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Representations of the singular braid monoid and group invariants of singular knots

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Abstract

It is well known that the Artin representation of the braid group may be used to calculate the fundamental group of knot complements while the Burau representation can be used to calculate the Alexander polynomial of knots. In this note we will study extensions of the Artin representation and the Burau representation to the singular braid monoid and the relation between them which are induced by Fox' free calculus.

Closing singular braids, we obtain singular knots as they appear in the theory of Vassiliev invariants. Thus the extensions of the Artin representation and the Burau presentation give rise to invariants of singular knots. Here, we will focus on the invariants coming from the extended Artin representations. We obtain an infinite family of group invariants, all of them in relation with the fundamental group of the knot complement. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A knot is represented by an embedding of one or several copies of S^1 into S^3. (We will not distinguish between knots and links and use the term knot throughout the article.) Two such embeddings represent the same knot if there is an ambient isotopy transforming one into the other.

A braid (on n strings) is represented by a subset of $\mathbb{C} \times [0,1]$ consisting of n disjoint arcs from $(i,1) \in \{1,\dots,n\} \times \{1\}$ to $(j,0) \in \{1,\dots,n\} \times \{0\}$, monotonic with respect to

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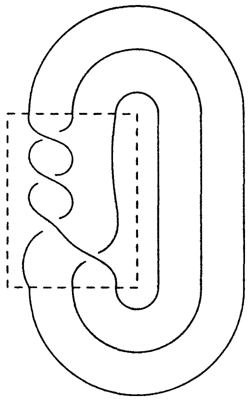

Fig. 1. Closing a braid.

the last coordinate. Two such subsets are considered to represent the same braid if there is an isotopy from one to the other through braids.

It is well known that the set B_n of braids on n strings forms a group, where the multiplication is given by attaching the upper ends of the second string to the lower ends of the first string.

Given a braid b we can construct a knot K_b out of it, by 'closing' the upper and lower ends with trivial non-crossing arcs (see Fig. 1). Alexander's theorem states that any knot can be obtained by closing a braid. Thus the closing operation is surjective. The question of injectivity is solved by Markov's theorem [3]: Two braids correspond to the same knot if they can be transformed one into the other by a finite sequence of so-called Markov moves.

This gives rise to an equivalence relation and as a consequence we can consider a knot as a Markov class of braids. This relation has been used to define knot invariants via representations of the braid group; the Jones polynomial [15] being the most prominent example. In this paper we will focus on some classical examples.

Let $\boldsymbol{F}^n$ be the free group on n generators $x_1, \ldots, x_n$. The Artin representation $\rho_n : B_n \to Aut(\boldsymbol{F}^n)$ from B_n into the automorphism group of $\boldsymbol{F}^n$ has already been defined in Artin's first paper on braid groups [1]. He uses it in order to calculate the fundamental group of the knot complement.

Theorem 1.1 (Artin's Theorem). *Let $b \in B_n$ be a braid, K_b the corresponding knot and $G_{\rho_n(b)}$ be the group presented by*

$$G_{\rho_n(b)} := \langle x_1, \ldots, x_n;\ x_1 = \bigl(\rho_n(b)\bigr)(x_1), \ldots, x_n = \bigl(\rho_n(b)\bigr)(x_n)\rangle.$$

Then we have $G_{\rho_n(b)} \cong \pi_1(S^3 \setminus K_b)$.

Applying Fox' free differential calculus [11] to Artin's representation, we can derive a linear representation, the Burau representation $\beta_n : B_n \to Gl_n(\mathbb{Z}[x^{\pm 1}])$, of the braid group. Its irreducible $(n-1)$-dimensional part may be used to calculate the Alexander polynomial of a knot [7].

Theorem 1.2 (Burau's Theorem). *Let $b \in B_n$ be a braid, K_b the corresponding knot and $\hat{\beta}_n : B_n \to Gl_{n-1}(\mathbb{Z}[x^{\pm 1}])$ the reduced Burau representation. The Alexander polynomial $\mathcal{A}(K_b)$ of K_b is given by*

$$\mathcal{A}(K_b) = (-1)^{n-1} \frac{\det(\hat{\beta}_n(b) - Id)}{x^{n-1} + x^{n-2} + \cdots + 1}.$$

Since Vassiliev invented the theory of Vassiliev invariants [17,5], singular knots, which were rarely considered before 1990, attracted a lot of interest. For singular knots, the embedding property is weakened: A knot is allowed to have a finite number of self-intersections, where two pieces of string intersect each other in a transverse manner. Singular knots are only distinguished up to a certain notion of isotopy, known as rigid vertex isotopy.

This notion of isotopy is translated to diagrams by introducing the following singular Reidemeister moves. So we might consider two singular knots as isotopic if they admit knot diagrams which can be transformed one into the other by a finite sequence of (possibly singular) Reidemeister moves. See Fig. 2.

In the same manner, we can generalize braids to singular braids, where a finite number of singularities are allowed. Since a singularity cannot be undone, the set SB_n of singular braids on n strings does not form a group any longer. However, it still forms a monoid. Algebraically it can be defined in the following way [4,2]:

Definition 1.3. The singular braid monoid SB_n on n strands is generated by the elements

$$s_1, \ldots, s_{n-1}, s_1^{-1}, \ldots, s_{n-1}^{-1}, t_1, \ldots, t_{n-1}$$

due to the relations

(1) $\forall i < n$: $s_i s_i^{-1} = e$.
(2) For $|i - j| > 1$:
  (a) $s_i s_j = s_j s_i$;
  (b) $t_i s_j = s_j t_i$;
  (c) $t_i t_j = t_j t_i$.
(3) $\forall i < n$: $s_i t_i = t_i s_i$.
(4) $\forall i < n - 1$:
  (a) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$;
  (b) $t_i s_{i+1} s_i = s_{i+1} s_i t_{i+1}$;
  (c) $t_{i+1} s_i s_{i+1} = s_i s_{i+1} t_i$.

Geometrically, the generators correspond to the following braids (see Fig. 3).

For more information about the structure of the singular braid monoid the reader might consult [10].

The braid group B_n is the group of all invertible elements in SB_n. Its classical presentation is obtained from the presentation of SB_n by omitting the generators t_i as well as all the relations in which they appear.

The relation between singular knots and singular braids is just the same as in the classical case.

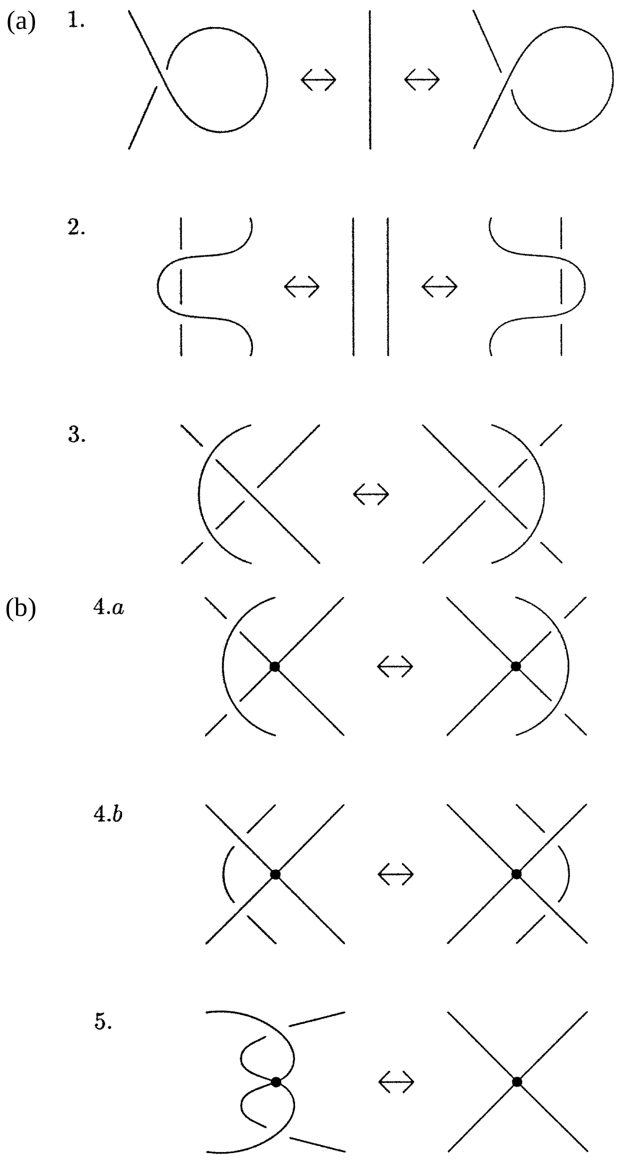

Fig. 2. (a) Regular Reidemeister moves. (b) Singular Reidemeister moves.

