



Generalized Arf invariants in algebraic L -theory

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Abstract

The difference between the quadratic L -groups $L_*(A)$ and the symmetric L -groups $L^*(A)$ of a ring with involution A is detected by generalized Arf invariants. The special case $A = \mathbb{Z}[x]$ gives a complete set of invariants for the Cappell UNil-groups $\text{UNil}_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ for the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, extending the results of Connolly and Ranicki [Adv. Math. 195 (2005) 205–258], Connolly and Davis [Geom. Topol. 8 (2004) 1043–1078, e-print <http://arXiv.org/abs/math/0306054>].

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0. Introduction

The invariant of Arf [1] is a basic ingredient in the isomorphism classification of quadratic forms over a field of characteristic 2. The algebraic L -groups of a ring with involution A are Witt groups of quadratic structures on A -modules and A -module chain complexes, or equivalently the cobordism groups of algebraic Poincaré complexes over A .

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The cobordism formulation of algebraic L -theory is used here to obtain generalized Arf invariants detecting the difference between the quadratic and symmetric L -groups of an arbitrary ring with involution A , with applications to the computation of the Cappell UNil-groups.

The (projective) *quadratic L -groups* of Wall [20] are 4-periodic groups

$$L_n(A) = L_{n+4}(A).$$

The $2k$ -dimensional L -group $L_{2k}(A)$ is the Witt group of nonsingular $(-1)^k$ -quadratic forms (K, ψ) over A , where K is a f.g. projective A -module and ψ is an equivalence class of A -module morphisms

$$\psi : K \rightarrow K^* = \text{Hom}_A(K, A)$$

such that $\psi + (-1)^k \psi^* : K \rightarrow K^*$ is an isomorphism, with

$$\psi \sim \psi + \chi + (-1)^{k+1} \chi^* \quad \text{for } \chi \in \text{Hom}_A(K, K^*).$$

A lagrangian L for (K, ψ) is a direct summand $L \subset K$ such that $L^\perp = L$, where

$$\begin{aligned} L^\perp &= \{x \in K \mid (\psi + (-1)^k \psi^*)(x)(y) = 0 \text{ for all } y \in L\}, \\ \psi(x)(x) &\in \{a + (-1)^{k+1} \bar{a} \mid a \in A\} \quad \text{for all } x \in L. \end{aligned}$$

A form (K, ψ) admits a lagrangian L if and only if it is isomorphic to the hyperbolic form $H_{(-1)^k}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, in which case

$$(K, \psi) = H_{(-1)^k}(L) = 0 \in L_{2k}(A).$$

The $(2k + 1)$ -dimensional L -group $L_{2k+1}(A)$ is the Witt group of $(-1)^k$ -quadratic formations $(K, \psi; L, L')$ over A , with $L, L' \subset K$ lagrangians for (K, ψ) .

The *symmetric L -groups* $L^n(A)$ of Mishchenko [13] are the cobordism groups of n -dimensional symmetric Poincaré complexes (C, ϕ) over A , with C an n -dimensional f.g. projective A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \rightarrow \cdots$$

and $\phi \in Q^n(C)$ an element of the n -dimensional symmetric Q -group of C (about which more in §1 below) such that $\phi_0 : C^{n-*} \rightarrow C$ is a chain equivalence. In particular, $L^0(A)$ is the Witt group of nonsingular symmetric forms (K, ϕ) over A , with

$$\phi = \phi^* : K \rightarrow K^*$$

an isomorphism, and $L^1(A)$ is the Witt group of symmetric formations $(K, \phi; L, L')$ over A . An analogous cobordism formulation of the quadratic L -groups was obtained in [15], expressing $L_n(A)$ as the cobordism group of n -dimensional quadratic Poincaré complexes (C, ψ) , with $\psi \in Q_n(C)$ an element of the n -dimensional quadratic Q -group of C such that $(1 + T)\psi_0 : C^{n-*} \rightarrow C$ is a chain equivalence. The hyperquadratic L -groups $\widehat{L}^n(A)$ of [15] are the cobordism groups of n -dimensional (symmetric, quadratic) Poincaré pairs $(f : C \rightarrow D, (\delta\phi, \psi))$ over A such that

$$(\delta\phi_0, (1 + T)\psi_0) : D^{n-*} \rightarrow \mathcal{C}(f)$$

is a chain equivalence, with $\mathcal{C}(f)$ the algebraic mapping cone of f . The various L -groups are related by an exact sequence

$$\cdots \longrightarrow L_n(A) \xrightarrow{1+T} L^n(A) \longrightarrow \widehat{L}^n(A) \xrightarrow{\partial} L_{n-1}(A) \longrightarrow \cdots .$$

The symmetrization maps $1 + T : L_*(A) \rightarrow L^*(A)$ are isomorphisms modulo 8-torsion, so that the hyperquadratic L -groups $\widehat{L}^*(A)$ are of exponent 8. The symmetric and hyperquadratic L -groups are not 4-periodic in general. However, there are defined natural maps

$$L^n(A) \rightarrow L^{n+4}(A), \quad \widehat{L}^n(A) \rightarrow \widehat{L}^{n+4}(A)$$

(which are isomorphisms modulo 8-torsion), and there are 4-periodic versions of the L -groups

$$L^{n+4*}(A) = \lim_{k \rightarrow \infty} L^{n+4k}(A), \quad \widehat{L}^{n+4*}(A) = \lim_{k \rightarrow \infty} \widehat{L}^{n+4k}(A).$$

The 4-periodic symmetric L -group $L^{n+4*}(A)$ is the cobordism group of n -dimensional symmetric Poincaré complexes (C, ϕ) over A with C a finite (but not necessarily n -dimensional) f.g. projective A -module chain complex, and similarly for $\widehat{L}^{n+4*}(A)$.

The Tate $\mathbb{Z}_2$ -cohomology groups of a ring with involution A ,

$$\widehat{H}^n(\mathbb{Z}_2; A) = \frac{\{x \in A \mid \bar{x} = (-1)^n x\}}{\{y + (-1)^n \bar{y} \mid y \in A\}} \quad (n \pmod{2})$$

are A -modules via

$$A \times \widehat{H}^n(\mathbb{Z}_2; A) \rightarrow \widehat{H}^n(\mathbb{Z}_2; A); \quad (a, x) \mapsto ax\bar{a}.$$

The Tate $\mathbb{Z}_2$ -cohomology A -modules give an indication of the difference between the quadratic and symmetric L -groups of A . If $\widehat{H}^*(\mathbb{Z}_2; A) = 0$ (e.g. if $\frac{1}{2} \in A$) then the symmetrization maps $1 + T : L_*(A) \rightarrow L^*(A)$ are isomorphisms and $\widehat{L}^*(A) = 0$. If A is such that $\widehat{H}^0(\mathbb{Z}_2; A)$ and $\widehat{H}^1(\mathbb{Z}_2; A)$ have one-dimensional f.g. projective A -module resolutions then the symmetric and hyperquadratic L -groups of A are 4-periodic (Proposition 30).

For any ring A define

$$A_2 = A/2A,$$

an additive group of exponent 2.

We shall say that a ring with the involution A is r -even for some $r \geq 1$ if

- (i) A is commutative with the identity involution, so that $\widehat{H}^0(\mathbb{Z}_2; A) = A_2$ as an additive group with

$$A \times \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow \widehat{H}^0(\mathbb{Z}_2; A); (a, x) \mapsto a^2x$$

and

$$\widehat{H}^1(\mathbb{Z}_2; A) = \{a \in A \mid 2a = 0\},$$

- (ii) $2 \in A$ is a nonzero divisor, so that $\widehat{H}^1(\mathbb{Z}_2; A) = 0$,
 (iii) $\widehat{H}^0(\mathbb{Z}_2; A)$ is a f.g. free A_2 -module of rank r with a basis $\{x_1 = 1, x_2, \dots, x_r\}$.

If A is r -even then $\widehat{H}^0(\mathbb{Z}_2; A)$ has a one-dimensional f.g. free A -module resolution

$$0 \rightarrow A^r \xrightarrow{2} A^r \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow 0,$$

so that the symmetric and hyperquadratic L -groups of A are 4-periodic (30).

Theorem A. *The hyperquadratic L -groups of a 1-even ring with involution A are given by*

$$\widehat{L}^n(A) = \begin{cases} \frac{\{a \in A \mid a - a^2 \in 2A\}}{\{8b + 4(c - c^2) \mid b, c \in A\}} & \text{if } n \equiv 0 \pmod{4}, \\ \frac{\{a \in A \mid a - a^2 \in 2A\}}{2A} & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ \frac{A}{\{2a + b - b^2 \mid a, b \in A\}} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The boundary maps $\partial : \widehat{L}^n(A) \rightarrow L_{n-1}(A)$ are given by

$$\begin{aligned}\partial : \widehat{L}^0(A) &\rightarrow L_{-1}(A); \quad a \mapsto \left(A \oplus A, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad A, \operatorname{im} \left(\begin{pmatrix} 1-a \\ a \end{pmatrix} : A \rightarrow A \oplus A \right) \right), \\ \partial : \widehat{L}^1(A) &\rightarrow L_0(A); \quad a \mapsto \left(A \oplus A, \begin{pmatrix} (a-a^2)/2 & 1-2a \\ 0 & -2 \end{pmatrix} \right), \\ \partial : \widehat{L}^3(A) &\rightarrow L_2(A); \quad a \mapsto \left(A \oplus A, \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \right).\end{aligned}$$

The map

$$L^0(A) \rightarrow \widehat{L}^0(A); \quad (K, \phi) \mapsto \phi(v, v)$$

is defined using any element $v \in K$ such that

$$\phi(u, u) = \phi(u, v) \in A_2 \quad (u \in K).$$

For any commutative ring A the squaring function on A_2 :

$$\psi^2 : A_2 \rightarrow A_2; \quad a \mapsto a^2$$

is a morphism of additive groups. If $2 \in A$ is a nonzero divisor then A is 1-even if and only if ψ^2 is an isomorphism, with

$$\begin{aligned}\widehat{L}^1(A) &= \ker(\psi^2 - 1 : A_2 \rightarrow A_2), \\ \widehat{L}^3(A) &= \operatorname{coker}(\psi^2 - 1 : A_2 \rightarrow A_2).\end{aligned}$$

In particular, if $2 \in A$ is a nonzero divisor and $\psi^2 = 1 : A_2 \rightarrow A_2$ (or equivalently $a - a^2 \in 2A$ for all $a \in A$) then A is 1-even. In this case Theorem A gives

$$\widehat{L}^n(A) = \begin{cases} A_8 & \text{if } n \equiv 0 \pmod{4}, \\ A_2 & \text{if } n \equiv 1, 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Thanks to Liam O’Carroll and Frans Clauwens for examples of 1-even rings A such that $\psi^2 \neq 1$, e.g. $A = \mathbb{Z}[x]/(x^3 - 1)$ with

$$\psi^2 : A_2 = \mathbb{Z}_2[x]/(x^3 - 1) \rightarrow A_2; \quad a + bx + cx^2 \mapsto (a + bx + cx^2)^2 = a + cx + bx^2.$$

Theorem A is proved in §2 (Corollary 61). In particular, $A = \mathbb{Z}$ is 1-even with $\psi^2 = 1$, and in this case Theorem A recovers the computation of $\widehat{L}^*(\mathbb{Z})$ obtained in [15]—the algebraic L -theory of $\mathbb{Z}$ is recalled further below in the Introduction.

Theorem B. If A is 1-even with $\psi^2 = 1$ then the polynomial ring $A[x]$ is 2-even, with $A[x]_2$ -module basis $\{1, x\}$ for $\widehat{H}^0(\mathbb{Z}_2; A[x])$. The hyperquadratic L -groups of $A[x]$ are given by

$$\widehat{L}^n(A[x]) = \begin{cases} A_8 \oplus A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 0 \pmod{4}, \\ A_2 & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ A_2[x] & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Theorems A and B are special cases of the following computation:

Theorem C. The hyperquadratic L -groups of an r -even ring with involution A are given by

$$\widehat{L}^n(A) = \begin{cases} \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A) + \{2(N + N^t) - 4N^t XN \mid N \in M_r(A)\}} & \text{if } n = 0, \\ \frac{\{N \in M_r(A) \mid N + N^t - 2N^t XN \in 2\text{Quad}_r(A)\}}{2M_r(A)} & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + \{L - LXL \mid L \in \text{Sym}_r(A)\}} & \text{if } n = 3, \end{cases}$$

with $\text{Sym}_r(A)$ the additive group of symmetric $r \times r$ matrices $(a_{ij}) = (a_{ji})$ in A , $\text{Quad}_r(A) \subset \text{Sym}_r(A)$ the subgroup of the matrices such that $a_{ii} \in 2A$, and

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & x_r \end{pmatrix} \in \text{Sym}_r(A)$$

for an A_2 -module basis $\{x_1 = 1, x_2, \dots, x_r\}$ of $\widehat{H}^0(\mathbb{Z}_2; A)$. The boundary maps $\partial : \widehat{L}^n(A) \rightarrow L_{n-1}(A)$ are given by

$$\partial : \widehat{L}^0(A) \rightarrow L_{-1}(A); \quad M \mapsto \left(H_-(A^r); A^r, \text{im} \left(\begin{pmatrix} 1 - XM \\ M \end{pmatrix} : A^r \rightarrow A^r \oplus (A^r)^* \right) \right),$$

$$\partial : \widehat{L}^1(A) \rightarrow L_0(A); \quad N \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^t XN) & 1 - 2NX \\ 0 & -2X \end{pmatrix} \right),$$

$$\partial : \widehat{L}^3(A) \rightarrow L_2(A); \quad M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix} \right).$$

In §1,2 we recall and extend the $\mathcal{Q}$ -groups and algebraic chain bundles of Ranicki [15,18] and Weiss [21]. Theorem C is proved in Theorem 60.

We shall be dealing with two types of generalized Arf invariant: for forms on f.g. projective modules, and for linking forms on homological dimension 1 torsion modules, which we shall be considering separately.

In §3 we define the *generalized Arf invariant* of a nonsingular $(-1)^k$ -quadratic form (K, ψ) over an arbitrary ring with involution A with a lagrangian $L \subset K$ for $(K, \psi + (-)^k \psi^*)$ to be an element

$$(K, \psi; L) \in \widehat{L}^{4*+2k+1}(A),$$

with image

$$\begin{aligned} (K, \psi) &\in \text{im}(\partial : \widehat{L}^{4*+2k+1}(A) \rightarrow L_{2k}(A)) \\ &= \ker(1 + T : L_{2k}(A) \rightarrow L^{4*+2k}(A)). \end{aligned}$$

Theorem 70 gives an explicit formula for the generalized Arf invariant $(K, \psi; L) \in \widehat{L}^3(A)$ for an r -even A . Generalizations of the Arf invariants in L -theory have been previously studied by Clauwens [7], Bak [2] and Wolters [22].

In §4 we consider a ring with involution A with a localization $S^{-1}A$ inverting a multiplicative subset $S \subset A$ of central nonzero divisors such that $\widehat{H}^*(\mathbb{Z}_2; S^{-1}A) = 0$ (e.g. if $2 \in S$). The relative L -group $L_{2k}(A, S)$ in the localization exact sequence

$$\cdots \rightarrow L_{2k}(A) \rightarrow L_{2k}(S^{-1}A) \rightarrow L_{2k}(A, S) \rightarrow L_{2k-1}(A) \rightarrow L_{2k-1}(S^{-1}A) \rightarrow \cdots$$

is the Witt group of nonsingular $(-1)^k$ -quadratic linking forms (T, λ, μ) over (A, S) , with T a homological dimension 1 S -torsion A -module, λ an A -module isomorphism

$$\lambda = (-1)^k \widehat{\lambda} : T \rightarrow T^\wedge = \text{Ext}_A^1(T, A) = \text{Hom}_A(T, S^{-1}A/A)$$

and

$$\mu : T \rightarrow Q_{(-1)^k}(A, S) = \frac{\{b \in S^{-1}A \mid \bar{b} = (-1)^k b\}}{\{a + (-1)^k \bar{a} \mid a \in A\}}$$

a $(-1)^k$ -quadratic function for λ . The *linking Arf invariant* of a nonsingular $(-1)^k$ -quadratic linking form (T, λ, μ) over (A, S) with a lagrangian $U \subset T$ for (T, λ) is defined to be an element

$$(T, \lambda, \mu; U) \in \widehat{L}^{4*+2k}(A),$$

with properties analogous to the generalized Arf invariant defined for forms in §3. Theorem 80 gives an explicit formula for the linking Arf invariant $(T, \lambda, \mu; U) \in \widehat{L}^{2k}(A)$

for an r -even A , using

$$S = (2)^\infty = \{2^i \mid i \geq 0\} \subset A, \quad S^{-1}A = A[1/2].$$

In §5 we apply the generalized and linking Arf invariants to the algebraic L -groups of a polynomial extension $A[x]$ ($\bar{x} = x$) of a ring with involution A , using the exact sequence

$$\cdots \longrightarrow L_n(A[x]) \xrightarrow{1+T} L^n(A[x]) \longrightarrow \widehat{L}^n(A[x]) \longrightarrow L_{n-1}(A[x]) \longrightarrow \cdots.$$

For a Dedekind ring A the quadratic L -groups of $A[x]$ are related to the $\text{UNil}_*(A)$ of Cappell [4] by the splitting formula of Connolly and Ranicki [10]

$$L_n(A[x]) = L_n(A) \oplus \text{UNil}_n(A)$$

and the symmetric and hyperquadratic L -groups of $A[x]$ are 4-periodic, and such that

$$L^n(A[x]) = L^n(A), \quad \widehat{L}^{n+1}(A[x]) = \widehat{L}^{n+1}(A) \oplus \text{UNil}_n(A).$$

Any computation of $\widehat{L}^*(A)$ and $\widehat{L}^*(A[x])$ thus gives a computation of $\text{UNil}_*(A)$. Combining the splitting formula with Theorems A, B gives:

Theorem D. *If A is a 1-even Dedekind ring with $\psi^2 = 1$ then*

$$\begin{aligned} \text{UNil}_n(A) &= \widehat{L}^{n+1}(A[x]) / \widehat{L}^{n+1}(A) \\ &= \begin{cases} 0 & \text{if } n \equiv 0, 1 \pmod{4}, \\ x A_2[x] & \text{if } n \equiv 2 \pmod{4}, \\ A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

In particular, Theorem D applies to $A = \mathbb{Z}$. The twisted quadratic Q -groups were first used in the partial computation of

$$\text{UNil}_n(\mathbb{Z}) = \widehat{L}^{n+1}(\mathbb{Z}[x]) / \widehat{L}^{n+1}(\mathbb{Z})$$

by Connolly and Ranicki [10]. The calculation in [10] was almost complete, except that $\text{UNil}_3(\mathbb{Z})$ was only obtained up to extensions. The calculation was first completed by Connolly and Davis [8], using linking forms. We are grateful to them for sending us a preliminary version of their paper. The calculation of $\text{UNil}_3(\mathbb{Z})$ in [8] uses the results of [10] and the classifications of quadratic and symmetric linking forms over $(\mathbb{Z}[x], (2)^\infty)$. The calculation of $\text{UNil}_3(\mathbb{Z})$ here uses the linking Arf invariant measuring

the difference between the Witt groups of quadratic and symmetric linking forms over $(\mathbb{Z}[x], (2)^\infty)$, developing further the Q -group strategy of [10].

The algebraic L -groups of $A = \mathbb{Z}_2$ are given by

$$L^n(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \text{ (rank (mod 2))} & \text{if } n \equiv 0(\text{mod } 2), \\ 0 & \text{if } n \equiv 1(\text{mod } 2), \end{cases}$$

$$L_n(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 0(\text{mod } 2), \\ 0 & \text{if } n \equiv 1(\text{mod } 2), \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $1 + T = 0 : L_n(\mathbb{Z}_2) \rightarrow L^n(\mathbb{Z}_2)$. The classical Arf invariant is defined for a nonsingular quadratic form (K, ψ) over $\mathbb{Z}_2$ and a lagrangian $L \subset K$ for the symmetric form $(K, \psi + \psi^*)$ to be

$$(K, \psi; L) = \sum_{i=1}^{\ell} \psi(e_i, e_i) \cdot \psi(e_i^*, e_i^*) \in \widehat{L}^1(\mathbb{Z}_2) = L_0(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $e_1, e_2, \dots, e_\ell$ any basis for $L \subset K$, and $e_1^*, e_2^*, \dots, e_\ell^*$ a basis for a direct summand $L^* \subset K$ such that

$$(\psi + \psi^*)(e_i^*, e_j^*) = 0, \quad (\psi + \psi^*)(e_i^*, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The Arf invariant is independent of the choices of L and L^* .

The algebraic L -groups of $A = \mathbb{Z}$ are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0(\text{mod } 4), \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1(\text{mod } 4), \\ 0 & \text{otherwise,} \end{cases}$$

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } n \equiv 0(\text{mod } 4), \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 2(\text{mod } 4), \\ 0 & \text{otherwise,} \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}) = \begin{cases} \mathbb{Z}_8 \text{ (signature (mod 8))} & \text{if } n \equiv 0(\text{mod } 4), \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1(\text{mod } 4), \\ 0 & \text{if } n \equiv 2(\text{mod } 4), \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 3(\text{mod } 4). \end{cases}$$

Given a nonsingular symmetric form (K, ϕ) over $\mathbb{Z}$ there is a congruence [19,12, Theorem 3.10]

$$\text{signature}(K, \phi) \equiv \phi(v, v) \pmod{8},$$

with $v \in K$ any element such that

$$\phi(u, v) \equiv \phi(u, u) \pmod{2} \quad (u \in K),$$

so that

$$\begin{aligned} (K, \phi) &= \text{signature}(K, \phi) = \phi(v, v) \\ &\in \text{coker}(1 + T : L_0(\mathbb{Z}) \rightarrow L^0(\mathbb{Z})) = \widehat{L}^0(\mathbb{Z}) = \text{coker}(8 : \mathbb{Z} \rightarrow \mathbb{Z}) \\ &= \mathbb{Z}_8. \end{aligned}$$

The projection $\mathbb{Z} \rightarrow \mathbb{Z}_2$ induces an isomorphism $L_2(\mathbb{Z}) \cong L_2(\mathbb{Z}_2)$, so that the Witt class of a nonsingular (-1) -quadratic form (K, ψ) over $\mathbb{Z}$ is given by the Arf invariant of the mod 2 reduction

$$(K, \psi; L) = \mathbb{Z}_2 \otimes_{\mathbb{Z}} (K, \psi; L) \in L_2(\mathbb{Z}) = L_2(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $L \subset K$ a lagrangian for the (-1) -symmetric form $(K, \psi - \psi^*)$. Again, this is independent of the choice of L .

