DOES THE JONES POLYNOMIAL DETERMINE THE SIGNATURE OF A KNOT?

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ABSTRACT. The signature function of a knot is a locally constant integer valued function with domain the unit circle. The jumps (i.e., the discontinuities) of the signature function can occur only at the roots of the Alexander polynomial on the unit circle. The latter are important in deforming $U(1)$ representations of knot groups to irreducible $SU(2)$ representations. Under the assumption that these roots are simple, we formulate a conjecture that explicitly computes the jumps of the signature function in terms of the Jones polynomial of a knot and its parallels. As evidence, we prove our conjecture for torus knots, and also (using computer calculations) for knots with at most 8 crossings. We also give a formula for the jump function at simple roots in terms of relative signs of Alexander polynomials.

CONTENTS

1. INTRODUCTION

1.1. **The signature function of a knot.** A celebrated invariant of a knot K in 3-space is its σ*ignature function*

$$\sigma(K) : S^1 \longrightarrow \mathbb{Z},$$

defined for complex numbers of absolute value 1, and taking values in the set of integers. The signature function of a knot is a concordance invariant, and plays a key role in the study of knots via surgery theory, [L].

It turns out that the signature function is a locally constant function away from the (possibly empty) set

$$\mathrm{Div}_{\Delta(K)} = \{\rho \in S^1 \,|\, \Delta(K)(\rho) = 0\}$$

of roots of the Alexander polynomial on the unit circle. In view of this, the interesting part of the signature function is its *jumping behavior* on the set $\mathrm{Div}_{\Delta(K)}$.

In other words, we may consider the associated *jump function*

$$\mathrm{j}(K) : \mathrm{Div}_{\Delta(K)} \longrightarrow \mathbb{Z}$$

defined by $\mathrm{j}_{\rho_0}(K) = \lim_{\rho \to \rho_0^+} \sigma_\rho(K) - \lim_{\rho \to \rho_0^-} \sigma_\rho(K)$.

We may identify the jump function with a *jump divisor* $\sum_{\rho \in \mathrm{Div}_{\Delta(K)}} \mathrm{j}_\rho(K)[\rho]$ in S^1.

Since $1 \notin \mathrm{Div}_{\Delta(K)}$ and $\sigma_1(K) = 0$, it follows that the jump function uniquely determines the signature function away from the set $\mathrm{Div}_{\Delta(K)}$. Since $-1 \notin \mathrm{Div}_{\Delta(K)}$, it follows in particular that $\mathrm{j}(K)$ determines the σ*ignature* of the knot $\sigma_{-1}(K)$.

The signature of a knot may be defined using a Seifert surface of a knot (see Section 2.1 below). An intrinsic definition of the jump function of a knot was given by Milnor [M1, M2], using the Blanchfield pairing of the universal abelian cover of a knot. This definition, among other things, makes evident the role played by the roots of the Alexander polynomial on the unit circle (as opposed to the rest of the roots of the Alexander polynomial, which are ignored).

From the point of view of gauge theory and mathematical physics, the signature function of a knot may be identified with the spectral flow of a 1-parameter family of the signature operator, twisted along abelian (that is, $U(1)$-valued) representations of the knot complement.

The moduli space of $U(1)$ representations of the knot complement is well understood; it may be identified with the unit circle. On the other hand, the moduli space of $SU(2)$ representations is less understood, and carries nontrivial topological information about the knot and its Dehn fillings, as was originally discovered by Casson (see [AM]) and also by X-S. Lin; see [Li].

One may ask to identify the $U(1)$ representations which *deform to* irreducible $SU(2)$ representations. Using a linearization argument, Klassen and Frohman showed that a necessary condition for a $U(1)$ representation ρ to deform is that $\Delta(K)(\rho^2) = 0$. This brings us to the (square of the) set $\mathrm{Div}_{\Delta(K)}$. Conversely, Frohman-Klassen proved sufficiency provided that the Alexander polynomial has *simple roots* on the unit circle; see [FK]. Herald proved sufficiency under the (more relaxed condition that) the jump function vanishes nowhere; see [H1, H2].

It is unknown at present whether sufficiency holds without any further assumptions.

Let us summarize the two key properties of the jump divisor $\mathrm{Div}_{\Delta(K)}(K)$, in the spirit of Mazur (see [Ma]):

- The jump divisor controls the signature function of a knot.
- The jump divisor controls (infinitesimally) deformations of $U(1)$ representations of the knot complement to irreducible $SU(2)$ representations.

1.2. The colored Jones function of a knot.
It is a long standing problem to find a formula for the σ*ignature* function of a knot in terms of its *colored Jones function*. The latter is a sequence of Jones polynomials associated to a knot. Recall that given a knot K and a positive integer n (which corresponds to an n-dimensional irreducible representation of $\mathfrak{sl}_2$), one can define a Laurent polynomial $J_n(K) \in \mathbb{Z}[q^{\pm}]$.