The following generalizations of Alexander's Theorem and Markov's Theorem may be found in [4,12].

Theorem 1.4 (Alexander's Theorem, singular version). *For any* (*singular*) *knot K there is a* (*singular*) *braid b such that K is isotopic to K_b.*

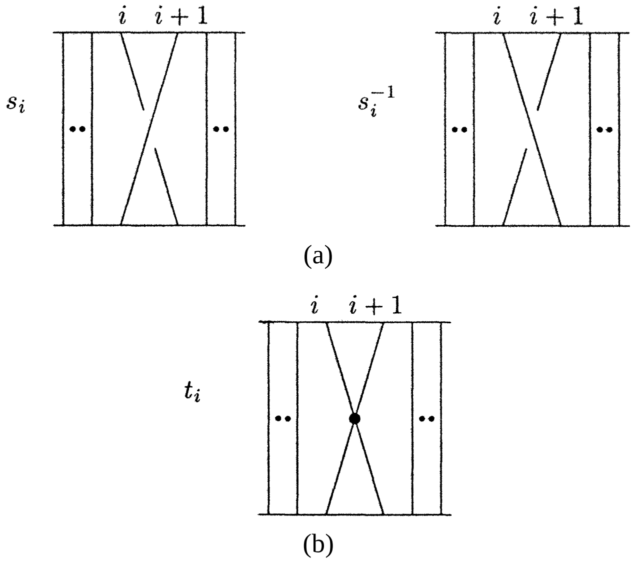

Fig. 3. (a) Generators of the braid group. (b) Singular generators.

Theorem 1.5 (Markov's Theorem, singular version). *If $b \in SB_n$ and $c \in SB_m$ are two* (*singular*) *braids with K_b isotopic to K_c then there is a finite sequence of* (*singular*) *Markov moves which transforms b into c. The singular Markov moves are given by*

(1) $b_1 b_2 \in SB_n \sim_M b_2 b_1 \in SB_n$;
(2) $b_1 \in SB_n \sim_M b_1 s_n^{\pm 1} \in SB_{n+1}$.

In this article we want to consider extensions of ρ_n and β_n to the singular braid monoid and examine the topological meanings of these extensions.

SB_n is a monoid and not a group and therefore it would be an unnecessary restriction if we would only consider extensions

$$\tilde{\rho}_n : SB_n \to Aut\big(\boldsymbol{F}^n\big) \quad \text{and} \quad \tilde{\beta}_n : SB_n \to Gl_n\big(\mathbb{Z}\big[x^{\pm 1}\big]\big).$$

Since the group $Aut(\boldsymbol{F}^n)$ embeds in the monoid $End(\boldsymbol{F}^n)$ of endomorphisms of $\boldsymbol{F}^n$ and the group $Gl_n(\mathbb{Z}[x^{\pm 1}])$ embeds in the monoid $M_n(\mathbb{Z}[x^{\pm 1}])$ of all matrices with entries in $\mathbb{Z}[x^{\pm 1}]$, it would be more interesting to look for extensions $\tilde{\rho}_n : SB_n \to End(F^n)$ and $\tilde{\beta}_n : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}])$. To summarize this, we regard the following diagram:

$$\begin{tikzcd}
Aut(\boldsymbol{F}^n) \arrow[d] \arrow[rr, bend left, "Fox"] & B_n \arrow[l, "\rho_n"'] \arrow[r, "\beta_n"] \arrow[d] & Gl_n(\mathbb{Z}[x^{\pm 1}]) \arrow[d] \\
End(\boldsymbol{F}^n) \arrow[rr, bend right, "?"'] & SB_n \arrow[l, "?"] \arrow[r, "?"'] & M_n(\mathbb{Z}[x^{\pm 1}])
\end{tikzcd}$$

Five questions arise naturally:

(1) Are there any representations $\tilde{\rho}_n : SB_n \to End(\boldsymbol{F}^n)$ which extend the Artin representation ρ_n?
This question will be answered in the affirmative in Section 3. In fact, we will find a whole family Λ_n of possible extensions. This family can be indexed in a natural way by the elements of the free group $\boldsymbol{F}^2$ on two generators.

(2) Are there any representations $\tilde{\beta}_n : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}])$ which extend the Burau representation β_n?
This question will be answered in Section 2. Once more, we will find a whole family λ_n of possible extensions. All these extensions arise as specifications of a universal extension $\beta_n^* : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}, y])$.

(3) Does Fox' free differential calculus relate the extensions in Λ_n to those in λ_n?
This question will be answered in Section 5. In fact, the free differential calculus induces a map $\phi_n : \Lambda_n \to \lambda_n$ which preserves a certain multiplicative structure on Λ_n and λ_n. This multiplicative structure will be defined in Section 4.

(4) Is there a generalizations of Burau's Theorem: Can the Alexander polynomial be generalized to singular knots by using extensions of the Burau representation?
This question is treated in a separate paper [13] where the universal extension β_n^* is used to define a generalized Alexander polynomial. A generalization of the Homfly polynomial and relations between these generalized polynomials and Vassiliev invariants are also treated in this paper.

(5) Is there a generalization of Artin's Theorem: Do we obtain group invariants of singular knots from the extensions of ρ_n which relate to the fundamental group of the knot complement?
Section 7 deals with this question.

This note summarizes some results of the author's Ph.D. thesis [14]. The other parts of it may be found in [13].

2. Extensions of the Burau representation

In this section we will examine possible extensions of the Burau representation β_n. We start by recalling the definition of the classical Burau representation [6].

Definition 2.1. The classical Burau representation $\beta_n : B_n \to Gl_n(\mathbb{Z}[x^{\pm 1}])$ is given by

$$\beta_n(s_i) = \left(\begin{array}{c|cc|c} Id_{i-1} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 1-x & x & 0 \\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & Id_{n-i-1} \end{array}\right).$$

The Burau representation satisfies a certain locality condition: The only non-trivial entries of $\beta_n(s_i)$ are on the intersections of the ith and $(i+1)$th row and column. This corresponds to the geometric fact, that the braid s_i does only affect the ith and $(i+1)$th string. It is therefore natural to impose an analogous locality condition on our extensions.

Definition 2.2. An extension $\tilde{\beta}_n : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}])$ of the Burau representation β_n is called local if

$$\tilde{\beta}_n(t_i) = \left(\begin{array}{c|c|c} Id_{i-1} & 0 & 0 \\ \hline 0 & M_i & 0 \\ \hline 0 & 0 & Id_{n-i-1} \end{array}\right),$$

where $M_i \in M_2(\mathbb{Z}[x^{\pm 1}])$. The set of all local extensions of β_n will be denoted by λ_n.

Note that for $n \geqslant 3$ there are extensions which are not local. For example, we get a non-local representation of SB_n by setting $\tilde{\beta}_n(t_i) = 0$ for all i. Obviously, this representation is not very interesting.

With the notation introduced above, we can answer the first question of the introduction:

Proposition 2.3. *There is a representation $\beta_n^* : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}, y])$ given by $\beta_n^*(\sigma_i) = \beta_n(\sigma_i)$ and*

$$\beta_n^*(t_i) = \left(\begin{array}{c|c|c} Id_{i-1} & 0 & 0 \\ \hline 0 & \begin{matrix} 1-xy & xy \\ y & 1-y \end{matrix} & 0 \\ \hline 0 & 0 & Id_{n-i-1} \end{array}\right).$$

All local extensions $\tilde{\beta}_n : SB_n \to M_n(\mathbb{Z}[x^{\pm 1}])$ are obtained from β_n^* by substitution of y by a Laurent polynomial in $\mathbb{Z}[x^{\pm 1}]$.