The Q -groups are defined for an A -module chain complex C to be $\mathbb{Z}_2$ -hyperhomology invariants of the $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex $C \otimes_A C$. The involution on A is used to define the tensor product over A of left A -module chain complexes C, D , the abelian group chain complex

$$C \otimes_A D = \frac{C \otimes_{\mathbb{Z}} D}{\{ax \otimes y - x \otimes \bar{a}y \mid a \in A, x \in C, y \in D\}}.$$

Let $C \otimes_A C$ denote the $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex defined by $C \otimes_A C$ via the transposition involution

$$T : C_p \otimes_A C_q \rightarrow C_q \otimes_A C_p; \quad x \otimes y \mapsto (-1)^{pq} y \otimes x.$$

The $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$ Q -groups of C are defined by

$$\begin{cases} Q^n(C) = H^n(\mathbb{Z}_2; C \otimes_A C), \\ Q_n(C) = H_n(\mathbb{Z}_2; C \otimes_A C), \\ \widehat{Q}^n(C) = \widehat{H}^n(\mathbb{Z}_2; C \otimes_A C). \end{cases}$$

The Q -groups are covariant in C , and are chain homotopy invariant. The Q -groups are related by an exact sequence

$$\cdots \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \longrightarrow \cdots.$$

A *chain bundle* (C, γ) over A is an A -module chain complex C together with an element $\gamma \in \widehat{Q}^0(C^{-*})$. The *twisted quadratic Q -groups* $Q_*(C, \gamma)$ were defined in [21] using simplicial abelian groups, to fit into an exact sequence

$$\cdots \longrightarrow Q_n(C, \gamma) \xrightarrow{N_\gamma} Q^n(C) \xrightarrow{J_\gamma} \widehat{Q}^n(C) \xrightarrow{H_\gamma} Q_{n-1}(C, \gamma) \longrightarrow \cdots,$$

with

$$J_\gamma : Q^n(C) \rightarrow \widehat{Q}^n(C); \quad \phi \mapsto J(\phi) - (\widehat{\phi}_0)^\%(\gamma).$$

An n -dimensional *algebraic normal complex* $(C, \phi, \gamma, \theta)$ over A is an n -dimensional symmetric complex (C, ϕ) together with a chain bundle $\gamma \in \widehat{Q}^0(C^{-*})$ and an element $(\phi, \theta) \in Q_n(C, \gamma)$ with image $\phi \in Q^n(C)$. Every n -dimensional symmetric Poincaré complex (C, ϕ) has the structure of an algebraic normal complex $(C, \phi, \gamma, \theta)$: the *Spivak normal chain bundle* (C, γ) is characterized by

$$(\widehat{\phi}_0)^\%(\gamma) = J(\phi) \in Q^n(C),$$

with

$$(\widehat{\phi}_0)^\% : \widehat{Q}^0(C^{-*}) = \widehat{Q}^n(C^{n-*}) \rightarrow \widehat{Q}^n(C),$$

the isomorphism induced by the Poincaré duality chain equivalence $\phi_0 : C^{n-*} \rightarrow C$, and the *algebraic normal invariant* $(\phi, \theta) \in Q_n(C, \gamma)$ is such that

$$N_\gamma(\phi, \theta) = \phi \in Q^n(C).$$

See [18, §7] for the one–one correspondence between the homotopy equivalence classes of n -dimensional (symmetric, quadratic) Poincaré pairs and n -dimensional algebraic normal complexes. Specifically, an n -dimensional algebraic normal complex $(C, \phi, \gamma, \theta)$ determines an n -dimensional (symmetric, quadratic) Poincaré pair $(\partial C \rightarrow C^{n-*}, (\delta\phi, \psi))$ with

$$\partial C = \mathcal{C}(\phi_0 : C^{n-*} \rightarrow C)_{*+1}.$$

Conversely, an n -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ determines an n -dimensional algebraic normal complex $(\mathcal{C}(f), \phi, \gamma, \theta)$, with $\gamma \in \widehat{Q}^0(\mathcal{C}(f)^{-*})$ the Spivak normal chain bundle and $\phi = \delta\phi/(1+T)\psi$; the class $(\phi, \theta) \in Q_n(\mathcal{C}(f), \gamma)$ is the algebraic normal invariant of $(f : C \rightarrow D, (\delta\phi, \psi))$. Thus $\widehat{L}^n(A)$ is the cobordism group of n -dimensional normal complexes over A .

Weiss [21] established that for any ring with involution A there exists a universal chain bundle (B^A, β^A) over A , such that every chain bundle (C, γ) is classified by a chain bundle map

$$(g, \chi) : (C, \gamma) \rightarrow (B^A, \beta^A),$$

with

$$H_*(B^A) = \widehat{H}^*(\mathbb{Z}_2; A).$$

The function

$$\widehat{L}^{n+4*}(A) \rightarrow Q_n(B^A, \beta^A); (C, \phi, \gamma, \theta) \mapsto (g, \chi)_{\%}(\phi, \theta)$$

was shown in [21] to be an isomorphism. Since the Q -groups are homological in nature (rather than of the Witt type) they are in principle effectively computable. The algebraic normal invariant defines the isomorphism

$$\begin{array}{ccc} \ker(1+T : L_n(A) \rightarrow L^{n+4*}(A)) & \xrightarrow{\cong} & \text{coker}(L^{n+4*+1}(A) \rightarrow Q_{n+1}(B^A, \beta^A)), \\ & & (C, \psi) \mapsto (g, \chi)_{\%}(\phi, \theta), \end{array}$$

with $(\phi, \theta) \in Q_{n+1}(\mathcal{C}(f), \gamma)$ the algebraic normal invariant of any $(n+1)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$, with classifying chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \rightarrow (B^A, \beta^A)$. For $n = 2k$ such a pair with $H_i(C) = H_i(D) = 0$ for $i \neq k$ is just a nonsingular $(-1)^k$ -quadratic form $(K = H^k(C), \psi)$ with a lagrangian

$$L = \text{im}(f^* : H^k(D) \rightarrow H^k(C)) \subset K = H^k(C)$$

for $(K, \psi + (-1)^k \psi^*)$, such that the generalized Arf invariant is the image of the algebraic normal invariant

$$(K, \psi; L) = (g, \chi)_{\%}(\phi, \theta) \in \widehat{L}^{4*+2k+1}(A) = Q_{2k+1}(B^A, \beta^A).$$

For $A = \mathbb{Z}_2$ and $n = 0$ this is just the classical Arf invariant isomorphism

$$\begin{aligned} L_0(\mathbb{Z}_2) &= \ker(1 + T = 0 : L_0(\mathbb{Z}_2) \rightarrow L^0(\mathbb{Z}_2)) \\ &\xrightarrow{\cong} \operatorname{coker}(L^1(\mathbb{Z}_2) = 0 \rightarrow Q_1(B^{\mathbb{Z}_2}, \beta^{\mathbb{Z}_2})) = \mathbb{Z}_2, \\ &\quad (K, \psi) \mapsto (K, \psi; L), \end{aligned}$$

with $L \subset K$ an arbitrary lagrangian of $(K, \psi + \psi^*)$. The isomorphism

$$\operatorname{coker}(1 + T : L_n(A) \rightarrow L^{n+4*}(A)) \xrightarrow{\cong} \ker(\partial : Q_n(B^A, \beta^A) \rightarrow L_{n-1}(A))$$

is a generalization from $A = \mathbb{Z}$, $n = 0$ to arbitrary A , n of the identity signature $(K, \phi) \equiv \phi(v, v) \pmod{8}$ described above.

(Here is some of the geometric background. Chain bundles are algebraic analogues of vector bundles and spherical fibrations, and the twisted Q -groups are the analogues of the homotopy groups of the Thom spaces. A $(k-1)$ -spherical fibration $v : X \rightarrow BG(k)$ over a connected CW complex X determines a chain bundle $(C(\tilde{X}), \gamma)$ over $\mathbb{Z}[\pi_1(X)]$, with $C(\tilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$ -module chain complex of the universal cover $\tilde{X}$, and there are defined Hurewicz-type morphisms

$$\pi_{n+k}(T(v)) \rightarrow Q_n(C(\tilde{X}), \gamma),$$

with $T(v)$ the Thom space. An n -dimensional normal space $(X, v : X \rightarrow BG(k), \rho : S^{n+k} \rightarrow T(v))$ [14] determines an n -dimensional algebraic normal complex $(C(\tilde{X}), \phi, \gamma, \theta)$ over $\mathbb{Z}[\pi_1(X)]$. An n -dimensional geometric Poincaré complex X has a Spivak normal structure (v, ρ) such that the composite of the Hurewicz map and the Thom isomorphism

$$\pi_{n+k}(T(v)) \rightarrow \tilde{H}_{n+k}(T(v)) \cong H_n(X)$$

sends ρ to the fundamental class $[X] \in H_n(X)$, and there is defined an n -dimensional symmetric Poincaré complex $(C(\tilde{X}), \phi)$ over $\mathbb{Z}[\pi_1(X)]$, with

$$\phi_0 = [X] \cap - : C(\tilde{X})^{n-*} \rightarrow C(\tilde{X}).$$

The symmetric signature of X is the symmetric Poincaré cobordism class

$$\sigma^*(X) = (C(\tilde{X}), \phi) \in L^n(\mathbb{Z}[\pi_1(X)]),$$

which is both a homotopy and a $K(\pi_1(X), 1)$ -bordism invariant. The algebraic normal invariant of a normal space (X, ν, ρ) ,

$$[\rho] = (\phi, \theta) \in Q_n(C(\tilde{X}), \gamma)$$

is a homotopy invariant. The classifying chain bundle map

$$(g, \chi) : (C(\tilde{X}), \gamma) \rightarrow (B^{\mathbb{Z}[\pi_1(X)]}, \beta^{\mathbb{Z}[\pi_1(X)]})$$

sends $[\rho]$ to the hyperquadratic signature of X :

$$\widehat{\sigma}^*(X) = [\phi, \theta] \in Q_n(B^{\mathbb{Z}[\pi_1(X)]}, \beta^{\mathbb{Z}[\pi_1(X)]}) = \widehat{L}^{n+4*}(\mathbb{Z}[\pi_1(X)]),$$

which is both a homotopy and a $K(\pi_1(X), 1)$ -bordism invariant. The (simply-connected) symmetric signature of a $4k$ -dimensional geometric Poincaré complex X is just the signature

$$\sigma^*(X) = \text{signature}(X) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$$

and the hyperquadratic signature is the mod 8 reduction of the signature

$$\widehat{\sigma}^*(X) = \text{signature}(X) \in \widehat{L}^{4k}(\mathbb{Z}) = \mathbb{Z}_8.$$

See [18] for a more extended discussion of the connections between chain bundles and their geometric models.)

1. The Q - and L -groups

1.1. Duality

Let $T \in \mathbb{Z}_2$ be the generator. The Tate $\mathbb{Z}_2$ -cohomology groups of a $\mathbb{Z}[\mathbb{Z}_2]$ -module M are given by

$$\widehat{H}^n(\mathbb{Z}_2; M) = \frac{\{x \in M \mid T(x) = (-1)^n x\}}{\{y + (-1)^n T(y) \mid y \in M\}}$$

and the $\begin{cases} \mathbb{Z}_2\text{-cohomology} \\ \mathbb{Z}_2\text{-homology} \end{cases}$ groups are given by

$$H^n(\mathbb{Z}_2; M) = \begin{cases} \{x \in M \mid T(x) = x\} & \text{if } n = 0, \\ \widehat{H}^n(\mathbb{Z}_2; M) & \text{if } n > 0, \\ 0 & \text{if } n < 0, \end{cases}$$

$$H_n(\mathbb{Z}_2; M) = \begin{cases} M/\{y - T(y) \mid y \in M\} & \text{if } n = 0, \\ \widehat{H}^{n+1}(\mathbb{Z}_2; M) & \text{if } n > 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We recall some standard properties of $\mathbb{Z}_2$ -(co)homology:

Proposition 1. *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$ -module.*

(i) *There is defined an exact sequence*

$$\cdots \rightarrow H_n(\mathbb{Z}_2; M) \xrightarrow{N} H^{-n}(\mathbb{Z}_2; M) \rightarrow \widehat{H}^n(\mathbb{Z}_2; M) \rightarrow H_{n-1}(\mathbb{Z}_2; M) \rightarrow \cdots,$$

with

$$N = 1 + T : H_0(\mathbb{Z}_2; M) \rightarrow H^0(\mathbb{Z}_2; M); \quad x \mapsto x + T(x).$$

(ii) *The Tate $\mathbb{Z}_2$ -cohomology groups are 2-periodic and of exponent 2,*

$$\widehat{H}^*(\mathbb{Z}_2; M) = \widehat{H}^{*+2}(\mathbb{Z}_2; M), \quad 2\widehat{H}^*(\mathbb{Z}_2; M) = 0.$$

(iii) $\widehat{H}^*(\mathbb{Z}_2; M) = 0$ if M is a free $\mathbb{Z}[\mathbb{Z}_2]$ -module.

Let A be an associative ring with 1, and with an involution

$$\bar{} : A \rightarrow A; \quad a \mapsto \bar{a},$$

such that

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{b}.\bar{a}, \quad \bar{1} = 1, \quad \bar{\bar{a}} = a.$$

When a ring A is declared to be commutative it is given the identity involution.

Definition 2. For a ring with involution A and $\varepsilon = \pm 1$ let (A, ε) denote the $\mathbb{Z}[\mathbb{Z}_2]$ -module given by A with $T \in \mathbb{Z}_2$ acting by

$$T_\varepsilon : A \rightarrow A; \quad a \mapsto \varepsilon \bar{a}.$$

For $\varepsilon = 1$ we shall write

$$\begin{aligned} \widehat{H}^*(\mathbb{Z}_2; A, 1) &= \widehat{H}^*(\mathbb{Z}_2; A), \\ H^*(\mathbb{Z}_2; A, 1) &= H^*(\mathbb{Z}_2; A), \quad H_*(\mathbb{Z}_2; A, 1) = H_*(\mathbb{Z}_2; A). \end{aligned}$$

The *dual* of a f.g. projective (left) A -module P is the f.g. projective A -module

$$P^* = \text{Hom}_A(P, A), \quad A \times P^* \rightarrow P^*; \quad (a, f) \mapsto (x \mapsto f(x)\bar{a}).$$

The natural A -module isomorphism

$$P \rightarrow P^{**}; \quad x \mapsto (f \mapsto \overline{f(x)})$$

is used to identify

$$P^{**} = P.$$

For any f.g. projective A -modules P, Q there is defined an isomorphism

$$P \otimes_A Q \rightarrow \text{Hom}_A(P^*, Q); \quad x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

regarding Q as a right A -module by

$$Q \times A \rightarrow Q; \quad (y, a) \mapsto \bar{a}y.$$

There is also a duality isomorphism

$$T : \text{Hom}_A(P, Q) \rightarrow \text{Hom}_A(Q^*, P^*); \quad f \mapsto f^*,$$

with

$$f^* : Q^* \rightarrow P^*; \quad g \mapsto (x \mapsto g(f(x))).$$

Definition 3. For any f.g. projective A -module P and $\varepsilon = \pm 1$ let $(S(P), T_\varepsilon)$ denote the $\mathbb{Z}[\mathbb{Z}_2]$ -module given by the abelian group

$$S(P) = \text{Hom}_A(P, P^*),$$

with $\mathbb{Z}_2$ -action by the ε -duality involution

$$T_\varepsilon : S(P) \rightarrow S(P); \quad \phi \mapsto \varepsilon\phi^*.$$

Furthermore, let

$$\text{Sym}(P, \varepsilon) = H^0(\mathbb{Z}_2; S(P), T_\varepsilon) = \{\phi \in S(P) \mid T_\varepsilon(\phi) = \phi\},$$

$$\text{Quad}(P, \varepsilon) = H_0(\mathbb{Z}_2; S(P), T_\varepsilon) = \frac{S(P)}{\{\theta \in S(P) \mid \theta - T_\varepsilon(\theta)\}}.$$

An element $\phi \in S(P)$ can be regarded as a sesquilinear form

$$\phi : P \times P \rightarrow A; \quad (x, y) \mapsto \langle x, y \rangle_\phi = \phi(x)(y)$$

such that

$$\langle ax, by \rangle_\phi = b \langle x, y \rangle_\phi \bar{a} \in A \quad (x, y \in P, a, b \in A),$$

with

$$\langle x, y \rangle_{T_\varepsilon(\phi)} = \varepsilon \overline{\langle y, x \rangle_\phi} \in A.$$

An A -module morphism $f : P \rightarrow Q$ induces contravariantly a $\mathbb{Z}[\mathbb{Z}_2]$ -module morphism

$$S(f) : (S(Q), T_\varepsilon) \rightarrow (S(P), T_\varepsilon); \quad \theta \mapsto f^* \theta f.$$

For a f.g. free A -module $P = A^r$ we shall use the A -module isomorphism

$$A^r \rightarrow (A^r)^*; \quad (a_1, a_2, \dots, a_r) \mapsto \left((b_1, b_2, \dots, b_r) \mapsto \sum_{i=1}^r b_i \bar{a}_i \right)$$

to identify

$$(A^r)^* = A^r, \quad \text{Hom}_A(A^r, (A^r)^*) = M_r(A),$$

noting that the duality involution T corresponds to the conjugate transposition of a matrix. We can thus identify

$$\begin{aligned} M_r(A) &= S(A^r) = \text{additive group of } r \times r \text{ matrices } (a_{ij}) \text{ with } a_{ij} \in A, \\ T : M_r(A) &\rightarrow M_r(A); \quad M = (a_{ij}) \mapsto M^t = (\bar{a}_{ji}), \\ \text{Sym}_r(A, \varepsilon) &= \text{Sym}(A^r, \varepsilon) = \{(a_{ij}) \in M_r(A) \mid a_{ij} = \varepsilon \bar{a}_{ji}\}, \\ \text{Quad}_r(A, \varepsilon) &= \text{Quad}(A^r, \varepsilon) = \frac{M_r(A)}{\{(a_{ij} - \varepsilon \bar{a}_{ji}) \mid (a_{ij}) \in M_r(A)\}}, \\ 1 + T_\varepsilon : \text{Quad}_r(A, \varepsilon) &\rightarrow \text{Sym}_r(A, \varepsilon); \quad M \mapsto M + \varepsilon M^t. \end{aligned}$$

The homology of the chain complex

$$\cdots \longrightarrow M_r(A) \xrightarrow{1-T} M_r(A) \xrightarrow{1+T} M_r(A) \xrightarrow{1-T} M_r(A) \longrightarrow \cdots$$

is given by

$$\frac{\ker(1 - (-1)^n T : M_r(A) \rightarrow M_r(A))}{\text{im}(1 + (-1)^n T : M_r(A) \rightarrow M_r(A))} = \widehat{H}^n(\mathbb{Z}_2; M_r(A)) = \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A).$$

The $(-1)^n$ -symmetrization map $1 + (-1)^n T : \text{Sym}_r(A) \rightarrow \text{Quad}_r(A)$ fits into an exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2; A) &\rightarrow \text{Quad}_r(A, (-1)^n) \\ &\xrightarrow{1+(-1)^n T} \text{Sym}_r(A, (-1)^n) \rightarrow \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A) \rightarrow 0. \end{aligned}$$

For $\varepsilon = 1$ we abbreviate

$$\begin{aligned} \text{Sym}(P, 1) &= \text{Sym}(P), \quad \text{Quad}(P, 1) = \text{Quad}(P), \\ \text{Sym}_r(A, 1) &= \text{Sym}_r(A), \quad \text{Quad}_r(A, 1) = \text{Quad}_r(A). \end{aligned}$$

Definition 4. An involution on a ring A is *even* if

$$\widehat{H}^1(\mathbb{Z}_2; A) = 0,$$

that is if

$$\{a \in A \mid a + \bar{a} = 0\} = \{b - \bar{b} \mid b \in A\}.$$

Proposition 5. (i) For any f.g. projective A -module P there is defined an exact sequence

$$0 \rightarrow \widehat{H}^1(\mathbb{Z}_2; S(P), T) \rightarrow \text{Quad}(P) \xrightarrow{1+T} \text{Sym}(P),$$

with

$$1 + T : \text{Quad}(P) \rightarrow \text{Sym}(P); \quad \psi \mapsto \psi + \psi^*.$$

(ii) If the involution on A is even the symmetrization $1 + T : \text{Quad}(P) \rightarrow \text{Sym}(P)$ is injective, and

$$\widehat{H}^n(\mathbb{Z}_2; S(P), T) = \begin{cases} \frac{\text{Sym}(P)}{\text{Quad}(P)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

identifying $\text{Quad}(P)$ with $\text{im}(1 + T) \subseteq \text{Sym}(P)$.

Proof. (i) This is a special case of 1(i).

(ii) If Q is a f.g. projective A -module such that $P \oplus Q = A^r$ is f.g. free then

$$\begin{aligned} \widehat{H}^1(\mathbb{Z}_2; S(P), T) \oplus \widehat{H}^1(\mathbb{Z}_2; S(Q), T) &= \widehat{H}^1(\mathbb{Z}_2; S(P \oplus Q), T) \\ &= \bigoplus_r \widehat{H}^1(\mathbb{Z}_2; A, -T) = 0 \end{aligned}$$

and so $\widehat{H}^1(\mathbb{Z}_2; S(P), T) = 0$. $\square$

In particular, if the involution on A is even there is defined an exact sequence

$$0 \rightarrow \text{Quad}_r(A) \xrightarrow{1+T} \text{Sym}_r(A) \rightarrow \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow 0$$

with

$$\text{Sym}_r(A) \rightarrow \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A); \quad (a_{ij}) \mapsto (a_{ii}).$$

For any involution on A , $\text{Sym}_r(A)$ is the additive group of symmetric $r \times r$ matrices $(a_{ij}) = (\bar{a}_{ji})$ with $a_{ij} \in A$. For an even involution $\text{Quad}_r(A) \subseteq \text{Sym}_r(A)$ is the subgroup of the matrices such that the diagonal terms are of the form $a_{ii} = b_i + \bar{b}_i$

for some $b_i \in A$, with

$$\frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A).$$

Definition 6. A ring A is *even* if $2 \in A$ is a nonzero divisor, i.e. $2 : A \rightarrow A$ is injective.

Example 7. (i) An integral domain A is even if and only if it has characteristic $\neq 2$.

(ii) The identity involution on a commutative ring A is even (4) if and only if the ring A is even (6), in which case

$$\widehat{H}^n(\mathbb{Z}_2; A) = \begin{cases} A_2 & \text{if } n \equiv 0 \pmod{2}, \\ 0 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

and

$$\text{Quad}_r(A) = \{(a_{ij}) \in \text{Sym}_r(A) \mid a_{ii} \in 2A\}.$$

Example 8. For any group π there is defined an involution on the group ring $\mathbb{Z}[\pi]$:

$$- : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi]; \quad \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1}.$$

If π has no 2-torsion this involution is even.

