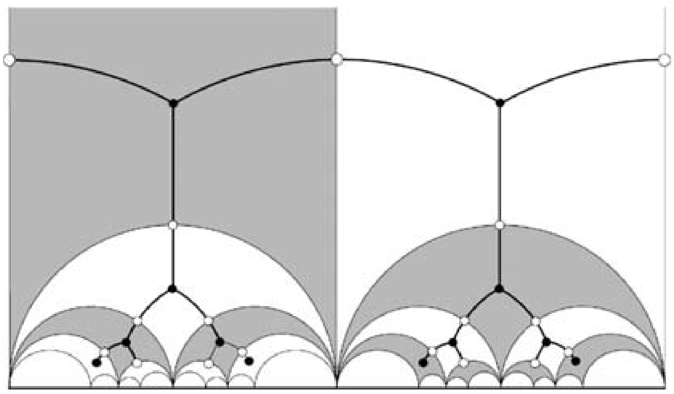

FIGURE 1. Tree and tessellation in $\mathbb{H}^2$.

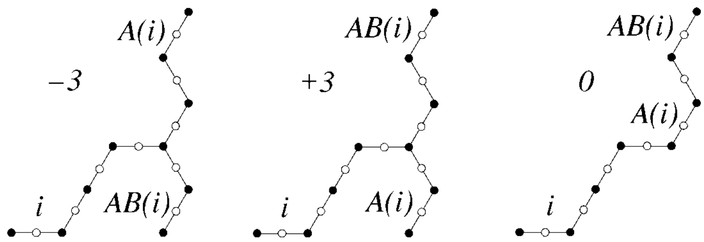

FIGURE 2. $\mathfrak{Radem}$ is a quasi-morphism.

The group $\mathrm{PSL}(2,\mathbb{Z})$ acts isometrically on Poincaré's upper half-space $\mathbb{H}^2$. The stabilizer of the point $i = \sqrt{-1}$ is a cyclic group of order 2 and the stabilizer of $j = (-1 + i\sqrt{3})/2$ is a cyclic group of order 3. Let γ be the geodesic arc connecting these two points. The union of the images of γ under the action of $\mathrm{PSL}(2,\mathbb{Z})$ is a tree T embedded in the half-space (see, for instance, [**21**]). This illustrates the decomposition of $\mathrm{PSL}(2,\mathbb{Z})$ as a free product $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ (see Figure 1).

Let A be an element of $\mathrm{PSL}(2,\mathbb{Z})$. Consider the two points i and $A(i)$ in the tree T. There exists a unique 'combinatorial geodesic' contained in T connecting these two points: this is a chain consisting of a succession of geodesic arcs, images of γ by some elements of $\mathrm{PSL}(2,\mathbb{Z})$. Travelling from i to $A(i)$ along this combinatorial geodesic, one successively meets bifurcations which can be left or right turns. Indeed, the three arcs going out of any triple point of the tree are cyclically ordered. Let us denote by $t_{\mathrm{left}}(A)$ and $t_{\mathrm{right}}(A)$ the number of left and right turns, respectively, from i to $A(i)$. Let us define the function

$$\mathfrak{Radem}(A) = t_{\mathrm{left}}(A) - t_{\mathrm{right}}(A).$$

Figure 2 shows that $\mathfrak{Radem}$ is a quasi-morphism and that $\mathfrak{Radem}(AB) - \mathfrak{Radem}(A) - \mathfrak{Radem}(B)$ is equal to $-3, 0$ or $+3$. This function $\mathfrak{Radem}$ is called the *Rademacher function*. See [**6**] for some motivation.

We consider the 2-sphere $\mathbb{S}^2$ as the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$. Given four distinct points z_1, z_2, z_3, z_4 of $\overline{\mathbb{C}}$, their cross ratio is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_3 - z_1)}{(z_3 - z_2)} \frac{(z_4 - z_2)}{(z_4 - z_1)} \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

Let g be an element of $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ and choose, as usual, an isotopy $(g_t)_{t \in [0,1]}$ from id to g. If z_1, z_2, z_3, z_4 are four distinct points of $\overline{\mathbb{C}}$, we can consider the curve

$$t \in [0, 1] \mapsto [g_t(z_1), g_t(z_2), g_t(z_3), g_t(z_4)] \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

This curve can be lifted to the universal cover of the sphere minus three points. Recall that this universal cover can be identified with the Poincaré upper half-space $\mathbb{H}^2$ (or the disc). More precisely, there is a covering map from $\mathbb{H}^2$ onto $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and the inverse images of points of $\overline{\mathbb{R}} \setminus \{0, 1, \infty\}$ define a tessellation of $\mathbb{H}^2$ by ideal triangles (see Figure 1).

The group of positive isometries of $\mathbb{H}^2$ preserving this tessellation globally is the group $\mathrm{PSL}(2, \mathbb{Z})$. The Galois group of the covering map is the fundamental group of the sphere minus three points, which is a free group F_2 on two generators, of index 6 in $\mathrm{PSL}(2, \mathbb{Z})$. The edges of the tessellation can be colored with three colors according to the component of $\overline{\mathbb{R}} \setminus \{0, 1, \infty\}$ on which they project and F_2 is the subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ consisting of elements which are color preserving. Considering barycenters of ideal triangles, and connecting them if the triangles are adjacent, one finds the tree T that we already considered, with its action of $\mathrm{PSL}(2, \mathbb{Z})$.

Coming back to the curve $[g_t(z_1), g_t(z_2), g_t(z_3), g_t(z_4)]$ on the sphere minus three points, its lift to the universal cover is a curve in Poincaré half-space $\mathbb{H}^2$. Let us assume, for simplicity, that $[z_1, z_2, z_3, z_4]$ and $[g(z_1), g(z_2), g(z_3), g(z_4)]$ are not real (i.e. that these points are not cocyclic). In this case, the endpoints of any lift of the curve are in the interior of two triangles and, therefore, define two vertices v_0 and v_1 of the tree T. The unique combinatorial geodesic in T from v_0 to v_1 makes a certain number t_{left} (respectively t_{right}) of left (respectively right) turns. Let us define

$$T(g; z_1, z_2, z_3, z_4) = t_{\mathrm{left}} - t_{\mathrm{right}}.$$

In order to justify this notation, we first have to check that this number is independent of the choice of the lift: this follows from the fact that two lifts differ by some deck transformation and that deck transformations are orientation preserving and map left and right turns to left and right turns. We also have to check that this does not depend on the choice of the isotopy g_t: this follows from the fact that any loop in $\mathrm{Diff}_0^\infty(\mathbb{S}^2)$ is homotopic to a loop in $\mathrm{SO}(3)$, hence preserving cross ratios of distinct 4-tuples.

Under composition, we have

$$\begin{aligned} T(gh; z_1, z_2, z_3, z_4) &- T(h; z_1, z_2, z_3, z_4) \\ &- T(g; h(z_1), h(z_2), h(z_3), h(z_4)) = -3, 0, \text{ or } +3 \end{aligned}$$

at least when all these numbers are defined, i.e. when the cross ratios $[z_1, z_2, z_3, z_4]$, $[h(z_1), h(z_2), h(z_3), h(z_4)]$ and $[gh(z_1), gh(z_2), gh(z_3), gh(z_4)]$ are not real. We then define

$$t(g) = \iiiint T(g; z_1, z_2, z_3, z_4)\, d\, \mathit{area}(z_1)\, d\, \mathit{area}(z_2)\, d\, \mathit{area}(z_3)\, d\, \mathit{area}(z_4).$$

We have to check that $T(g; z_1, z_2, z_3, z_4)$ is integrable as a function of the four points. First, we observe that this function is indeed defined almost everywhere: on the set of 4-tuples of distinct points (z_1, z_2, z_3, z_4) such that $[z_1, z_2, z_3, z_4]$ and $[g(z_1), g(z_2), g(z_3), g(z_4)]$ are not real. The following lemma ensures integrability.