Proof. Checking the relations of Definition 1.3, we easily see that β_n^* is a representation of SB_n. We have to show that any local extension of β_n is obtained by substitution of y.

By relation 4(c) of the singular braid monoid we have $t_2 = s_1 s_2 t_1 s_2^{-1} s_1^{-1}$. This implies that $M_2 = M_1$. By induction we see that $M_i = M_1$ for all i. So assume that

$$M_1 = \begin{pmatrix} u & v \\ y & z \end{pmatrix}$$

with $u, v, y, z \in \mathbb{Z}[x^{\pm 1}]$. Now, relation (3) implies that we must have $v = xy$ and $u = y - xy + z$ while relation 4(b) implies that $x + z = 1$. These equations prove the proposition. $\square$

The local extension corresponding to the Laurent polynomial p will be denoted by β_n^p. As an immediate consequence of the last proposition we get:

Corollary 2.4. *The set λ_n is in natural bijection with $\mathbb{Z}[x^{\pm 1}]$.*

We have already mentioned in the introduction that the universal extension β_n^* may be used to generalize the Alexander polynomial to singular knots [13]. Still, there are also some other applications of β_n^*: In [9] the faithfulness of β_3^* is used to show that the Birman conjecture [4] is true for $n = 3$.

3. Extensions of the Artin representation

In this section we will consider extensions of the Artin representation ρ_n [1]. Once more, we start by recalling the classical case. Let $\boldsymbol{F}^n$ denote the free group generated by $x_1, \ldots, x_n$. We will consider the group $\boldsymbol{F}^n$ as canonically embedded in $\boldsymbol{F}^m$ for $n \leqslant m$. By $\langle a_1, a_2, \ldots, a_k\rangle$ we will denote the subgroup of $\boldsymbol{F}^n$ generated by $a_1, a_2, \ldots, a_k$.

Definition 3.1. The Artin Representation $\rho_n : B_n \to Aut(\boldsymbol{F}^n)$ is given by $\rho_n(s_i) = S_i$ where S_i is given by the following formulas:

$$\begin{aligned} S_i : x_i &\mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\mapsto x_i \\ x_j &\mapsto x_j \quad \text{for } j \neq i, i+1. \end{aligned}$$

Remark. In the braid group $b_1 b_2$ usually denotes the braid where b_1 is applied first and b_2 second. In the automorphism group $\rho_n' \circ \rho_n''$ denotes the automorphism where ρ_n'' is applied first and ρ_n' second. We will stick to this notation even though we get the rather unusual formula $\rho_n(b_1 b_2) = \rho_n(b_2) \circ \rho_n(b_1)$ for homomorphisms.

As the Burau representation, the Artin representation does also fulfill a locality condition. We therefore impose the analogous condition on our extensions $\tilde{\rho}_n$ as well.

Definition 3.2. An extension $\tilde{\rho}_n : SB_n \to End(\boldsymbol{F}^n)$ of ρ_n is called local, if

$$\begin{aligned} &\big(\tilde{\rho}_n(t_i)\big)(x_i) \in \langle x_i, x_{i+1}\rangle; \\ &\big(\tilde{\rho}_n(t_i)\big)(x_{i+1}) \in \langle x_i, x_{i+1}\rangle; \\ &\big(\tilde{\rho}_n(t_i)\big)(x_j) = x_j \quad \text{for } j \neq i, i, +1. \end{aligned}$$

The set of all local extensions of ρ_n will be denoted by Λ_n.

As in the case of local extensions of β, we can give a complete classification of all local extensions of ρ_n. To do this, we need the following two lemmas.

Lemma 3.3. *Let $\tau_i : \langle x_1, x_2\rangle \to \langle x_i, x_{i+1}\rangle$ be the homomorphism given by $\tau_i(x_j) = x_{i+j-1}$ for $j = 1, 2$. If $\tilde{\rho}_n : SB_n \to End(\boldsymbol{F}^n)$ is a local extension, we have*

$$\begin{aligned} &\tilde{\rho}_n(t_i)(x_i) = \tau_i\big(\tilde{\rho}_n(t_1)(x_1)\big); \\ &\tilde{\rho}_n(t_i)(x_{i+1}) = \tau_i\big(\tilde{\rho}_n(t_1)(x_2)\big). \end{aligned}$$

Proof. The assertion follows easily by induction on i, using relation 4(c) of the presentation of SB_n. $\square$

The last lemma means geometrically that the effect of a singularity does not depend on the place where it is situated in the braid. This translation invariance implies that we know the local extension $\tilde{\rho}_n$ completely, if we know its effect on t_1, that is, if we know $(\tilde{\rho}_n(t_1))(x_1)$ and $(\tilde{\rho}_n(t_1))(x_2)$.

Lemma 3.4. *Let $w, v \in \boldsymbol{F}^2 \subset \boldsymbol{F}^n$ be two words in $x_1^{\pm 1}$ and $x_2^{\pm 1}$. Let S_i be the automorphism of $\boldsymbol{F}^n$ defined by $\rho_n(s_i)$. Let $S_1(v) = w$ and $wv = x_1x_2$ in $\boldsymbol{F}^n$. Then there is a local extension $\tilde{\rho}_n : SB_n \to End(\boldsymbol{F}^n)$ with*

$$\big(\tilde{\rho}_n(t_i)\big)(x_i) = \tau_i(w) \quad \text{and} \quad \big(\tilde{\rho}_n(t_i)\big)(x_{i+1}) = \tau_i(v).$$

Proof. We have to show that all the relations of SB_n are satisfied by $\tilde{\rho}_n$. The relations (1), (2) and (4a) are relations in B_n. Therefore they are automatically satisfied because $\tilde{\rho}_n|_{B_n} = \rho_n$. Relations (2b) and (2c) hold by the locality condition. Hence we only have to deal with the relations (3), (4b) and (4c). In fact, translation invariance ensures that we just have to consider the following three relations:

(1) $s_1t_1 = t_1s_1$;
(2) $t_2s_1s_2 = s_1s_2t_1$;
(3) $t_1s_2s_1 = s_2s_1t_2$.

We will check these relations one after another. In the sequel we will denote the endomorphism $\tilde{\rho}_n(t_i)$ by T_i.

We start with the first relation. Since only the subgroup $\boldsymbol{F}^2$ of $\boldsymbol{F}^n$ is affected, we have to check the relation only on this subgroup. We see that:

$$\begin{aligned} T_1 \circ S_1(x_1x_2) &= T_1(x_1x_2) = wv = x_1x_2 \\ &= S_1(x_1x_2) = S_1(wv) = S_1 \circ T_1(x_1x_2) \end{aligned}$$

and that

$$T_1 \circ S_1(x_2) = T_1(x_1) = w = S_1(v) = S_1 \circ T_1(x_2).$$

Since x_1x_2 and x_2 generate $\boldsymbol{F}^2$ we have shown the first relation.

We go on with the second relation. This time we have to consider $\boldsymbol{F}^3 \subset \boldsymbol{F}^n$. We compute:

$$\begin{aligned} S_2 \circ S_1 \circ T_2(x_1) &= S_2 \circ S_1(x_1) = x_1x_2x_3x_2^{-1}x_1^{-1} = wvx_3v^{-1}w^{-1} \\ &= T_1\big(x_1x_2x_3x_2^{-1}x_1^{-1}\big) = T_1 \circ S_2 \circ S_1(x_1). \end{aligned}$$

In an analogous manner we see that

$$\begin{aligned} S_2 \circ S_1 \circ T_2(x_1x_2x_3) &= S_2 \circ S_1(x_1x_2x_3) = x_1x_2x_3 = T_1(x_1x_2x_3) \\ &= T_1 \circ S_2 \circ S_1(x_1x_2x_3). \end{aligned}$$

Since x_1, x_2 and $x_1x_2x_3$ generate $\boldsymbol{F}^3$, it remains to show that $T_1 \circ S_2 \circ S_1(x_2) = T_1 \circ S_2(x_1) = T_1(x_1) = w$ equals $S_2 \circ S_1 \circ T_2(x_2)$. By translation invariance we have $T_2(x_2) =$

$\tau_2(w)$. Thus we have to show that $S_2 \circ S_1 \circ \tau_2$ restricted to $\boldsymbol{F}^2$ is the identity. We easily check:

$$S_2 \circ S_1 \circ \tau_2(x_1) = S_2 \circ S_1(x_2) = x_1;$$
$$S_2 \circ S_1 \circ \tau_2(x_2) = S_2 \circ S_1(x_3) = x_2.$$

Hence we have shown that the second relation holds. The third relation holds by a similar argument. □

We are now in a position to state the main result of this section. For this purpose we introduce the automorphism r of $\boldsymbol{F}^2$. It is defined by $r(x_1) = x_2$ and $r(x_2) = x_1$.