1.2. The hyperquadratic Q -groups

Let C be a finite (left) f.g. projective A -module chain complex. The dual of the f.g. projective A -module C_p is written

$$C^p = (C_p)^* = \text{Hom}_A(C_p, A).$$

The dual A -module chain complex C^{-*} is defined by

$$d_{C^{-*}} = (d_C)^* : (C^{-*})_r = C^{-r} \rightarrow (C^{-*})_{r-1} = C^{-r+1}.$$

The n -dual A -module chain complex C^{n-*} is defined by

$$d_{C^{n-*}} = (-1)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \rightarrow (C^{n-*})_{r-1} = C^{n-r+1}.$$

Identify

$$C \otimes_A C = \text{Hom}_A(C^{-*}, C),$$

noting that a cycle $\phi \in (C \otimes_A C)_n$ is a chain map $\phi : C^{n-*} \rightarrow C$. For $\varepsilon = \pm 1$ the ε -transposition involution T_ε on $C \otimes_A C$ corresponds to the ε -duality involution on $\text{Hom}_A(C^{-*}, C)$,

$$T_\varepsilon : \text{Hom}_A(C^p, C_q) \rightarrow \text{Hom}_A(C^q, C_p); \quad \phi \mapsto (-1)^{pq} \varepsilon \phi^*.$$

Let $\widehat{W}$ be the complete resolution of the $\mathbb{Z}[\mathbb{Z}_2]$ -module $\mathbb{Z}$:

$$\widehat{W} : \cdots \rightarrow \widehat{W}_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_0 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \widehat{W}_{-1} = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_{-2} = \mathbb{Z}[\mathbb{Z}_2] \rightarrow \cdots.$$

If we set

$$\widehat{W}^\% C = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, \text{Hom}_A(C^{-*}, C)),$$

then an n -dimensional ε -hyperquadratic structure on C is a cycle $\theta \in (\widehat{W}^\% C)_n$, which is just a collection $\{\theta_s \in \text{Hom}_A(C^{n-r+s}, C_r) \mid r, s \in \mathbb{Z}\}$ such that

$$d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s-1} (\theta_{s-1} + (-1)^s T_\varepsilon \theta_{s-1}) = 0 : C^{n-r+s-1} \rightarrow C_r.$$

Definition 9. The n -dimensional ε -hyperquadratic Q -group $\widehat{Q}^n(C, \varepsilon)$ is the abelian group of equivalence classes of n -dimensional ε -hyperquadratic structures on C , that is,

$$\widehat{Q}^n(C, \varepsilon) = H_n(\widehat{W}^\% C).$$

The ε -hyperquadratic Q -groups are 2-periodic and of exponent 2

$$\widehat{Q}^*(C, \varepsilon) \cong \widehat{Q}^{*+2}(C, \varepsilon), \quad 2\widehat{Q}^*(C, \varepsilon) = 0.$$

More precisely, there are defined isomorphisms

$$\widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C, \varepsilon); \quad \{\theta_s\} \mapsto \{\theta_{s+2}\}$$

and for any n -dimensional ε -hyperquadratic structure $\{\theta_s\}$,

$$2\theta_s = d\chi_s + (-1)^r \chi_s d^* + (-1)^{n+s} (\chi_{s-1} + (-1)^s T_\varepsilon \chi_{s-1}) : C^{n-r+s} \rightarrow C_r,$$

with $\chi_s = (-1)^{n+s-1} \theta_{s+1}$. There are also defined suspension isomorphisms

$$S : \widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+1}(C_{*-1}, \varepsilon); \{\theta_s\} \mapsto \{\theta_{s-1}\}$$

and skew-suspension isomorphisms

$$\overline{S} : \widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C_{*-1}, -\varepsilon); \{\theta_s\} \mapsto \{\theta_s\}.$$

Proposition 10. *Let C be a f.g. projective A -module chain complex which is concentrated in degree k*

$$C : \cdots \rightarrow 0 \rightarrow C_k \rightarrow 0 \rightarrow \cdots .$$

The ε -hyperquadratic Q -groups of C are given by

$$\widehat{Q}^n(C, \varepsilon) = \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon)$$

(with $S(C^k) = \text{Hom}_A(C^k, C_k)$).

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex $V = \text{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \rightarrow V_{2k+1} = 0 \rightarrow V_{2k} = S(C^k) \rightarrow V_{2k-1} = 0 \rightarrow \cdots$$

and

$$(\widehat{W}^{\%} C)_j = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, V_{2k}) = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, S(C^k)).$$

Thus the chain complex $\widehat{W}^{\%}C$ is of the form

$$\begin{array}{rcl}
 (\widehat{W}^{\%}C)_{2k+1} & = & \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{-1}, V_{2k}) = S(C^k), \\
 \downarrow d_{2k+1}=1+(-1)^k T_{\varepsilon} & & \\
 (\widehat{W}^{\%}C)_{2k} & = & \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_0, V_{2k}) = S(C^k), \\
 \downarrow d_{2k}=1+(-1)^{k+1} T_{\varepsilon} & & \\
 (\widehat{W}^{\%}C)_{2k-1} & = & \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_1, V_{2k}) = S(C^k), \\
 \downarrow d_{2k-1}=1+(-1)^k T_{\varepsilon} & & \\
 (\widehat{W}^{\%}C)_{2k-2} & = & \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_2, V_{2k}) = S(C^k) \\
 \downarrow & &
 \end{array}$$

and

$$\widehat{Q}^n(C, \varepsilon) = H_n(\widehat{W}^{\%}C) = \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_{\varepsilon}). \quad \square$$

Example 11. The ε -hyperquadratic Q -groups of a zero-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_0 = A^r \rightarrow 0 \rightarrow \cdots$$

are given by

$$\widehat{Q}^n(C, \varepsilon) = \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon).$$

The *algebraic mapping cone* $\mathcal{C}(f)$ of a chain map $f : C \rightarrow D$ is the chain complex defined as usual by

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-1)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \rightarrow \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

The relative homology groups

$$H_n(f) = H_n(\mathcal{C}(f))$$

fit into an exact sequence

$$\cdots \rightarrow H_n(C) \xrightarrow{f_*} H_n(D) \rightarrow H_n(f) \rightarrow H_{n-1}(C) \rightarrow \cdots .$$

An A -module chain map $f : C \rightarrow D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map

$$\begin{aligned} f \otimes f &= \text{Hom}_A(f^*, f) : C \otimes_A C = \text{Hom}_A(C^{-*}, C) \rightarrow D \otimes_A D \\ &= \text{Hom}_A(D^{-*}, D) \end{aligned}$$

and hence a $\mathbb{Z}$ -module chain map

$$\widehat{f}^{\%} = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_{\widehat{W}}, f \otimes_A f) : \widehat{W}^{\%} C \longrightarrow \widehat{W}^{\%} D,$$

which induces

$$\widehat{f}^{\%} : \widehat{Q}^n(C, \varepsilon) \longrightarrow \widehat{Q}^n(D, \varepsilon)$$

on homology. The *relative ε -hyperquadratic Q -group*

$$\widehat{Q}^n(f, \varepsilon) = H_n(\widehat{f}^{\%} : \widehat{W}^{\%} C \rightarrow \widehat{W}^{\%} D)$$

fits into a long exact sequence

$$\cdots \longrightarrow \widehat{Q}^n(C, \varepsilon) \xrightarrow{\widehat{f}^{\%}} \widehat{Q}^n(D, \varepsilon) \longrightarrow \widehat{Q}^n(f, \varepsilon) \longrightarrow \widehat{Q}^{n-1}(C, \varepsilon) \longrightarrow \cdots .$$

As in [15, §1] define a $\mathbb{Z}_2$ -isovariant chain map $f : C \rightarrow D$ of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes C, D to be a collection

$$\{f_s \in \text{Hom}_{\mathbb{Z}}(C_r, D_{r+s}) | r \in \mathbb{Z}, s \geq 0\}$$

such that

$$\begin{aligned} d_D f_s + (-1)^{s-1} f_s d_C + (-1)^{s-1} (f_{s-1} + (-1)^s T_D f_{s-1} T_C) \\ = 0 : C_r \rightarrow D_{r+s-1} \quad (f_{-1} = 0), \end{aligned}$$

so that $f_0 : C \rightarrow D$ is a $\mathbb{Z}$ -module chain map, $f_1 : f_0 \simeq T_D f_0 : C \rightarrow D$ is a $\mathbb{Z}$ -module chain map, etc. There is a corresponding notion of $\mathbb{Z}_2$ -isovariant chain homotopy.

For any A -module chain complexes C, D a $\mathbb{Z}_2$ -isovariant chain map $F : C \otimes_A C \rightarrow D \otimes_A D$ induces morphisms of the ε -hyperquadratic Q -groups

$$\widehat{F}^{\%} : \widehat{Q}^n(C, \varepsilon) \rightarrow \widehat{Q}^n(D, \varepsilon); \quad \theta \mapsto \widehat{F}^{\%}(\theta), \quad \widehat{F}^{\%}(\theta)_s = \sum_{r=0}^{\infty} \pm F_r(T^r \theta_{s-r}).$$

If F_0 is a chain equivalence the morphisms $\widehat{F}^{\%}$ are isomorphisms. An A -module chain map $f : C \rightarrow D$ determines a $\mathbb{Z}_2$ -isovariant chain map

$$f \otimes_A f : C \otimes_A C \rightarrow D \otimes_A D,$$

with $(f \otimes_A f)_s = 0$ for $s \geq 1$.

Proposition 12 (Ranicki [15, Propositions 1.1, 1.4] Weiss [21, Theorem 1.1]). (i) *The relative ε -hyperquadratic Q -groups of an A -module chain map $f : C \rightarrow D$ are isomorphic to the absolute ε -hyperquadratic Q -groups of the algebraic mapping cone $\mathcal{C}(f)$,*

$$\widehat{Q}^*(f, \varepsilon) \cong \widehat{Q}^*(\mathcal{C}(f), \varepsilon).$$

(ii) *The ε -hyperquadratic Q -groups are additive: for any collection $\{C(i) \mid i \in \mathbb{Z}\}$ of f.g. projective A -module chain complexes $C(i)$,*

$$\widehat{Q}^n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i \widehat{Q}^n(C(i), \varepsilon).$$

(iii) *If $f : C \rightarrow D$ is a chain equivalence the morphisms $\widehat{f}^{\%} : \widehat{Q}^*(C, \varepsilon) \rightarrow \widehat{Q}^*(D, \varepsilon)$ are isomorphisms, and*

$$\widehat{Q}^*(f, \varepsilon) = 0.$$

Proof. (i) The $\mathbb{Z}_2$ -isovariant chain map $t : \mathcal{C}(f \otimes_A f) \rightarrow \mathcal{C}(f) \otimes_A \mathcal{C}(f)$ defined by

$$t_0(\theta, \partial\theta) = \theta + (f \otimes 1)\partial\theta, \quad t_1(\theta, \partial\theta) = \partial\theta, \quad t_s = 0 \quad (s \geq 2)$$

induces the algebraic Thom construction maps

$$\widehat{t}^{\%} : \widehat{Q}^n(f, \varepsilon) \rightarrow \widehat{Q}^n(\mathcal{C}(f), \varepsilon); \quad (\theta, \partial\theta) \mapsto \theta/\partial\theta,$$

with

$$(\theta/\partial\theta)_s = \begin{pmatrix} \theta_s & 0 \\ \pm\partial\theta_s f^* & \pm T_\varepsilon \partial\theta_{s-1} \end{pmatrix} :$$

$$\mathcal{C}(f)^{n-r+s} = D^{n-r+s} \oplus C^{n-r+s-1} \rightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1} \quad (r, s \in \mathbb{Z}).$$

Define a free $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex

$$E = (C_{*-1} \otimes_A \mathcal{C}(f)) \oplus (\mathcal{C}(f) \otimes_A C_{*-1}),$$

with

$$T : E \rightarrow E; \quad (a \otimes b, x \otimes y) \mapsto (y \otimes x, b \otimes a),$$

such that

$$H_*(\widehat{W} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} E) = H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, E)) = 0.$$

Let $p : \mathcal{C}(f) \rightarrow C_{*-1}$ be the projection. The chain map

$$\begin{pmatrix} p \otimes 1 \\ 1 \otimes p \end{pmatrix} : \mathcal{C}(f) \otimes_A \mathcal{C}(f) \rightarrow E$$

induces a chain equivalence

$$\mathcal{C}(t_0 : \mathcal{C}(f \otimes f) \rightarrow \mathcal{C}(f) \otimes_A \mathcal{C}(f)) \simeq E$$

so that the morphisms $\widehat{t}^{\%} : \widehat{Q}^*(f, \varepsilon) \cong \widehat{Q}^*(\mathcal{C}(f), \varepsilon)$ are isomorphisms.

(ii) $\widehat{Q}^*(C(1) \oplus C(2)) = \widehat{Q}^*(C(1)) \oplus \widehat{Q}^*(C(2))$ is the special case of (i) with $f = 0 : C(1)_{*+1} \rightarrow C(2)$.

(iii) An A -module chain homotopy $g : f \simeq f' : C \rightarrow D$ determines a $\mathbb{Z}_2$ -isovariant chain homotopy

$$h : f \otimes_A f \simeq f' \otimes_A f' : C \otimes_A C \rightarrow D \otimes_A D,$$

with

$$h_0 = f \otimes_A g \pm g \otimes_A f, \quad h_1 = \pm g \otimes_A g, \quad h_s = 0 \quad (s \geq 2),$$

so that

$$\widehat{f}^{\%} = \widehat{f'}^{\%} : \widehat{Q}^n(C, \varepsilon) \rightarrow \widehat{Q}^n(D, \varepsilon).$$

(See the proof of [15, Proposition 1.1(ii)] for the signs.) In particular, if f is a chain equivalence the morphisms $\widehat{f}^{\%}$ are isomorphisms. $\square$

Proposition 13. *Let C be a f.g. projective A -module chain complex which is concentrated in degrees $k, k+1$,*

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \xrightarrow{d} C_k \rightarrow 0 \rightarrow \cdots .$$

(i) *The ε -hyperquadratic Q -groups of C are the relative Tate $\mathbb{Z}_2$ -cohomology groups in the exact sequence*

$$\begin{aligned} \cdots \rightarrow \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) &\xrightarrow{\widehat{d}^{\%}} \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\ &\rightarrow \widehat{Q}^n(C, \varepsilon) \rightarrow \widehat{H}^{n-2k-1}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \rightarrow \cdots \end{aligned}$$

that is

$$\widehat{Q}^n(C, \varepsilon) = \frac{\{(\phi, \theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi^* = (-1)^{n+k-1} \varepsilon \phi, d\phi d^* = \theta + (-1)^{n+k-1} \varepsilon \theta^*\}}{\{(\sigma + (-1)^{n+k-1} \varepsilon \sigma^*, d\sigma d^* + \tau + (-1)^{n+k} \varepsilon \tau^*) \mid (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}},$$

with (ϕ, θ) corresponding to the cycle $\beta \in (\widehat{W}^{\%} C)_n$ given by

$$\begin{aligned} \beta_{2k-n+2} &= \theta : C^{k+1} \rightarrow C_{k+1}, \quad \beta_{2k-n} = \phi : C^k \rightarrow C_k, \\ \beta_{2k-n+1} &= \begin{cases} d\phi : C^{k+1} \rightarrow C_k, \\ 0 : C^k \rightarrow C_{k+1}. \end{cases} \end{aligned}$$

(ii) *If the involution on A is even then*

$$\widehat{Q}^n(C) = \begin{cases} \text{coker} \left(\widehat{d}^{\%} : \frac{\text{Sym}(C^{k+1})}{\text{Quad}(C^{k+1})} \rightarrow \frac{\text{Sym}(C^k)}{\text{Quad}(C^k)} \right) & \text{if } n-k \text{ is even,} \\ \text{ker} \left(\widehat{d}^{\%} : \frac{\text{Sym}(C^{k+1})}{\text{Quad}(C^{k+1})} \rightarrow \frac{\text{Sym}(C^k)}{\text{Quad}(C^k)} \right) & \text{if } n-k \text{ is odd.} \end{cases}$$

Proof. (i) Immediate from Proposition 12.

(ii) Combine (i) and the vanishing $\widehat{H}^1(\mathbb{Z}_2; S(P), T) = 0$ given by Proposition 5(ii). $\square$

For $\varepsilon = 1$ we write

$$T_\varepsilon = T, \quad \widehat{Q}^n(C, \varepsilon) = \widehat{Q}^n(C), \quad \varepsilon\text{-hyperquadratic} = \text{hyperquadratic}.$$

Example 14. Let A be a ring with an involution which is even (6), i.e. such that $2 \in A$ is a nonzero divisor.

(i) The hyperquadratic Q -groups of a one-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_1 = A^q \xrightarrow{d} C_0 = A^r \rightarrow 0 \rightarrow \cdots$$

are given by

$$\widehat{Q}^n(C) = \frac{\{(\phi, \theta) \in M_q(A) \oplus M_r(A) \mid \phi^* = (-1)^{n-1} \phi, d\phi d^* = \theta + (-1)^{n-1} \theta^*\}}{\{(\sigma + (-1)^{n-1} \sigma^*, d\sigma d^* + \tau + (-1)^n \tau^* \mid (\sigma, \tau) \in M_q(A) \oplus M_r(A)\}}.$$

Example 11 and Proposition 13 give an exact sequence

$$\begin{aligned} \widehat{H}^1(\mathbb{Z}_2; S(C^1), T) &= 0 \rightarrow \widehat{Q}^1(C) \\ &\longrightarrow \widehat{H}^0(\mathbb{Z}_2; S(C^1), T) = \bigoplus_q \widehat{H}^0(\mathbb{Z}_2; A) \\ &\xrightarrow{\widehat{d}^{\%}} \widehat{H}^0(\mathbb{Z}_2; S(C^0), T) = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \\ &\longrightarrow \widehat{Q}^0(C) \rightarrow \widehat{H}^{-1}(\mathbb{Z}_2; S(C^1), T) = 0. \end{aligned}$$

(ii) If A is an even commutative ring and

$$d = 2 : C_1 = A^r \rightarrow C_0 = A^r,$$

then $\widehat{d}^{\%} = 0$ and there are defined isomorphisms

$$\begin{aligned} \widehat{Q}^0(C) &\xrightarrow{\cong} \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} = \bigoplus_r A_2; \quad (\phi, \theta) \mapsto \theta = (\theta_{ii})_{1 \leq i \leq r}, \\ \widehat{Q}^1(C) &\xrightarrow{\cong} \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} = \bigoplus_r A_2; \quad (\phi, \theta) \mapsto \phi = (\phi_{ii})_{1 \leq i \leq r}. \end{aligned}$$

1.3. The symmetric Q -groups

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of $\mathbb{Z}$:

$$W : \cdots \rightarrow W_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} W_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_0 = \mathbb{Z}[\mathbb{Z}_2] \rightarrow 0.$$

Given a f.g. projective A -module chain complex C we set

$$W^{\%}C = \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \operatorname{Hom}_A(C^{-*}, C)),$$

with $T \in \mathbb{Z}_2$ acting on $C \otimes_A C = \operatorname{Hom}_A(C^{-*}, C)$ by the ε -duality involution T_ε . An n -dimensional ε -symmetric structure on C is a cycle $\phi \in (W^{\%}C)_n$, which is just a collection $\{\phi_s \in \operatorname{Hom}_A(C^r, C_{n-r+s}) \mid r \in \mathbb{Z}, s \geq 0\}$ such that

$$d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1} (\phi_{s-1} + (-1)^s T_\varepsilon \phi_{s-1}) = 0 : C^r \rightarrow C_{n-r+s-1} \\ (r \in \mathbb{Z}, s \geq 0, \phi_{-1} = 0).$$

Definition 15. The n -dimensional ε -symmetric Q -group $Q^n(C, \varepsilon)$ is the abelian group of equivalence classes of n -dimensional ε -symmetric structures on C , that is,

$$Q^n(C, \varepsilon) = H_n(W^{\%}C).$$

Note that there are defined skew-suspension isomorphisms

$$\bar{S} : Q^n(C, \varepsilon) \xrightarrow{\cong} Q^{n+2}(C_{*-1}, -\varepsilon); \{\phi_s\} \mapsto \{\phi_s\}.$$

Proposition 16. The ε -symmetric Q -groups of a f.g. projective A -module chain complex concentrated in degree k ,

$$C : \cdots \rightarrow 0 \rightarrow C_k \rightarrow 0 \rightarrow \cdots$$

are given by

$$Q^n(C, \varepsilon) = H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\ = \begin{cases} \widehat{H}^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n \leq 2k-1, \\ H^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n = 2k, \\ 0 & \text{if } n \geq 2k+1. \end{cases}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex $V = \operatorname{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \rightarrow V_{2k+1} = 0 \rightarrow V_{2k} = S(C^k) \rightarrow V_{2k-1} = 0 \rightarrow \cdots$$

and

$$(W^{\%}C)_j = \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, V_{2k}) = \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, S(C^k)),$$

which vanishes for $j > 2k$. Thus the chain complex $W^{\%}C$ is of the form

$$\begin{array}{rcl}
 (W^{\%}C)_{2k+1} & = & 0, \\
 \downarrow d_{2k+1} & & \\
 (W^{\%}C)_{2k} & = & \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_0, V_{2k}) = S(C^k), \\
 \downarrow d_{2k}=1+(-1)^{k+1}T_{\varepsilon} & & \\
 (W^{\%}C)_{2k-1} & = & \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_1, V_{2k}) = S(C^k), \\
 \downarrow d_{2k-1}=1+(-1)^kT_{\varepsilon} & & \\
 (W^{\%}C)_{2k-2} & = & \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_2, V_{2k}) = S(C^k) \\
 \downarrow & &
 \end{array}$$

and

$$Q^n(C, \varepsilon) = H_n(W^{\%}C) = H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_{\varepsilon}). \quad \square$$

For $\varepsilon = 1$ we write

$$T_{\varepsilon} = T, \quad Q^n(C, \varepsilon) = Q^n(C), \quad \varepsilon\text{-symmetric} = \text{symmetric}.$$

Example 17. The symmetric Q -groups of a zero-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_0 = A^r \rightarrow 0 \rightarrow \cdots$$

are given by

$$Q^n(C) = \begin{cases} \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A) & \text{if } n < 0, \\ \operatorname{Sym}_r(A) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An A -module chain map $f : C \rightarrow D$ induces a chain map

$$\mathrm{Hom}_A(f^*, f) : \mathrm{Hom}_A(C^{-*}, C) \rightarrow \mathrm{Hom}_A(D^{-*}, D); \quad \phi \mapsto f\phi f^*$$

and thus a chain map

$$f^{\%} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_W, \mathrm{Hom}_A(f^*, f)) : W^{\%}C \longrightarrow W^{\%}D,$$

which induces

$$f^{\%} : Q^n(C, \varepsilon) \longrightarrow Q^n(D, \varepsilon)$$

on homology. The *relative ε -symmetric Q -group*

$$Q^n(f, \varepsilon) = H_n(f^{\%} : W^{\%}C \rightarrow W^{\%}D)$$

fits into a long exact sequence

$$\cdots \longrightarrow Q^n(C, \varepsilon) \xrightarrow{f^{\%}} Q^n(D, \varepsilon) \longrightarrow Q^n(f, \varepsilon) \longrightarrow Q^{n-1}(C, \varepsilon) \longrightarrow \cdots$$

Proposition 18. (i) *The relative ε -symmetric Q -groups of an A -module chain map $f : C \rightarrow D$ are related to the absolute ε -symmetric Q -groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \rightarrow H_n(\mathcal{C}(f) \otimes_A C) \xrightarrow{F} Q^n(f, \varepsilon) \xrightarrow{t} Q^n(\mathcal{C}(f), \varepsilon) \rightarrow H_{n-1}(\mathcal{C}(f) \otimes_A C) \rightarrow \cdots,$$

with

$$t : Q^n(f, \varepsilon) \rightarrow Q^n(\mathcal{C}(f), \varepsilon); \quad (\phi, \partial\phi) \mapsto \phi/\partial\phi$$

the algebraic Thom construction

$$(\phi/\partial\phi)_s = \begin{pmatrix} \phi_s & 0 \\ \pm\partial\phi_s f^* & \pm T_\varepsilon \partial\phi_{s-1} \end{pmatrix} :$$

$$\mathcal{C}(f)^{n-r+s} = D^{n-r+s} \oplus C^{n-r+s-1} \rightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1} \quad (r \in \mathbb{Z}, s \geq 0, \phi_{-1} = 0).$$

An element $(g, h) \in H_n(\mathcal{C}(f) \otimes_A C)$ is represented by a chain map $g : C^{n-1-*} \rightarrow C$ together with a chain homotopy $h : fg \simeq 0 : C^{n-1-*} \rightarrow D$, and

$$F : H_n(\mathcal{C}(f) \otimes_A C) \rightarrow Q^n(f, \varepsilon); \quad (g, h) \mapsto (\phi, \partial\phi),$$

with

$$\partial\phi_s = \begin{cases} (1 + T_\varepsilon)g & \text{if } s = 0, \\ 0 & \text{if } s \geq 1, \end{cases} \quad \phi_s = \begin{cases} (1 + T_\varepsilon)hf^* & \text{if } s = 0, \\ 0 & \text{if } s \geq 1. \end{cases}$$

The map

$$Q^n(\mathcal{C}(f), \varepsilon) \rightarrow H_{n-1}(\mathcal{C}(f) \otimes_A C); \quad \phi \mapsto p\phi_0$$

is defined using $p = \text{projection} : \mathcal{C}(f) \rightarrow C_{*-1}$.