In [R2], Rozansky considered a repackaging of the sequence $\{J_n(K)\}$. Namely, he defined a sequence of rational functions $Q_k(K) \in \mathbb{Q}(q)$ for $k \geq 0$ with the following properties:

- $Q_k(K) = P_k(K)/\Delta^{2k+1}(K)$ for some polynomials $P_k(K) \in \mathbb{Z}[q, q^{-1}]$ with $P_0(K) = 1$ and such that $P_k(K)(q) = P_k(K)(q^{-1})$.
- For every n we have:

$$\text{(1)} \qquad \frac{J_n(K)(q)}{J_n(\mathrm{unknot})(q)} = \sum_{k=0}^{\infty} Q_k(q^n)(q-1)^k \in \mathbb{Q}[[q-1]]$$

where $\mathbb{Q}[[q-1]]$ is the ring of formal power series in $q-1$ with rational coefficients

Equation (1) is often called the *Euler expansion* of the colored Jones function. In physical terms, the above expansion is an asymptotic expansion of the Chern-Simons path integral of the knot complement, expanded around a backround $U(1)$ flat connection. Thus, philosophically, it should not be a surprise to discover that this expansion has something to do with the signature of the knot.

For the curious reader, let us point out that Rozansky conjectured such an expansion for the full Kontsevich integral of a knot, graded by the negative Euler characteristic of graphs (thus the name, Euler expansion). This conjecture was proven by Kricker and the author; [GK1]. Furthermore, a close relation was discovered between residues of the rational functions Q_k at roots of unity and the LMO invariant of cyclic branched coverings of the knot; [GK2]. In an attempt to understand the Euler expansion, a theory of finite type invariants of knots (different from the usual theory of Vassiliev invariants) was proposed in [GR]. According to that theory, two knots are 0-equivalent iff they are S-equivalent; [GR]. Moreover, Q_k is a finite type invariant of type $2k$.

Technically, the Euler expansion of the colored Jones function is an integrality statement. Namely, it is easy to see that there exist unique sequence of power series $Q_k(K)(q) \in \mathbb{Q}[[q - 1]]$ for $k \geq 0$ that satisfies Equation (1). The hard part is to show that these power series are Taylor series expansions of rational functions with integer coefficients and prescribed denominators.

The statement $P_0(K) = 1$ in the leading term of the Euler expansion is nothing but the Melvin-Morton-Rozansky conjecture, proven by Bar-Natan and the author in [BG]. Thus, the leading order term in the Euler expansion is a well-understood topological invariant of knots. Ever since the Euler expansion was established, it has been a question to establish a topological understanding of the lower order terms.

1.3. **The conjecture.** Consider $Q(K)(t) = \frac{P(K)(t)}{\Delta^2(K)(t)} \in \mathbb{Q}(t)$ where $P(K) = P_1(K)$. We will think of $Q(K)$ as a function (with singularities) defined on the unit circle.

If $\rho = e^{i\theta_0}$ is a root of the Alexander polynomial on S^1, we may expand $Q(K)(e^{i\theta})$ around $\theta = \theta_0$. The result is a power series with lowest term $c_\rho(\theta - \theta_0)^{m_\rho}$, for some integer m_ρ and some nonzero real number c_ρ.

Definition 1.1. Let us define the *Jones jump function* of a knot K

$$\mathrm{jj}(K) : \mathrm{Div}_{\Delta(K)} \longrightarrow \mathbb{Z}$$

by

$$\mathrm{jj}_\rho(K) = \mathrm{sgn}(c_\rho) \max\{0, -m_\rho\} \, \mathrm{sgn}(\mathrm{Im}(\rho))$$

where $\mathrm{Im}(z)$ is the imaginary part of a complex number z and $\mathrm{sgn}(x)$ is the *sign* of a real number x is defined by $\mathrm{sgn}(x) = +1, 0$ or -1 according to $x > 0$, $x = 0$ or $x < 0$ respectively.

Definition 1.2. We say that a knot K is *simple* if its Alexander polynomial $\Delta(K)$ has simple roots on the unit circle.

Conjecture 1. *If K is simple, then $\mathrm{j}(K) = \mathrm{jj}(K)$.*

A modest corollary is:

Corollary 1.3. *If K is simple, Conjecture 1 implies that the colored Jones function of K determines the signature $\sigma_{-1}(K)$.*

Remark 1.4. Notice that $\mathrm{j}_\rho(K) = -\mathrm{j}_{\bar\rho}(K)$ and $\mathrm{jj}_\rho(K) = -\mathrm{jj}_{\bar\rho}(K)$. Thus, it suffices to check the conjecture on the upper semicircle.

Remark 1.5. The conjecture is false if $\Delta(K)$ has multiple roots (of odd or even multiplicity). For example, consider the connected sum $\sharp^n K$ of n right trefoils. Then, $Q(\sharp^n K) = nQ(K)$ and $\Delta(\sharp K) = \Delta(K)^n$.