Theorem 3.5. *If $\tilde{\rho}_n : SB_n \to End(\boldsymbol{F}^n)$ is a local extension of ρ_n, then there exists $\omega \in \boldsymbol{F}^2$ such that*
(1) $(\tilde{\rho}_n(t_i))(x_i) = \tau_i(x_1 r(\omega)\omega^{-1})$;
(2) $(\tilde{\rho}_n(t_i))(x_{i+1}) = \tau_i(\omega r(\omega^{-1})x_2)$.
Moreover, any element $\omega \in \boldsymbol{F}^2$ induces a local extension in the indicated way.

Proof. We start by showing that the formulas above define a local extension for any $\omega \in \boldsymbol{F}^2$. In view of Lemmas 3.3 and 3.4 we only have to show that
(1) $x_1 r(\omega)\omega^{-1}\omega r(\omega)^{-1}x_2 = x_1x_2$;
(2) $S_1(\omega r(\omega)^{-1}x_2) = x_1 r(\omega)\omega^{-1}$.
The first equation being obvious, we only have to consider the second equation. It is easy to observe that $S_1 \circ r$, restricted to $\boldsymbol{F}^2$, is just conjugation with x_1. Therefore we have

$$S_1\big(\omega r(\omega)^{-1}x_2\big) = S_1\big(r\big(r(\omega)\omega^{-1}x_1\big)\big) = x_1 r(\omega)\omega^{-1}$$

which proves the first part of the theorem.

Now let us show that for any local extension $\tilde{\rho}_n$ we have $\omega \in \boldsymbol{F}^2$ as described above. Let $\tilde{\rho}_n$ be an arbitrary local extension and set $T_i = \tilde{\rho}_n(t_i)$. Define w, v to be $w = T_1(x_1)$ and $v = T_1(x_2)$. Using the relations holding in SB_n we compute that $T_1 \circ S_2 \circ S_1(x_1) = wvx_3v^{-1}w^{-1}$ and $S_2 \circ S_1 \circ T_2(x_1) = x_1x_2x_3x_2^{-1}x_1^{-1}$. By locality w and v are in $\boldsymbol{F}^2$. Hence we get $wv = x_1x_2$. This implies that w and v may be written in the following form:

$$w = x_1\widetilde{w}^{-1}, \qquad v = \widetilde{w}x_2,$$

for some $\widetilde{w} \in \boldsymbol{F}^2$. With this notation we observe that

$$x_2x_1r\big(\widetilde{w}\big)x_2^{-1} = r \circ T_1 \circ S_1(x_1) = r \circ S_1 \circ T_1(x_1) = x_2x_1x_2^{-1}r \circ S_1\big(\widetilde{w}^{-1}\big).$$

Since $r \circ S_1$ is conjugation with x_2 this implies that $\widetilde{w} = r(\widetilde{w}^{-1})$. It is now easy to derive that there exists ω as required in the assertion. This completes the proof. □

As an immediate corollary we get:

Corollary 3.6. *The elements of Λ_n are in natural bijection with the elements $\boldsymbol{F}^2$.*

The simplest examples of local extensions will be given explicitly in the following example.

Example 3.7. The simplest local extension, corresponding to the identity of $\boldsymbol{F}^2$, is given by setting $\rho_n^e(t_i) = id$. The element $x_1 \in \boldsymbol{F}^2$ corresponds to the extension $\rho_n^{x_1}$ defined by $\rho_n^{x_1}(t_1) = \rho_n(s_i)$ and the element x_2^{-1} corresponds to the extension $\rho_n^{x_2^{-1}}$ given by $\rho_n^{x_2^{-1}}(t_1) = \rho_n(s_1^{-1})$. Two other local extensions are given by

$$\rho_n^{x_2}(t_1)(x_1) = x_1x_1x_2^{-1}, \qquad \rho_n^{x_2}(t_1)(x_2) = x_2x_1^{-1}x_2,$$

where $\omega = x_2$ and

$$\rho_n^{x_1^{-1}}(t_1)(x_1) = x_1x_2^{-1}x_1, \qquad \rho_n^{x_1^{-1}}(t_1)(x_2) = x_1^{-1}x_2x_2,$$

where $\omega = x_1^{-1}$.

It is worth noting that while $\rho_n : B_n \to Aut(\boldsymbol{F}^n)$ is known to be faithful, we do not know if any of our extensions ρ_n^ω is faithful. Even if this is not the case, it seems very likely that, given two different singular braids b_1 and b_2, there is an extension ρ_n^ω with $\rho_n^\omega(b_1) \neq \rho_n^\omega(b_2)$.

4. Multiplicative structures on λ_n and Λ_n

In this section we will equip λ_n and Λ_n with multiplicative structures. We start by a simple observation:

Remark that the projection $p_n^k : SB_n \to B_n$ given by $p_n^k(s_i) = s_i$ and $p_n^k(t_i) = s_i^k$ is a homomorphism (of monoids) for any $k \in \mathbb{Z}$. Hence, given any representation $\gamma_n : B_n \to M$ of B_n into some monoid M, we get representations $\gamma_n^k : SB_n \to M$ by setting $\gamma_n^k = \gamma_n \circ p_n^k$. Moreover we have $\gamma_n^k(t_i) \cdot \gamma_n^l(t_i) = \gamma_n^{k+l}(t_i)$. This formula gives rise to a multiplicative structure on the representations of SB_n.

Proposition 4.1. *Let M be a monoid and $\phi_n : B_n \to M$ a representation. Let $\phi_n', \phi_n'' : SB_n \to M$ be two extensions of ϕ_n with the commutator property that*

$$\phi_n'(t_i)\phi_n''(t_j) = \phi_n''(t_j)\phi_n'(t_i) \quad \textit{for all } |i-j| \geqslant 2.$$

Then we have another representation $\psi_n = \phi_n' \odot \phi_n''$ of SB_n into M given by:

$$\psi_n(s_i) = \phi_n(s_i); \qquad \psi_n(t_i) = \phi_n'(t_i) \cdot \phi_n''(t_i).$$

Proof. We just have to check that $\phi_n' \odot \phi_n''$ satisfies the relations in SB_n. Since $(\phi_n' \odot \phi_n'')|_{B_n} = \phi_n$ we do not have to check the relations (1), (2a) and (4a).

(2b) For $|i-j| \geqslant 2$ we have

$$\begin{aligned}(\phi_n' \odot \phi_n'')(t_is_j) &= (\phi_n' \odot \phi_n'')(t_i) \cdot (\phi_n' \odot \phi_n'')(s_j) = \phi_n'(t_i)\phi_n''(t_i)\phi_n(s_j) \\ &= \phi_n'(t_i)\phi_n(s_j)\phi_n''(t_i) = \phi_n(s_j)\phi_n'(t_j)\phi_n''(t_i) = (\phi_n' \odot \phi_n'')(s_jt_i).\end{aligned}$$

(2c) For $|i - j| \geqslant 2$ we have

$$\begin{aligned}(\phi_n' \odot \phi_n'')(t_i t_j) &= \phi_n'(t_i)\phi_n''(t_i)\phi_n'(t_j)\phi_n''(t_j)\\ &= \phi_n'(t_j)\phi_n''(t_j)\phi_n'(t_i)\phi_n''(t_i) = (\phi_n' \odot \phi_n'')(t_j t_i).\end{aligned}$$

Here we have used the commutator property. The relations (3), (4b) and (4c) may be treated in a similar way. □

Thus, we can multiply representations of SB_n under certain circumstances. By the definition of our locality conditions, any pair of representations in Λ_n (respectively λ_n) can be multiplied with each other.