(ii) If $f : C \rightarrow D$ is a chain equivalence the morphisms $f^\% : Q^*(C, \varepsilon) \rightarrow Q^*(D, \varepsilon)$ are isomorphisms, and

$$Q^*(\mathcal{C}(f), \varepsilon) = Q^*(f, \varepsilon) = 0.$$

(iii) For any collection $\{C(i) \mid i \in \mathbb{Z}\}$ of f.g. projective A -module chain complexes $C(i)$

$$Q^n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i Q^n(C(i), \varepsilon) \oplus \bigoplus_{i < j} H_n(C(i) \otimes_A C(j)).$$

Proof. (i) As in Proposition 12 there is defined a chain equivalence

$$\mathcal{C}(t_0 : \mathcal{C}(f \otimes f) \rightarrow \mathcal{C}(f) \otimes_A \mathcal{C}(f)) \simeq E,$$

with

$$E = (C_{*-1} \otimes_A \mathcal{C}(f)) \oplus (\mathcal{C}(f) \otimes_A C_{*-1}), \\ H_*(W^\% E) = H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, E)) = H_{*-1}(C \otimes_A \mathcal{C}(f)).$$

(ii)+(iii) See [15, Propositions 1.1, 1.4]. $\square$

Proposition 19. Let C be a f.g. projective A -module chain complex which is concentrated in degrees $k, k+1$:

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \xrightarrow{d} C_k \rightarrow 0 \rightarrow \cdots$$

The absolute ε -symmetric Q -groups $Q^*(C, \varepsilon)$ and the relative ε -symmetric Q -groups $Q^*(d, \varepsilon)$ of $d : C_{k+1} \rightarrow C_k$ regarded as a morphism of chain complexes concentrated

in degree k are given as follows:

(i) For $n \neq 2k, 2k+1, 2k+2$:

$$Q^n(C, \varepsilon) = Q^n(d, \varepsilon) = \begin{cases} \widehat{Q}^n(d, \varepsilon) = \widehat{Q}^n(C, \varepsilon) & \text{if } n \leq 2k-1, \\ 0 & \text{if } n \geq 2k+3, \end{cases}$$

with $\widehat{Q}^n(C, \varepsilon)$ as given by Proposition 13.

(ii) For $n = 2k, 2k+1, 2k+2$ there are exact sequences

$$\begin{aligned} 0 &\longrightarrow Q^{2k+1}(d, \varepsilon) \longrightarrow Q^{2k}(C_{k+1}, \varepsilon) = H^0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\ &\xrightarrow{d^\%} Q^{2k}(C_k, \varepsilon) = H^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \longrightarrow Q^{2k}(d, \varepsilon) \\ &\longrightarrow Q^{2k-1}(C_{k+1}, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\ &\xrightarrow{d^\%} Q^{2k-1}(C_k, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon), \\ Q^{2k+2}(d, \varepsilon) &= 0 \rightarrow Q^{2k+2}(C, \varepsilon) \rightarrow C_{k+1} \otimes_A H_{k+1}(C) \xrightarrow{F} Q^{2k+1}(d, \varepsilon) \\ &\xrightarrow{t} Q^{2k+1}(C, \varepsilon) \rightarrow C_{k+1} \otimes_A H_k(C) \xrightarrow{F} Q^{2k}(d, \varepsilon) \\ &\xrightarrow{t} Q^{2k}(C, \varepsilon) \rightarrow 0. \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex $V = \text{Hom}_A(C^{-*}, C)$ is such that

$$V_n = \begin{cases} S(C^k) & \text{if } n = 2k, \\ \text{Hom}_A(C^k, C_{k+1}) \oplus \text{Hom}_A(C^{k+1}, C_k) & \text{if } n = 2k+1, \\ S(C^{k+1}) & \text{if } n = 2k+2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(W^\% C)_n = \sum_{s=0}^{\infty} \text{Hom}_A(W_s, V_{n+s}) = 0 \quad \text{for } n \geq 2k+3. \quad \square$$

Example 20. Let C be a one-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_1 = A^q \xrightarrow{d} C_0 = A^r \rightarrow 0 \rightarrow \cdots,$$

so that $C = \mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d : C_1 \rightarrow C_0$ of zero-dimensional complexes, with

$$d^{\%} : \operatorname{Hom}_A(C^1, C_1) = M_q(A) \rightarrow \operatorname{Hom}_A(C^0, C_0) = M_r(A); \quad \phi \mapsto d\phi d^*.$$

Example 17 and Proposition 19 give exact sequences

$$\begin{array}{ccccccc} Q^1(C_0) = 0 \rightarrow Q^1(d) & \longrightarrow & Q^0(C_1) = \operatorname{Sym}_q(A) & \xrightarrow{d^{\%}} & Q^0(C_0) = \operatorname{Sym}_r(A) \\ & & & & \\ & \longrightarrow & Q^0(d) & \longrightarrow & Q^{-1}(C_1) = \bigoplus_q \hat{H}^1(\mathbb{Z}_2; A) & \xrightarrow{d^{\%}} & Q^{-1}(C_0) = \bigoplus_r \hat{H}^1(\mathbb{Z}_2; A) \\ & & & & \\ H_1(C) \otimes_A C_1 & \xrightarrow{F} & Q^1(d) & \xrightarrow{t} & Q^1(C) \rightarrow H_0(C) \otimes_A C_1 & \xrightarrow{F} & Q^0(d) \xrightarrow{t} \\ & & & & & & Q^0(C) \rightarrow 0. \end{array}$$

In particular, if A is an even commutative ring and

$$d = 2 : C_1 = A^r \rightarrow C_0 = A^r,$$

then $d^{\%} = 4$ and

$$\begin{aligned} Q^0(d) &= \frac{\operatorname{Sym}_r(A)}{4\operatorname{Sym}_r(A)}, \quad Q^1(d) = 0, \\ Q^0(C) &= \operatorname{coker} \left(2(1+T) : M_r(A) \rightarrow \frac{\operatorname{Sym}_r(A)}{4\operatorname{Sym}_r(A)} \right) = \frac{\operatorname{Sym}_r(A)}{2\operatorname{Quad}_r(A)}, \\ Q^1(C) &= \ker \left(2(1+T) : \frac{M_r(A)}{2M_r(A)} \rightarrow \frac{\operatorname{Sym}_r(A)}{4\operatorname{Sym}_r(A)} \right) \\ &= \frac{\{(a_{ij}) \in M_r(A) \mid a_{ij} + a_{ji} \in 2A\}}{2M_r(A)} = \frac{\operatorname{Sym}_r(A)}{2\operatorname{Sym}_r(A)}. \end{aligned}$$

We refer to [15] for the one–one correspondence between highly-connected algebraic Poincaré complexes/pairs and forms, lagrangians and formations.

1.4. The quadratic Q -groups

Given a f.g. projective A -module chain complex C we set

$$W_{\%}C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \operatorname{Hom}_A(C^{-*}, C),$$

with $T \in \mathbb{Z}_2$ acting on $C \otimes_A C = \operatorname{Hom}_A(C^{-*}, C)$ by the ε -duality involution T_{ε} . An n -dimensional ε -quadratic structure on C is a cycle $\psi \in (W_{\%}C)_n$, a collection

$\{\psi_s \in \text{Hom}_A(C^r, C_{n-r-s}) \mid r \in \mathbb{Z}, s \geq 0\}$ such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1} (\psi_{s+1} + (-1)^{s+1} T_\varepsilon \psi_{s+1}) = 0 : C^r \rightarrow C_{n-r-s-1}.$$

Definition 21. The n -dimensional ε -quadratic Q -group $Q_n(C, \varepsilon)$ is the abelian group of equivalence classes of n -dimensional ε -quadratic structures on C , that is,

$$Q_n(C, \varepsilon) = H_n(W_{\%} C).$$

Note that there are defined skew-suspension isomorphisms

$$\bar{S} : Q_n(C, \varepsilon) \xrightarrow{\cong} Q_{n+2}(C_{*-1}, -\varepsilon); \{\psi_s\} \mapsto \{\psi_s\}.$$

Proposition 22. The ε -quadratic Q -groups of a f.g. projective A -module chain complex concentrated in degree k ,

$$C : \cdots \rightarrow 0 \rightarrow C_k \rightarrow 0 \rightarrow \cdots$$

are given by

$$Q_n(C, \varepsilon) = H_{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\ = \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n \geq 2k+1, \\ H_0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n = 2k, \\ 0 & \text{if } n \leq 2k-1. \end{cases}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex $V = \text{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \rightarrow V_{2k+1} = 0 \rightarrow V_{2k} = \text{Hom}_A(C^k, C_k) \rightarrow V_{2k-1} = 0 \rightarrow \cdots$$

and

$$(W_{\%} C)_j = W_{j-2k} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, S(C^k)),$$

which vanishes for $j < 2k$. Thus the chain complex $W_{\%}C$ is of the form

$$\begin{array}{rcl}
 (W_{\%}C)_{2k+2} & = & W_2 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
 \downarrow d_{2k+2}=1+(-1)^k T_{\varepsilon} & & \\
 (W_{\%}C)_{2k+1} & = & W_1 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
 \downarrow d_{2k+1}=1+(-1)^{k+1} T_{\varepsilon} & & \\
 (W_{\%}C)_{2k} & = & W_0 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
 \downarrow & & \\
 (W_{\%}C)_{2k-1} & = & 0
 \end{array}$$

and

$$Q_n(C, \varepsilon) = H_n(W_{\%}C) = H_{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_{\varepsilon}). \quad \square$$

Example 23. The ε -quadratic Q -groups of the zero-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_0 = A^r \rightarrow 0 \rightarrow \cdots$$

are given by

$$Q_n(C) = \begin{cases} \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2; A) & \text{if } n > 0, \\ \text{Quad}_r(A) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An A -module chain map $f : C \rightarrow D$ induces a chain map

$$f_{\%} = 1_W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \text{Hom}_A(f^*, f) : W_{\%}C \longrightarrow W_{\%}D,$$

which induces

$$f_{\%} : Q_n(C, \varepsilon) \longrightarrow Q_n(D, \varepsilon)$$

on homology. The *relative ε -quadratic Q -group* $Q_n(f, \varepsilon)$ is designed to fit into a long exact sequence

$$\cdots \longrightarrow Q_n(C, \varepsilon) \xrightarrow{f_\%} Q_n(D, \varepsilon) \longrightarrow Q_n(f, \varepsilon) \longrightarrow Q_{n-1}(C, \varepsilon) \longrightarrow \cdots,$$

that is, $Q_n(f, \varepsilon)$ is defined as the n th homology group of the mapping cone of $f_\%$,

$$Q_n(f, \varepsilon) = H_n(f_\% : W_\% C \longrightarrow W_\% D).$$

Proposition 24. (i) *The relative ε -quadratic Q -groups of $f : C \rightarrow D$ are related to the absolute ε -quadratic Q -groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \rightarrow H_n(\mathcal{C}(f) \otimes_A C) \xrightarrow{F} Q_n(f, \varepsilon) \xrightarrow{t} Q_n(\mathcal{C}(f), \varepsilon) \rightarrow H_{n-1}(\mathcal{C}(f) \otimes_A C) \rightarrow \cdots.$$

(ii) *If $f : C \rightarrow D$ is a chain equivalence the morphisms $f_\% : Q_*(C) \rightarrow Q_*(D)$ are isomorphisms, and*

$$Q_*(\mathcal{C}(f), \varepsilon) = Q_*(f, \varepsilon) = 0.$$

(iii) *For any collection $\{C(i) \mid i \in \mathbb{Z}\}$ of f.g. projective A -module chain complexes $C(i)$*

$$Q_n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i Q_n(C(i), \varepsilon) \oplus \bigoplus_{i < j} H_n(C(i) \otimes_A C(j)).$$

Proposition 25. *Let C be a f.g. projective A -module chain complex which is concentrated in degrees $k, k+1$:*

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \xrightarrow{d} C_k \rightarrow 0 \rightarrow \cdots.$$

The absolute ε -quadratic Q -groups $Q_(C, \varepsilon)$ and the relative ε -quadratic Q -groups $Q_*(d, \varepsilon)$ of $d : C_{k+1} \rightarrow C_k$ regarded as a morphism of chain complexes concentrated in degree k are given as follows:*

(i) *For $n \neq 2k, 2k+1, 2k+2$*

$$Q_n(C, \varepsilon) = Q_n(d, \varepsilon) = \begin{cases} \widehat{Q}^{n+1}(d, \varepsilon) = \widehat{Q}^{n+1}(C, \varepsilon) & \text{if } n \geq 2k+3, \\ 0 & \text{if } n \leq 2k-1, \end{cases}$$

with $\widehat{Q}^n(C, \varepsilon)$ as given by Proposition 13.

(ii) For $n = 2k, 2k + 1, 2k + 2$ there are exact sequences

$$\begin{array}{ccccccc}
 Q_{2k+2}(C_{k+1}, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & \xrightarrow{d_{\%}} & Q_{2k+2}(C_k, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & & & & \\
 & \longrightarrow & Q_{2k+2}(d, \varepsilon) = \widehat{Q}^{2k+3}(C, \varepsilon) & \xrightarrow{d_{\%}} & Q_{2k+1}(C_{k+1}, \varepsilon) & & \\
 = \widehat{H}^0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & & & & & & \\
 & \xrightarrow{d_{\%}} & Q_{2k+1}(C_k, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \longrightarrow & Q_{2k+1}(d, \varepsilon) & & \\
 & \longrightarrow & Q_{2k}(C_{k+1}, \varepsilon) = H_0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & & & & \\
 & \xrightarrow{d_{\%}} & Q_{2k}(C_k, \varepsilon) = H_0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \longrightarrow & Q_{2k}(d, \varepsilon) & & \\
 \longrightarrow & Q_{2k-1}(C_{k+1}) = 0, & & & & & \\
 0 \rightarrow Q_{2k+2}(d, \varepsilon) & \xrightarrow{t} & Q_{2k+2}(C, \varepsilon) & \longrightarrow & H_{k+1}(C) \otimes_A C_{k+1} & & \\
 & \xrightarrow{F} & Q_{2k+1}(d, \varepsilon) & & & & \\
 & & \xrightarrow{t} & Q_{2k+1}(C, \varepsilon) & \longrightarrow & C_{k+1} \otimes_A H_k(C) & \xrightarrow{F} Q_{2k}(d, \varepsilon) \\
 & & \xrightarrow{t} & Q_{2k}(C, \varepsilon) \rightarrow 0. & & &
 \end{array}$$

For $\varepsilon = 1$ we write

$$T_\varepsilon = T, \quad Q_n(C, \varepsilon) = Q_n(C), \quad \varepsilon\text{-quadratic} = \text{quadratic}.$$

Example 26. Let C be a one-dimensional f.g. free A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_1 = A^q \xrightarrow{d} C_0 = A^r \rightarrow 0 \rightarrow \cdots,$$

so that $C = \mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d : C_1 \rightarrow C_0$ of zero-dimensional complexes, with

$$d^{\%} : \text{Hom}_A(C^1, C_1) = M_q(A) \rightarrow \text{Hom}_A(C^0, C_0) = M_r(A); \quad \phi \mapsto d\phi d^*.$$

Example 23 and Proposition 25 give exact sequences

$$\begin{aligned}
 Q_1(C_1) &= \bigoplus_q \widehat{H}^0(\mathbb{Z}_2; A) \xrightarrow{\widehat{d}^{\%}} Q_1(C_0) = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow Q_1(d) \\
 &\rightarrow Q_0(C_1) = \text{Quad}_q(A) \xrightarrow{d_{\%}} Q_0(C_0) = \text{Quad}_r(A) \rightarrow Q_0(d) \\
 &\rightarrow Q_{-1}(C_1) = 0, H_1(C) \otimes_A C_1 \rightarrow Q_1(d) \\
 &\rightarrow Q_1(C) \rightarrow H_0(C) \otimes_A C_1 \rightarrow Q_0(d) \rightarrow Q_0(C) \rightarrow 0.
 \end{aligned}$$

In particular, if A is an even commutative ring and

$$d = 2 : C_1 = A^r \rightarrow C_0 = A^r,$$

then $d_{\%} = 4$ and

$$\begin{aligned}
 Q_0(d) &= \frac{\text{Quad}_r(A)}{4\text{Quad}_r(A)}, \\
 Q_1(d) &= \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + 4\text{Sym}_r(A)}, \\
 Q_0(C) &= \text{coker} \left(2(1+T) : \frac{M_r(A)}{2M_r(A)} \rightarrow \frac{\text{Quad}_r(A)}{4\text{Quad}_r(A)} \right) = \frac{\text{Quad}_r(A)}{2\text{Quad}_r(A)}, \\
 Q_1(C) &= \frac{\{(\psi_0, \psi_1) \in M_r(A) \oplus M_r(A) \mid 2\psi_0 = \psi_1 - \psi_1^*\}}{\{(2(\chi_0 - \chi_0^*), 4\chi_0 + \chi_2 + \chi_2^*) \mid (\chi_0, \chi_2) \in M_r(A) \oplus M_r(A)\}} = \bigoplus_{\frac{r(r+1)}{2}} A_2.
 \end{aligned}$$

1.5. L -groups

An n -dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complex $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$ over A is an n -dimensional f.g. projective A -module chain complex

$$C : \cdots \rightarrow 0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \rightarrow \cdots$$

together with an element $\begin{cases} \phi \in Q^n(C, \varepsilon) \\ \psi \in Q_n(C, \varepsilon) \end{cases}$ such that the A -module chain map

$$\begin{cases} \phi_0 : C^{n-*} \rightarrow C \\ (1 + T_{\varepsilon})\psi_0 : C^{n-*} \rightarrow C \end{cases}$$

is a chain equivalence. We refer to [18] for the detailed definition of the n -dimensional

$\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ L -group $\begin{cases} L^n(A, \varepsilon) \\ L_n(A, \varepsilon) \end{cases}$ as the cobordism group of n -dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complexes over A .

Definition 27. (i) The *relative* (ε -symmetric, ε -quadratic) Q -group $Q_n^n(f, \varepsilon)$ of a chain map $f : C \rightarrow D$ of f.g. projective A -module chain complexes is the relative group in the exact sequence

$$\cdots \rightarrow Q_n(C, \varepsilon) \xrightarrow{(1+T_\varepsilon)f_\%} Q^n(D, \varepsilon) \rightarrow Q_n^n(f, \varepsilon) \rightarrow Q_{n-1}(C, \varepsilon) \rightarrow \cdots.$$

An element $(\delta\phi, \psi) \in Q_n^n(f, \varepsilon)$ is an equivalence class of pairs

$$(\delta\phi, \psi) \in (W^\% D)_n \oplus (W_\% C)_{n-1},$$

such that

$$d(\psi) = 0 \in (W_\% C)_{n-2}, \quad (1 + T_\varepsilon)f_\% \psi = d(\delta\phi) \in (W^\% D)_{n-1}.$$

(ii) An n -dimensional (ε -symmetric, ε -quadratic) pair over A $(f : C \rightarrow D, (\delta\phi, \psi))$ is a chain map f together with a class $(\delta\phi, \psi) \in Q_n^n(f, \varepsilon)$ such that the chain map

$$(\delta\phi, (1 + T_\varepsilon)\psi)_0 : D^{n-*} \rightarrow \mathcal{C}(f)$$

defined by

$$(\delta\phi, (1 + T_\varepsilon)\psi)_0 = \begin{pmatrix} \delta\phi_0 \\ (1 + T_\varepsilon)\psi_0 f^* \end{pmatrix} : D^{n-r} \rightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is a chain equivalence.