LEMMA 4.1. *For every g, the function $(z_1, z_2, z_3, z_4) \mapsto T(g; z_1, z_2, z_3, z_4)$ is bounded on its domain of definition.*

Proof. Since the group $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ is generated by any neighborhood of the identity and the function $T(g; z_1, z_2, z_3, z_4)$ is quasi-additive under composition, we can assume that g is close to the identity. We are going to construct an isotopy adapted to the projective geometry of $\overline{\mathbb{C}}$. Observe that the complement of a point in $\overline{\mathbb{C}}$ can be identified to $\mathbb{C}$ up to similitude. Hence, if z_1 and z_2 are two points in $\overline{\mathbb{C}} \setminus \{z_3\}$ and $t \in [0, 1]$, one can consider the barycenter of z_1 and z_2 with coefficients t and $1 - t$ in the affine plane $\overline{\mathbb{C}} \setminus \{z_3\}$. We shall denote this point by $[tz_1 + (1 - t)z_2]_{z_3}$. When we identify the 2-sphere to $\overline{\mathbb{C}}$ via stereographic projection, the antipody is transformed into the involution $z \mapsto -1/\overline{z}$. Let g be an element of $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ which is close enough to the identity so that $g(z) \neq -1/\overline{z}$ for every point z. Define g_t by $g_t(z) = [tg(z) + (1 - t)z]_{(-1/\overline{z})}$. When g is sufficiently C^1-close to the identity, g_t is an isotopy. Observe that, in the definition of $T(g; z_1, z_2, z_3, z_4)$, we could use any isotopy, not necessarily through area-preserving diffeomorphisms. Let us consider the cross-ratio curve $\rho(t) = [g_t(z_1), g_t(z_2), g_t(z_3), g_t(z_4)]$ for this particular choice of isotopy. Observe that each curve $g_t(z_i)$ has the form $(a(z_i)t + b(z_i))/(c(z_i)t + d(z_i))$ so that the computation of $\rho(t)$ leads to some expression of the form $p(t)/q(t)$ where $p(t)$ and $q(t)$ are polynomials of degrees, at most, 8. The values of t for which $\rho(t)$ is a real number are the roots of $p(t)\overline{q(t)}$. Hence, there are at most 16 values of t for which $\rho(t)$ is real (unless $\rho(t)$ is real for every t but we assumed that $\rho(0)$ is not real). It follows that the curve $\rho(t)$ crosses, at most, 16 times the real axis and its lift to the universal cover crosses, at most, 16 ideal triangles. It follows that in this specific neighborhood of the identity $T(g; z_1, z_2, z_3, z_4)$ is bounded by 16. □

Finally, by homogenization, we define

$$\mathfrak{Turn}(g) = \lim_{p \to +\infty} \frac{1}{p} t(g^p).$$

This is a homogeneous quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$.

We have to show that $\mathfrak{Turn}$ is non-trivial. Before computing $\mathfrak{Turn}$ on some explicit examples, let us study the effect of a permutation of (z_1, z_2, z_3, z_4) on the number $T(g; z_1, z_2, z_3, z_4)$. By elementary projective geometry, there is a representation f of the symmetric group $\mathfrak{S}(4)$ on four points to the projective group $\mathrm{PGL}(2; \mathbb{C})$ such that $[z_{\sigma(z_1)}, z_{\sigma(z_2)}, z_{\sigma(z_3)}, z_{\sigma(z_4)}] = f(\sigma)([z_1, z_2, z_3, z_4])$. The image group $f(\mathfrak{S}(4))$ is a group with six elements, isomorphic to $\mathfrak{S}(3)$, which permutes the three components of $\overline{\mathbb{R}} \setminus \{0, 1, \infty\}$. It follows that, for every permutation σ, the two arcs $[g_t(z_1), g_t(z_2), g_t(z_3), g_t(z_4)]$ and $[g_t(z_{\sigma(1)}), g_t(z_{\sigma(2)}), g_t(z_{\sigma(3)}), g_t(z_{\sigma(4)})]$ in $\mathbb{H}^2$ differ by the action of some element of $f(\mathfrak{S}(4))$ and their lifts to $\mathbb{H}^2$ by some element of $\mathrm{PSL}(2, \mathbb{Z})$. We conclude that *the number $T(g; z_1, z_2, z_3, z_4)$ is invariant under permutation of the four points.*

Consider a disc D_r in $\mathbb{C}$ centered at the origin and of radius $r > 0$. Under the identification of the 2-sphere with $\overline{\mathbb{C}}$, this disc can be considered as a disc in the 2-sphere. Of course, we choose the usual (normalized) area form on the 2-sphere so that the pullback of the area form on D_r is invariant by rotations around the origin and have total mass $a(r)$ (of course, the exact formula for $a(r)$ is not relevant for our discussion but an easy computation shows that $a(r) = r^2/(1+r^2)$, if one uses standard stereographic projection from the north pole on the equatorial plane). Choose, as before, a function $\omega : [0, r] \to \mathbb{R}$ which vanishes in the neighborhood of r and is constant in a neighborhood of 0. Define a diffeomorphism F_ω of $\overline{\mathbb{C}}$ by

$$F_\omega(z) = \exp(2i\pi\omega(|z|))z$$

for $|z| < r$ and which is the identity outside of D_r. This is an area-preserving diffeomorphism which is isotopic to the identity (one can choose $F_{t\omega}$ as an isotopy). The following lemma shows that $\mathfrak{Turn}$ is indeed non-trivial.

LEMMA 4.2. $\mathfrak{Turn}(F_\omega) = 12\int_0^r a(t)(1-a(t))(1-2a(t))a'(t)\omega(t)\,dt.$

Proof. Consider four circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ centered at the origin and with radii $r_1 < r_2 < r_3 < r_4$ and the cross-ratio map

$$(z_1, z_2, z_3, z_4) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \mapsto \frac{(z_3 - z_1)}{(z_3 - z_2)}\frac{(z_4 - z_2)}{(z_4 - z_1)} \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

We first analyze the induced mapping at the level of fundamental groups. It is easy to check that when z_2, z_3, z_4 are fixed and z_1 describes $\mathcal{C}_1$, the image curve in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ is homotopic to a point. The same is true if z_1, z_2, z_3 are fixed and if z_4 describes $\mathcal{C}_4$. When z_1, z_3, z_4 are fixed and z_2 describes $\mathcal{C}_2$ positively, the image curve is homotopic to a small simple positive loop around the point 1. Finally, when z_1, z_2, z_4 are fixed and z_3 describes $\mathcal{C}_3$ positively, the image curve is homotopic to small simple negative loop around the point 1. We can now compute $T(g^p; z_1, z_2, z_3, z_4)$ when the z_i are in the circles $\mathcal{C}_i$, using the isotopy $F_{tp\omega}$. The cross-ratio curve $\rho(t)$ is the composition of the map

$$\begin{aligned} &\gamma : [0, 1] \to \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \\ &t \mapsto (\exp(2i\pi tp\omega(|z_1|)z_1), \exp(2i\pi tp\omega(|z_2|)z_2), \exp(2i\pi tp\omega(|z_3|)z_3), \\ &\qquad \exp(2i\pi tp\omega(|z_4|)z_4)) \end{aligned}$$

with the map

$$(z_1, z_2, z_3, z_4) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \mapsto \frac{(z_3 - z_1)}{(z_3 - z_2)}\frac{(z_4 - z_2)}{(z_4 - z_1)} \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

The arc $\gamma([0, 1])$ in the four-dimensional torus can be turned into a closed loop by connecting the two end points by some arc of bounded length (independently of p). It follows that the cross-ratio arc, when lifted to $\mathbb{H}^2$ and completed by an arc of bounded length, is the lift of some loop in the four-dimensional torus. We have seen that the images of these loops in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ are homotopic to loops around the point 1. In particular, lifts of such loops only make right turns or left turns depending on whether they are positive or negative. More precisely, there is a constant $C(\omega, r_1, r_2, r_3, r_4)$ such that, for all p,

$$|T(g^p; z_1, z_2, z_3, z_4) - p(\omega(r_2) - \omega(r_3))| \leq C.$$

Hence,

$$\lim_{p\to+\infty}\frac{1}{p}T(g^p; z_1, z_2, z_3, z_4) = \omega(r_2) - \omega(r_3).$$

In order to compute $\mathfrak{Turn}(g)$, we have to integrate over all 4-tuples of points but we know that $T(g; z_1, z_2, z_3, z_4)$ is invariant under permutations of the points so that we can compute $4! = 24$ times the integral on 4-tuples for which $|z_1| < |z_2| < |z_3| < |z_4|$. Therefore,

$$\begin{aligned}
&\mathfrak{Turn}(g)\\
&= \lim_{p\to\infty}\frac{1}{p}\iiiint T(g^p; z_1, z_2, z_3, z_4)\, d\,area(z_1)\, d\,area(z_2)\, d\,area(z_3)\, d\,area(z_4)\\
&= \iiiint \lim_{p\to\infty}\frac{1}{p}T(g^p; z_1, z_2, z_3, z_4)\, d\,area(z_1)\, d\,area(z_2)\, d\,area(z_3) d\,area(z_4)\\
&= 24\iiiint_{r_1<r_2<r_3<r_4}(\omega(r_2)-\omega(r_3))\, da(r_1)\, da(r_2)\, da(r_3)\, da(r_4)\\
&= 24\bigg(\int_0^r a(r_2)\frac{(1-a(r_2))^2}{2}\omega(r_2)\, da(r_2) - \int_0^r \frac{a(r_3)^2}{2}(1-a(r_3))\omega(r_3)\, da(r_3)\bigg)\\
&= 12\int_0^r a(t)(1-a(t))(1-2a(t))\omega(t)\, da(t)\bigg(=24\int_0^r \frac{t^3(1-t^2)}{(1+t^2)^4}\omega(t)\, dt\bigg).
\end{aligned}$$

Note that the permutation of the limit and the integral is justified by dominated convergence. This proves the lemma. □

For future reference, we note that the change of variable $a = (1-u)/2$ and $\omega(t) = \overline{\omega}(u(t))$ leads to the formula $\mathfrak{Turn}(g) = \frac{3}{2}\int_{-1}^{+1}(u-u^3)\overline{\omega}(u)\, du$.