Corollary 4.2. *The sets Λ_n and λ_n are monoids with respect to the $\odot$-product.*

Attention: The multiplicative structures given by the $\odot$-product do not correspond to the multiplicative structure on $\boldsymbol{F}^2$ (respectively $\mathbb{Z}[x^{\pm 1}]$) induced by the bijections of Corollaries 2.4 and 3.6.

It may be interesting to understand the structures of the monoids Λ_n and λ_n. We will give some first results here.

Proposition 4.3. *The monoid λ_n is commutative. Its invertible elements are exactly those of the form $\tilde{\beta}_n = \beta_n \circ p_n^k$ (with p_n^k defined as above). Hence, the group of all invertible elements of λ_n is isomorphic to $\mathbb{Z}$.*

Proof. Let p_1 and p_2 be two polynomials in $\mathbb{Z}[x^{\pm 1}]$ and $\beta_n^{p_1}$ and $\beta_n^{p_2}$ be the two corresponding local extensions of λ_n. Easy calculations show that

$$\beta_n^{p_1} \odot \beta_n^{p_2} = \beta_n^q = \beta_n^{p_2} \odot \beta_n^1,$$

where $q = p_1 + p_2 - (x-1)p_1 p_2$. Thus, $(\lambda_n, \odot)$ is commutative.

Since it is clear that the extensions of the form $\beta_n \circ p_n^k$ are invertible, it remains to show that all invertible elements are of this form.

Now, if $\beta_n^{p_1}$ is invertible, then $\det(\beta_n^{p_1}(t_1)) = 1 - p_1 - x p_1$ has to be invertible in $\mathbb{Z}[x^{\pm 1}]$. Thus, there has to be an $m \in \mathbb{Z}$ such that $1 - p_1 - x p_1 = \pm x^m$. Easy calculations show that these equations are only satisfied for polynomials p_1 corresponding to the extensions described above. □

The structure of Λ_n is much more complicated than the one of λ_n. We only give the following result.

Proposition 4.4. *The monoid Λ_n is not commutative.*

Proof. It is easy to work out that the two extensions $\rho_n^{x_2}$ and $\rho_n^{x_1^{-1}}$ corresponding to $x_2 \in \boldsymbol{F}^2$ and $x_1^{-1} \in \boldsymbol{F}^2$ do not commute with respect to the $\odot$-product. □

5. The Fox homomorphism $\Phi_n : \Lambda_n \to \lambda_n$

In this section we will show that Fox' free differential calculus induces a homomorphism of monoids Φ_n from Λ_n to λ_n.

We briefly recall some facts about the free differential calculus. A detailed treatment may be found in [11] or [8].

Denote the group ring of $\boldsymbol{F}^n$ by $\mathbb{Z}[\boldsymbol{F}^n]$. The group $\boldsymbol{F}^n$ is canonically included in $\mathbb{Z}[\boldsymbol{F}^n]$. The partial derivative

$$\frac{\partial}{\partial x_i} : \boldsymbol{F}^n \to \mathbb{Z}[\boldsymbol{F}^n]$$

is uniquely defined by the following conditions:

(1) $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker symbol.

(2) $\frac{\partial}{\partial x_i}(w_1 w_2) = \frac{\partial}{\partial x_i}(w_1) + w_1 \frac{\partial}{\partial x_i}(w_2)$, where w_1, w_2 are elements in $\boldsymbol{F}^n$.

As an easy consequence we derive that $\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$.

Let $\pi : \mathbb{Z}[\boldsymbol{F}^n] \to \mathbb{Z}[x^{\pm 1}]$ be the ring homomorphism defined by $\pi(x_i) = x$ for all $i = 1, \ldots, n$. (By abuse of language we will denote the restriction of π to $\boldsymbol{F}^n$ with π as well.) By $M_n(\mathbb{Z}[x^{\pm 1}])$ we denote the ring of $(n \times n)$-matrices with entries in $\mathbb{Z}[x^{\pm 1}]$.

With this notation we can state the following result which relates representations into $\mathit{End}(\boldsymbol{F}^n)$ and linear representations. It is a slight generalization of analogous result concerning representations of groups into $\mathit{Aut}(\boldsymbol{F}_n)$ which may be found in [3] or [16]. The proof of the classical case also works for this generalization.

Proposition 5.1. *Let M be a monoid and let $\psi_n : M \to \mathit{Aut}(\boldsymbol{F}^n)$ be a representation of M such that $\pi((\psi_n(m))(x_i)) = x$ for all $m \in M$ and for all x_i. Then we obtain a representation $\Phi_n(\psi_n) : M \to M_n(\mathbb{Z}[x^{\pm 1}])$ by setting*:

$$\big(\Phi_n(\psi_n)\big)(m) = \begin{pmatrix} \pi\Big(\frac{\partial(\psi_n(m))(x_1)}{\partial x_1}\Big) & \cdots & \pi\Big(\frac{\partial(\psi_n(m))(x_1)}{\partial x_n}\Big) \\ \vdots & & \vdots \\ \pi\Big(\frac{\partial(\psi_n(m))(x_n)}{\partial x_1}\Big) & \cdots & \pi\Big(\frac{\partial(\psi_n(m))(x_n)}{\partial x_n}\Big) \end{pmatrix}.$$

Applying Φ_n to the representation ρ_n we get the following well known fact [3]:

Proposition 5.2. *With notations as above we have $\Phi_n(\rho_n) = \beta_n$.*

We are interested in an analogous result for local extensions. The following proposition answers the third question of the introduction.

Proposition 5.3. *The map $\Phi_n : \Lambda_n \to \lambda_n$ is a monoid homomorphism.*

Proof. Theorem 3.5 implies directly that the conditions of Proposition 5.1 are satisfied for all elements in Λ_n. Moreover, the locality condition of the local extension $\tilde{\rho}_n$ implies the locality condition for $\Phi_n(\tilde{\rho}_n)$. Hence, we have a well defined map $\Phi_n : \Lambda_n \to \lambda_n$. The fact that this map is a monoid homomorphism follows easily from Proposition 5.1. $\square$

Obviously, the monoid homomorphism Φ_n cannot be injective since Λ_n is not commutative while λ_n is. As an example the non-trivial local extension of ρ_n corresponding to the element

$$x_2^2 x_1^{-1} x_2^{-1} x_1^2 x_2^{-1} x_1^{-1} \in \boldsymbol{F}^2$$

maps to the unit element in λ_n under Φ_n.

Surprisingly, the map Φ_n is not surjective neither. The following proposition determines its image completely.

Proposition 5.4. *Let β_n^p be a Burau-like extension of β_n and $p = \sum p_i x^i \in \mathbb{Z}[x^{\pm 1}]$ the associated polynomial. Let $\hat{p} \in \mathbb{Z}/2\mathbb{Z}[x^{\pm 1}]$ polynomial p reduced modulo* 2. *Then β_n^p is the image of some $\rho'_n \in \Lambda_p^n$ if and only if $\hat{p}$ is of the following form*:

(1) $\hat{p} = 0$;
(2) $\hat{p} = \sum_{i=0}^{m} x^i$ *with* $m \geqslant 0$;
(3) $\hat{p} = \sum_{i=-m}^{-1} x^i$ *with* $m \geqslant 1$.

Proof. For notational reasons, we consider the map Φ_n as a map from $\boldsymbol{F}^2$ to $\mathbb{Z}[x^{\pm 1}]$ rather than a map from Λ_n to λ_n. By definition of Φ_n and the bijections of Corollaries 2.4 and 3.6 this means that

$$\Phi_n(\omega) = \pi\left(\frac{\partial(\omega r(\omega^{-1})x_2)}{\partial x_1}\right).$$

So we have to prove that for any $\omega \in \boldsymbol{F}^2$, the polynomial $p = \pi(\partial(\omega r(\omega^{-1})x_2)/\partial x_1)$ is of the form presented in the proposition. This can be done by induction on the minimal word length of ω. The induction step requires long and technical computations. Therefore it is omitted here. A detailed proof may be found in [14].