Proposition 28. The relative (ε -symmetric, ε -quadratic) Q -groups $Q_n^n(f, \varepsilon)$ of a chain map $f : C \rightarrow D$ fit into a commutative braid of exact sequences

$$\begin{array}{ccccc} & & \xrightarrow{(1+T_\varepsilon)f_\%} & & \\ & Q_n(C, \varepsilon) & & Q^n(D, \varepsilon) & \xrightarrow{J} \hat{Q}^n(D, \varepsilon) \\ & \searrow f_\% & \nearrow 1+T_\varepsilon & \searrow & \nearrow J_f \\ & Q_n(D, \varepsilon) & & Q_n^n(f, \varepsilon) & \\ \hat{Q}^{n+1}(D, \varepsilon) & \nearrow H & \searrow & \nearrow & \searrow \\ & Q_n(f, \varepsilon) & & Q_{n-1}(C, \varepsilon) & \end{array}$$

with

$$J_f : \mathcal{Q}_n^n(f, \varepsilon) \rightarrow \widehat{\mathcal{Q}}^n(D, \varepsilon); (\delta\phi, \psi) \mapsto \alpha,$$

$$\alpha_s = \begin{cases} \delta\phi_s & \text{if } s \geq 0 \\ f\psi_{-s-1}f^* & \text{if } s \leq -1 \end{cases} : D^r \rightarrow D_{n-r+s}.$$

The n -dimensional ε -hyperquadratic L -group $\widehat{L}^n(A, \varepsilon)$ is the cobordism group of n -dimensional $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pairs $(f : C \rightarrow D, (\phi, \psi))$ over A . As in [15], there is defined an exact sequence

$$\cdots \longrightarrow L_n(A, \varepsilon) \xrightarrow{1+T_\varepsilon} L^n(A, \varepsilon) \longrightarrow \widehat{L}^n(A, \varepsilon) \longrightarrow L_{n-1}(A, \varepsilon) \longrightarrow \cdots.$$

The skew-suspension maps in the $\pm\varepsilon$ -quadratic L -groups are isomorphisms

$$\overline{S} : L_n(A, \varepsilon) \xrightarrow{\cong} L_{n+2}(A, -\varepsilon); (C, \{\psi_s\}) \mapsto (C_{*-1}, \{\psi_s\}),$$

so the ε -quadratic L -groups are 4-periodic

$$L_n(A, \varepsilon) = L_{n+2}(A, -\varepsilon) = L_{n+4}(A, \varepsilon).$$

The skew-suspension maps in ε -symmetric and ε -hyperquadratic L -groups and $\pm\varepsilon$ -hyperquadratic L -groups

$$\begin{aligned} \overline{S} : L^n(A, \varepsilon) &\rightarrow L^{n+2}(A, -\varepsilon); (C, \{\phi_s\}) \mapsto (C_{*-1}, \{\phi_s\}), \\ \overline{S} : \widehat{L}^n(A, \varepsilon) &\rightarrow \widehat{L}^{n+2}(A, -\varepsilon); (f : C \rightarrow D, \{\psi_s, \phi_s\}) \\ &\mapsto (f : C_{*-1} \rightarrow D_{*-1}, \{(\psi_s, \phi_s)\}) \end{aligned}$$

are not isomorphisms in general, so the ε -symmetric and ε -hyperquadratic L -groups need not be 4-periodic. We shall write the 4-periodic versions of the ε -symmetric and ε -hyperquadratic L -groups of A as

$$L^{n+4*}(A, \varepsilon) = \lim_{k \rightarrow \infty} L^{n+4k}(A, \varepsilon), \quad \widehat{L}^{n+4*}(A, \varepsilon) = \lim_{k \rightarrow \infty} \widehat{L}^{n+4k}(A, \varepsilon),$$

noting that there is defined an exact sequence

$$\cdots \rightarrow L_n(A, \varepsilon) \rightarrow L^{n+4*}(A, \varepsilon) \rightarrow \widehat{L}^{n+4*}(A, \varepsilon) \rightarrow L_{n-1}(A, \varepsilon) \rightarrow \cdots.$$

Definition 29. The Wu classes of an n -dimensional ε -symmetric complex (C, ϕ) over A are the A -module morphisms

$$\widehat{v}_k(\phi) : H^{n-k}(C) \rightarrow \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon); \quad x \mapsto \phi_{n-2k}(x)(x) \quad (k \in \mathbb{Z}).$$

For an n -dimensional ε -symmetric Poincaré complex (C, ϕ) over A the evaluation of the Wu class $\widehat{v}_k(\phi)(x) \in \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)$ is the obstruction to killing $x \in H^{n-k}(C) \cong H_k(C)$ by algebraic surgery [15, §4].

Proposition 30. (i) If $\widehat{H}^0(\mathbb{Z}_2; A, \varepsilon)$ has a one-dimensional f.g. projective A -module resolution then the skew-suspension maps

$$\overline{S} : L^{n-2}(A, -\varepsilon) \rightarrow L^n(A, \varepsilon), \quad \overline{S} : \widehat{L}^{n-2}(A, -\varepsilon) \rightarrow \widehat{L}^n(A, \varepsilon) \quad (n \geq 2)$$

are isomorphisms. Thus if $\widehat{H}^1(\mathbb{Z}_2; A, \varepsilon)$ also has a one-dimensional f.g. projective A -module resolution the ε -symmetric and ε -hyperquadratic L -groups of A are 4-periodic

$$\begin{aligned} L^n(A, \varepsilon) &= L^{n+2}(A, -\varepsilon) = L^{n+4}(A, \varepsilon), \\ \widehat{L}^n(A, \varepsilon) &= \widehat{L}^{n+2}(A, -\varepsilon) = \widehat{L}^{n+4}(A, \varepsilon). \end{aligned}$$

(ii) If A is a Dedekind ring then the ε -symmetric L -groups are ‘homotopy invariant’

$$L^n(A[x], \varepsilon) = L^n(A, \varepsilon)$$

and the ε -symmetric and ε -hyperquadratic L -groups of A and $A[x]$ are 4-periodic.

Proof. (i) Let D be a one-dimensional f.g. projective A -module resolution of $\widehat{H}^0(\mathbb{Z}_2; A, \varepsilon)$:

$$0 \rightarrow D_1 \rightarrow D_0 \rightarrow \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow 0.$$

Given an n -dimensional ε -symmetric Poincaré complex (C, ϕ) over A resolve the A -module morphism

$$\widehat{v}_n(\phi)(\phi_0)^{-1} : H_0(C) \cong H^n(C) \rightarrow H_0(D) = \widehat{H}^0(\mathbb{Z}_2; A, \varepsilon); \quad u \mapsto (\phi_0)^{-1}(u)(u)$$

by an A -module chain map $f : C \rightarrow D$, defining an $(n+1)$ -dimensional ε -symmetric pair $(f : C \rightarrow D, (\delta\phi, \phi))$. The effect of algebraic surgery on (C, ϕ) using $(f : C \rightarrow D, (\delta\phi, \phi))$ is a cobordant n -dimensional ε -symmetric Poincaré complex (C', ϕ') such that there are defined an exact sequence:

$$0 \rightarrow H^n(C') \rightarrow H^n(C) \xrightarrow{\widehat{v}_n(\phi)} \widehat{H}^0(\mathbb{Z}_2; A, \varepsilon) \rightarrow H^{n+1}(C') \rightarrow 0$$

and an $(n + 1)$ -dimensional ε -symmetric pair $(f' : C' \rightarrow D', (\delta\phi', \phi'))$ with f' the projection onto the quotient complex of C' defined by

$$D' : \cdots \rightarrow 0 \rightarrow D'_{n+1} = C'_{n+1} \rightarrow D'_n = C'_n \rightarrow 0 \rightarrow \cdots .$$

The effect of algebraic surgery on (C', ϕ') using $(f' : C' \rightarrow D', (\delta\phi', \phi'))$ is a cobordant n -dimensional ε -symmetric Poincaré complex (C'', ϕ'') with $H_n(C'') = 0$, so that it is (homotopy equivalent to) the skew-suspension of an $(n - 2)$ -dimensional $(-\varepsilon)$ -symmetric Poincaré complex.

(ii) The 4-periodicity $L^*(A, \varepsilon) = L^{*+4}(A, \varepsilon)$ was proved in [15, §7]. The ‘homotopy invariance’ $L^*(A[x], \varepsilon) = L^*(A, \varepsilon)$ was proved in [17, 41.3]; [10, 2.1]. The 4-periodicity of the ε -symmetric and ε -hyperquadratic L -groups for A and $A[x]$ now follows from the 4-periodicity of the ε -quadratic L -groups $L_*(A, \varepsilon) = L_{*+4}(A, \varepsilon)$. $\square$

2. Chain bundle theory

2.1. Chain bundles

Definition 31. (i) An ε -bundle over an A -module chain complex C is a zero-dimensional ε -hyperquadratic structure γ on C^{0-*} , that is, a cycle

$$\gamma \in (\widehat{W}^{\%} C^{0-*})_0$$

as given by a collection of A -module morphisms

$$\{\gamma_s \in \text{Hom}_A(C_{r-s}, C^{-r}) \mid r, s \in \mathbb{Z}\},$$

such that

$$(-1)^{r+1} d^* \gamma_s + (-1)^s \gamma_s d + (-1)^{s-1} (\gamma_{s-1} + (-1)^s T_\varepsilon \gamma_{s-1}) = 0 : C_{r-s+1} \rightarrow C^{-r}.$$

(ii) An *equivalence of ε -bundles* over C ,

$$\chi : \gamma \longrightarrow \gamma'$$

is an equivalence of ε -hyperquadratic structures.

(iii) A *chain ε -bundle* (C, γ) over A is an A -module chain complex C together with an ε -bundle $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$.

Let (D, δ) be a chain ε -bundle and $f : C \rightarrow D$ a chain map. The dual of f

$$f^* : D^{0-*} \longrightarrow C^{0-*}$$

induces a map

$$(\widehat{f^*})_0^{\%} : (\widehat{W}^{\%} D^{0-*})_0 \longrightarrow (\widehat{W}^{\%} C^{0-*})_0.$$

Definition 32. (i) The *pullback chain ε -bundle* $(C, f^* \delta)$ is defined to be

$$f^* \delta = (\widehat{f^*})_0^{\%}(\delta) \in (\widehat{W}^{\%} C^{0-*})_0.$$

(ii) A *map of chain ε -bundles*

$$(f, \chi) : (C, \gamma) \longrightarrow (D, \delta)$$

is a chain map $f : C \rightarrow D$ together with an equivalence of ε -bundles over C :

$$\chi : \gamma \longrightarrow f^* \delta.$$

The ε -hyperquadratic Q -group $\widehat{Q}^0(C^{0-*}, \varepsilon)$ is thus the group of equivalence classes of chain ε -bundles on the chain complex C , the algebraic analogue of the topological K -group of a space. The Tate $\mathbb{Z}_2$ -cohomology groups

$$\widehat{H}^n(\mathbb{Z}_2; A, \varepsilon) = \frac{\{a \in A \mid \bar{a} = (-1)^n \varepsilon a\}}{\{b + (-1)^n \varepsilon \bar{b} \mid b \in A\}}$$

are A -modules via

$$A \times \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon) \rightarrow \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon); (a, x) \mapsto ax\bar{a}.$$

Definition 33. The *Wu classes* of a chain ε -bundle (C, γ) are the A -module morphisms

$$\widehat{v}_k(\gamma) : H_k(C) \rightarrow \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon); x \mapsto \gamma_{-2k}(x)(x) \quad (k \in \mathbb{Z}).$$

An n -dimensional ε -symmetric Poincaré complex (C, ϕ) with Wu classes (29)

$$\widehat{v}_k(\phi) : H^{n-k}(C) \rightarrow \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon); y \mapsto \phi_{n-2k}(y)(y) \quad (k \in \mathbb{Z})$$

has a Spivak normal ε -bundle [15]

$$\gamma = S^{-n}(\phi_0^{\%})^{-1}(J(\phi)) \in \widehat{Q}^0(C^{0-*}, \varepsilon),$$

such that

$$\widehat{v}_k(\phi) = \widehat{v}_k(\gamma)\phi_0 : H^{n-k}(C) \cong H_k(C) \rightarrow \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon) \quad (k \in \mathbb{Z}),$$

the abstract analogue of the formulae of Wu and Thom.

For any A -module chain map $f : C \rightarrow D$ Proposition 12(i) gives an exact sequence

$$\cdots \rightarrow \widehat{Q}^1(C^{0-*}, \varepsilon) \rightarrow \widehat{Q}^0(\mathcal{C}(f)^{0-*}, \varepsilon) \rightarrow \widehat{Q}^0(D^{0-*}, \varepsilon) \xrightarrow{(\widehat{f}^*)^\%} \widehat{Q}^0(C^{0-*}, \varepsilon) \rightarrow \cdots,$$

motivating the following construction of chain ε -bundles:

Definition 34. The *cone* of a chain ε -bundle map $(f, \chi) : (C, 0) \rightarrow (D, \delta)$ is the chain ε -bundle

$$(B, \beta) = \mathcal{C}(f, \chi),$$

with $B = \mathcal{C}(f)$ the algebraic mapping cone of $f : C \rightarrow D$ and

$$\beta_s = \begin{pmatrix} \delta_s & 0 \\ f^*\delta_{s+1} & \chi_{s+1} \end{pmatrix} : B_{r-s} = D_{r-s} \oplus C_{r-s-1} \rightarrow B^{-r} = D^{-r} \oplus C^{-r-1}.$$

Note that $(D, \delta) = g^*(B, \beta)$ is the pullback of (B, β) along the inclusion $g : D \rightarrow B$.

Proposition 35. For a f.g. projective A -module chain complex concentrated in degree k :

$$C : \cdots \rightarrow 0 \rightarrow C_k \rightarrow 0 \rightarrow \cdots,$$

the k th Wu class defines an isomorphism

$$\widehat{v}_k : \widehat{Q}^0(C^{0-*}, \varepsilon) \xrightarrow{\cong} \text{Hom}_A(C_k, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)); \quad \gamma \mapsto \widehat{v}_k(\gamma).$$

Proof. By construction. $\square$

Proposition 36. For a f.g. projective A -module chain complex concentrated in degrees $k, k+1$,

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \xrightarrow{d} C_k \rightarrow 0 \rightarrow \cdots$$

there is defined an exact sequence

$$\begin{aligned} \mathrm{Hom}_A(C_k, \widehat{H}^{k+1}(\mathbb{Z}_2; A, \varepsilon)) &\xrightarrow{d^*} \mathrm{Hom}_A(C_{k+1}, \widehat{H}^{k+1}(\mathbb{Z}_2; A, \varepsilon)) \\ &\longrightarrow \widehat{Q}^0(C^{0-*}, \varepsilon) \xrightarrow{p^*\widehat{v}_k} \mathrm{Hom}_A(C_k, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)) \\ &\xrightarrow{d^*} \mathrm{Hom}_A(C_{k+1}, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)), \end{aligned}$$

with $p : C_k \rightarrow H_k(C)$ the projection. Thus every chain ε -bundle (C, γ) is equivalent to the cone $\mathcal{C}(d, \chi)$ (34) of a chain ε -bundle map $(d, \chi) : (C_{k+1}, 0) \rightarrow (C_k, \delta)$, regarding $d : C_{k+1} \rightarrow C_k$ as a map of chain complexes concentrated in degree k , with

$$\begin{aligned} \delta^* &= (-1)^k \delta : C_k \rightarrow C^k, \quad d^* \delta d = \chi + (-1)^k \chi^* : C_{k+1} \rightarrow C^{k+1}, \\ \gamma_{-2k} &= \delta : C_k \rightarrow C^k, \quad \gamma_{-2k-1} = \begin{cases} d^* \delta : C_k \rightarrow C^{k+1} \\ 0 : C_{k+1} \rightarrow C^k \end{cases}, \\ \gamma_{-2k-2} &= \chi : C_{k+1} \rightarrow C^{k+1}. \end{aligned}$$

Proof. This follows from Proposition 35 and the algebraic Thom isomorphisms

$$\widehat{t} : \widehat{Q}^*(d, \varepsilon) \cong \widehat{Q}^*(C, \varepsilon)$$

of Proposition 12. $\square$

2.2. The twisted quadratic Q -groups

For any f.g. projective A -module chain complex C there is defined a $\mathbb{Z}$ -module chain map

$$\begin{aligned} 1 + T_\varepsilon : W^\% C; \quad \psi &\mapsto (1 + T_\varepsilon)\psi, \\ ((1 + T_\varepsilon)\psi)_s &= \begin{cases} (1 + T_\varepsilon)(\psi_0) & \text{if } s = 0, \\ 0 & \text{if } s \geq 1, \end{cases} \end{aligned}$$

with algebraic mapping cone

$$\mathcal{C}(1 + T_\varepsilon) = \widehat{W}^\% C.$$

Write the inclusion as

$$J : W^\% C \rightarrow \widehat{W}^\% C; \quad \phi \mapsto J\phi, \quad (J\phi)_s = \begin{cases} \phi_s & \text{if } s \geq 0, \\ 0 & \text{if } s \leq -1. \end{cases}$$

The sequence of $\mathbb{Z}$ -module chain complexes

$$0 \rightarrow W_{\%}C \xrightarrow{1+T_{\varepsilon}} W^{\%}C \xrightarrow{J} \widehat{W}^{\%}C \rightarrow 0$$

induces the long exact sequence of Ranicki [15] relating the ε -symmetric, ε -quadratic and ε -hyperquadratic Q -groups of C ,

$$\cdots \rightarrow \widehat{Q}^{n+1}(C, \varepsilon) \xrightarrow{H} Q_n(C, \varepsilon) \xrightarrow{1+T_{\varepsilon}} Q^n(C, \varepsilon) \xrightarrow{J} \widehat{Q}^n(C, \varepsilon) \rightarrow \cdots,$$

with

$$H : \widehat{W}^{\%}C \rightarrow (W_{\%}C)_{*-1}; \quad \theta \mapsto H\theta, \quad (H\theta)_s = \theta_{-s-1} \quad (s \geq 0).$$

Weiss [21] used simplicial abelian groups to defined the twisted quadratic Q -groups $Q_*(C, \gamma, \varepsilon)$ of a chain ε -bundle (C, γ) , to fit into the exact sequence

$$\cdots \rightarrow \widehat{Q}^{n+1}(C, \varepsilon) \xrightarrow{H_{\gamma}} Q_n(C, \gamma, \varepsilon) \xrightarrow{N_{\gamma}} Q^n(C, \varepsilon) \xrightarrow{J_{\gamma}} \widehat{Q}^n(C, \varepsilon) \rightarrow \cdots.$$

The morphisms

$$J_{\gamma} : Q^n(C, \varepsilon) \rightarrow \widehat{Q}^n(C, \varepsilon); \quad \phi \mapsto J_{\gamma}\phi, \quad (J_{\gamma}\phi)_s = J(\phi) - (\phi_0)^{\%}(S^n\gamma)$$

are induced by a morphism of simplicial abelian groups, where

$$S^n : \widehat{Q}^0(C^{0-*}, \varepsilon) \xrightarrow{\cong} \widehat{Q}^n(C^{n-*}, \varepsilon); \quad \{\theta_s\} \mapsto \{(S^n\theta)_s = \theta_{s-n}\}$$

are the n -fold suspension isomorphisms.

The Kan-Dold theory associates to a chain complex C a simplicial abelian group $K(C)$ such that

$$\pi_*(K(C)) = H_*(C).$$

For any chain complexes C, D a simplicial map $f : K(C) \rightarrow K(D)$ has a mapping fibre $K(f)$. The relative homology groups of f are defined by

$$H_*(f) = \pi_{*-1}(K(f))$$

and the fibration sequence of simplicial abelian groups

$$K(f) \longrightarrow K(C) \xrightarrow{f} K(D)$$

induces a long exact sequence in homology

$$\cdots \rightarrow H_n(C) \rightarrow H_n(D) \rightarrow H_n(f) \rightarrow H_{n-1}(C) \rightarrow \cdots$$

For a chain map $f : C \rightarrow D$,

$$K(f) = K(\mathcal{C}(f)).$$

The applications involve simplicial maps which are not chain maps, and the *triad homology groups*: given a homotopy-commutative square of simplicial abelian groups

$$\Phi : \begin{array}{ccc} K(C) & \longrightarrow & K(D) \\ \downarrow & \rightsquigarrow & \downarrow \\ K(E) & \longrightarrow & K(F) \end{array}$$

(with $\rightsquigarrow$ denoting an explicit homotopy) the triad homology groups of Φ are the homotopy groups of the mapping fibre of the map of mapping fibres

$$H_*(\Phi) = \pi_{*-1}(K(C \rightarrow D) \rightarrow K(E \rightarrow F)),$$

which fit into a commutative diagram of exact sequences

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ \cdots & \longrightarrow & H_{n+1}(D) & \longrightarrow & H_{n+1}(C \rightarrow D) & \longrightarrow & H_n(C) & \longrightarrow & H_n(D) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{n+1}(F) & \longrightarrow & H_{n+1}(E \rightarrow F) & \longrightarrow & H_n(E) & \longrightarrow & H_n(F) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{n+1}(D \rightarrow F) & \longrightarrow & H_{n+1}(\Phi) & \longrightarrow & H_n(C \rightarrow E) & \longrightarrow & H_n(D \rightarrow F) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_n(D) & \longrightarrow & H_n(C \rightarrow D) & \longrightarrow & H_{n-1}(C) & \longrightarrow & H_{n-1}(D) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

If $H_*(\Phi) = 0$ there is a commutative braid of exact sequences

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowright & \\
 H_{n+1}(C \rightarrow D) & & H_n(E) & & H_n(C \rightarrow E) \\
 & \searrow & \swarrow & \searrow & \swarrow \\
 & H_n(C) & & H_n(F) & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 H_{n+1}(C \rightarrow E) & & H_n(D) & & H_n(C \rightarrow D) \\
 & \curvearrowright & & \curvearrowright &
 \end{array}$$

The twisted ε -quadratic Q -groups were defined in [21] to be the relative homology groups of a simplicial map

$$J_\gamma : K(W^{\%}C) \rightarrow K(\widehat{W}^{\%}C),$$

with

$$Q_n(C, \gamma, \varepsilon) = \pi_{n+1}(J_\gamma).$$

A more explicit description of the twisted quadratic Q -groups was then obtained in [18], as equivalence classes of ε -symmetric structures on the chain ε -bundle.

Definition 37. (i) An ε -symmetric structure on a chain ε -bundle (C, γ) is a pair (ϕ, θ) with $\phi \in (W^{\%}C)_n$ a cycle and $\theta \in (\widehat{W}^{\%}C)_{n+1}$ such that

$$d\theta = J_\gamma(\phi),$$

or equivalently

$$\begin{aligned}
 d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1} (\phi_{s-1} + (-1)^s T_\varepsilon \phi_{s-1}) &= 0 : C^r \rightarrow C_{n-r+s-1}, \\
 \phi_s - \phi_0^* \gamma_{s-n} \phi_0 &= d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s} (\theta_{s-1} + (-1)^s T_\varepsilon \theta_{s-1}) : C^r \rightarrow C_{n-r+s} \\
 (r, s \in \mathbb{Z}, \phi_s &= 0 \text{ for } s \leq -1).
 \end{aligned}$$

(ii) Two structures (ϕ, θ) and (ϕ', θ') are *equivalent* if there exist $\xi \in (W^{\%}C)_{n+1}$, $\eta \in (\widehat{W}^{\%}C)_{n+2}$ such that

$$d\xi = \phi' - \phi, \quad d\eta = \theta' - \theta + J(\xi) + (\xi_0, \phi_0, \phi'_0)^{\%}(S^n \gamma),$$

where $(\xi_0, \phi_0, \phi'_0)^{\%} : (\widehat{W}^{\%}C^{-*})_n \rightarrow (\widehat{W}^{\%}C)_{n+1}$ is the chain homotopy from $(\phi_0)^{\%}$ to $(\phi'_0)^{\%}$ induced by ξ_0 . (See [15, 1.1] for the precise formula.)