We present the following evidence for the conjecture:

Theorem 1. (a) *Conjecture 1 is true for torus knots, and for knots with at most 8 crossings.*
(b) *The Conjecture is compatible with the operations of mirror image, connected sum (assuming the resulting knot is simple) and $(n, 1)$ parallels of knots.*

En route to establish our results, we give a skein formula that uniquely characterizes the jump function of simple knots; see Theorem 3.

Let us compare Conjecture 1 with existing conjectures about the structure of the colored Jones function. At the time of the writing, there are two conjectures that relate the colored Jones function to hyperbolic geometry. Namely,

- The *Hyperbolic Volume Conjecture*, after Kashaev and J&J.Murakami, which states that for a hyperbolic knot K,

$$\lim_{n \to \infty} \frac{\log |J'_n(K)(e^{2\pi i/n})|}{n} = c \operatorname{vol}(S^3 - K)$$

where $J'_n(K) = J_n(K)/J_n(\text{unknot})$.
- The *Characteristic equals deformation variety Conjecture*, due to the author, which compares the deformation curve of $\mathrm{SL}_2(\mathbb{C})$ representations of a knot complement (viewed from the boundary) with a complex curve which is defined using the recursion relations (with respect to n) of the sequence $\{J_n(K)\}$; see [GL] and [Ga3].

The Hyperbolic Volume Conjecture is an analytic statement, which involves the existence and identification of a sequence of real numbers.

On the other hand, the Characteristic equals Deformation Variety conjecture is an algebraic statement, since it is equivalent to the equality of two polynomials with integer coefficients, one of which is obtained by noncommutative elimination, and the other obtained by commutative elimination.

Conjecture 1 appears to be an analytic conjecture, since its basic ingredients are signs of real numbers. In the field of Quantum Topology, analytic conjectures have held the longest.

Let us end the introduction with the following

Question 1. Understand the underlying geometry and perturbative quantum field theory behind the Taylor expansion of the Q function (and more generally, Euler expansion (1) of the colored Jones function). In particular, use the higher order terms Q_k in the expansion (1) to formulate a conjecture for the jump function of all knots.

2. The signature and the jump function

2.1. **Symmetries of the jump function.** Given a Seifert matrix V of a knot K, consider the *Hermitian matrix* $B(t) = (1-t)V + (1-\bar{t})V^T$, for $t \in S^1$. The eigenvalues of $B(t)$ are real, and we define $\sigma_t(K) = \sigma(B(t))$, where $\sigma(M)$ denotes the *signature* of a Hermitian matrix M. It turns out that $\sigma(K)$ is independent of the Seifert surface V chosen. Since $B(t) = (t^{1/2} - t^{-1/2})A(t)$, where $A(t) = t^{1/2}V - t^{-1/2}V^T$, and $\det(A(t)) = D(K)(t)$ is the *symmetrized Alexander polynomial* of K, it follows that $\sigma(K)$ is a locally constant function with possible jumps along the set $\operatorname{Div}_{\Delta(K)}$.

The next lemma, which follows from the proof of [H1, Corollary 2], summarizes the symmetries of the jump function.

Lemma 2.1. *If ρ is a root of the Alexander polynomial on S^1, then $|\mathrm{j}_\rho(K)| = 2\,a_\rho$, where*

- α_ρ *is an integer*
- $a_\rho \leq \operatorname{mult}(\rho, \Delta(K))$, *where* $\operatorname{mult}(\rho, \Delta(K))$ *is the multiplicity of ρ in $\Delta(K)$, and*
- $\alpha_\rho \equiv \operatorname{mult}(\rho, \Delta(K)) \bmod 2$.

Moreover, $\mathrm{j}_\rho(K) = -\mathrm{j}_{\bar\rho}(K)$.

In particular, if K is simple, $\mathrm{j}(K)$ takes values in the set $\{-2, 2\}$. For a precise formula for the jump function in that case, see Theorem 2.

2.2. **A skein theory for the signature and the jump function.** Let us begin with a useful definition. A triple of links (L^+, L^-, L^0) is called *bordered* if there is an embedded ball D^3 in S^3 that locally intersects them as in figure 1.

If we choose planar projection and a crossing, then a bordered triple corresponds to replacing the crossing by a positive, negative or smoothening. Notice that if L^+ is a link with 1 component, then L^- and L^0 are links with 1 and 2 components respectively.

The next lemma computes the change of the signature function with respect to the change of a crossing, in terms of the sign of the Alexander polynomials.
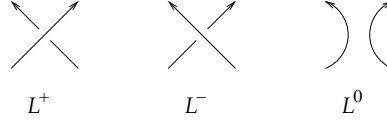
$$L^+ \qquad L^- \qquad L^0$$

Figure 1. A bordered triple of links (L^+, L^-, L^0).