5. *More quasi-morphisms on the disc and the sphere*

So far, we have constructed explicit homogeneous quasi-morphisms which are non-trivial when restricted to the kernels of the corresponding Calabi's homomorphisms. In this section, we show that there exists an infinite number of linearly independent homogeneous quasi-morphisms on the disc. From these quasi-morphisms, we derive an infinite number of linearly independent homogeneous quasi-morphisms on the sphere concluding in this way, the proof of Theorem 1.2 for the sphere.

5.1. *Closed braids and their signatures.* Let us fix n distinct points $x_1^0, x_2^0, \ldots, x_n^0$ in the interior of the disc $\mathbb{D}^2$ and let us denote by $X_n(\mathbb{D}^2)$ the space of n-tuples of distinct points of $\mathbb{D}^2$. The fundamental group of $X_n(\mathbb{D}^2)$ based at $(x_1^0, x_2^0, \ldots, x_n^0)$ is the group of *pure braids of the disc* $\mathbb{D}^2$, denoted by $P_n(\mathbb{D}^2)$. Any braid γ in $P_n(\mathbb{D}^2)$ is represented by a loop $t \in [0,1] \mapsto (x_1^t, x_2^t, \ldots, x_n^t) \in X_n(\mathbb{D}^2)$, i.e. by a system of n disjoint arcs $t \mapsto (t, x_i^t)$ in the cylinder $[0,1]\times\mathbb{D}^2$. The identification $(0,x) \approx (1,x)$ for all x in $\mathbb{D}^2$ produces n disjoint oriented circles in the solid torus $\mathbb{R}/\mathbb{Z}\times\mathbb{D}^2$, images of the arcs $t \mapsto (t, x_i^t)$. The standard embedding of the solid torus in 3-space $\mathbb{R}^3$ allows us to associate with any pure braid γ, an oriented link i.e. a collection of n disjoint embeddings of an oriented circle in 3-space, called the *closed pure braid* associated with γ and denoted $\hat{\gamma}$.

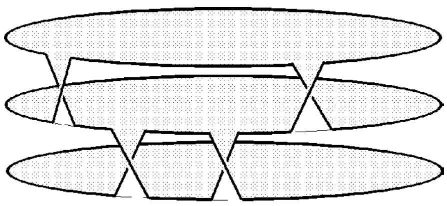

FIGURE 3. Seifert surface of a braid.

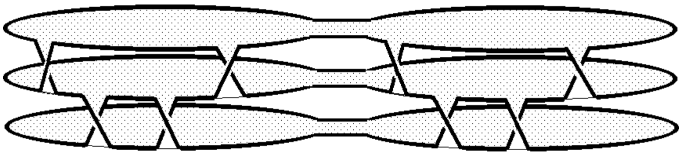

FIGURE 4. Signature is a quasi-morphism.

An important invariant of an oriented link is its *signature*. Let $\lambda \subset \mathbb{R}^3$ be such a link and let us choose a *Seifert surface*: an oriented surface S_λ embedded in $\mathbb{R}^3$ whose oriented boundary is λ. The first homology group $H_1(S_\lambda, \mathbb{Z})$ is equipped with a bilinear form B_λ in the following way. If x and y are two oriented curves on S_λ, one defines $B_\lambda(x, y)$ as being the linking number between x and a curve $y^\star$ obtained from y by pushing y a little away from S along the positive direction transverse to S. Clearly $B_\lambda(x, y)$ only depends on the homology classes of x and y on S. Turning B_λ into a symmetric bilinear form $\tilde{B}_\lambda(x, y) = B_\lambda(x, y) + B_\lambda(y, x)$, and tensoring by $\mathbb{R}$, we get a symmetric bilinear form on the vector space $H_1(S_\lambda, \mathbb{R})$. It turns out that the signature of this symmetric bilinear form is independent of the choice of the Seifert surface: it is the signature $sign(\lambda) \in \mathbb{Z}$ of the link λ. For these definitions and additional information, see [**8, 17, 18**].

Combining these two constructions, we get a map from the pure braid group $P_n(\mathbb{D}^2)$ to $\mathbb{Z}$ which associates with each braid γ the signature $sign(\hat{\gamma})$ of the link $\hat{\gamma}$.

PROPOSITION 5.1. *The mapping $\gamma \in P_n(\mathbb{D}^2) \mapsto sign(\hat{\gamma}) \in \mathbb{Z}$ is a quasi-morphism.*

Proof. Consider a pure braid α in $P_n(\mathbb{D}^2)$. We can manage so that, outside a cylinder in $\mathbb{R}^3$, a Seifert surface $S_{\hat{\alpha}}$ associated with the closed pure braid $\hat{\alpha}$ is a collection of n disjoint discs $D_{\alpha,x_1^0}, \ldots, D_{\alpha,x_n^0}$ (see Figure 3). Consider another pure braid β in $P_n(\mathbb{D}^2)$ and its Seifert surface $S_{\hat{\beta}}$. A simple way to construct a Seifert surface $S_{\widehat{\alpha.\beta}}$ associated with the closed pure braid $\widehat{\alpha\beta}$ consists in gluing each disc D_{α,x_i^0} with the disc D_{β,x_i^0} along an interval along their boundary (see Figure 4). Note that the Seifert surface $S_{\widehat{\alpha.\beta}}$ has n boundary components and that its first homology group of $S_{\widehat{\alpha.\beta}}$ contains a copy of the sum of the first homology groups of $S_{\hat{\alpha}}$ and $S_{\hat{\beta}}$ with codimension, at most, $n-1$ (by the Mayer–Vietoris exact sequence). Observe that if one considers the restriction of a quadratic form to a subspace of codimension q, then the signature of the quadratic form changes, at most, by q. Hence, the signature of $S_{\widehat{\alpha.\beta}}$ differs from the sum of the signatures of $S_{\hat{\alpha}}$ and $S_{\hat{\beta}}$ by, at most, $n - 1$. □

5.2. *More quasi-morphisms on the disc.* We now construct many independent homogeneous quasi-morphisms on the group $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$ using the motion of n points.

We start with an element g in $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$, an isotopy g_t from id to g and n distinct points $(x_1, x_2, \dots, x_n)$ in the disc $\mathbb{D}^2$. We consider a pure braid γ in $P_n(\mathbb{D}^2)$ obtained by the concatenation of three parts:

- $t \in [0, 1/3] \mapsto ((1-3t)x_i^0 + 3tx_i)_{i=1,\dots,n} \in X_n(\mathbb{D}^2)$;
- $t \in [1/3, 2/3] \mapsto (g_{3t-1}(x_i))_{i=1,\dots,n} \in X_n(\mathbb{D}^2)$;
- $t \in [2/3, 1] \mapsto ((3-3t)g(x_i) + (3t-2)x_i^0)_{i=1,\dots,n} \in X_n(\mathbb{D}^2)$.

For almost every $(x_1, x_2, \dots, x_n)$, this is indeed a loop in $X_n(\mathbb{D}^2)$ and defines a pure braid in $P_n(\mathbb{D}^2)$. Since $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$ is contractible, this braid does not depend on the isotopy: let us denote it by $\gamma(g; x_1, \dots, x_n)$. Clearly, for almost every $(x_1, x_2, \dots, x_n)$, we have

$$\gamma(gh; x_1, \dots, x_n) = \gamma(h; x_1, \dots, x_n).\gamma(g; h(x_1), \dots, h(x_n))$$

and, therefore,

$$|sign(\gamma(gh; \widehat{x_1, \dots}, x_n)) - sign(\gamma(h; \widehat{x_1, \dots}, x_n)) - sign(\gamma(g; h(\widehat{x_1), \dots}, h(x_n)))| \leq n-1.$$

As usual, we set

$$sign_n(g) = \int \dots \int sign(\gamma(g; \widehat{x_1, \dots}, x_n))\, d\, area(x_1) \dots d\, area(x_n) \in \mathbb{R}$$

and

$$\mathfrak{Sign}_{n,\mathbb{D}^2}(g) = \lim_{p \to +\infty} \frac{1}{p} sign_n(g^p).$$

Consider the homogeneous quasi-morphism *Sign* equivalent to *sign* and defined by $Sign(\alpha) = \lim_{p\to+\infty} \frac{1}{p} sign(\alpha^p)$. By setting

$$Sign_n(g) = \int \dots \int Sign(\gamma(g; \widehat{x_1, \dots}, x_n))\, d\, area(x_1) \dots d\, area(x_n) \in \mathbb{R},$$

we also have

$$\mathfrak{Sign}_{n,\mathbb{D}^2}(g) = \lim_{p \to +\infty} \frac{1}{p} Sign_n(g^p).$$

Clearly all these $\mathfrak{Sign}_{n,\mathbb{D}^2}$ are homogeneous quasi-morphisms on $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$. We shall show that they are linearly independent.