(iii) The n -dimensional twisted ε -quadratic Q -group $Q_n(C, \gamma, \varepsilon)$ is the abelian group of equivalence classes of n -dimensional ε -symmetric structures on (C, γ) with addition by

$$(\phi, \theta) + (\phi', \theta') = (\phi + \phi', \theta + \theta' + \zeta), \text{ where } \zeta_s = \phi_0 \gamma_{s-n+1} \phi'_0.$$

As for the $\pm\varepsilon$ -symmetric and $\pm\varepsilon$ -quadratic Q -groups, there are defined skew-suspension isomorphisms of twisted $\pm\varepsilon$ -quadratic Q -groups

$$\bar{S} : Q_n(C, \gamma, \varepsilon) \xrightarrow{\cong} Q_{n+2}(C_{*-1}, \gamma, -\varepsilon); (\{\phi_s\}, \{\theta_s\}) \mapsto (\{\phi_s\}, \{\theta_s\}).$$

Proposition 38. (i) The twisted ε -quadratic Q -groups $Q_*(C, \gamma, \varepsilon)$ are related to the ε -symmetric Q -groups $Q^*(C, \varepsilon)$ and the ε -hyperquadratic Q -groups $\widehat{Q}^*(C, \varepsilon)$ by the exact sequence

$$\dots \rightarrow \widehat{Q}^{n+1}(C, \varepsilon) \xrightarrow{H_\gamma} Q_n(C, \gamma, \varepsilon) \xrightarrow{N_\gamma} Q^n(C, \varepsilon) \xrightarrow{J_\gamma} \widehat{Q}^n(C, \varepsilon) \rightarrow \dots,$$

with

$$\begin{aligned} H_\gamma : \widehat{Q}^{n+1}(C, \varepsilon) &\rightarrow Q_n(C, \gamma, \varepsilon); \theta \mapsto (0, \theta), \\ N_\gamma : Q_n(C, \gamma, \varepsilon) &\rightarrow Q^n(C, \varepsilon); (\phi, \theta) \mapsto \phi. \end{aligned}$$

(ii) For a chain ε -bundle (C, γ) such that C splits as

$$C = \sum_{i=-\infty}^{\infty} C(i),$$

the ε -hyperquadratic Q -groups split as

$$\widehat{Q}^n(C, \varepsilon) = \sum_{i=-\infty}^{\infty} \widehat{Q}^n(C(i), \varepsilon)$$

and

$$\gamma = \sum_{i=-\infty}^{\infty} \gamma(i) \in \widehat{Q}^0(C^{-*}, \varepsilon) = \sum_{i=-\infty}^{\infty} \widehat{Q}^0(C(i)^{-*}, \varepsilon).$$

The twisted ε -quadratic Q -groups of (C, γ) fit into the exact sequence

$$\begin{array}{ccccccc} \cdots \rightarrow \sum_i Q_n(C(i), \gamma(i), \varepsilon) & \xrightarrow{q} & Q_n(C, \gamma, \varepsilon) & \xrightarrow{p} & \sum_{i < j} H_n(C(i) \otimes_A C(j)) \\ & & & \searrow \delta & \\ & & & \sum_i Q_{n-1}(C(i), \gamma(i), \varepsilon) \rightarrow \cdots, \end{array}$$

with

$$\begin{aligned} p : Q_n(C, \gamma, \varepsilon) &\rightarrow \sum_{i < j} H_n(C(i) \otimes_A C(j)); \quad (\phi, \theta) \mapsto \sum_{i < j} (p(i) \otimes p(j))(\phi_0) \\ (p(i) &= \text{projection} : C \rightarrow C(i)), \\ q &= \sum_i q(i)_{\%} : \sum_i Q_n(C(i), \gamma(i), \varepsilon) \rightarrow Q_n(C, \gamma, \varepsilon) \\ (q(i) &= \text{inclusion} : C(i) \rightarrow C), \\ \delta : \sum_{i < j} H_n(C(i) \otimes_A C(j)) &\rightarrow \sum_i Q_{n-1}(C(i), \gamma(i), \varepsilon); \\ \sum_{i < j} h(i, j) &\mapsto \left(0, \sum_{i \neq j} \widehat{h(i, j)}_{\%} (S^n \gamma(j)) \right) \quad (h(i, j) : C(j)^{n-*} \rightarrow C(i)), \end{aligned}$$

with $h(j, i) = h(i, j)^*$ for $i < j$.

Proof. (i) See [21].

(ii) See [18, p. 26]. $\square$

Example 39. The twisted ε -quadratic Q -groups of the zero chain ε -bundle $(C, 0)$ are just the ε -quadratic Q -groups of C , with isomorphisms

$$Q_n(C, \varepsilon) \rightarrow Q_n(C, 0, \varepsilon); \quad \psi \mapsto ((1 + T)\psi, \theta)$$

defined by

$$\theta_s = \begin{cases} \psi_{-s-1} : C^{n-r+s+1} \rightarrow C_r & \text{if } s \leq -1, \\ 0 & \text{if } s \geq 0 \end{cases}$$

and with an exact sequence

$$\cdots \rightarrow \widehat{Q}^{n+1}(C, \varepsilon) \xrightarrow{H} Q_n(C, \varepsilon) \xrightarrow{N} Q^n(C, \varepsilon) \xrightarrow{J} \widehat{Q}^n(C, \varepsilon) \rightarrow \cdots$$

For $\varepsilon = 1$ we write

$$\text{chain 1-bundle} = \text{chain bundle}, \quad Q_n(C, \gamma, 1) = Q_n(C, \gamma).$$

2.3. The algebraic normal invariant

Fix a chain ε -bundle (B, β) over A .

Definition 40. (i) An *algebraic normal structure* (γ, ϕ, θ) on an n -dimensional $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ is a chain ε -bundle $(\mathcal{C}(f), \gamma)$ together with an ε -symmetric structure (ϕ, θ) , where $\phi = \delta\phi/(1 + T_\varepsilon)\psi \in (W^{\text{go}}\mathcal{C}(f))_n$ is the ε -symmetric structure on $\mathcal{C}(f)$ given by the algebraic Thom construction on $(\delta\phi, (1 + T_\varepsilon)\psi)$ (18).

(ii) A (B, β) -*structure* $(\gamma, \phi, \theta, g, \chi)$ on an n -dimensional $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ is an algebraic normal structure (γ, ϕ, θ) with $\phi = \delta\phi/(1 + T_\varepsilon)\psi$, together with a chain ε -bundle map

$$(g, \chi) : (\mathcal{C}(f), \gamma) \rightarrow (B, \beta).$$

(iii) The n -dimensional (B, β) -*structure ε -symmetric L -group* $L\langle B, \beta \rangle^n(A, \varepsilon)$ is the cobordism group of n -dimensional ε -symmetric Poincaré complexes $(D, \delta\phi)$ over A together with a (B, β) -structure $(\gamma, \delta\phi, \theta, g, \chi)$ (so $(C, \psi) = (0, 0)$).

(iv) The n -dimensional (B, β) -*structure ε -hyperquadratic L -group* $\widehat{L}\langle B, \beta \rangle^n(A, \varepsilon)$ is the cobordism group of n -dimensional $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pairs $(f : C \rightarrow D, (\delta\phi, \psi))$ over A together with a (B, β) -structure $(\gamma, \delta\phi/(1 + T_\varepsilon)\psi, \theta, g, \chi)$.

There are defined skew-suspension maps in the (B, β) -structure ε -symmetric and ε -hyperquadratic L -groups

$$\begin{aligned} \overline{S} : L\langle B, \beta \rangle^n(A, \varepsilon) &\rightarrow L\langle B_{*-1}, \beta_{*-1} \rangle^{n+2}(A, -\varepsilon), \\ \overline{S} : \widehat{L}\langle B, \beta \rangle^n(A, \varepsilon) &\rightarrow \widehat{L}\langle B_{*-1}, \beta_{*-1} \rangle^{n+2}(A, -\varepsilon) \end{aligned}$$

given by $C \mapsto C_{*-1}$ on the chain complexes, with (B_{*-1}, β_{*-1}) a chain $(-\varepsilon)$ -bundle. We shall write the 4-periodic versions of the (B, β) -structure L -groups as

$$\begin{aligned} L\langle B, \beta \rangle^{n+4*}(A, \varepsilon) &= \lim_{k \rightarrow \infty} L\langle B, \beta \rangle^{n+4k}(A, \varepsilon), \\ \widehat{L}\langle B, \beta \rangle^{n+4*}(A, \varepsilon) &= \lim_{k \rightarrow \infty} \widehat{L}\langle B, \beta \rangle^{n+4k}(A, \varepsilon). \end{aligned}$$

Example 41. An $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pair with a $(0, 0)$ -structure is essentially the same as an ε -quadratic Poincaré pair. In particular, an ε -symmetric Poincaré complex with a $(0, 0)$ -structure is essentially the same as an ε -quadratic Poincaré complex. The $(0, 0)$ -structure L -groups are given by

$$L\langle 0, 0 \rangle^n(A, \varepsilon) = L_n(A, \varepsilon), \quad \widehat{L}\langle 0, 0 \rangle^n(A, \varepsilon) = 0.$$

Proposition 42 (Ranicki [18, §7]). (i) An n -dimensional ε -symmetric structure $(\phi, \theta) \in \mathcal{Q}_n(B, \beta, \varepsilon)$ on a chain ε -bundle (B, β) determines an n -dimensional (ε -symmetric, ε -quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ with

$$f = \text{proj.} : C = \mathcal{C}(\phi_0 : B^{n-*} \rightarrow B)_{*+1} \rightarrow D = B^{n-*},$$

$$\psi_0 = \begin{pmatrix} \theta_0 & 0 \\ 1 + \beta_{-n}\phi_0^* & \beta_{-n-1}^* \end{pmatrix} :$$

$$C^r = B^{r+1} \oplus B_{n-r} \rightarrow C_{n-r-1} = B_{n-r} \oplus B^{r+1},$$

$$\psi_s = \begin{pmatrix} \theta_{-s} & 0 \\ \beta_{-n-s}\phi_0^* & \beta_{-n-s-1}^* \end{pmatrix} :$$

$$C^r = B^{r+1} \oplus B_{n-r} \rightarrow C_{n-r-s-1} = B_{n-r-s} \oplus B^{r+s+1} \quad (s \geq 1),$$

$$\delta\phi_s = \beta_{s-n} : D^r = B_{n-r} \rightarrow D_{n-r+s} = B^{r-s} \quad (s \geq 0)$$

(up to signs) such that $(\mathcal{C}(f), \gamma) \simeq (B, \beta)$.

(ii) An n -dimensional (ε -symmetric, ε -quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi) \in \mathcal{Q}_n^*(f, \varepsilon))$ has a canonical equivalence class of ‘algebraic Spivak normal structures’ (γ, ϕ, θ) with γ a chain ε -bundle over $\mathcal{C}(f)$ and (ϕ, θ) an n -dimensional ε -symmetric structure on γ representing an element

$$(\phi, \theta) \in \mathcal{Q}_n(\mathcal{C}(f), \gamma, \varepsilon),$$

with $\phi = \delta\phi/(1 + T_\varepsilon)\psi$. The construction of (i) applied to (ϕ, θ) gives an n -dimensional (ε -symmetric, ε -quadratic) Poincaré pair homotopy equivalent to $(f : C \rightarrow D, (\delta\phi, \psi) \in \mathcal{Q}_n^*(f, \varepsilon))$.

Proof. (i) By construction.

(ii) The equivalence class $\phi = \delta\phi/(1 + T_\varepsilon)\psi \in \mathcal{Q}_n^*(\mathcal{C}(f))$ is given by the algebraic Thom construction

$$\phi_s = \begin{cases} \begin{pmatrix} \delta\phi_0 & 0 \\ (1 + T_\varepsilon)\psi_0 f^* & 0 \end{pmatrix} & \text{if } s = 0, \\ \begin{pmatrix} \delta\phi_1 & 0 \\ 0 & (1 + T_\varepsilon)\psi_0 \end{pmatrix} & \text{if } s = 1, \\ \begin{pmatrix} \delta\phi_s & 0 \\ 0 & 0 \end{pmatrix} & \text{if } s \geq 2, \end{cases}$$

$$: \mathcal{C}(f)^r = D^r \oplus C^{r-1} \rightarrow \mathcal{C}(f)_{n-r+s} = D_{n-r+s} \oplus C_{n-r+s-1},$$

such that

$$\phi_0 : \mathcal{C}(f)^{n-*} \rightarrow D^{n-*} \xrightarrow[\simeq]{(\delta\phi, (1+T_\varepsilon)\psi)_0} \mathcal{C}(f).$$

The equivalence class $\gamma \in \widehat{Q}^0(\mathcal{C}(f)^{0-*}, \varepsilon)$ of the Spivak normal chain bundle is the image of $(\delta\phi, \psi) \in \mathcal{Q}_n^n(f, \varepsilon)$ under the composite

$$\mathcal{Q}_n^n(f, \varepsilon) \xrightarrow{J_f} \widehat{Q}^n(D, \varepsilon) \xrightarrow{((\delta\phi, (1+T_\varepsilon)\psi)_0^\varepsilon)^{-1}} \widehat{Q}^n(\mathcal{C}(f)^{n-*}, \varepsilon) \xrightarrow{s^{-n}_-} \widehat{Q}^0(\mathcal{C}(f)^{0-*}, \varepsilon).$$

□

Definition 43. (i) The *boundary* of an n -dimensional ε -symmetric structure $(\phi, \theta) \in \mathcal{Q}_n(B, \beta, \varepsilon)$ on a chain ε -bundle (B, β) over A is the ε -symmetric null-cobordant $(n-1)$ -dimensional ε -quadratic Poincaré complex over A :

$$\partial(\phi, \theta) = (C, \psi)$$

defined in Proposition 42(i) above, with $C = \mathcal{C}(\phi_0 : B^{n-*} \rightarrow B)_{*+1}$.

(ii) The *algebraic normal invariant* of an n -dimensional $(\varepsilon$ -symmetric, ε -quadratic) Poincaré pair over A $(f : C \rightarrow D, (\delta\phi, \psi) \in \mathcal{Q}_n^n(f, \varepsilon))$ is the class

$$(\phi, \theta) \in \mathcal{Q}_n(\mathcal{C}(f), \gamma, \varepsilon)$$

defined in Proposition 42(ii) above.

Proposition 44. Let (B, β) be a chain ε -bundle over A such that B is concentrated in degree k ,

$$B : \cdots \rightarrow 0 \rightarrow B_k \rightarrow 0 \rightarrow \cdots.$$

The boundary map $\partial : \mathcal{Q}_{2k}(B, \beta, \varepsilon) \rightarrow L_{2k-1}(A, \varepsilon)$ sends an ε -symmetric structure $(\phi, \theta) \in \mathcal{Q}_{2k}(B, \beta, \varepsilon)$ to the Witt class of the $(-1)^{k-1}$ - ε -quadratic formation

$$\partial(\phi, \theta) = \left(H_{(-1)^{k-1}\varepsilon}(B^k); B^k, \operatorname{im} \left(\begin{pmatrix} 1 - \beta\phi \\ \phi \end{pmatrix} : B^k \rightarrow B^k \oplus B_k \right) \right),$$

with

$$H_{(-1)^{k-1}\varepsilon}(B^k) = \left(B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right),$$

the hyperbolic $(-1)^{k-1}\varepsilon$ -quadratic form.

Proof. The chain ε -bundle (equivalence class)

$$\beta \in \widehat{Q}^0(B^{0-*}, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B_k), (-1)^k \varepsilon)$$

is represented by a $(-1)^k \varepsilon$ -symmetric form (B_k, β) . An ε -symmetric structure $(\phi, \theta) \in Q_{2k}(B, \beta, \varepsilon)$ is represented by an $(-1)^k \varepsilon$ -symmetric form (B^k, ϕ) together with $\theta \in S(B_k)$ such that

$$\phi - \phi\beta\phi = \theta + (-1)^k \varepsilon \theta^* \in H^0(\mathbb{Z}_2; S(B^k), (-1)^k \varepsilon).$$

The boundary of (ϕ, θ) is the ε -symmetric null-cobordant $(2k-1)$ -dimensional ε -quadratic Poincaré complex $\partial(\phi, \theta) = (C, \psi)$ concentrated in degrees $k-1, k$ corresponding to the formation in the statement. $\square$

Proposition 45. Let (B, β) be a chain ε -bundle over A such that B is concentrated in degrees $k, k+1$,

$$B : \cdots \rightarrow 0 \rightarrow B_{k+1} \xrightarrow{d} B_k \rightarrow 0 \rightarrow \cdots.$$

The boundary map $\partial : Q_{2k+1}(B, \beta, \varepsilon) \rightarrow L_{2k}(A, \varepsilon)$ sends an ε -symmetric structure $(\phi, \theta) \in Q_{2k+1}(B, \beta, \varepsilon)$ to the Witt class of the nonsingular $(-1)^k \varepsilon$ -quadratic form over A

$$\left(\text{coker} \left(\begin{pmatrix} -d^* \\ \phi_0^* \\ 1 - \beta_{-2k} d \phi_0^* \end{pmatrix} : B^k \rightarrow B^{k+1} \oplus B_{k+1} \oplus B^k \right), \begin{pmatrix} \theta_0 & 0 & \phi_0 \\ 1 & \beta_{-2k-2}^* & d^* \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Proof. This is an application of the instant surgery obstruction of [15, 4.3], which identifies the cobordism class $(C, \psi) \in L_{2k}(A, \varepsilon)$ of a $2k$ -dimensional ε -quadratic Poincaré complex (C, ψ) with the Witt class of the nonsingular ε -quadratic form

$$I(C, \psi) = \left(\text{coker} \left(\begin{pmatrix} d^* \\ (-1)^{k+1}(1 + T_\varepsilon)\psi_0 \end{pmatrix} : C^{k-1} \rightarrow C^k \oplus C_{k+1} \right), \begin{pmatrix} \psi_0 & d \\ 0 & 0 \end{pmatrix} \right).$$

By Proposition 36 the chain ε -bundle β can be taken to be the cone of a chain ε -bundle map

$$(d, \beta_{-2k-2}) : (B_{k+1}, 0) \rightarrow (B_k, \beta_{-2k}),$$

with

$$\begin{aligned}\beta_{-2k}^* &= (-1)^k \varepsilon \beta_{-2k} : B_k \rightarrow B^k, \\ d^* \beta_{-2k} d &= \beta_{-2k-2} + (-1)^k \varepsilon \beta_{-2k-2}^* : B_{k+1} \rightarrow B^{k+1}, \\ \beta_{-2k-1} &= \begin{cases} \beta_{-2k} d : B_{k+1} \rightarrow B^k, \\ 0 : B_k \rightarrow B^{k+1}. \end{cases}\end{aligned}$$

An ε -symmetric structure $(\phi, \theta) \in \mathcal{Q}_{2k+1}(B, \beta, \varepsilon)$ is represented by A -module morphisms

$$\begin{aligned}\phi_0 : B^k &\rightarrow B_{k+1}, \quad \tilde{\phi}_0 : B^{k+1} \rightarrow B_k, \quad \phi_1 : B^{k+1} \rightarrow B_{k+1}, \\ \theta_0 : B^{k+1} &\rightarrow B_{k+1}, \quad \theta_{-1} : B^k \rightarrow B_{k+1}, \quad \tilde{\theta}_{-1} : B^{k+1} \rightarrow B_k, \quad \theta_{-2} : B^k \rightarrow B_k\end{aligned}$$

such that

$$\begin{aligned}d\phi_0 + (-1)^k \tilde{\phi}_0 d^* &= 0 : B^k \rightarrow B_k, \\ \phi_0 - \varepsilon \tilde{\phi}_0^* + (-1)^{k+1} \phi_1 d^* &= 0 : B^k \rightarrow B_{k+1}, \\ \phi_1 + (-1)^{k+1} \varepsilon \phi_1^* &= 0 : B^{k+1} \rightarrow B_{k+1}, \\ \phi_0 - \phi_0 \beta_{-2k} d \tilde{\phi}_0^* &= (-1)^k \theta_0 d^* - \theta_{-1} - \varepsilon \tilde{\theta}_{-1}^* : B^k \rightarrow B_{k+1}, \\ \tilde{\phi}_0 &= d\theta_0 - \tilde{\theta}_{-1} - \varepsilon \theta_{-1}^* : B^{k+1} \rightarrow B_k, \\ -\tilde{\phi}_0 \beta_{-2k-2} \tilde{\phi}_0^* &= \theta_{-2} + (-1)^{k+1} \varepsilon \theta_{-2}^* : B^k \rightarrow B_k, \\ \phi_1 - \phi_0 \beta_{-2k} \phi_0^* &= \theta_0 + (-1)^k \varepsilon \theta_0^* : B^{k+1} \rightarrow B_{k+1}.\end{aligned}$$

The boundary of (ϕ, θ) given by 43(i) is an ε -symmetric null-cobordant $2k$ -dimensional ε -quadratic Poincaré complex $\partial(\phi, \theta) = (C, \psi)$ concentrated in degrees $k-1, k, k+1$, with $I(C, \psi)$ the instant surgery obstruction form (45) in the statement. $\square$

The ε -quadratic L -groups and the (B, β) -structure L -groups fit into an evident exact sequence

$$\cdots \rightarrow L_n(A, \varepsilon) \rightarrow L\langle B, \beta \rangle^n(A, \varepsilon) \rightarrow \widehat{L}\langle B, \beta \rangle^n(A, \varepsilon) \xrightarrow{\partial} L_{n-1}(A, \varepsilon) \rightarrow \cdots$$

and similarly for the 4-periodic versions

$$\cdots \rightarrow L_n(A, \varepsilon) \rightarrow L\langle B, \beta \rangle^{n+4*}(A, \varepsilon) \rightarrow \widehat{L}\langle B, \beta \rangle^{n+4*}(A, \varepsilon) \xrightarrow{\partial} L_{n-1}(A, \varepsilon) \rightarrow \cdots$$