Hence, any local extension in the image of Φ_n is covered by our proposition. It remains to prove that every polynomial p, which is of the given form, has a preimage in $\boldsymbol{F}^2$. We start by constructing some examples:

$\omega \in \boldsymbol{F}^2$	Polynomial $\Phi_n(\omega)$
$\nu_1^k = x_1^{k+1}$	$\mu_1^k = \sum_{i=0}^{k} x^i$
$\nu_2^k = x_2^{-k}$	$\mu_2^k = \sum_{i=-k}^{-1} x^i$
$\nu_3^k = x_1^{k+1} x_2^{-1} x_1^{-k}$	$\mu_3^k = 2x^k$
$\nu_4^k = x_1^k x_2 x_1^{-k-1}$	$\mu_4^k = -2x^k$
$\nu_5^k = x_2^{-k} x_1 x_2^{k-1}$	$\mu_5^k = 2x^{-k}$
$\nu_6^k = x_1^{-k} x_2 x_1^{k-1}$	$\mu_6^k = -2x^{-k}$

We observe that for $i \geqslant 3$ and any $\omega \in \boldsymbol{F}^2$ we have

$$\Phi_n(\nu_i^k \omega) = \mu_i^k + \Phi_n(\omega).$$

Since any polynomial p of the form given above may be written as a sum of the μ_i^k in which only one term of the sum is of the form μ_1^k or μ_2^k, this formula allows us to construct inductively an element in the preimage of p. $\square$

6. The fundamental group of the complement of a singular knot

The fundamental group of a knot complement is a strong invariant in the regular case and many algorithms which yield presentations of these groups are known. One of the most famous such algorithms is due to Wirtinger [8]. It starts off from an arbitrary knot diagram and produces the so-called Wirtinger presentation. In this section we will generalize this method to singular knots. Let K be a (possibly) singular knot and D_K be a diagram of this knot. We decompose K in components by cutting it open at

(1) any undercrossing;
(2) any singular point. Locally, four components arise from a singular point.

To each of the constructed components we assign a generator of the Wirtinger presentation. Any singular point of K and any other double point of D_K gives a relation of the Wirtinger presentation. These relations are depicted in Fig. 4.

Definition 6.1. The Wirtinger group $W(K_D)$ of a knot diagram D_K is the group presented by the Wirtinger presentation.

Example 6.2. We want to calculate the Wirtinger presentation associated to the knot diagram drawn below. We get:

$$\begin{aligned} W(D_K) &\cong \langle a,b,c,d;\, dc=ba,\, cb=dc,\, ba=cb\rangle \\ &\cong \langle a,b,c,d;\, d=bac^{-1},\, cb=dc,\, a=b^{-1}cb\rangle \\ &\cong \langle b,c;\, bc=bc\rangle \\ &\cong \mathbb{Z}*\mathbb{Z}. \end{aligned}$$

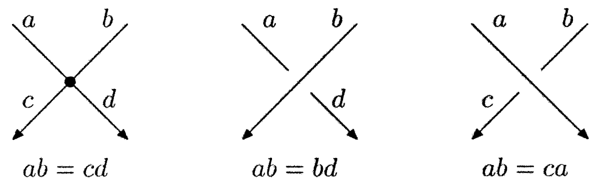

Fig. 4. Relations of the Wirtinger presentation.

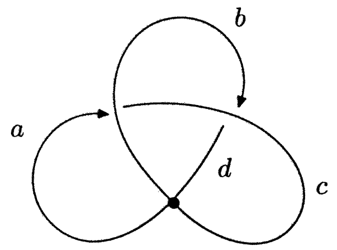

Fig. 5. The Wirtinger presentation of $3^{*!}$.

Regarding singular Reidemeister moves, it is easy to check that the Wirtinger group does not depend on the diagram D_K but only on the knot itself. Moreover, we get:

Proposition 6.3. *Let K be a (possibly) singular knot, D_K a diagram of K and $W(D_K)$ its Wirtinger group. Then we have $W(D_K) \cong \pi_1(S^3 \setminus K)$.*

The proof is an easy adaptation of Crowell and Fox' proof in the regular case [8]. Therefore it will be omitted here.

7. Group invariants of singular knots

In this section we will study group invariants of singular knots coming from local extensions of ρ_n. We will see that these group invariants are related to the Wirtinger group of a singular knot. We start with the following proposition which assures that we get a group invariant for any local extension $\tilde{\rho} \in \Lambda_n$.

Proposition 7.1. *Let ω be an element in $\boldsymbol{F}^2$ and ρ_n^ω be the local extension of ρ_n corresponding to ω. Let $b \in SB_n$ be a braid. The group $G_{\rho_n^\omega(b)}$ presented by*

$$G_{\rho_n^\omega(b)} = \big\langle x_1, \dots, x_n;\ x_1 = \big(\rho_n^\omega(b)\big)(x_1), \dots, x_n = \big(\rho_n^\omega(b)\big)(x_n)\big\rangle$$

does only depend on the (singular) Markov class of b. We therefore can define a group invariant G_ω of singular knots by setting $G_\omega(K_b) = G_{\rho_n^\omega(b)}$.

Proof. We have to show that the group $G_{\rho_n^\omega(b)}$ is invariant under Markov's moves, that is we have to show that

(1) $G_{\rho_n^\omega(b_1b_2)} \cong G_{\rho_n^\omega(b_2b_1)}$;

(2) $G_{\rho_n^\omega(b)} \cong G_{\rho_{n+1}^\omega(bs_n^{\pm 1})}$.

By definition we have

$$G_{\rho_n^\omega(b_1b_2)} \cong \big\langle x_1, \dots, x_n\colon x_1 = \rho_n^\omega(b_1b_2)(x_1), \dots, x_n = \rho_n^\omega(b_1b_2)(x_n)\big\rangle;$$
$$G_{\rho_n^\omega(b_2b_1)} \cong \big\langle \tilde{x}_1, \dots, \tilde{x}_n\colon \tilde{x}_1 = \rho_n^\omega(b_2b_1)(\tilde{x}_1), \dots, \tilde{x}_n = \rho_n^\omega(b_2b_1)(\tilde{x}_n)\big\rangle.$$

Now, we define homomorphisms $\phi: G_{\rho_n^\omega(b_1b_2)} \to G_{\rho_n^\omega(b_2b_1)}$ and $\psi: G_{\rho_n^\omega(b_2b_1)} \to G_{\rho_n^\omega(b_1b_2)}$ by setting

$$\phi(x_i) = \rho_n^\omega(b_1)(\tilde{x}_i)$$

and

$$\psi(\tilde{x}_i) = \rho_n^\omega(b_2)(x_i).$$

It is easy to check that these homomorphisms are well defined and that we have $\psi \circ \phi = id$ and $\phi \circ \psi = id$. Therefore we have $G_{\rho_n^\omega(b_1b_2)} \cong G_{\rho_n^\omega(b_2b_1)}$ and we are done with the first case. So, we are left with the second case. By definition we have

$$G_{\rho_{n+1}^\omega(bs_n)} \cong \big\langle x_1, \dots, x_{n+1};\ x_1 = \rho_{n+1}^\omega\big(bs_n^{\pm 1}\big)(x_1), \dots, x_{n+1} = \rho_{n+1}^\omega\big(bs_n^{\pm 1}\big)(x_{n+1})\big\rangle.$$

Since

$$\begin{aligned}\rho^{\omega}_{n+1}\big(bs_n^{\pm 1}\big)(x_{n+1}) &= \rho^{\omega}_{n+1}\big(s_n^{\pm 1}\big)\big(\rho^{\omega}_{n+1}(b)(x_{n+1})\big)\\ &= \rho^{\omega}_{n+1}\big(s_n^{\pm 1}\big)(x_{n+1}) = x_n\end{aligned}$$

we get

$$\begin{aligned}G_{\rho^{\omega}_{n+1}(bs_n^{\pm 1})} \cong \big\langle & x_1,\dots,x_{n+1};\ x_1 = \rho^{\omega}_{n+1}\big(s_n^{\pm 1}\big)\big(\rho^{\omega}_{n+1}(b)(x_1)\big),\dots,\\ & x_n = \rho^{\omega}_{n+1}\big(s_n^{\pm 1}\big)\big(\rho^{\omega}_{n+1}(b)(x_n)\big), x_{n+1} = x_n\big\rangle.\end{aligned}$$