Lemma 2.2. *If K is a knot, $\rho = e^{i\theta} \in S^1$ such that $\Delta(K^+)(\rho)\Delta(K^-)(\rho) \neq 0$, then*

(2)
$$\sigma_\rho(K^-) - \sigma_\rho(K^+) = \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho)\Delta(K^-)(\rho) < 0 \\ 0 & \text{if} \quad \Delta(K^+)(\rho)\Delta(K^-)(\rho) > 0. \end{cases}$$

Proof. We can choose Seifert surfaces (V_+, V_-, V_0) for (K^+, K^-, K^0) such that

$$V_+ = \begin{pmatrix} a & v_1 \\ v_2^T & V_0 \end{pmatrix} \qquad V_- = \begin{pmatrix} a+1 & v_1 \\ v_2^T & V_0 \end{pmatrix}$$

where v_1 and v_2 are some row vectors. Hermitianizing, we get:

$$B_+ = \begin{pmatrix} 2a & v \\ v^T & B_0 \end{pmatrix} \qquad B_- = \begin{pmatrix} 2a+2-2\cos\theta & v \\ v^T & B_0 \end{pmatrix}.$$

Let us call a triple of Hermitian matrices (A_+, A_-, A_0) ρ-*bordered* if

$$A_+ = \begin{pmatrix} a & v \\ v^T & A_0 \end{pmatrix} \qquad A_- = \begin{pmatrix} a+2-2\cos\theta & v \\ v^T & A_0 \end{pmatrix},$$

for $\rho = e^{i\theta}$ and some row vector v. Using Lemma 2.3 the result follows. $\qquad\square$

Lemma 2.3. *If (A_+, A_-, A_0) is a ρ-bordered triple, and $\det(A_+)\det(A_-) \neq 0$, then*

$$\sigma(A_-) - \sigma(A_+) = \begin{cases} 2 & \text{if} \quad \det(A_+)\det(A_-) < 0 \\ 0 & \text{if} \quad \det(A_+)\det(A_-) > 0. \end{cases}$$

Proof. This is well-known for $\rho = 1$; [C] and also [Ga1, Lemma 3.1]. We give a proof here for all ρ.

By a similarity transformation (that is a replacement of B by $P^\star BP$ where P is an invertible matrix, and $P^\star$ is the conjugate transpose of P), we can assume that

$$A_+ = \begin{pmatrix} a & v \\ v^T & 0 \end{pmatrix} \oplus D \qquad A_- = \begin{pmatrix} a+2-2\cos\theta & v \\ v^T & 0 \end{pmatrix} \oplus D, \qquad A_0 = [0]^r \oplus D,$$

where D is a nonsingular diagonal matrix, $[0]^r$ is the zero $r \times r$ matrix, v is a $1 \times r$ vector and a a real number.

Since the nullity (that is, the dimension of the kernel) and the signature of the matrix $\begin{pmatrix} b & v \\ v^T & 0 \end{pmatrix}$ are given by:

	$v = b = 0$	$v = 0, b \neq 0$	$v \neq 0$
nullity	$r+1$	r	$r-1$
signature	0	$\text{sgn}(a)$	0

the result follows by a case-by-case argument. $\qquad\square$

The next theorem computes the jump function of a simple knot in terms of a relative sign of Alexander polynomials. First, a preliminary definition.

Definition 2.4. If $f(x)$ is a real-valued analytic function of x in a neighborhood of a, we define *the sign of f at a* $\text{sgn}(f, a)$ to be the sign of the first nonvanishing Taylor series coefficient (around a), if there is such, and zero otherwise. In other words, if $f \neq 0$, we have:

$$\text{sgn}(f, a) = \text{sgn}(f^{(n)}(a)) \in \{-1, 1\},$$

where $f^{(k)}(a) = 0$ for $k < n$ and $f^{(n)}(a) \neq 0$.

Remark 2.5. Notice that if $f(a) \neq 0$, then $\operatorname{sgn}(f, a) = \operatorname{sgn}(f(a))$, and that if a is a simple root, then $\operatorname{sgn}(f, a) = \operatorname{sgn}(f(a + \delta)) = -\operatorname{sgn}(f(a - \delta))$ where δ is sufficiently small and positive.

Fix a simple knot K and a complex number $\rho = e^{i\theta} \in \operatorname{Div}_{\Delta(K)}$. Choose a planar projection of K and a crossing (positive or negative). Then, $K = K^\epsilon$, where $\epsilon \in \{+, -\}$ is the sign of the chosen crossing. Suppose that $\Delta(K^{-\epsilon})(\rho) \neq 0$. Such a projection and choice of crossing will be called (ρ, K)-*good*.

Theorem 2. *Fix* (ρ, K) *as above. For every* (ρ, K)-*good projection, we have*

$$\mathrm{j}_\rho(K) = 2\epsilon \operatorname{sgn}(\Delta(K^+), \theta) \operatorname{sgn}(\Delta(K^-), \theta) \in \{-2, 2\}.$$

Proof. Without loss of generality, let us assume $K = K^-$, that is $\epsilon = -1$. We will apply Lemma 2.2 twice to $\rho' = e^{i(\theta + \delta)}$ and $\rho'' = e^{i(\theta - \delta)}$ for sufficiently small positive δ.