First, observe that the group $P_2(\mathbb{D}^2)$ is isomorphic to an infinite cyclic group $\mathbb{Z}$ whose generator is the pure braid ζ_2 on two strands 'turning once in the positive direction'. For all p in $\mathbb{Z}$, we have (see [**18**])

$$sign(\zeta_2^p) = 1 - 2p.$$

Thus, the mapping $\gamma \in P_2(\mathbb{D}^2) \simeq \mathbb{Z} \mapsto Sign(\widehat{\gamma}) \in \mathbb{Z}$ is a *homomorphism* and is the multiplication by -2. This homomorphism is also twice the linking number of the two strands. It follows that $\mathfrak{Sign}_{2,\mathbb{D}^2}$ is a *homomorphism* from $\mathrm{Diff}_0^\infty(\mathbb{D}^2, \partial\mathbb{D}^2, area)$ to $\mathbb{R}$. This is the homomorphism constructed by Calabi in a different form, explained by Fathi in his (unpublished) thesis, and detailed in [**12**]. Banyaga showed that the kernel of $\mathfrak{Sign}_{2,\mathbb{D}^2}$ is a simple group [**4**].

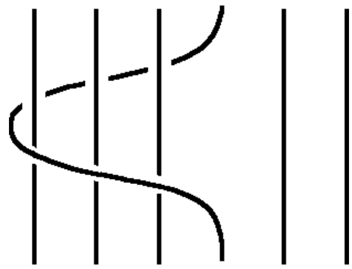

FIGURE 5. The braid $\eta_{4,6}$.

In order to show that the quasi-morphisms are independent, we shall evaluate them on the family of diffeomorphisms:

$$F_\omega : z \in \mathbb{D}^2 \mapsto \exp(2i\pi\omega(|z|))z \in \mathbb{D}^2$$

where $\omega : [0,1] \to \mathbb{R}$ is a function which vanishes in a neighborhood of 1 and is constant in a neighborhood of 0.

Let $x_1, \dots, x_n$ be n points in the disc such that $|x_1| < |x_2| < \cdots < |x_n|$ and, for $i = 2, \dots, n$, let $\eta_{i,n}$ be the pure braid where x_i makes one loop in the positive direction around $x_1 \dots, x_{i-1}$ (see Figure 5). Note that the $\eta_{i,n}$'s are commuting pure braids. A standard computation (using [**18**, p. 148]) shows that, for $i = 2, \dots, n$, $Sign(\eta_{i,n}) = 1-i$ when i is odd and $Sign(\eta_{i,n}) = -i$ when i is even.

As in §4, we introduce the spherical normalized area $a(r)$ of the disc of radius r in $\mathbb{C}$ centered at the origin. We also introduce the function u by the relation $a = (1-u)/2$. Finally, we define the function $\overline{\omega}$ on $[-1,+1]$ by $\omega(r) = \overline{\omega}(u(r))$.

LEMMA 5.2. $\mathfrak{Sign}_{n,\mathbb{D}^2}(F_\omega) = (n/4)\int_{-1}^{+1}(u^{n-1} + (n-1)u - n)\overline{\omega}(u)\,du.$

Proof. There exists $M(n) > 0$ such that, for each $p > 0$, and each $x_1, \dots, x_n$ such that $|x_1| < |x_2| < \cdots < |x_n|$, the pure braid $\gamma(F_\omega^p; x_1, \dots, x_n)$ reads as

$$\gamma(F_\omega^p; x_1, \dots, x_n) = \gamma_1.\eta_{i,n}^{[\omega(|x_2|)]} \dots \eta_{n,n}^{[\omega(|x_n|)]} \cdot \gamma_2$$

where $|Sign(\gamma_1)|$ and $|Sign(\gamma_2)|$ are smaller than $M(n)$ and $[-]$ stands for the integer part. Using the fact that homogeneous quasi-morphisms restrict to homomorphisms on Abelian subgroups, we get

$$|Sign(\gamma(F_\omega^p; \widehat{x_1, \dots}, x_n)) - \sum_{i=2}^{i=n} Sign(\eta_{i,n})[\omega(|x_i|)p]| \le 2M(n) + 2(n-1).$$

This yields

$$\mathfrak{Sign}_{n,\mathbb{D}^2}(F_\omega) = (n!)\int \dots \int_{|x_1|<\cdots<|x_n|} \sum_{i=2}^{i=n} Sign(\eta_{i,n})\omega(|x_i|)\,d\,area(x_1) \dots d\,area(x_n),$$

and, thus,

$$\mathfrak{Sign}_{n,\mathbb{D}^2}(F_\omega) = \int_0^1 \sum_{i=2}^{i=n} Sign(\eta_{i,n}) i \binom{n}{i} (a(r))^{i-1}(1-a(r))^{n-i}\omega(r)\,da(r).$$

The change of variable $a = (1-u)/2$ and the value $Sign(\eta_{i,n}) = -i$ for even i and $1-i$ for odd i produces a formula which simplifies (miraculously?) and gives the lemma. $\square$

In order to prove that the $\mathfrak{Sign}_{n,\mathbb{D}^2}$ are linearly independent, it is enough to observe that the polynomials of degree n appearing in the previous lemma have non-zero terms of degree n and are, therefore, linearly independent.

5.3. *More on the 2-sphere.* For $n \geq 2$, denote by $X_n(\mathbb{S}^2)$ the space of n-tuples of distinct points on the 2-sphere. Consider $n-1$ distinct points $x_1^0, \dots, x_{n-1}^0$ on $\mathbb{S}^2$ all of them different from ∞. The *pure braid group of the sphere* $\mathbb{S}^2$, denoted by $P_n(\mathbb{S}^2)$, is the fundamental group of the space $X_n(\mathbb{S}^2)$ where we choose $(\infty, x_1^0, \dots, x_{n-1}^0)$ as a base point. Adding the point at infinity, one can consider $X_{n-1}(\mathbb{D}^2)$ as a subset of $X_n(\mathbb{S}^2)$. The induced map on fundamental groups

$$P_{n-1}(\mathbb{D}^2) \to P_n(\mathbb{S}^2)$$

is onto and its kernel is the infinite cyclic group $\mathbb{Z} \cdot \zeta_{n-1}^2$ generated by the pure braid ζ_{n-1}^2 which is a double turn:

$$\zeta_{n-1}^2(t) = (\exp(4i\pi t) \cdot x_1^0, \dots, \exp(4i\pi t) \cdot x_{n-1}^0).$$

Recall that the fundamental group of $\mathrm{Diff}_0^\infty(\mathbb{S}^2)$ is isomorphic to the fundamental group of $\mathrm{SO}(3)$, which is a cyclic group of order 2 generated by one simple turn. Note also that ζ_{n-1} is a generator of the center of the pure braid group $P_{n-1}(\mathbb{D}^2)$. Hence, the group $P_n(\mathbb{S}^2)$ is isomorphic to the quotient of $P_{n-1}(\mathbb{D}^2)$ by the central subgroup $\mathbb{Z} \cdot \zeta_{n-1}^2$ (see [**8**, Ch. 4]).

There is a canonical *homomorphism* $lk_{n-1} : P_{n-1}(\mathbb{D}^2) \to \mathbb{Z}$ mapping every pure braid to the half-sum of the linking numbers of all pairs of strands chosen in the braid. On has $lk_{n-1}(\zeta_{n-1}) = (n-1)(n-2)$. Moreover, since ζ_{n-1}^2 is in the center of $P_{n-1}(\mathbb{D}^2)$ and *Sign* is a homogeneous quasi-morphism, we have, for any pure braid γ in $P_{n-1}(\mathbb{D}^2)$,

$$Sign(\widehat{\gamma \cdot \zeta_{n-1}^2}) = Sign(\hat{\gamma}) + Sign(\zeta_{n-1}^2).$$

The mapping

$$s_{n-1} : \gamma \in P_{n-1}(\mathbb{D}^2) \mapsto Sign(\hat{\gamma}) - \frac{Sign(\zeta_{n-1})}{lk_{n-1}(\zeta_{n-1})} lk_{n-1}(\gamma) \in \mathbb{Z}$$

descends to a quasi-morphism $\tilde{s}_n : P_n(\mathbb{S}^2) \to \mathbb{Z}$. Note that $\tilde{s}_2$ vanishes on $P_2(\mathbb{S}^2)$ and $\tilde{s}_3$ on $P_3(\mathbb{S}^2)$.

We proceed as in the case of the disc. We start with an element g in $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ and an isotopy g_t. We construct an element $\gamma(g; x_1, \dots, x_n)$ in $P_n(\mathbb{S}^2)$, for almost every n-tuple of distinct points $x_1, \dots, x_n$ on the sphere. Then, we use the quasi-morphism $\tilde{s}_n$ and integrate $\tilde{s}_n(\gamma(g; x_1, \dots, x_n))$ over the space of n-tuples to get a quasi-morphism $\tilde{s}_n$ on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ and use homogenization so that we finally get a homogeneous quasi-morphism

$$\mathfrak{Sign}_{n,\mathbb{S}^2} : \mathrm{Diff}_0^\infty(\mathbb{S}^2, area) \to \mathbb{R},$$

for each $n \geq 2$.