The equation $x_{n+1} = x_n$ implies that $\rho^{\omega}_n(s_n^{\pm 1})$ is the identity. Hence

$$\begin{aligned}G_{\rho^{\omega}_{n+1}(bs_n^{\pm 1})} &\cong \big\langle x_1,\dots,x_n;\ x_1 = \rho^{\omega}_n(b)(x_1),\dots,x_n = \rho^{\omega}_{n+1}(b)(x_n)\big\rangle\\ &\cong G_{\rho^{\omega}_n(b)}.\end{aligned}$$

This completes the proof. $\square$

Thus we have defined an infinite family of group invariants for singular knots. If K is a regular knot, Artin's Theorem assures that $G_\omega(K) \cong \pi_1(S^3 \setminus K)$ for all ω. If K is singular, the group invariants corresponding to different $\omega \in \boldsymbol{F}^2$ are different as the following example shows:

Example 7.2. Let K be a singular knot. Let K_+ be the regular knot obtained from K by resolving all the singularities in the right-handed manner and K_- be the regular knot obtained from K by resolving all the singularities in the left-handed way. Then we have:

$$G_{x_1}(K) \cong \pi_1\big(S^3 \setminus K_+\big);$$
$$G_{x_2^{-1}}(K) \cong \pi_1\big(S^3 \setminus K_-\big).$$

Table 1 shows the value of the group invariants G_{x_2} and $G_{x_1^{-1}}$ on the three knots drawn below. More examples may be found in Appendix B.

Thus, for a singular knot K the group $G_\omega(K)$ is not the fundamental group of the knot complement (at least not for all ω). However, there is still a close relationship between our group invariants and $\pi_1(S^3 \setminus K)$.

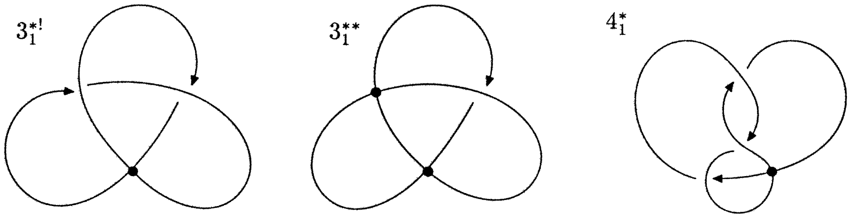

Fig. 6. Group invariants of certain singular knots.

Table 1

Knot K	Local extension	Corresponding element ω	$G_\omega(K)$
$3_1^{*!}$	$\rho_n^{x_2}$	x_2	$\langle a, b;\ a^2b = b^2a\rangle$
$3_1^{*!}$	$\rho_n^{x_1^{-1}}$	x_1^{-1}	$\langle a, b;\ bab^{-1} = aba^{-1}\rangle$
3_2^{**}	$\rho_n^{x_2}$	x_2	$\langle a, b;\ a^2b^{-1}a^2 = b^2a^{-1}b^2\rangle$
3_2^{**}	$\rho_n^{x_1^{-1}}$	x_1^{-1}	$\langle a, b;\ ba^{-1}ba^{-2} = ab^{-1}ab^{-2}\rangle$
4_1^{*}	$\rho_n^{x_2}$	x_2	$\langle a, b;\ (ba)^{-1}aba^{-1} = ab^{-1}bab^{-1}\rangle$
4_1^{*}	$\rho_n^{x_1^{-1}}$	x_1^{-1}	$\langle a, b;\ a^2b = b^2a\rangle$

Proposition 7.3. *Let ω be an element of $\boldsymbol{F}^2$ and K be a singular knot. The group $G_\omega(K)$ is a quotient group (depending on ω) of the fundamental group $\pi_1(S^3 \setminus K)$ of the knot complement.*

Proof. Let $b \in SB_n$ be a braid with $K_b \sim K$. Write $b = b_1 b_2 \dots b_m$ in the standard generators ($b_i \in \{s_j^{\pm 1}, t_j\}$) and consider the Wirtinger presentation of the closed braid K_b.

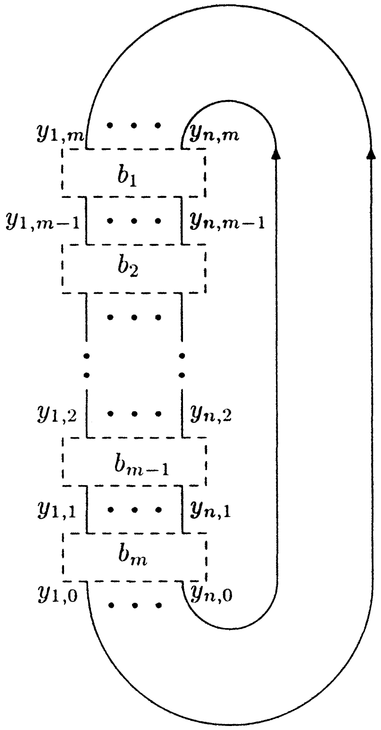

Fig. 7. Group invariants and the Wirtinger presentation.

Slightly modifying the Wirtinger presentation, we see that $\pi_1(S^3 \setminus K_b)$ is generated by the $y_{i,j}$ (with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$), due to the relations:

(1) $y_{i,m} = y_{i,0}$ for $1 \leqslant i \leqslant n$;
(2) If $b_{m-j} = s_i$:
  (a) $y_{i,j} = y_{i+1,j+1}$;
  (b) $y_{i,j+1} y_{i+1,j+1} = y_{i,j} y_{i+1,j}$;
  (c) $y_{k,j} = y_{k,j+1}$ for $k \neq i, i+1$;
(3) If $b_{m-j} = s_i^{-1}$:
  (a) $y_{i+1,j} = y_{i,j+1}$;
  (b) $y_{i,j+1} y_{i+1,j+1} = y_{i,j} y_{i+1,j}$;
  (c) $y_{k,j} = y_{k,j+1}$ for $k \neq i, i+1$;
(4) If $b_{m-j} = t_i$:
  (a) $y_{i,j+1} y_{i+1,j+1} = y_{i,j} y_{i+1,j}$;
  (b) $y_{k,j} = y_{k,j+1}$ for $k \neq i, i+1$;

The group $G_{\rho_n^\omega}(b)$ is given by

$$\bigl\langle x_1, \ldots, x_n;\ x_1 = \rho_n^\omega(b)(x_1), \ldots, x_n = \rho_n^\omega(b)(x_n)\bigr\rangle.$$

We have to show that there is a surjective group homomorphism

$$\psi : \pi_1\bigl(S^3 \setminus K_b\bigr) \to G_{\rho_n^\omega(b)}.$$

We define it by $\psi(y_{i,j}) = \rho_n^\omega(b_{m+1-j} b_{m+2-j} \ldots b_m)(x_i)$. Especially, this means that $\psi(y_i, 0) = \rho_n^\omega(e)(x_i) = x_i$. Thus, if ψ is well defined, then it is surjective. We therefore have to show that the relations of $\pi_1(S^3 \setminus V)$ are respected by ψ.

(1) Since

$$\psi(y_{i,m}) = \rho_n^\omega(b_1 \ldots b_m)(x_i) = \rho_n^\omega(b)(x_i) = x_i = \psi(y_{i,0})$$

the relations of (1) hold.