Under these assumptions, we have that $\Delta(K^-)(\rho') \neq 0$ (since ρ is an isolated root of a polynomial) and $\Delta(K^+)(\rho') \neq 0$ (since $\Delta(K^+)(\rho) \neq 0$ by assumption), and similarly for ρ''. Thus, the hypothesis of Lemma 2.2 are satisfied. Applying Lemma 2.2 twice, we get

$$\sigma_{\rho'}(K^-) - \sigma_{\rho'}(K^+) = \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho')\Delta(K^-)(\rho') < 0 \\ 0 & \text{if} \quad \Delta(K^+)(\rho')\Delta(K^-)(\rho') > 0 \end{cases}$$

and

$$\sigma_{\rho''}(K^-) - \sigma_{\rho''}(K^+) = \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho'')\Delta(K^-)(\rho'') < 0 \\ 0 & \text{if} \quad \Delta(K^+)(\rho'')\Delta(K^-)(\rho'') > 0 \end{cases}$$

Now, subtract and remember that $\sigma(K^+)$ is continuous at ρ since $\Delta(K^+)(\rho) \neq 0$. We get

$$\mathrm{j}_\rho(K^-) = \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho')\Delta(K^-)(\rho') < 0 \\ 0 & \text{if} \quad \Delta(K^+)(\rho')\Delta(K^-)(\rho') > 0 \end{cases} - \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho'')\Delta(K^-)(\rho'') < 0 \\ 0 & \text{if} \quad \Delta(K^+)(\rho'')\Delta(K^-)(\rho'') > 0 \end{cases}$$

Since K is simple, it follows that $\Delta(K^-)(\rho')\Delta(K^-)(\rho'') < 0$, thus the cases $2 - 2$ or $0 - 0$ do not occur above. Thus,

$$\mathrm{j}_\rho(K^-) = \begin{cases} 2 & \text{if} \quad \Delta(K^+)(\rho')\Delta(K^-)(\rho') < 0 \\ -2 & \text{if} \quad \Delta(K^+)(\rho'')\Delta(K^-)(\rho'') < 0 \end{cases}$$

The result follows using Remark 2.5. Indeed, $\operatorname{sgn}(\Delta(K^+), \rho) = \operatorname{sgn}(\Delta(K^+)(\rho))$ and $\operatorname{sgn}(\Delta(K^-), \rho) = \operatorname{sgn}(\Delta(K^-)(\rho'')) = -\operatorname{sgn}(\Delta(K^-)(\rho')$. $\qquad\square$

Theorem 3. *There is a unique invariant j defined for a simple knot K and $\rho \in \operatorname{Div}_{\Delta(K)}$ such that for every* (ρ, K)-*good projection we have:*

$$j_\rho(K) = 2\epsilon \operatorname{sgn}(\Delta(K^+), \theta) \operatorname{sgn}(\Delta(K^-), \theta).$$

Proof. In view of Theorem 2, we need to prove that there is at most one such invariant.

Fix a simple knot K and a complex number $\rho = e^{i\theta} \in \operatorname{Div}_{\Delta(K)}$. We need to prove that there exists a (ρ, K)-good projection.

Start with any planar projection of K and a crossing. If it is not good, apply Reidemaster moves II, which Frohman-Klassen call *threading* and improve it to be good, using the proof of [FK, Theorem 6.2]. $\qquad\square$

Thus, Conjecture 1 is equivalent to the following:

Conjecture 2. (a) *For every simple knot K, and every $\rho = e^{i\theta} \in \operatorname{Div}_{\Delta(K)}$, we have $P(K)(\rho) \neq 0$.*
(b) *Moreover, for every* (ρ, K)-*good projection we have:*

$$\operatorname{sgn}(P(K), \theta) = \epsilon \operatorname{sgn}(\Delta(K^+), \theta) \operatorname{sgn}(\Delta(K^-), \theta).$$

3. Evidence

3.1. Torus knots. In this Section we will prove Conjecture 1 for torus knots. Let $T_{a,b}$ denote the (a,b) *torus knot*, where a,b are coprime natural numbers. For example, $T(2,3)$ is the right-hand trefoil.