In order to show that these homogeneous quasi-morphisms generate an infinite dimensional vector space, we shall evaluate the quasi-morphisms $\mathfrak{Sign}_{n,\mathbb{S}^2}$ on the family of diffeomorphisms:

$$F_\omega : \mathbb{S}^2 \to \mathbb{S}^2$$

such that $F_\omega(\infty) = \infty$, and defined on $\mathbb{C}$ by

$$z \in \mathbb{C} \mapsto \exp(2i\pi\omega(|z|))z \in \mathbb{C},$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ is a function which is constant in a neighborhood of 0 and outside some compact set. Introducing again the parametrization $u = 1 - 2a(r)$ where $a(r)$ is the spherical area of the disc in $\mathbb{C}$ with radius r centered at 0 and $\overline{\omega}(u) = \omega(r)$, we have the following lemma.

LEMMA 5.3. *For each even integer $n \geq 4$,*

$$\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n}{4}\int_{-1}^{+1}(u^{n-1} - u)\overline{\omega}(u)\,du$$

and, for each odd integer $n \geq 5$,

$$\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n-1}{4}\int_{-1}^{+1}(u^{n-2} - u)\overline{\omega}(u)\,du.$$

Proof. Consider n points $x_1 \dots x_n$ in $\mathbb{S}^2$. We denote by $l(x_1, \dots, x_n)$ the limit

$$\lim_{p\to+\infty}\frac{1}{p}\tilde{s}_n(F_\omega^p; x_1, \dots, x_n).$$

Note that $l(x_1, \dots, x_n)$ might depend of the ordering of the x_i. In order to compute the value of $\tilde{s}_n$ on a pure braid in $P_n(\mathbb{S}^2)$, one has first to send the first strand to infinity and then to compute the value of s_{n-1} on the resulting braid in $P_{n-1}(\mathbb{D}^2)$ which is well defined up to a power of ζ_{n-1}^2. We shall use the following notation. We assume that $|x_1| < |x_2| < \dots < |x_n|$ and we denote by $l_j(x_1, \dots, x_n)$ the value of $l(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation σ of $\{1, \dots, n\}$ which is such that $\sigma(1) = j$, i.e. such that the strand sent to infinity is x_j. Our first goal is to compute $l_j(x_1, \dots, x_n)$ for $j = 1, \dots, n$.

Looking at the image of the inversion $z \mapsto 1/(z - z_j)$, we conclude that there exists a positive constant $M'(n)$ such that, for each $p > 0$ and each $(x_1, \dots, x_n)$, the pure braid $\gamma(F_\omega^p; x_{\sigma(1)}, \dots, x_{\sigma(n)})$ in $P_n(\mathbb{S}^2)$ is conjugate to the image by the morphism $P_{n-1}(\mathbb{D}^2) \to P_n(\mathbb{S}^2)$ of a braid in $P_{n-1}(\mathbb{D}^2)$ which reads as

$$\gamma_1 \cdot \eta_{2,n-1}^{p[\omega(|x_2|)-\omega(|x_n|)]} \dots \eta_{j-1,n-1}^{-p[\omega(|x_{j-1}|)-\omega(|x_j|)]}\eta_{2,n-1}^{p[\omega(|x_j|)-\omega(|x_{n-1}|)]} \dots \eta_{n-j,n-1}^{p[\omega(|x_j|)-\omega(|x_{j+1}|)]} \cdot \gamma_2$$

where $|\,Sign(\gamma_1)|$ and $|\,Sign(\gamma_2)|$ are smaller than $M'(n)$.

Recall that, for i even, $Sign(\eta_{i,n}) = -i$ and, for i odd, $Sign(\eta_{i,n}) = 1 - i$ so that it is not difficult to compute the values of $s_n(\eta_{i,n})$. We get

$$s_n(\eta_{i,n}) = \begin{cases} \dfrac{i-n}{n-1} & \text{if } n \text{ is even and } i \text{ even,} \\ \dfrac{i-1}{n-1} & \text{if } n \text{ is even and } i \text{ odd,} \\ \dfrac{i-n-1}{n} & \text{if } n \text{ is odd and } i \text{ even,} \\ \dfrac{i-1}{n} & \text{if } n \text{ is odd and } i \text{ odd.} \end{cases}$$

Suppose first that n is even. We get

$$l_j = \sum_{i=2}^{j-1} s_{n-1}(\eta_{j,n-1})(\omega(|x_i|) - \omega(|x_j|)) + \sum_{i=j+1}^{n-1} s_{n-1}(\eta_{n+1-j,n-1})(\omega(|x_j|) - \omega(|x_i|)).$$

Taking into consideration the facts that, for n even and $i_1 + i_2 = n + 1$, we have $s_{n-1}(\eta_{i_1,n-1}) + s_{n-1}(\eta_{i_2,n-1}) = 0$ and that $\sum_{i=2}^{n-1} s_{n-1}(\eta_{i,n-1}) = 0$, we get

$$l_j = \sum_{i=2}^{n-1} s_{n-1}(\eta_{j,n-1})\omega(|x_i|).$$

Note that, in this case, the expression l_j is independent of j. Substituting the values of the s_{n-1}, we find that

$$l_j = \frac{1}{n-1}\left(\sum_{i=1}^{(n-2)/2} (2i-n)\omega(|x_{2i}|)\right) + \left(\sum_{i=1}^{(n-2)/2} (2i)\omega(|x_{2i+1}|)\right).$$

We now have to integrate the value of $l(x_1, \ldots, x_n)$ over the space of all n-tuples of points (not necessarily ordered by increasing modulus). We get

$$\begin{aligned} \mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n!}{n-1}\int_{\mathbb{S}^2} \Bigg(&\sum_{i=1}^{(n-2)/2} \frac{(2i-n)}{(2i-1)!(n-2i)!} a(|x|)^{2i-1}(1-a(|x|))^{n-2i} \\ &+ \sum_{i=1}^{(n-2)/2} \frac{2i}{(2i)!(n-2i-1)!} a(|x|)^{2i}(1-a(|x|))^{n-2i-1}\Bigg)\omega(|x|)\, d\, area(x). \end{aligned}$$

Changing variables $a = (1-u)/2$, we get

$$\begin{aligned} \mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n}{n-1}\int_{-1}^{+1} \Bigg(&\sum_{i=1}^{(n-2)/2} (2i-n)\binom{n-1}{2i-1}(1-u)^{2i-1}(1+u)^{n-2i} \\ &+ \sum_{i=1}^{(n-2)/2} (2i)\binom{n-1}{2i}(1-u)^{2i}(1+u)^{n-2i-1}\Bigg) 2^{-n}\, du. \end{aligned}$$

After standard 'manipulations', we finally get

$$\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n}{4}\int_{-1}^{+1} (u^{n-1} - u)\overline{\omega}(u)\, du.$$

This proves the proposition for n even.

The case where n is odd is a similar (boring) computation. We first compute l_j starting with the same formula as before:

$$l_j = \sum_{i=2}^{j-1} s_{n-1}(\eta_{j,n-1})(\omega(|x_i|) - \omega(|x_j|)) + \sum_{i=j+1}^{n-1} s_{n-1}(\eta_{n+1-j,n-1})(\omega(|x_j|) - \omega(|x_i|)).$$

Using the facts that, for n odd and $i_1 + i_2 = n$, we have $s_{n-1}(\eta_{i_1,n-1}) + s_{n-1}(\eta_{i_1,n-1}) = 0$ and that $\sum_{i=2}^{n-1} s_{n-1}(\eta_{i,n-1}) = 0$, we get

$$l_j = \sum_{i=2}^{j-1} s_{n-1}(\eta_{i,n-1})\omega(|x_i|) + \sum_{i=j+1}^{n-1} s_{n-1}(\eta_{i-1,n-1})\omega(|x_i|).$$

Note that, in this case, l_j does depend on j. Fixing the x_i's and averaging the l_j for j from 1 to n, we get, after substituting the values of the s_{n-1},

$$\frac{1}{n}(l_1 + \cdots + l_n) = \frac{1}{n} \sum_{i=1}^{(n-1)/2} (4i - n - 1)\omega(|x_{2i}|).$$

We now have to integrate over all n-tuples of points and get

$$\begin{aligned}\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n!}{n} \int_{\mathbb{S}^2} \sum_{i=1}^{(n-1)/2} & \frac{(4i - n - 1)}{(2i - 1)!(n - 2i)!} \\ & \times a(|x|)^{2i-1}(1 - a(|x|)^{n-2i})\omega(|x|)\, d\, area(x).\end{aligned}$$

Using the variable u, we have

$$\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \int_{-1}^{+1} \sum_{i=1}^{(n-1)/2} (4i - n - 1)\binom{n-1}{2i-1}(1-u)^{2i-1}(1+u)^{n-2i}2^{-n}\overline{\omega}(u)\, du.$$

Again this simplifies to give the lemma for n odd:

$$\mathfrak{Sign}_{n,\mathbb{S}^2}(F_\omega) = \frac{n-1}{4} \int_{-1}^{+1} (u^{n-2} - u)\overline{\omega}(u)\, du.$$

The lemma is established. □

We deduce from the lemma that the homogeneous quasi-morphisms $\mathfrak{Sign}_{2n,\mathbb{S}^2}$ are linearly independent and this concludes the proof of Theorem 1.2 for the sphere.