Proposition 46 (Weiss [21]). (i) *The function*

$$Q_n(B, \beta, \varepsilon) \rightarrow \widehat{L}\langle B, \beta \rangle^{n+4*}(A, \varepsilon); (\phi, \theta) \mapsto (f : C \rightarrow D, (\delta\phi, \psi)) \quad (42(\text{ii}))$$

is an isomorphism, with inverse given by the algebraic normal invariant. The ε -quadratic L -groups of A , the 4-periodic (B, β) -structure ε -symmetric L -groups of A and the twisted ε -quadratic Q -groups of (B, β) are thus related by an exact sequence

$$\cdots \rightarrow L_n(A, \varepsilon) \xrightarrow{1+T} L\langle B, \beta \rangle^{n+4*}(A, \varepsilon) \rightarrow Q_n(B, \beta, \varepsilon) \xrightarrow{\partial} L_{n-1}(A, \varepsilon) \rightarrow \cdots$$

(ii) *The cobordism class of an n -dimensional (ε -symmetric, ε -quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ over A with a (B, β) -structure $(\gamma, \phi, \theta, g, \chi)$ is the image of the algebraic normal invariant $(\phi, \theta) \in Q_n(C(f), \gamma, \varepsilon)$*

$$(f : C \rightarrow D, (\delta\phi, \psi)) = (g, \chi) \% (\phi, \theta) \in Q_n(B, \beta, \varepsilon).$$

Proof. The ε -symmetrization of an n -dimensional ε -quadratic Poincaré complex (C, ψ) is an n -dimensional ε -symmetric Poincaré complex $(C, (1 + T_\varepsilon)\psi)$ with (B, β) -structure $(0, (1 + T)\psi, \theta, 0, 0)$ given by

$$\theta_s = \begin{cases} \psi_{-s-1} \in \text{Hom}_A(C^{-*}, C)_{n+s+1} & \text{if } s \leq -1, \\ 0 & \text{if } s \geq 0. \end{cases}$$

The relative groups of the symmetrization map

$$1 + T_\varepsilon : L_n(A, \varepsilon) \rightarrow L\langle B, \beta \rangle^n(A, \varepsilon); (C, \psi) \mapsto (C, (1 + T_\varepsilon)\psi)$$

are the cobordism groups of n -dimensional (ε -symmetric, ε -quadratic) Poincaré pairs $(f : C \rightarrow D, (\delta\phi, \psi))$ together with a (B, β) -structure $(\gamma, \phi, \theta, g, \chi)$. $\square$

Proposition 47. *Let (B, β) be a chain ε -bundle over A with B concentrated in degree k*

$$B : \cdots \rightarrow 0 \rightarrow B_k \rightarrow 0 \rightarrow \cdots$$

so that $\beta \in \widehat{Q}^0(B^{0-}, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon)$ is represented by an element*

$$\beta_{-2k} = (-1)^k \varepsilon \beta_{-2k}^* \in S(B^k).$$

The twisted ε -quadratic Q -groups $Q_n(B, \beta, \varepsilon)$ are given as follows:

(i) For $n \neq 2k - 1, 2k$:

$$\begin{aligned} Q_n(B, \beta, \varepsilon) &= Q_n(B, \varepsilon) \\ &= \begin{cases} \widehat{Q}^{n+1}(B, \varepsilon) = \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) & \text{if } n \geq 2k + 1, \\ 0 & \text{if } n \leq 2k - 2. \end{cases} \end{aligned}$$

(ii) For $n = 2k$:

$$Q_{2k}(B, \beta, \varepsilon) = \frac{\{(\phi, \theta) \in S(B^k) \oplus S(B^k) \mid \phi = (-1)^k \varepsilon \phi^*, \phi - \phi \beta_{-2k} \phi^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(0, \eta + (-1)^{k+1} \varepsilon \eta^*) \mid \eta \in S(B^k)\}},$$

with addition by

$$(\phi, \theta) + (\phi', \theta') = (\phi + \phi', \theta + \theta' + \phi' \beta_{-2k} \phi^*).$$

The boundary of $(\phi, \theta) \in Q_{2k}(B, \beta, \varepsilon)$ is the $(2k - 1)$ -dimensional ε -quadratic Poincaré complex over A concentrated in degrees $k - 1, k$ corresponding to the $(-1)^{k+1} \varepsilon$ -quadratic formation over A ,

$$\partial(\phi, \theta) = \left(H_{(-1)^{k+1} \varepsilon}(B^k); B^k, \operatorname{im} \left(\begin{pmatrix} 1 - \beta_{-2k} \phi \\ \phi \end{pmatrix} : B^k \rightarrow B^k \oplus B_k \right) \right).$$

(iii) For $n = 2k - 1$:

$$\begin{aligned} Q_{2k-1}(B, \beta, \varepsilon) &= \operatorname{coker}(J_\beta : Q^{2k}(B, \varepsilon) \rightarrow \widehat{Q}^{2k}(B, \varepsilon)) \\ &= \frac{\{\sigma \in S(B^k) \mid \sigma = (-1)^k \varepsilon \sigma^*\}}{\{\phi - \phi \beta_{-2k} \phi^* - (\theta + (-1)^k \varepsilon \theta^*) \mid \phi = (-1)^k \varepsilon \phi^*, \theta \in S(B^k)\}}. \end{aligned}$$

The boundary of $\sigma \in Q_{2k-1}(B, \beta, \varepsilon)$ is the $(2k - 2)$ -dimensional ε -quadratic Poincaré complex over A concentrated in degree $k - 1$ corresponding to the $(-1)^{k+1} \varepsilon$ -quadratic form over A ,

$$\partial(\sigma) = \left(B^k \oplus B_k, \begin{pmatrix} \sigma & 1 \\ 0 & \beta_{-2k} \end{pmatrix} \right),$$

with

$$(1 + T_{(-1)^{k+1} \varepsilon}) \partial(\sigma) = \left(B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} \varepsilon & 0 \end{pmatrix} \right).$$

(iv) *The maps in the exact sequence*

$$\begin{array}{ccccccc} 0 \rightarrow \widehat{Q}^{2k+1}(B, \varepsilon) & \xrightarrow{H_\beta} & Q_{2k}(B, \beta, \varepsilon) & \xrightarrow{N_\beta} & Q^{2k}(B, \varepsilon) & & \\ & & & \xrightarrow{J_\beta} & \widehat{Q}^{2k}(B, \varepsilon) & \xrightarrow{H_\beta} & Q_{2k-1}(B, \beta, \varepsilon) \rightarrow 0 \end{array}$$

are given by

$$\begin{aligned} H_\beta : \widehat{Q}^{2k+1}(B, \varepsilon) &= \widehat{H}^1(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \rightarrow Q_{2k}(B, \beta, \varepsilon); \quad \theta \mapsto (0, \theta), \\ N_\beta : Q_{2k}(B, \beta, \varepsilon) &\rightarrow Q^{2k}(B, \varepsilon) = H^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon); \quad (\phi, \theta) \mapsto \phi, \\ J_\beta : Q^{2k}(B, \varepsilon) &= H^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \rightarrow \widehat{Q}^{2k}(B, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon); \\ &\quad \phi \mapsto \phi - \phi\beta_{-2k}\phi^*, \\ H_\beta : \widehat{Q}^{2k}(B, \varepsilon) &= \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \rightarrow Q_{2k-1}(B, \beta, \varepsilon); \quad \sigma \mapsto \sigma. \end{aligned}$$

Example 48. Let (K, λ) be a nonsingular ε -symmetric form over A , which may be regarded as a zero-dimensional ε -symmetric Poincaré complex (D, ϕ) over A with

$$\phi_0 = \lambda : D^0 = K \rightarrow D_0 = K^*.$$

The composite

$$Q^0(D, \varepsilon) = H^0(\mathbb{Z}_2; S(K), \varepsilon) \xrightarrow{J} \widehat{Q}^0(D, \varepsilon) \xrightarrow{(\phi_0)^{-1}} \widehat{Q}^0(D^{0-*}, \varepsilon)$$

sends $\phi \in Q^0(D, \varepsilon)$ to the algebraic Spivak normal chain bundle

$$\gamma \in \widehat{Q}^0(D^{0-*}, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(K^*), \varepsilon),$$

with

$$\gamma_0 = \varepsilon\lambda^{-1} : D_0 = K^* \rightarrow D^0 = K.$$

By Proposition 47

$$Q_0(D, \gamma, \varepsilon) = \frac{\{(\kappa, \theta) \in S(K) \oplus S(K) \mid \kappa = \varepsilon\kappa^*, \kappa - \kappa\gamma_0\kappa^* = \theta + \varepsilon\theta^*\}}{\{(0, \eta - \varepsilon\eta^*) \mid \eta \in S(K)\}},$$

with addition by

$$(\kappa, \theta) + (\kappa', \theta') = (\kappa + \kappa', \theta + \theta' + \kappa' \gamma_0 \kappa^*).$$

The algebraic normal invariant of (D, ϕ) is given by

$$(\phi, 0) \in Q_0(D, \gamma, \varepsilon).$$

Example 49. Let A be a ring with even involution (4), and let C be concentrated in degree k with $C_k = A^r$. For odd $k = 2j + 1$,

$$\widehat{Q}^0(C^{0-*}) = 0$$

and there is only one chain ε -bundle $\gamma = 0$ over C , with

$$Q_n(C, \gamma) = Q_n(C) = \begin{cases} \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) & \text{if } n \geq 4j + 2, \ n \equiv 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

For even $k = 2j$,

$$\widehat{Q}^0(C^{0-*}) = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A),$$

a chain ε -bundle $\gamma \in \widehat{Q}^0(C^{0-*})$ is represented by a diagonal matrix

$$\gamma = X = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_r \end{pmatrix} \in \text{Sym}_r(A),$$

with $\bar{x}_i = x_i \in A$, and there is defined an exact sequence

$$\widehat{Q}^{4j+1}(C) = 0 \rightarrow Q_{4j}(C, \gamma) \rightarrow Q^{4j}(C) \xrightarrow{J_\gamma} \widehat{Q}^{4j}(C) \rightarrow Q_{4j-1}(C, \gamma) \rightarrow Q^{4j-1}(C) = 0,$$

with

$$J_\gamma : Q^{4j}(C) = \text{Sym}_r(A) \rightarrow \widehat{Q}^{4j}(C) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}; \quad M \mapsto M - MXM,$$

so that

$$Q_n(C, \gamma) = \begin{cases} \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) & \text{if } n \geq 4j + 1, \\ & \text{and } n \equiv 1 \pmod{2}, \\ \{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\} & \text{if } n = 4j, \\ M_r(A)/\{M - MXM - (N + N^t) \mid M \in \text{Sym}_r(A), N \in M_r(A)\} & \text{if } n = 4j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Proposition 38(ii) gives an exact sequence

$$0 \rightarrow \bigoplus_{i=1}^r Q_{4j}(B, x_i) \rightarrow Q_{4j}(C, \gamma) \rightarrow \bigoplus_{r(r-1)/2} A \rightarrow \bigoplus_{i=1}^r Q_{4j-1}(B, x_i) \rightarrow Q_{4j-1}(C, \gamma) \rightarrow 0$$

with B concentrated in degree $2j$ with $B_{2j} = A$.

2.4. The relative twisted quadratic Q -groups

Let $(f, \chi) : (C, \gamma) \rightarrow (D, \delta)$ be a map of chain ε -bundles, and let (ϕ, θ) be an n -dimensional ε -symmetric structure on (C, γ) , so that $\chi \in (\widehat{W}^{\%}C)_1$, $\phi \in (W^{\%}C)_n$ and $\theta \in (\widehat{W}^{\%}C)_{n+1}$. Composing the chain map $\phi_0 : C^{n-*} \rightarrow C$ with f , we get an induced map

$$(\widehat{f\phi_0})^{\%} : \widehat{W}^{\%}C^{n-*} \rightarrow \widehat{W}^{\%}D.$$

The morphisms of twisted quadratic Q -groups

$$(f, \chi)^{\%} : Q_n(C, \gamma, \varepsilon) \rightarrow Q_n(D, \delta, \varepsilon); \quad (\phi, \theta) \mapsto (f^{\%}(\phi), \widehat{f}^{\%}(\theta) + (\widehat{f\phi_0})^{\%}(S^n \chi))$$

are induced by a simplicial map of simplicial abelian groups. The relative homotopy groups are the *relative twisted ε -quadratic Q -groups* $Q_n(f, \chi, \varepsilon)$, designed to fit into a long exact sequence

$$\cdots \longrightarrow Q_n(C, \gamma, \varepsilon) \xrightarrow{(f, \chi)^{\%}} Q_n(D, \delta, \varepsilon) \longrightarrow Q_n(f, \chi, \varepsilon) \longrightarrow Q_{n-1}(C, \gamma, \varepsilon) \longrightarrow \cdots$$

Proposition 50. For any chain ε -bundle map $(f, \chi) : (C, \gamma) \rightarrow (D, \delta)$ the various Q -groups fit into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots \\
 \cdots \longrightarrow & \widehat{Q}^{n+1}(C, \epsilon) & \xrightarrow{H_\gamma} & Q_n(C, \gamma, \epsilon) & \xrightarrow{N_\gamma} & Q^n(C, \epsilon) & \xrightarrow{J_\gamma} & \widehat{Q}^n(C, \epsilon) \longrightarrow \cdots \\
 & \downarrow \widehat{f}^\% & & \downarrow (f, \chi)^\% & & \downarrow f^\% & & \downarrow \widehat{f}^\% \\
 \cdots \longrightarrow & \widehat{Q}^{n+1}(D, \epsilon) & \xrightarrow{H_\delta} & Q_n(D, \delta, \epsilon) & \xrightarrow{N_\delta} & Q^n(D, \epsilon) & \xrightarrow{J_\delta} & \widehat{Q}^n(D, \epsilon) \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots \longrightarrow & \widehat{Q}^{n+1}(f) & \xrightarrow{H_\chi} & Q_n(f, \chi, \epsilon) & \xrightarrow{N_\chi} & Q^n(f, \epsilon) & \xrightarrow{J_\chi} & \widehat{Q}^n(f, \epsilon) \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots \longrightarrow & \widehat{Q}^n(C, \epsilon) & \xrightarrow{H_\gamma} & Q_{n-1}(C, \gamma, \epsilon) & \xrightarrow{N_\delta} & Q^{n-1}(C, \epsilon) & \xrightarrow{J_\gamma} & \widehat{Q}^{n-1}(C, \epsilon) \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

Proof. These are the exact sequences of the homotopy groups of the simplicial abelian groups in the commutative diagram of fibration sequences

$$\begin{array}{ccccc}
 K(J_\gamma) & \longrightarrow & K(W^\% C) & \xrightarrow{J_\gamma} & K(\widehat{W}^\% C) \\
 \downarrow & & \downarrow f^\% & & \downarrow \widehat{f}^\% \\
 K(J_\delta) & \longrightarrow & K(W^\% D) & \xrightarrow{J_\delta} & K(\widehat{W}^\% D) \\
 \downarrow & & \downarrow & & \downarrow \\
 K(J_\chi) & \longrightarrow & K(W^\% \mathcal{C}(f)) & \xrightarrow{J_\chi} & K(\widehat{W}^\% \mathcal{C}(f))
 \end{array}$$

with

$$\pi_n(K(J_\chi)) = Q_n(f, \chi, \epsilon). \quad \square$$

There is also a twisted ε -quadratic Q -group version of the algebraic Thom constructions (12, 18, 24):

Proposition 51. Let $(f, \chi) : (C, 0) \rightarrow (D, \delta)$ be a chain ε -bundle map, and let $(B, \beta) = \mathcal{C}(f, \chi)$ be the cone chain ε -bundle (34). The relative twisted ε -quadratic Q -groups

$Q_*(f, \chi, \varepsilon)$ are related to the (absolute) twisted ε -quadratic Q -groups $Q_*(B, \beta, \varepsilon)$ by a commutative braid of exact sequences

$$\begin{array}{ccccc}
 \widehat{Q}^{n+1}(B, \varepsilon) & & Q_n(B, \beta, \varepsilon) & & H_{n-1}(B \otimes_A C) \\
 \searrow & \nearrow t & \searrow & \nearrow & \\
 Q_n(f, \chi, \varepsilon) & & Q^n(B, \varepsilon) & & \\
 \nearrow F & \searrow & \nearrow t & \searrow & \\
 H_n(B \otimes_A C) & & Q^n(f, \varepsilon) & & \widehat{Q}^n(B, \varepsilon) \\
 \searrow F & \nearrow & \searrow & \nearrow & \\
 & & & &
 \end{array}$$

involving the exact sequence of 18,

$$\cdots \rightarrow H_n(B \otimes_A C) \xrightarrow{F} Q^n(f, \varepsilon) \xrightarrow{t} Q^n(B, \varepsilon) \rightarrow H_{n-1}(B \otimes_A C) \rightarrow \cdots$$

Proof. The identity

$$\widehat{f}^{*\%}(\delta) = d\chi \in (\widehat{W}C^{0-*})_0$$

determines a homotopy $\rightsquigarrow$ in the square

$$\begin{array}{ccc}
 K(W^{\%}C) & \xrightarrow{J} & K(\widehat{W}^{\%}C) \\
 \downarrow f^{\%} & \rightsquigarrow & \downarrow \widehat{f}^{\%} \\
 K(W^{\%}D) & \xrightarrow{J_\delta} & K(\widehat{W}^{\%}D)
 \end{array}$$

(with $J = J_0$) and hence maps of the mapping fibres

$$J_\chi : K(\mathcal{C}(f^{\%})) \rightarrow K(\mathcal{C}(\widehat{f}^{\%})), \quad (f, \chi)^{\%} : K(J) \rightarrow K(J_\delta).$$

The map J_χ is related to $J_\beta : K(W^{\%}B) \rightarrow K(\widehat{W}^{\%}B)$ by a homotopy commutative diagram

$$\begin{array}{ccc}
 K(\mathcal{C}(f^{\%})) & \xrightarrow{J_\chi} & K(\mathcal{C}(\widehat{f}^{\%})) \\
 \downarrow t & \rightsquigarrow & \downarrow \widehat{t} \\
 K(W^{\%}B) & \xrightarrow{J_\beta} & K(\widehat{W}^{\%}B)
 \end{array}$$

with $\widehat{t} : K(\mathcal{C}(\widehat{f}^{\%})) \simeq K(\widehat{W}^{\%}B)$ a simplicial homotopy equivalence inducing the algebraic Thom isomorphisms $\widehat{t} : \widehat{Q}^*(f, \varepsilon) \cong \widehat{Q}^*(B, \varepsilon)$ of Proposition 12, and $t :$

$K(\mathcal{C}(f^\%)) \rightarrow K(W^\%B)$ a simplicial map inducing the algebraic Thom maps $t : Q^*(f, \varepsilon) \rightarrow Q^*(B, \varepsilon)$ of Proposition 18, with mapping fibre $K(t) \simeq K(B \otimes_A C)$. The braid in the statement is the commutative braid of homotopy groups induced by the homotopy commutative braid of fibrations

$$\begin{array}{ccccc}
 & & K(J_\beta) & & \\
 & \nearrow & & \searrow & \\
 & K(J_\chi) & \rightsquigarrow & K(W^\%B) & \\
 \nearrow F & & & & \searrow J_\beta \\
 K(B \otimes_A C) & & K(\mathcal{C}(f^\%)) & \xrightarrow{t} & K(\widehat{W}^\%B) \\
 \searrow F & & \nearrow \hat{t}J_\chi & & \\
 & & & &
 \end{array} \quad \square$$

Proposition 52. Let (C, γ) be a chain ε -bundle over a f.g. projective A -module chain complex which is concentrated in degrees $k, k+1$,

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \xrightarrow{d} C_k \rightarrow 0 \rightarrow \cdots,$$

so that (C, γ) can be taken (up to equivalence) to be the cone $\mathcal{C}(d, \chi)$ of a chain ε -bundle map $(d, \chi) : (C_{k+1}, 0) \rightarrow (C_k, \delta)$ (36), regarding C_k, C_{k+1} as chain complexes concentrated in degree k . The relative twisted ε -quadratic Q -groups $Q_*(d, \chi, \varepsilon)$ and the absolute twisted ε -quadratic Q -groups $Q_*(C, \gamma, \varepsilon)$ are given as follows:

(i) For $n \neq 2k-1, 2k, 2k+1, 2k+2$

$$Q_n(C, \gamma, \varepsilon) = Q_n(d, \chi, \varepsilon) = Q_n(C, \varepsilon) = \begin{cases} \widehat{Q}^{n+1}(C, \varepsilon) & \text{if } n \geq 2k+3, \\ 0 & \text{if } n \leq 2k-2, \end{cases}$$

with

$$\widehat{Q}^{n+1}(C, \varepsilon) = \frac{\{(\phi, \theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi = (-1)^{n+k} \varepsilon \phi^*, d\phi d^* = \theta + (-1)^{n+k} \varepsilon \theta^*\}}{\{(\sigma + (-1)^{n+k} \varepsilon \sigma^*, d\sigma d^* + \tau + (-1)^{n+k+1} \varepsilon \tau^* \mid (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}}$$

as given by Proposition 13.