The Alexander polynomial of torus knots is given by:

$$\Delta(T_{a,b})(t) \quad = \quad \frac{(t^{ab/2} - t^{-ab/2})(t^{1/2} - t^{-1/2})}{(t^{a/2} - t^{-a/2})(t^{b/2} - t^{-b/2})}.$$

The roots of $\Delta(T_{a,b})$ on the unit circle are ab complex roots of unity which are not a or b order roots of unity. They are all simple. Using a useful parametrization of them, following Kearton [K2, Sec.13], we obtain that

$$\mathrm{Roots}_{\Delta(T_{a,b})} = \{t(m,n) := e^{2\pi i(m/a + n/b)} \,|\, 0 < m < a, \ 0 < n < b\}.$$

Since the jump function satisfies $j_\rho(K) = -j_{\bar\rho}(K)$, we need only compute the jump at the points $t(m,n)$ where $0 < m < a$, $0 < n < b$ and $m/a + n/b < 1$. In [K2, p.177] Kearton computes the jump function of torus knots by

$$j_{t_{m,n}}(T_{a,b}) = \begin{cases} 2 & \text{if} \quad m/a + n/b < \frac{1}{2} \\ -2 & \text{if} \quad \frac{1}{2} < m/a + n/b < 1. \end{cases}$$

In other words, we have:

$$j_\rho(T_{a,b}) = \begin{cases} -2 & \text{if} \quad \mathrm{Im}(\rho) > 0 \\ 2 & \text{if} \quad \mathrm{Im}(\rho) < 0. \end{cases}$$

Now we discuss the Q function of torus knots, which was originally computed by Rozansky (see [R1, Eqn.(2.2)]), and most recently, it has been recomputed by Marché and Ohtsuki; see [Mr, Oh]. We understand that Bar-Natan has unpublished computations of the Euler expansion of the Kontsevich integral of torus knots.

According to [R1, Eqn.(2.2)], the Q function of torus knots is given by:

$$Q(T_{a,b})(t) \quad = \quad \frac{1}{4}\left(ab - \frac{a}{b} - \frac{b}{a}\right) + \frac{1}{ab}\frac{\Delta(T_{a,b})(t)}{(t^{1/2} - t^{-1/2})}\frac{\partial^2}{\partial x^2}\Big|_{x=0} \frac{t^{1/2}e^{x/2} - t^{-1/2}e^{-x/2}}{\Delta(T_{a,b})(te^x)}$$

Given an analytic function $f(t)$ let us define

$$g(t) = \frac{f(t)}{(t^{1/2} - t^{-1/2})}\frac{\partial^2}{\partial x^2}\Big|_{x=0} \frac{t^{1/2}e^{x/2} - t^{-1/2}e^{-x/2}}{f(te^x)}$$

We have that

$$g(t) = \frac{1}{8(t^{1/2} - t^{-1/2})}\frac{tf(t)^2 - f(t)^2 + 4t^2 f(t)f''(t) - 8t^2 f(t)f'(t) - 4t^3 f(t)f''(t) - 8t^2(f'(t))^2 + 8t^3(f'(t))^2}{t^{1/2}f(t)^2}$$

When we expand $g(e^{i\theta})$ around a root $\rho = e^{i\theta_0}$, only the last two terms of the numerator contribute to the coefficient of $(\theta - \theta_0)^2$. That is,

$$\mathrm{coeff}(g(e^{i\theta}), (\theta - \theta_0)^2) = \frac{1}{8(t^{1/2} - t^{-1/2})}\frac{-8t^2(f'(t))^2 + 8t^3(f'(t))^2}{t^{1/2}f(t)^2}\Big|_{t=e^{i\theta_0}} = \frac{t^2(f'(t))^2}{f(t)^2}\Big|_{t=e^{i\theta_0}}.$$

Now, suppose that $f(t)$ is a Laurent polynomial with real coefficients that satisfies $f(t) = f(t^{-1})$. Then, $f(t) = \sum_k a_k(t^k + t^{-k})$. Thus,

$$t^2(f'(t))^2 = t^2\left(\sum_k ka_k(t^{k-1} - t^{-k-1})\right)^2 = \left(\sum_k ka_k(t^k - t^{-k})\right)^2$$

and if we substitute $t = e^{i\theta_0}$, we get

$$t^2(f'(t))^2\big|_{t=e^{i\theta_0}} = -4\left(\sum_k ka_k \sin k\theta\right)^2 \le 0.$$

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If θ_0 is a simple root of $f(e^{i\theta})$ on the unit circle (as is the case for the Alexander polynomial of torus knots), then the above real number is negative.

This proves that

$$\text{jj}_\rho(T_{a,b}) = \begin{cases} -2 & \text{if} \quad \text{Im}(\rho) > 0 \\ 2 & \text{if} \quad \text{Im}(\rho) < 0 \end{cases}$$

and confirms Conjecture 1 for torus knots.

3.2. **Operations on knots that preserve Conjecture 1.** Let f denote either the Q function or the σignature function of a knot. The following list describes some well-known properties of f.

- If $-K$ denote the knot K with opposite orientation, then $f(-K) = f(K)$.
- If $K^!$ denote the mirror image of K, then $f(K^!) = -f(K)$.
- If $\sharp$ denotes the connected sum of knots, then $f(K_1 \sharp K_2) = f(K_1) + f(K_2)$.
- If $K^{(n)}$ denote the $(n, 1)$ parallel of a knot K with zero framing, then $f(K^{(n)})(t) = f(K)(t^n)$.

The stated behavior of the signature function under $(n, 1)$ parallel was proven by Kearton [K1], and for the Q function was proven by Ohtsuki [Oh, Prop. 3.1].