The lemma also suggests that $\mathfrak{Sign}_{2n,\mathbb{S}^2}$ and $\mathfrak{Sign}_{2n+1,\mathbb{S}^2}$ might coincide but we did not try to establish this (note that in §6.3 we construct some examples of non-trivial quasi-morphisms which vanish on all diffeomorphisms of the form F_ω).

To finish this section, we relate $\mathfrak{Sign}_{4,\mathbb{S}^2}$ to the quasi-morphism $\mathfrak{Turn}$ constructed in §4.

PROPOSITION 5.4. *The homogeneous quasi-morphisms* $\mathfrak{Turn}$ *and* $-\frac{3}{2}\mathfrak{Sign}_{4,\mathbb{S}^2}$ *coincide.*

Proof. We only sketch the main ideas of the proof: this is somehow contained 'between the lines' in [**3, 6, 16**] and, moreover, this is a very special case of a result described by the authors in a forthcoming paper [**13**].

Recall that if (M^3, τ) is a closed oriented 3-manifold equipped with a trivialization of its tangent bundle, one can define an integer *def*(M^3, τ) (called the signature defect) as the difference $3sign(W) - p_1(W)$ where W is any 4-manifold with boundary M^3 and $sign(W)$ is the signature of W and $p_1(W)$ is the first Pontryagin number of W relative to the trivialization on the boundary given by τ (see, for instance, [**3**]).

There is a canonical identification between the braid group on 3-strands $B_3(\mathbb{D}^2)$ and the group $\widetilde{\mathrm{PSL}(2,\mathbb{Z})}$, inverse image of $\mathrm{PSL}(2,\mathbb{Z})$ in the universal cover of $\mathrm{PSL}(2,\mathbb{R})$. One way to see this identification is to associate to each triple of points in a disc the 2-fold cover of the sphere $\mathbb{S}^2$ ramified on these points and at infinity, which is a torus. In this way, a 3-braid yields a torus fibration over the circle equipped with a trivialization of its tangent bundle, which is the same object as an element of $\widetilde{\mathrm{PSL}(2,\mathbb{Z})}$.

The main point is to show that the signature map on $B_3(\mathbb{D}^2)$ coincide s with the signature defect $-def/3$ on $\widetilde{\mathrm{PSL}(2,\mathbb{Z})}$: this is the statement that one can 'almost find' in [**3, 6, 16**] and that we shall not detail it more precisely here.

The center of $\widetilde{\mathrm{PSL}(2,\mathbb{Z})}$ is generated by a single element z and there is a homomorphism $v : \widetilde{\mathrm{PSL}(2,\mathbb{Z})} \to \mathbb{Z}$ such that $v(z) = 1$. The map $def/12 - v$ descends to a map on the quotient of $\widetilde{\mathrm{PSL}(2,\mathbb{Z})}$ by its center, i.e. on $\mathrm{PSL}(2,\mathbb{Z})$. In [**6**], this map has been identified (up to a bounded term) with one-sixth of the Rademacher function $\mathfrak{Radem}$ defined in §4, using left and right turns in the tree. We know that the quotient of $P_3(\mathbb{D}^2)$ by a subgroup of index 2 in its center is isomorphic to $P_4(\mathbb{S}^2)$. All these identifications imply that the quasi-morphisms $s_3 = (Sign - \frac{2}{3}lk_3)$ and $-(def/3 - 4v)$ (defined on the 'same group') are equivalent. The first map has been used to define the quasi-morphism $\mathfrak{Sign}_{4,\mathbb{S}^2}$ and the second one to define $\mathfrak{Turn}$. This implies the proposition. □

5.4. *More on the sphere: a Ruelle type quasi-morphism.* In this section we construct a quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ in the spirit of Ruelle's quasi-morphisms.

The definitions of Ruelle's quasi-morphisms on the disc and the torus use the triviality of the tangent bundle of theses surfaces and the definition on higher-genus surfaces uses some kind of 'quasi' triviality of the tangent bundle given by the circle at infinity. We now try to give a definition in the case of the sphere. Instead of using the action on tangent vectors, we use the action on *pairs* of tangent vectors. Denote by $T_2(\mathbb{S}^2)$ the space of pairs of non-zero tangent vectors $(\delta x_1, \delta x_2)$ at distinct points x_1, x_2 of the sphere. Observe that the fundamental group of $T_2(\mathbb{S}^2)$ is infinite cyclic so that it does make sense to say that a curve in $T_2(\mathbb{S}^2)$ turns. In order to give a quantitative statement, we again identify the sphere with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The complex differential form

$$\theta = \frac{dx_1\, dx_2}{(x_1 - x_2)^2}$$

can be seen as a holomorphic 1-form on the space of pairs of distinct points on $\mathbb{C}P^1$, or as a function on $T_2(\mathbb{S}^2)$. Note that this form is invariant under the projective action of $\mathrm{PGL}(2;\mathbb{C})$ and, in particular, θ is well defined and non-singular when x_1 or x_2 is at infinity. As for the geometrical meaning of θ, observe that θ is the 'cross ratio of the four points $x_1, x_1 + \delta x_1, x_1, x_2 + \delta x_2$'. Given a curve $c : [0,1] \to T_2(\mathbb{S}^2)$ we define the rotation angle $Ang(c) \in \mathbb{R}$ as the variation the argument of the complex number $\theta(c)$. This is invariant under homotopies fixing the endpoints.

We can now proceed as in the case of the disc. Start with an element g in $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ and choose an isotopy $(g_t)_{t\in[0,1]}$ and an element $v = (x_1, \delta x_1; x_2, \delta x_2)$ of $T_2(\mathbb{S}^2)$. We can consider the image v_t of v by the differential of g_t. This gives a curve in $T_2(\mathbb{S}^2)$ and, therefore, some rotation angle $Ang_g(x_1, \delta x_1; x_2, \delta x_2)$. Fixing x_1 and x_2 and changing the tangent vectors $\delta x_1, \delta x_2$ changes this rotation angle by at most two full turns. We can, therefore, define $Ang_g(x_1, x_2)$ as the minimum of $Ang(x_1, \delta x_1; x_2, \delta x_2)$ over all choices of $\delta x_1, \delta x_2$. We now set

$$r(g) = \int_{\mathbb{S}^2\times\mathbb{S}^2} Ang_g(x_1, x_2)\, d\, area(x_1)\, d\, area(x_2)$$

and, by homogenization,

$$\mathfrak{Ruelle}_{\mathbb{S}^2}(g) = \lim_{p\to+\infty} \frac{1}{p} r(g^p).$$

Clearly this defines a homogeneous quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ that we call *Ruelle's quasi-morphism* on the sphere.

When g has a support contained in a disc, we relate this invariant to previously defined invariants.

LEMMA 5.5. *Let g be an element of* $\mathrm{Diff}_0^\infty(\mathbb{S}^2, area)$ *which is the identity outside of a disc* $D \subset \mathbb{C} \subset \mathbb{S}^2$. *Then*

$$\mathfrak{Ruelle}_{\mathbb{S}^2}(g) = 2\mathfrak{Ruelle}_D(g_{|D}) + \mathfrak{Sign}_{2,D}(g_{|g}).$$

Proof. Choose an isotopy supported in D. If both x_1 and x_2 are outside D, the angle $Ang_g(x_1, x_2)$ is obviously zero. If x_1 is outside the disc D and x_2 inside, the variation of the argument of $dg_t(\delta x_1)dg_t(\delta x_2)/(g_t(x_1) - g_t(x_2))^2$ is equal to the variation of the argument of $dg_t(\delta x_2)$ up to 1 since the vector $(g_t(x_1) - g_t(x_2))$ remains in some sector. In this case, we have

$$|Ang_g(x_1, x_2) - Ang_g(x_2)| < 1.$$

If both x_1 and x_2 are in the disc, we see that $Ang_g(x_1, x_2)$ is equal (up to 2) to $Ang_g(x_1) + Ang_g(x_2)$ minus twice the variation of the argument of the vector $(g_t(x_1) - g_t(x_2))$. The latter is precisely the quantity which is used to define the invariant $\mathfrak{Sign}_{2,D}(g_{|g})$.

Summing up all parts, we get the lemma. □

6. *More quasi-morphisms on the torus and on surfaces of genus at least 2*

We could try to generalize the previous constructions on surfaces Σ of genus at least 1, but the structure of the corresponding braid groups $P_n(\Sigma)$ (the fundamental groups of the spaces of n-tuples of distinct points in Σ) are much more complicated. Instead, we prefer to choose a different route in order to construct infinitely many linearly independent quasi-morphisms on the groups $\mathrm{Diff}_0^\infty(\Sigma, area)$.

6.1. *Surfaces of genus at least 2.*

Let Σ be an oriented closed surface of genus at least 2. Choose a metric of curvature -1 on Σ so that the universal cover of Σ can be identified with the Poincaré disc $\mathbb{D}^2$.

The construction that we present is inspired by the construction of quasi-morphisms on the fundamental group of Σ presented in [**5**]. Let α be a (not necessarily closed) differential form of degree 1 on Σ and denote by $\tilde{\alpha}$ its lift to the universal cover.