(2) The relations for $b_{m-j} = s_i$:

(a) Relation (2a) holds because

$$\begin{aligned} \psi(y_{i+1,j+1}) &= \rho_n^\omega(b_{m-j} b_{m+1-j} \ldots b_m)(x_{i+1}) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)\bigl(\rho_n^\omega(b_{m-j})(x_{i+1})\bigr) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)\bigl(\rho_n^\omega(s_i)(x_{i+1})\bigr) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)(x_i) = \psi(y_{i,j}). \end{aligned}$$

(b) Relation (2b) holds because

$$\begin{aligned} \psi(y_{i,j+1} y_{i+1,j+1}) &= \rho_n^\omega(b_{m-j} b_{m+1-j} \ldots b_m)(x_i x_{i+1}) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)\bigl(\rho_n^\omega(b_{m-j})(x_i x_{i+1})\bigr) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)\bigl(\rho_n^\omega(s_i)(x_i x_{i+1})\bigr) \\ &= \rho_n^\omega(b_{m+1-j} \ldots b_m)(x_i x_{i+1}) = \psi(y_{i,j} y_{i+1,j}). \end{aligned}$$

(c) For relation (2c) we get the following equality:

$$\begin{aligned}\psi(y_{k,j+1}) &= \rho_n^\omega(b_{m-j}b_{m+1-j}\dots b_m)(x_k)\\ &= \rho_n^\omega(b_{m+1-j}\dots b_m)\big(\rho_n^\omega(b_{m-j})(x_k)\big)\\ &= \rho_n^\omega(b_{m+1-j}\dots b_m)\big(\rho_n^\omega(s_i)(x_k)\big)\\ &= \rho_n^\omega(b_{m+1-j}\dots b_m)(x_k) = \psi(y_{k,j}).\end{aligned}$$

(3) The relations for $b_{m-j} = s_i^{-1}$ may be treated in the same way.

(4) The relations for $b_{m-j} = t_i$:

(a) Since we have $\rho_n^\omega(t_i)(x_i x_{i+1}) = x_i r(\tau_i(\omega))\tau_i(\omega^{-1})\tau_i(\omega)r(\tau_i(\omega^{-1}))x_{i+1} = x_i x_{i+1}$ we can treat relation (4a) in the same way as relation (2b).

(b) Relation (4b) may be checked in the same way as relation (2c).

Hence, all relations are respected by ψ and the proof is complete. □

The last proposition gives another geometric interpretation for the generalized Alexander polynomial introduced in [13]: In fact, if $\Phi_n(\rho_n^\omega) = \beta_n^p$ then $\beta_n^p - Id$ is just the Alexander matrix corresponding to the group presentation of $G_{\rho_n^\omega(b)} = G_\omega(K_b)$. Since the generalized Alexander polynomial of K is defined by using the universal local extension β_n^* of β_n, it contains information about certain subgroups of $\pi_1(S^3 \setminus K)$.

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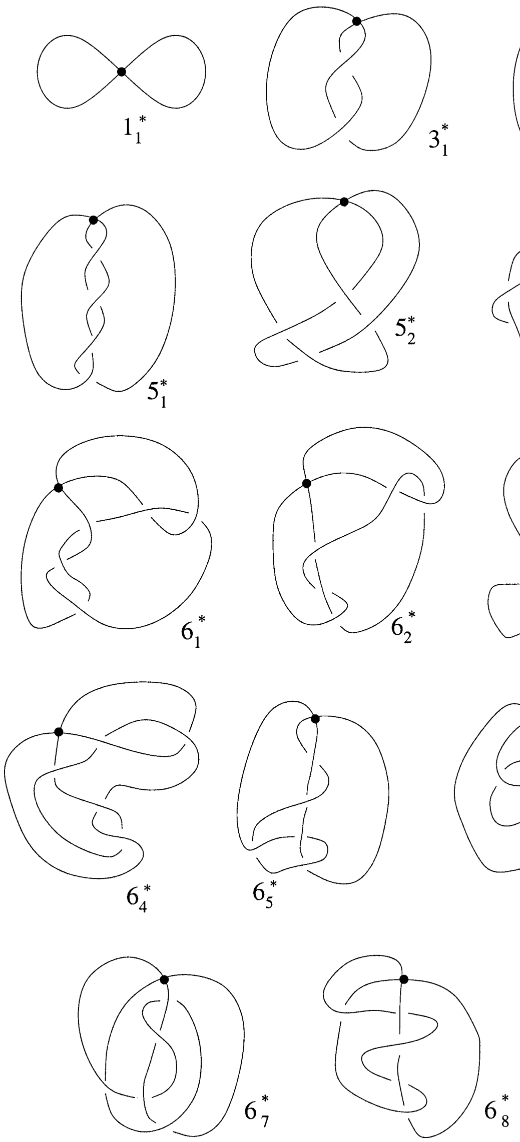
Appendix A. Singular knots

The following figure shows all irreducible knots with one singularity and admitting a projection with at most 6 double points.



Appendix B. Fundamental group and group invariants

For any knot K depicted in Appendix A, the following table gives the fundamental group of its complement, as well as the group invariants $G_{x_2}(K)$ and $G_{x_1^{-1}}(K)$

Knot	$\pi_1(S^3 \setminus K)$ $G_{x_2}(K)$ $G_{x_1^{-1}}(K)$
1_1^*	$\mathbb{Z} * \mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}$
3_1^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b;\ ab^2 = ba^2\rangle$ $\langle a, b; ab^{-1}a = b^{-1}ab\rangle$
4_1^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b;\ ab^2 = ba^2\rangle$ $\langle a, b;\ aba^{-1}b^{-1}a = babab^{-1}\rangle$
5_1^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b;\ abab^2 = baba^2\rangle$ $\langle a, b;\ ababa^{-1} = babab^{-1}\rangle$
5_2^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b;\ b^{-1}a^{-1}ba^{-1}bab^{-1} = ba^{-1}b^{-1}ab^{-1}aba^{-1}\rangle$ $\langle a, b;\ a^{-1}b^2ab^{-1} = b^{-1}a^2ba^{-1}\rangle$
5_3^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b;\ bab^{-1}a^{-1}bab = aba^{-1}b^{-1}aba\rangle$ $\mathbb{Z}$
6_1^*	$\langle a, b, c;\ abca = babc\rangle$ $\langle a, b, c; abca = babc, aabcac = baba^{-1}ca\rangle$ $\langle a, b, c; abca = babc, abcaca^{-1}c^{-1}b^{-1} = baba^{-1}b^{-1}a^1\rangle$
6_2^*	$\langle a, b, c;\ abca = babc\rangle$ $\langle a, b, c; abca = babc, a^2ba = bcab\rangle$ $\langle a, b, c; abca = babc, abab^{-1}a^{-1}b^{-1} = bcb^{-1}a^{-1}\rangle$
6_3^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b; aba^{-1}aba^{-1}ba = bab^{-1}a^{-1}bab^{-1}ab\rangle$ $\langle a, b; ba^{-1}b^{-1}ab = a^{-1}b^{-1}aba^{-1}\rangle$

Knot	$\pi_1(S^3 \setminus K)$ $G_{x_2}(K)$ $G_{x_1^{-1}}(K)$
6_4^*	$\langle a, b, c; abca = babc\rangle$ $\langle a, b, c; abca = babc,\ bca = abac^2\rangle$ $\langle a, b, c; abca = babc,\ bac^{-1}a^{-1}c = abacca^{-1}\rangle$
6_5^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b; bab^{-1}aba^{-1}b^2a = aba^{-1}bab^{-1}a^2b\rangle$ $\langle a, b; bab^{-1}aba^{-1}bab^{-1}a^{-1}b = aba^{-1}bab^{-1}aba^{-1}b^{-1}a\rangle$
6_6^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b; a^{-1}b^{-1}a^{-1}bab^{-1}a^{-1}b^1abab^{-1}a^{-1} = b^{-1}a^1b^{-1}aba^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}\rangle$ $\langle a, b; aba^2b^{-1}a^{-1}bab = bab^2a^{-1}b^{-1}aba\rangle$
6_7^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b; a^2bab = b^2aba\rangle$ $\langle a, b; (ab)^2(ba)^{-2}aba = (ba)^2(ab)^{-2}bab\rangle$
6_8^*	$\mathbb{Z} * \mathbb{Z}$ $\langle a, b; a^{-1}b^2ab^{-1} = b^{-1}a^2ba^{-1}\rangle$ $\langle a, b; a^{-1}bab^{-1}ab^{-1}a^{-1}ba^{-1} = b^{-1}aba^{-1}ba^{-1}b^{-1}ab^{-1}\rangle$

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