From this, it follows that if Conjecture 1 is true for a simple knot K, then it is true for $-K$, $K^!$, $K^{(n)}$ (for all n). Furthermore, if $K_1 \sharp K_2$ is simple, and Conjecture 1 is true for K_1 and K_2, then it is also true for $K_1 \sharp K_2$.

3.3. **Knots with at most 8 crossings.** In this section we will verify Conjecture 1 by computer calculations.

Rozansky has written a `Maple` program that computes the Q function of a knot; see [R2]. We will use a minor modification `Qfunction.mws` of Rozansky's program, adopted for our needs.

In `Qfunction.mws`, the knot is described by a braid word. For example, $[-1, 3, 3, 3, 2, 1, 1, -3, 2]$ represents the braid $\sigma_1^{-1}\sigma_3^3\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2$ whose closure is the 7_2 knot in classical notation. The command $br1([-1, 3, 3, 3, 2, 1, 1, -3, 2])$ gives a list whose first, second and third entries are the braid word, the polynomials $P(K)$ and $\Delta(K)$, where $z = t^{1/2} - t^{-1/2}$. A sample output of the program is:

```
> # the right trefoil 3_1
> br1([1,1,1]);
>

                               2      2     4
                  [[1, 1, 1], 1 + z , 2 z  + z ]

> # the 4_1 knot
> br1([1,-2,1,-2]);
>

                                        2
                  [[1, -2, 1, -2], 1 - z , 0]
> # the 7_2 knot
> br1([-1,3,3,3,2,1,1,-3,2]);
>

                                     2      2      4
      [[-1, 3, 3, 3, 2, 1, 1, -3, 2], 1 + 3 z , 12 z  + 14 z ]
> # 7_3
> br1([1,1,2,-1,2,2,2,2]);
>

                                   2     4
    [[1, 1, 2, -1, 2, 2, 2, 2], 1 + 5 z  + 2 z ,

             2       4       6      8
        22 z  + 65 z  + 46 z  + 9 z ]
```

For example, for the right hand trefoil, we have:

$$\begin{aligned} \Delta(K) &= 1 + z^2 = t + \bar{t} - 1 \\ P(K) &= 2z^2 + z^4 = t^2 - 2t + 2 - 2\bar{t} + \bar{t}^2 \\ Q(K) &= \frac{2z^2 + z^4}{(1 + z^2)^2} = \frac{t^2 - 2t + 2 - 2\bar{t} + \bar{t}^2}{(t + \bar{t} - 1)^2}. \end{aligned}$$

The `Mathematica program` `JJump.m` computes the jj function. For example, we may launch the `JJump.m` program from a Mathematica session.

```
(math100)/home/stavros: math
Mathematica 5.0 for Sun Solaris (UltraSPARC)
Copyright 1988-2003 Wolfram Research, Inc.
 -- Motif graphics initialized --

In[1]:= << JJump.m

In[2]:= Poles[1+z^2,2z^2+z^4]

Solve::ifun: Inverse functions are being used by Solve, so some solutions may
      not be found; use Reduce for complete solution information.

Out[2]= {{0.16666666666666666667, -0.008443431970019481429}}
```

We learn that the coefficient of $(\theta - \theta_0)^{-2}$ of $Q(3_1)(e^{2\pi i\theta})$ (where 3_1 is the right trefoil) around the root $\theta_0 = 0.1666666667$, is -0.00844343197019481429. This computes that $\mathrm{jj}_{e^{2\pi i\theta_0}}(3_1) = -2$, as needed.

Similarly,

```
In[4]:= Poles[1+5z^2+2z^4,22z^2+65z^4+46z^6+9z^8]

Solve::ifun: Inverse functions are being used by Solve, so some solutions may
      not be found; use Reduce for complete solution information.

Out[4]= {{0.075216475230034463796, -0.00388836700144941422},

>     {0.27241752919082620707, -0.005424244178920663095}}
```

We learn that the coefficient of $(\theta - \theta_i)^{-2}$ of $Q(7_3)(e^{2\pi i\theta})$ around the roots $\theta_0 = 0.0752164$ and $\theta_1 = 0.27241752$ are -0.003888367 and -0.0054242417 respectively. This computes the jump function $\mathrm{jj}_{e^{2\pi i\theta_j}}(7_3) = -2$ for $j = 0, 1$.

Now, let us compute the jump function of a knot. In [Or] Orevkov gives a `Mathematica` program `sm.mat` which takes as input a braid presentation of a knot, and gives as output a Seifert surface of a knot. Launching the `Jump.m` version of it in a Mathematica session produces

```
(math100)/home/stavros: math
Mathematica 5.0 for Sun Solaris (UltraSPARC)
Copyright 1988-2003 Wolfram Research, Inc.
 -- Motif graphics initialized --

In[1]:= << Jump.m

In[2]:= Jump[{1,1,1}]

InverseFunction::ifun:
   Inverse functions are being used. Values may be lost for multivalued
     inverses.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may
      not be found; use Reduce for complete solution information.
```

```
Out[2]= {-2}
```

which computes the jump function on the upper semicircle for the right trefoil 3_1.

```
In[3]:= Jump[{1,1,2,-1,2,2,2,2}]

InverseFunction::ifun:
    Inverse functions are being used. Values may be lost for multivalued
      inverses.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may
      not be found; use Reduce for complete solution information.

Out[3]= {-2, -2}
```

which computes the jump function on the upper semicircle for the 7_3 knot.