As usual, if g is an element of $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$, we choose an isotopy g_t from id to g and we lift it as an isotopy $\tilde{g}_t$ of the Poincaré disc, from id to some lift $\tilde{g}$. For every point $\tilde{x}$ in the disc, consider the curve $t \in [0,1] \mapsto \tilde{g}_t(x) \in \mathbb{D}^2$. Let us denote by $\delta(g, \tilde{x})$ the *geodesic* arc in $\mathbb{D}^2$ connecting the endpoints of this curve and note that this geodesic only depends on g and $\tilde{x}$ as the notation suggests. Now define

$$I_\alpha(g, \tilde{x}) = \int_{\delta(g,\tilde{x})} \tilde{\alpha} \in \mathbb{R}.$$

We claim that there is a constant $C_\alpha > 0$ such that, for every g, h and $\tilde{x}$,

$$|I_\alpha(gh, \tilde{x}) - I_\alpha(h, \tilde{x}) - I_\alpha(g, \tilde{h}_1(\tilde{x}))| \leq C_\alpha.$$

Indeed, by Stokes' theorem, this difference is the integral of the 2-form $d\tilde{\alpha}$ on a geodesic triangle and we know that a geodesic triangle in the Poincaré disc has an area which is bounded by π so that this integral is bounded by π multiplied by the supremum of the norm of $d\alpha$. Observe also that for each g, the function $\tilde{x} \mapsto I_\alpha(g, \tilde{x})$ is clearly invariant by deck transformations and defines a function $x \mapsto I_\alpha(g, x)$ on Σ. We define

$$i_\alpha(g) = \int_\Sigma I_\alpha(g, x)\, d\ \mathit{area}(x)$$

and

$$\mathfrak{Calabi}_{\alpha,\Sigma}(g) = \lim_{p \to +\infty} \frac{1}{p}\, i_\alpha(g^p).$$

Note that when α is an exact form, this invariant vanishes and that when α is closed, we recover the original Calabi's homomorphism described in §3.

We claim that these homogeneous quasi-morphisms $\mathfrak{Calabi}_{\alpha,\Sigma}$ span an infinite dimensional vector space.

Let c be a *simple* closed geodesic in Σ and choose an embedded collar $\iota : (s,t) \in [-\epsilon, +\epsilon] \times \mathbb{R}/\mathbb{Z} \mapsto \iota(s,t) \in \Sigma$ such that the restriction of ι to $\{0\} \times \mathbb{R}/\mathbb{Z}$ is the curve c and that the form $\iota^\star \mathit{area}$ is $ds \wedge dt$. Now let $\omega : [-\epsilon, \epsilon] \to \mathbb{R}$ be a map which vanishes in the neighborhood of $\pm\epsilon$ and define a diffeomorphism $F_{c,\iota,\omega}$ of Σ which is the identity outside the collar and which is given in the collar by

$$F_{c,\iota,\omega}(s,t) = (s, t + \omega(s)).$$

This map preserves the form $ds \wedge dt$ and is isotopic to the identity.

Choose a lift $\tilde{\iota} : [-\epsilon, +\epsilon] \times \mathbb{R} \to \mathbb{D}^2$ whose restriction to $\{0\} \times \mathbb{R}$ is a lift $\tilde{c} : \mathbb{R} \to \mathbb{D}^2$. If we choose a point $\tilde{x}$ of the form $\tilde{\iota}(s, \tilde{t})$, the sequence of geodesic arcs $\delta(F^p_{c,\iota,\omega}, \tilde{x})$ when the integer p goes to infinity approximates the sequence of geodesic arcs $\tilde{c}([\tilde{t}, \tilde{t} + p\omega(s)])$. Hence, we deduce that

$$\mathfrak{Calabi}_{\alpha,\Sigma}(F_{c,\iota,\omega}) = \left(\int_c \alpha\right)\left(\int_{-\epsilon}^{+\epsilon} \omega(s)\, ds\right).$$

Now it is easy to show that the $\mathfrak{Calabi}_{\alpha,\Sigma}$ span an infinite-dimensional vector space. Indeed, choose k distinct closed simple geodesics $c_1, c_2, \dots, c_k$ on Σ and k (not closed!) differential forms $\alpha_1, \dots, \alpha_k$ such that the integral of α_i on c_j is the Kronecker symbol $\delta_{i,j}$. Choose collars ι_i and functions ω_i in order to define diffeomorphisms $F_i = F_{c,\iota_i,\omega_i}$. With a suitable choice of ω_i, we have $\mathfrak{Calabi}_{\alpha_i,\Sigma}(F_j) = \delta_{i,j}$ so that the vector space generated by the $\mathfrak{Calabi}_{\alpha,\Sigma}$ has dimension at least k, for every integer k.

6.2. *The case of the torus.* The torus does not admit any negatively curved metric so that the previous construction does not apply in this case.

Consider the space $X_2(\mathbb{T}^2)$ of ordered pairs of distinct points (x, y) in $\mathbb{T}^2$. Using the group structure on the torus, we have a natural map $\pi : (x, y) \in X_2(\mathbb{T}^2) \mapsto x - y \in \mathbb{T}^2 \setminus \{0\}$. Choose a complete Riemannian metric with curvature -1 on $\mathbb{T}^2 \setminus \{0\}$ so that its universal cover can be identified with the Poincaré disc $\mathbb{D}^2$.

Let α be a 1-form with compact support in $\mathbb{T}^2 \setminus \{0\}$. Let g be an element of $\mathrm{Diff}_0^\infty(\mathbb{T}^2, \mathit{area})$, g_t an isotopy from id to g. If (x, y) is a point in $X_2(\mathbb{T}^2)$, we can consider the curve $g_t(x) - g_t(y)$ in $\mathbb{T}^2 \setminus \{0\}$, choose a lift to the Poincaré disc, draw the geodesic arc with the same endpoints and finally integrate the form $\tilde{\alpha}$ on this arc to produce a number $I_\alpha(g; x, y)$. Finally, define

$$i_\alpha(g) = \iint I_\alpha(g; x, y)\, d\,\mathit{area}(x)\, d\,\mathit{area}(y)$$

and

$$\mathfrak{Calabi}_{\alpha,\mathbb{T}^2}(g) = \lim_{p\to+\infty} \frac{1}{p} i_\alpha(g^p).$$

We claim that the homogeneous quasi-morphisms $\mathfrak{Calabi}_{\alpha,\mathbb{T}^2}$ span an infinite dimensional vector space.

Let D and D_0 be two small disjoint Euclidean discs in the torus, the disc D being centered at the origin of the torus and D_0 at some point x_0. Choose a closed curve $c : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2 \setminus \{0\}$, not necessarily simple, which is a geodesic for the given hyperbolic metric. If the discs D, D_0 are small enough, we can find a closed curve $f : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2$ such that $f(0) = 0$, all discs $D_t = f(t) + D_0$ are disjoint from D and the closed curve $f(t) + x_0$ is homotopic to c in the punctured torus $\mathbb{T}^2 \setminus \{0\}$. Choose some isotopy $(g_t)_{t\in[0,1]}$ of the torus, such that $g_0 = \mathrm{id}$, the restriction of g_t to D is the identity and the restriction of g_t to D_0 is the translation by $f(t)$. Denote g_1 by g and note that the restriction of g on the two discs D and D_0 is the identity.

Let us evaluate $\mathfrak{Calabi}_{\alpha,\mathbb{T}^2}(g)$. If x and y belong both to the same disc, we clearly have

$$I_\alpha(g; x, y) = 0$$

since $g_t(x) - g_t(y)$ is constant in this case. By construction, it is clear that if x belongs to D_0 and y belongs to D, we have

$$\lim_{p\to+\infty} \frac{1}{p} I_\alpha(g^p; x, y) = \int_c \alpha.$$

If x belongs to D and y belongs to D_0, we have

$$\lim_{p\to+\infty} \frac{1}{p} I_\alpha(g^p; x, y) = \int_{\bar{c}} \alpha$$

where $\overline{c}$ is the image of the geodesic c by the map $x \mapsto -x$ in the punctured torus (we may assume that this map is an isometry for the chosen hyperbolic metric). When x or y is outside of the discs, the only estimate that we can write is that $|I_\alpha(g; x, y)| \le K(g)\|\alpha\|$ where $\|\alpha\|$ is the sup norm of the form α and $K(g)$ is the supremum over x, y of the hyperbolic distance between the endpoints of the lifts of the curves $t \in [0,1] \mapsto g_t(x) - g_t(y) \in \mathbb{T}^2 \setminus \{0\}$. This gives the following estimate:

$$\left|\mathfrak{Calabi}_{\alpha,\mathbb{T}^2}(g) - area(D)area(D_0)\left(\int_c \alpha + \int_{\overline{c}} \alpha\right)\right| \le 2(1 - area(D) - area(D_0))K(g)\|\alpha\|.$$

These computations are valid for any area form which is invariant under g, not necessarily the Lebesgue measure on the torus. Note that if we multiply *area* by a function which is constant outside the union of the two discs, we get another area form which is invariant by g. In particular, we can find a sequence of area forms $area_n$, of total area 1, which are invariant by g and such that the $area_n$ outside the discs is less than $1/n$. Since we are using several area forms, let us momentarily introduce it explicitly in the notation, i.e. we denote by $\mathfrak{Calabi}_{\alpha,\mathbb{T}^2,area}$ the homogeneous quasi-morphism under study of $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$. We have, therefore, the estimate

$$\left|\mathfrak{Calabi}_{\alpha,\mathbb{T}^2,area_n}(g) - area_n(D)\ area_n(D_0)\left(\int_c \alpha + \int_{\overline{c}} \alpha\right)\right| \le \frac{2}{n}K(g)\|\alpha\|.$$

We can at last prove the fact that our quasi-morphisms span an infinite dimensional vector space. Choose an integer k and k closed geodesics $c_1, c_2, \ldots, c_k$ in the punctured torus. We can assume that the $2k$ geodesics $c_1, \overline{c_1}, c_2, \overline{c_2}, \ldots, c_k, \overline{c_k}$ are different. Note that the symmetry $x \mapsto -x$ on the torus reverses many closed geodesics: those which correspond to palindromic words in the two generators of the fundamental group, for instance *simple* closed geodesics, and this is the reason why we do have to use non-simple curves, unlike the previous case when the genus of Σ is at least 2. Choose k differential forms $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\int_{c_i} \alpha_j = \delta_{i,j}$ and $\int_{\overline{c_i}} \alpha_j = 0$. Choose two discs D, D_0 which are small enough. Using the previous construction, we get k area-preserving diffeomorphisms $g_1, \ldots, g_k$. Choose the area form $area_n$ as before. Denote by K the maximum of $K(g_1), \ldots, K(g_k)$ and by A the maximum of $\|\alpha_1\|, \|\alpha_2\|, \ldots, \|\alpha_k\|$. Our estimate yields

$$|\mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2,area_n}(g_j) - area_n(D)area_n(D_0)\delta_{i,j}| \le \frac{2}{n}KA.$$

By Moser's lemma, there is a diffeomorphism F_n of the torus which is isotopic to the identity and satisfies $F_n^\star area = area_n$. Of course the diffeomorphisms $g_{i,n} = F_n g_i F_n^{-1}$ preserve *area* and $\mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2,area_n}(g_j) = \mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2,area}(g_{j,n})$. Hence, the k elements $g_{i,n} = F_n g_i F_n^{-1}$ of the group $\mathrm{Diff}_0^\infty(\mathbb{T}^2, area)$ satisfy

$$|\mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2,area}(g_{j,n}) - area_n(D)area_n(D_0)\delta_{i,j}| \le \frac{2}{n}KA.$$

Choosing n large enough, we conclude that the k homogeneous quasi-morphisms $\mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2,area}$ (or $\mathfrak{Calabi}_{\alpha_i,\mathbb{T}^2}$, to come back to our original notation) are linearly independent. This proves that the vector space of these homogeneous quasi-morphisms is infinite dimensional.

6.3. *Still more on the sphere!* As a final remark for this section, we observe that, using similar ideas, we can construct more independent homogeneous quasi-morphisms on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$.

Choose a 1-form α with compact support on the sphere minus three points $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and consider the complete metric with curvature -1 on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ (unique up to isometry). Denote by $\tilde{\alpha}$ the lift of α to the universal cover, identified with the Poincaré disc $\mathbb{D}^2$.

As we have seen in §4, four distinct points z_1, z_2, z_3, z_4 of $\overline{\mathbb{C}}$ and an isotopy g_t define a cross-ratio curve

$$c : t \in [0, 1] \mapsto [g_t(z_1), g_t(z_2), g_t(z_3), g_t(z_4)] \in \overline{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

As before, we can lift it as a curve $\tilde{c}$ in $\mathbb{D}^2$ and integrate $\tilde{\alpha}$ along the geodesic arc of $\mathbb{D}^2$ with the same endpoints as $\tilde{c}$. This produces a number $I_\alpha(g; z_1, z_2, z_3, z_4)$ from which we can extract, as usual, a homogeneous quasi-morphism $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$ on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$. We have already observed that the symmetric group on four letters $\mathfrak{S}(4)$ acts (isometrically) on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and we shall assume that the form α is invariant under this action so that $I_\alpha(g; z_1, z_2, z_3, z_4)$ is a symmetric function of (z_1, z_2, z_3, z_4). Exactly as before, we can show that $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$ does not change if one replaces α by $\alpha + df$ where f is a function which is invariant by $\mathfrak{S}(4)$. Note that a closed 1-form on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ which is invariant under $\mathfrak{S}(4)$ is exact, so that this procedure does not construct *homomorphisms* on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ (which would be in contradiction with Banyaga's theorem!). Using the same ideas as in the previous section, it is easy to show that the $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$'s span an infinite-dimensional space of homogeneous quasi-morphisms.

The quasi-morphisms $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$ have an interesting feature: they vanish on the set of diffeomorphisms that we have studied several times:

$$F_\omega : z \in \mathbb{D}^2 \mapsto \exp(2i\pi\omega(|z|))z \in \mathbb{D}^2.$$

Indeed, we have noted that in such a situation, the corresponding cross-ratio curves c simply rotate around one of the punctures of $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ so that the geodesic arc in the same homotopy class relative to the endpoints 'escapes' in the cusps as the number p of iterations goes to infinity. More precisely, since α has compact support, $I_\alpha(g^p; z_1, z_2, z_3, z_4)$ is bounded as a function of p (for fixed z_i) and it follows that $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}(F_\omega) = 0$.

A smooth function H on the sphere defines a Hamiltonian vector field X_H whose time 1 is an element of $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$. We shall call these special diffeomorphisms *autonomous Hamiltonian diffeomorphisms*. Of course, the F_ω's are autonomous Hamiltonian diffeomorphisms. Note that every orbit of X_H rotates on some level curve of H so that it is not difficult in this case to analyze the cross-ratio curves c and to see that the phenomenon that we observed for F_ω generalizes to all autonomous Hamiltonian diffeomorphisms: the curves c rotate around the cusps so that $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$ vanishes on autonomous diffeomorphisms.

It follows from the simplicity of $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ that every area-preserving diffeomorphism g of the sphere can be written as a product of autonomous Hamiltonian diffeomorphisms: denote by $\mathit{ham}(g)$ the minimum length of such a product. This defines an integral valued function *ham* on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$. Our observation on the quasi-morphisms $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$ implies that *this Hamiltonian length ham is unbounded.*

The reader has probably seen enough examples of quasi-morphisms in the case of the sphere! We finish this section however, by mentioning that there is a combinatorial analog of the $\mathfrak{Calabi}_{\alpha,\mathbb{S}^2}$. The construction of $\mathfrak{Turn}$ used the explicit right–left quasi-morphism $\mathfrak{Radem}$ defined on $\mathrm{PSL}(2,\mathbb{Z})$. There are many other quasi-morphisms on this group: the so-called *Brooks quasi-morphisms* described, for instance, in [**7**] (note that these Brooks quasi-morphisms are defined in general for free groups but one can obviously generalize to free products like $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$). Each quasi-morphism on $\mathrm{PSL}(2,\mathbb{Z})$ defines a quasi-morphism on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ and it would be possible to prove that this produces an infinite number of linearly independent homogeneous quasi-morphisms (using ideas similar to those developed in §6.2). We leave the (elementary) details to the reader.

7. *Final remarks*

In [**10**], Entov and Polterovich construct a quasi-morphism $\mathfrak{EntPol}$ on $\mathrm{Diff}_0^\infty(\mathbb{S}^2, \mathit{area})$ which has the additional property that, for every embedded disc $D \subset \mathbb{S}^2$ with area less than $1/2$, the restriction of $\mathfrak{EntPol}$ to the subgroup $\mathrm{Diff}_0^\infty(D, \partial D, \mathit{area})$ is precisely Calabi's homomorphism $\mathfrak{Sign}_{2,D}$. It would be interesting to find a construction of $\mathfrak{EntPol}$ in the spirit of our paper.

For simplicity, we have assumed all diffeomorphisms to be of class C^∞ even though our constructions are obviously valid in the C^1 category. However, the C^0 case seems to be open. As a typical open question, one does not know whether or not the group $\mathrm{Homeo}_0(D, \partial D, \mathit{area})$ is simple. The difficulty in generalizing Calabi-type invariants stems from the fact that functions like, for example, $T(g; z_1, z_2, z_3, z_4)$ are not integrable for the most general homeomorphism so that one cannot define invariants by integration, like we did in this paper.

However, it is not difficult to prove that all homogeneous quasi-morphisms Φ introduced in this paper are topological invariants. In other words, if two elements of $\mathrm{Diff}_0^\infty(\Sigma, \mathit{area})$ are conjugate by some area-preserving homeomorphism isotopic to the identity, they have the same value of Φ. We gave the proof of this fact in the special case of Calabi's homomorphism and Ruelle's quasi-morphisms on the disc in [**12**] and the proof can be adapted with no difficulty to this more general context.

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