This confirms the conjecture for the 3_1 and 7_3 knots.

In the appendix, We give the source code of two Mathematica programs, `Jump.m` and `JJump.m` which compute the j and the jj function of knots.

APPENDIX A. THE JJump.m PROGRAM

```
    (* Poles[P,AP] computes the poles of the rational functions P/AP^2    *)
    (* at the roots of AP=0 on the unit circle. P,AP are polynomials in z *)
    (* Poles2[P,AP] lists the coefficients of the Taylor expansion at     *)
    (* (t-a)^{-2}.                                                        *)
    (* Poles[P,AP] lists {a,coefficient of Taylor expansion at (t-a)^{-2}} *)


FF[x_]:=x[[2]];

Poles[AP_,P_]:=Module[
  {quotient,APt,roots,poles,k},
  quotient=Simplify[P/AP^2  /. (z->z^{1/2})  /. (z->2 Cos[2*Pi*t]-2 )];
  APt= Simplify[AP /. (z->z^{1/2})  /. (z->2 Cos[2*Pi*t]-2 )];
  roots=Select[Map[FF, Flatten[
      NSolve[APt == 0, t, 20]] ],
      1/2 > # > 0 &];
  poles={};
  Table[Flatten[{roots[[k]], Coefficient[Series[quotient,{t,roots[[k]],0}],
      t-roots[[k]],-2]}], {k,Length[roots]}]
]

    (* For the 3_1 knot:          Poles[1+z^2,2z^2+z^4]                    *)
    (* For the 4_1 knot:          Poles[1-z^2,0]                          *)
    (* For the 7_2 knot:          Poles[1+3z^2,12z^2+14z^4]               *)
    (* For the 7_3 knot:          Poles[1+5z^2+2z^4,22z^2+65z^4+46z^6+9z^8] *)
```

APPENDIX B. THE Jump.m PROGRAM

```
    (* Computing the signature and jump function of knots presented as   *)
    (* closures of braids.                                               *)
    (* The signature of the right trefoil is SignatureBraid[{1,1,1}]=-2   *)
    (* SignatureM[A] of a matrix A is the signature of A+A^*             *)
    (* Jump[{1,1,1}] is the jumps of the signature of the right trefoil  *)

<< LinearAlgebra`MatrixManipulation`

<< sm.mat;

SignatureM[A_]:=Module[
```

10

```
    {eigen},
    eigen=Eigenvalues[N[A+ Transpose[Conjugate @ A],20]];
    Count[Sign @ eigen, 1]-Count[Sign @ eigen, -1]
]

SignatureBraid[brd_]:=Module[
    {m,V,eigen},
    m=Max[Abs @ brd]+1;
    V=N[SeifertMatrix[m,brd],20];
    SignatureM[V]
]

FF[x_]:=x[[2]];

Jump[brd_]:=Module[
    {m,V,APs,hermitian,roots,k},
    m=Max[Abs @ brd]+1;
    V=N[SeifertMatrix[m,brd]];
    hermitian=(1-Exp[2*Pi*I*s])V+(1-Exp[-2*Pi*I*s]) Transpose[V];
    APs=N[Det[(Cos[2*Pi*s/2]+I Sin[2*Pi*s/2])V-(Cos[2*Pi*s/2]-I
        Sin[2*Pi*s/2]) Transpose[V]],20];
    roots=Select[Map[FF, Flatten[
        NSolve[{APs == 0, Im[s]==0}, s, 15]] ], 1/2 > # > 0 &];
    If[Length[roots]==0, {}, Flatten[Table[SignatureM[hermitian /.
        s->(roots[[k]]+1/1000) ] -SignatureM[hermitian /.
s->(roots[[k]]-1/1000) ], {k,Length[roots]}]]]
]

    (* 7_3 knot    SignatureBraid[{1,1,2,-1,2,2,2,2}]    *)
    (* 7_5 knot    SignatureBraid[{1,1,1,1,2,-1,2,2}]    *)
    (* 8_2 knot    SignatureBraid[{-1,2,2,2,2,2,-1,2}]   *)
    (* 8_5 knot    SignatureBraid[{1,1,1,-2,1,1,1,-2}]   *)
    (* 8_15 knot   SignatureBraid[{1,1,-2,1,3,3,2,2,3}]  *)
    (* 7_3, 7_5, 8_2, 8_5, 8_15 have signature      -4   *)
```

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`http://www.math.yale.edu/~rozansky`

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