(ii) For $n = 2k-1, 2k, 2k+1, 2k+2$ the relative twisted ε -quadratic Q -groups are given by

$$\begin{aligned}
 & Q_n(d, \chi, \varepsilon) \\
 &= \begin{cases} \frac{\{(\phi, \theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi = (-1)^k \varepsilon \phi^*, d\phi d^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(\sigma + (-1)^k \varepsilon \sigma^*, d\sigma d^* + \tau + (-1)^{k+1} \varepsilon \tau^* \mid (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+2, \\ \frac{\{(\psi, \eta) \in S(C^{k+1}) \oplus S(C^k) \mid (d, \chi)_\%(\psi) = (0, \eta + (-1)^{k+1} \varepsilon \eta^*)\}}{\{(\sigma + (-1)^{k+1} \varepsilon \sigma^*, d\sigma d^* + \tau + (-1)^k \varepsilon \tau^* \mid (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+1, \\ \text{coker}((d, \chi)_\% : Q_{2k}(C_{k+1}, \varepsilon) \rightarrow Q_{2k}(C_k, \delta, \varepsilon)) & \text{if } n = 2k, \\ Q_{2k-1}(C_k, \delta, \varepsilon) & \text{if } n = 2k-1, \end{cases}
 \end{aligned}$$

with

$$Q_{2k}(C_k, \delta, \varepsilon) = \frac{\{(\phi, \theta) \in S(C^k) \oplus S(C^k) \mid \phi = (-1)^k \varepsilon \phi^*, \phi - \phi \delta \phi^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(0, \eta + (-1)^{k+1} \varepsilon \eta^*) \mid \eta \in S(C^k)\}},$$

$$Q_{2k-1}(C_k, \delta, \varepsilon) = \frac{\{\sigma \in S(C^k) \mid \sigma = (-1)^k \varepsilon \sigma^*\}}{\{\phi - \phi \delta \phi^* - (\theta + (-1)^k \varepsilon \theta^*) \mid \phi = (-1)^k \varepsilon \phi^*, \theta \in S(C^k)\}},$$

$$(d, \chi)_\% : Q_{2k}(C_{k+1}, \varepsilon) = H_0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \rightarrow Q_{2k}(C_k, \delta, \varepsilon);$$

$$\psi \mapsto (d(\psi + (-1)^k \varepsilon \psi^*)d^*, d\psi d^* - d(\psi + (-1)^k \varepsilon \psi^*)\chi(\psi^* + (-1)^k \varepsilon \psi)d^*).$$

The absolute twisted quadratic Q -groups are such that

$$Q_{2k-1}(C, \gamma, \varepsilon) = Q_{2k-1}(d, \chi, \varepsilon) = Q_{2k-1}(C_k, \delta, \varepsilon)$$

and there is defined an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & Q_{2k+2}(d, \chi, \varepsilon) & \xrightarrow{t} & Q_{2k+2}(C, \gamma, \varepsilon) & & \\ & & & & & & \\ & \rightarrow & H_{k+1}(C) \otimes_A C_{k+1} & \xrightarrow{F} & Q_{2k+1}(d, \chi, \varepsilon) & \xrightarrow{t} & Q_{2k+1}(C, \gamma, \varepsilon) \\ & & & & & & \\ & \rightarrow & H_k(C) \otimes_A C_{k+1} & \xrightarrow{F} & Q_{2k}(d, \chi, \varepsilon) & \xrightarrow{t} & Q_{2k}(C, \gamma, \varepsilon) \rightarrow 0, \end{array}$$

with

$$\begin{aligned} F : H_k(C) \otimes_A C_{k+1} &= \text{coker}(d^* : \text{Hom}_A(C^{k+1}, C_{k+1}) \rightarrow \text{Hom}_A(C^k, C_{k+1})) \\ &\rightarrow Q_{2k}(d, \chi); \quad \lambda \mapsto (\lambda d^* + (-1)^k \varepsilon d \lambda^* - d \lambda^* \delta \lambda d^*, \\ &\lambda d^* - \lambda \chi \lambda^* - d \lambda^* \delta \lambda \chi \lambda^* \delta \lambda d^* - d \lambda^* \delta (\lambda d^* + (-1)^k \varepsilon d \lambda^*) \\ &\quad - (\lambda d^* + (-1)^k \varepsilon d \lambda^*) \delta d \lambda^* \delta \lambda d^*). \end{aligned}$$

Proof. The absolute and relative twisted ε -quadratic Q -groups are related by the exact sequence of 51

$$\begin{array}{ccccccc} \cdots & \rightarrow & Q_n(d, \chi, \varepsilon) & \xrightarrow{t} & Q_n(C, \gamma, \varepsilon) & \rightarrow & H_{n-k-1}(C) \otimes_A C_{k+1} \\ & & & & & & \\ & & & & & & \\ & \xrightarrow{F} & Q_{n-1}(d, \chi, \varepsilon) & \rightarrow & \cdots & . \end{array}$$

The twisted ε -quadratic Q -groups of $(C_{k+1}, 0)$ are given by Proposition 22

$$\begin{aligned} Q_n(C_{k+1}, 0, \varepsilon) &= Q_n(C_{k+1}, \varepsilon) = H_{n-2k}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\ &= \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & \text{if } n \geq 2k+1, \\ H_0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & \text{if } n = 2k, \\ 0 & \text{if } n \leq 2k-1. \end{cases} \end{aligned}$$

The twisted ε -quadratic Q -groups of (C_k, δ) are given by Proposition 47

$$Q_n(C_k, \delta, \varepsilon) = \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n \geq 2k+1, \\ \frac{\{(\phi, \theta) \in S(C^k) \oplus S(C^k) \mid \phi = (-1)^k \varepsilon \phi^*, \phi - \phi \delta \phi^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(0, \eta + (-1)^{k+1} \varepsilon \eta^*) \mid \eta \in S(C^k)\}} & \text{if } n = 2k, \\ \frac{\{\sigma \in S(C^k) \mid \sigma = (-1)^k \varepsilon \sigma^*\}}{\{\phi - \phi \delta \phi^* - (\theta + (-1)^k \varepsilon \theta^*) \mid \phi = (-1)^k \varepsilon \phi^*, \theta \in S(C^k)\}} & \text{if } n = 2k-1, \\ 0 & \text{if } n \leq 2k-2. \end{cases}$$

The twisted ε -quadratic Q -groups of (d, χ) fit into the exact sequence

$$\cdots \longrightarrow Q_n(C_{k+1}, \varepsilon) \xrightarrow{(d, \chi)_{\%}} Q_n(C_k, \delta, \varepsilon) \longrightarrow Q_n(d, \chi, \varepsilon) \longrightarrow Q_{n-1}(C_{k+1}, \varepsilon) \longrightarrow \cdots$$

giving the expressions in the statements of (i) and (ii). $\square$

2.5. The computation of $Q_*(C(X), \gamma(X))$

In this section, we compute the twisted quadratic Q -groups $Q_*(C(X), \gamma(X))$ of the following chain bundles over an even commutative ring A .

Definition 53. For $X \in \text{Sym}_r(A)$ let

$$(C(X), \gamma(X)) = \mathcal{C}(d, \chi)$$

be the cone of the chain bundle map over A ,

$$(d, \chi) : (C(X)_1, 0) \rightarrow (C(X)_0, \delta)$$

defined by

$$\begin{aligned} d &= 2 : C(X)_1 = A^r \rightarrow C(X)_0 = A^r, \\ \delta &= X : C(X)_0 = A^r \rightarrow C(X)^0 = A^r, \\ \chi &= 2X : C(X)_1 = A^r \rightarrow C(X)^1 = A^r. \end{aligned}$$

By Proposition 36 every chain bundle (C, γ) with $C_1 = A^r \xrightarrow{2} C_0 = A^r$ is of the form $(C(X), \gamma(X))$ for some $X = (x_{ij}) \in \text{Sym}_r(A)$, with the equivalence class given by

$$\gamma = \gamma(X) = X = (x_{11}, x_{22}, \dots, x_{rr})$$

$$\in \widehat{Q}^0(C(X)^{-*}) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \quad (14).$$

The 0th Wu class of $(C(X), \gamma(X))$ is the A -module morphism

$$\widehat{v}_0(\gamma(X)) : H_0(C(X)) = (A_2)^r \rightarrow \widehat{H}^0(\mathbb{Z}_2; A);$$

$$a = (a_1, a_2, \dots, a_r) \mapsto aXa^t = \sum_{i=1}^r a_i x_{ij} a_j = \sum_{i=1}^r (a_i)^2 x_{ii}.$$

In Theorem 60 below the universal chain bundle (B^A, β^A) of a commutative even ring A with $\widehat{H}^0(\mathbb{Z}_2; A)$ a f.g. free A_2 -module will be constructed from $(C(X), \gamma(X))$ for a diagonal $X \in \text{Sym}_r(A)$ with $\widehat{v}_0(\gamma(X))$ an isomorphism, and the twisted quadratic Q -groups $Q_*(B^A, \beta^A)$ will be computed using the following computation of $Q_*(C(X), \gamma(X))$ (which holds for arbitrary X).

Theorem 54. *Let A be an even commutative ring, and let $X \in \text{Sym}_r(A)$.*

(i) *The twisted quadratic Q -groups of $(C(X), \gamma(X))$ are given by*

$$Q_n(C(X), \gamma(X))$$

$$= \begin{cases} 0 & \text{if } n \leq -2, \\ \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + \{M - MXM \mid M \in \text{Sym}_r(A)\}} & \text{if } n = -1, \\ \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A) + \{2(N + N^t) - 4N^t X N \mid N \in M_r(A)\}} & \text{if } n = 0, \\ \frac{\{N \in M_r(A) \mid N + N^t - 2N^t X N \in 2\text{Quad}_r(A)\}}{2M_r(A)} \oplus \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} & \text{if } n = 1, \\ \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} & \text{if } n \geq 2. \end{cases}$$

(ii) The boundary maps $\partial : Q_n(C(X), \gamma(X)) \rightarrow L_{n-1}(A)$ are given by

$$\begin{aligned}\partial : Q_{-1}(C(X), \gamma(X)) &\rightarrow L_{-2}(A); \quad M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix} \right), \\ \partial : Q_0(C(X), \gamma(X)) &\rightarrow L_{-1}(A); \quad M \mapsto (H_-(A^r); A^r, \operatorname{im} \begin{pmatrix} 1 - XM \\ M \end{pmatrix} : A^r \rightarrow A^r \oplus (A^r)^*), \\ \partial : Q_1(C(X), \gamma(X)) &\rightarrow L_0(A); \quad (N, P) \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^t X N) & 1 - 2NX \\ 0 & -2X \end{pmatrix} \right).\end{aligned}$$

(iii) The twisted quadratic Q -groups of the chain bundles

$$(B(i), \beta(i)) = (C(X), \gamma(X))_{*+2i} \quad (i \in \mathbb{Z})$$

are just the twisted quadratic Q -groups of $(C(X), \gamma(X))$ with a dimension shift

$$Q_n(B(i), \beta(i)) = Q_{n-4i}(C(X), \gamma(X)).$$

Proof. (i) Proposition 52(i) and Example 14(ii) give

$$Q_n(C(X), \gamma(X)) = \begin{cases} 0 & \text{if } n \leq -2, \\ \widehat{Q}^{n+1}(C(X)) = \frac{\operatorname{Sym}_r(A)}{\operatorname{Quad}_r(A)} & \text{if } n \geq 3. \end{cases}$$

For $-1 \leq n \leq 2$ Examples 14, 20, 49 and Proposition 52(ii) show that the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} Q^2(d) & \xrightarrow{J_\chi} & \widehat{Q}^2(d) & \xrightarrow{H_\chi} & Q_1(d, \chi) & \xrightarrow{N_\chi} & Q^1(d) & \xrightarrow{J_\chi} & \widehat{Q}^1(d) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Q^1(C(X)_1) & \xrightarrow{J} & \widehat{Q}^1(C(X)_1) & \xrightarrow{H} & Q_0(C(X)_1) & \xrightarrow{1+T} & Q^0(C(X)_1) & \xrightarrow{J} & \widehat{Q}^0(C(X)_1) \\ \downarrow d^\% & & \downarrow \widehat{d}^\% & & \downarrow (d, \chi)^\% & & \downarrow d^\% & & \downarrow \widehat{d}^\% \\ Q^1(C(X)_0) & \xrightarrow{J_\delta} & \widehat{Q}^1(C(X)_0) & \xrightarrow{H_\delta} & Q_0(C(X)_0, \delta) & \xrightarrow{N_\delta} & Q^0(C(X)_0) & \xrightarrow{J_\delta} & \widehat{Q}^0(C(X)_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Q^1(d) & \xrightarrow{J_\chi} & \widehat{Q}^1(d) & \xrightarrow{H_\chi} & Q_0(d, \chi) & \xrightarrow{N_\chi} & Q^0(d) & \xrightarrow{J_\chi} & \widehat{Q}^0(d) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Q^0(C(X)_1) & \xrightarrow{J} & \widehat{Q}^0(C(X)_1) & \xrightarrow{H} & Q_{-1}(C(X)_1) & \xrightarrow{1+T} & Q^{-1}(C(X)_1) & \xrightarrow{J} & \widehat{Q}^{-1}(C(X)_1) \end{array}$$

is given by

$$\begin{array}{ccccccc}
 0 & \xrightarrow{\quad \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \quad} & \xrightarrow{1} & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} & \longrightarrow & 0 & \longrightarrow & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \\
 \downarrow & & & \downarrow 0 & & \downarrow & & \downarrow 1 \\
 0 & \longrightarrow & 0 & \longrightarrow & \text{Quad}_r(A) & \longrightarrow & \text{Sym}_r(A) & \longrightarrow & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \\
 \downarrow & & & \downarrow 4 & & \downarrow 4 & & \downarrow 0 \\
 0 & \longrightarrow & 0 & \longrightarrow & Q_0(C(X)_0, \delta) & \longrightarrow & \text{Sym}_r(A) & \xrightarrow{J_\delta} & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \\
 \downarrow & & & \downarrow & & \downarrow & & \downarrow 1 \\
 0 & \longrightarrow & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} & \xrightarrow{4} & Q_0(d, \chi) & \longrightarrow & \frac{\text{Sym}_r(A)}{4\text{Sym}_r(A)} & \longrightarrow & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \\
 \downarrow & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow \\
 \text{Sym}_r(A) & \longrightarrow & \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

with

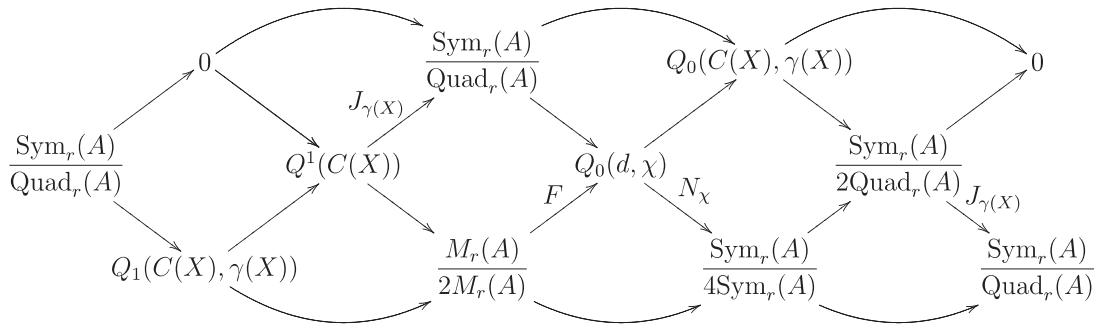
$$\begin{aligned}
 J_\delta : \text{Sym}_r(A) &\rightarrow \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}; \quad M \mapsto M - MXM, \\
 Q_0(C(X)_0, \delta) &= \ker(J_\delta) = \{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}, \\
 Q_0(d, \chi) &= \text{coker}((d, \chi)_\% : Q_0(C(X)_1) \rightarrow Q_0(C(X)_0, \delta)) \\
 &= \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A)}, \\
 N_\chi : Q_0(d, \chi) &\rightarrow Q^0(d) = \frac{\text{Sym}_r(A)}{4\text{Sym}_r(A)}; \quad M \mapsto M.
 \end{aligned}$$

Furthermore, the commutative braid of exact sequences

$$\begin{array}{ccccccc}
 & & Q^1(d) & & \hat{Q}^1(C(X)) & & Q_0(C(X), \gamma(X)) & & H_{-1}(C(X) \otimes_A C(X)_1) \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 Q_1(d, \chi) & & Q^1(C(X)) & & Q_0(d, \chi) & & Q^0(C(X)) & & \hat{Q}^0(C(X)) \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & Q_1(C(X), \gamma(X)) & & H_0(C(X) \otimes_A C(X)_1) & & Q^0(d) & &
 \end{array}$$

$\xrightarrow{J_{\gamma(X)}} \quad \xrightarrow{F} \quad \xrightarrow{J_{\gamma(X)}} \quad \xrightarrow{H_{-1}(C(X) \otimes_A C(X)_1)}$

is given by



with

$$\begin{aligned} \frac{\text{Sym}_r(A)}{2\text{Quad}_r(A)} &\cong Q^0(C(X)); \quad M \mapsto \phi \quad (\text{where } \phi_0 = M : C^0 \rightarrow C(X)_0), \\ J_{\gamma(X)} : \frac{\text{Sym}_r(A)}{2\text{Quad}_r(A)} &\rightarrow \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}; \quad M \mapsto M - MXM, \\ F : H_0(C(X) \otimes_A C(X)_1) &= \frac{M_r(A)}{2M_r(A)} \rightarrow Q_0(d, \chi); \quad N \mapsto 2(N + N^t) - 4N^t XN, \\ Q^1(C(X)) &= \ker(N_\chi F : H_0(C(X) \otimes_A C(X)_1) \rightarrow Q^0(d)) \\ &= \frac{\{N \in M_r(A) \mid N + N^t \in 2\text{Sym}_r(A)\}}{2M_r(A)} \\ &\quad (\text{where } \phi \in Q^1(C(X)) \text{ corresponds to } N = \phi_0 \in M_r(A)), \\ J_{\gamma(X)} : Q^1(C(X)) &\rightarrow \widehat{Q}^1(C(X)) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}; \quad N \mapsto \frac{1}{2}(N + N^t) - N^t XN. \end{aligned}$$

It follows that

$$\begin{aligned} Q_0(C(X), \gamma(X)) &= \text{coker} \left(F : \frac{M_r(A)}{2M_r(A)} \rightarrow Q_0(d, \chi) \right) \\ &= \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A) + \{2(N + N^t) - 4N^t XN \mid N \in M_r(A)\}}, \\ Q_{-1}(C(X), \gamma(X)) &= Q_{-1}(d, \chi) \\ &= \text{coker}(J_{\gamma(X)} : Q^0(C(X)) \rightarrow \widehat{Q}^0(C(X))) \\ &= \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + \{M - MXM \mid M \in \text{Sym}_r(A)\}}, \end{aligned}$$

with

$$\begin{aligned}\widehat{\mathcal{Q}}^1(C(X)) &= \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \rightarrow \mathcal{Q}_0(C(X), \gamma(X)); \quad M \mapsto 4M, \\ \widehat{\mathcal{Q}}^0(C(X)) &= \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \rightarrow \mathcal{Q}_{-1}(C(X), \gamma(X)); \quad M \mapsto M.\end{aligned}$$

Also

$$\begin{aligned}(d, \chi)_{\%} = 0 : \mathcal{Q}_2(d, \chi) &= \mathcal{Q}_1(C(X)_1) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \\ &\rightarrow \mathcal{Q}_1(C(X)_0, \delta) = \widehat{\mathcal{Q}}^2(C(X)_0) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}, \\ \mathcal{Q}^1(C(X)) &= \ker(N_{\chi}F : H_0(C(X) \otimes_A C(X)_1) \rightarrow \mathcal{Q}^0(d)) \\ &= \frac{\{N \in M_r(A) \mid N + N^t \in 2\text{Sym}_r(A)\}}{2M_r(A)}, \\ J_{\gamma(X)} : \mathcal{Q}^1(C(X)) &\rightarrow \widehat{\mathcal{Q}}^1(C(X)) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}; \quad N \mapsto \frac{1}{2}(N + N^t) - N^t X N, \\ \mathcal{Q}_2(C(X), \gamma(X)) &= \mathcal{Q}_2(d, \chi) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}.\end{aligned}$$

From the definition, an element $(\phi, \theta) \in \mathcal{Q}_1(C(X), \gamma(X))$ is represented by a collection of A -module morphisms

$$\begin{aligned}\phi_0 : C(X)^0 &\rightarrow C(X)_1, \quad \widetilde{\phi}_0 : C(X)^1 \rightarrow C(X)_0, \quad \phi_1 : C(X)^1 \rightarrow C(X)_0, \\ \theta_0 : C(X)^1 &\rightarrow C(X)_1, \quad \theta_{-1} : C(X)^0 \rightarrow C(X)_1, \quad \widetilde{\theta}_{-1} : C(X)^1 \rightarrow C(X)_0, \\ \theta_{-2} : C(X)^0 &\rightarrow C(X)_0\end{aligned}$$

such that

$$\begin{aligned}d\phi_0 + \widetilde{\phi}_0 d^* &= 0 : C^0 \rightarrow C_1, \\ \phi_0 - \widetilde{\phi}_0^* + \phi_1 d^* &= 0 : C(X)^0 \rightarrow C(X)_1, \\ \widetilde{\phi}_0 - \phi_0^* - d\phi_1 &= 0 : C(X)^1 \rightarrow C(X)_0, \\ \phi_1 - \phi_1^* &= 0 : C(X)^1 \rightarrow C(X)_1, \\ \phi_0 - \phi_0 \gamma(X)_{-1} \widetilde{\phi}_0^* &= -\theta_0 d^* - \theta_{-1} - \widetilde{\theta}_{-1}^* : C(X)^0 \rightarrow C(X)_1, \\ \widetilde{\phi}_0 - \widetilde{\phi}_0 \widetilde{\gamma}(X)_{-1} \phi_0^* &= d\theta_0 - \theta_{-1}^* - \widetilde{\theta}_{-1} : C(X)^1 \rightarrow C(X)_0, \\ \phi_1 - \phi_0 \gamma(X)_0 \phi_0^* &= \theta_0 + \theta_0^* : C(X)^1 \rightarrow C(X)_1, \\ -\widetilde{\phi}_0 \gamma(X)_{-2} \widetilde{\phi}_0^* &= d\theta_{-1} + \widetilde{\theta}_{-1} d^* + \theta_{-2} - \theta_{-2}^* : C(X)^0 \rightarrow C(X)_0,\end{aligned}$$

where

$$\gamma(X)_0 = X, \quad \gamma(X)_{-1} = 0, \quad \tilde{\gamma}(X)_{-1} = -2X, \quad \gamma(X)_{-2} = 0.$$

The maps in the exact sequence

$$\begin{aligned} 0 \rightarrow \widehat{Q}^2(C(X)) &= \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)} \rightarrow Q_1(C(X), \gamma(X)) \\ &\rightarrow \ker(J_{\gamma(X)} : Q^1(C(X)) \rightarrow \widehat{Q}^1(C(X))) \\ &= \frac{\{N \in M_r(A) \mid N + N^t - 2N^t X N \in 2\text{Quad}_r(A)\}}{2M_r(A)} \rightarrow 0 \end{aligned}$$

are defined by

$$\begin{aligned} Q_1(C(X), \gamma(X)) &\rightarrow \ker(J_{\gamma(X)}); \quad (\phi, \theta) \mapsto N = \phi_0, \\ \widehat{Q}^2(C(X)) &\rightarrow Q_1(C(X), \gamma(X)); \quad \theta_{-2} \mapsto (0, \theta) \quad (\theta_0 = 0, \quad \theta_{-1} = 0, \quad \tilde{\theta}_{-1} = 0), \end{aligned}$$

with $\widehat{Q}^2(C(X)) \rightarrow Q_1(C(X), \gamma(X))$ split by

$$Q_1(C(X), \gamma(X)) \rightarrow \widehat{Q}^2(C(X)); \quad (\phi, \theta) \mapsto \theta_{-2}.$$

(ii) The expressions for $\partial : Q_n(C(X), \gamma(X)) \rightarrow L_{n-1}(A)$ are given by the boundary construction of Proposition 43 and its expression in terms of forms and formations (44, 45). The form in the case $n = -1$ (resp. the formation in the case $n = 0$) is given by 45 (resp. 44) applied to the n -dimensional symmetric structure $(\phi, \theta) \in Q_n(C(X), \gamma(X))$ corresponding to $M \in \text{Sym}_r(A)$. For $n = 1$ the boundary of the one-dimensional symmetric structure $(\phi, \theta) \in Q_1(C(X), \gamma(X))$ corresponding to $N \in M_r(A)$ with

$$N + N^t \in 2\text{Sym}_r(A), \quad \frac{1}{2}(N + N^t) - N^t X N \in \text{Quad}_r(A)$$

is a zero-dimensional quadratic Poincaré complex (C, ψ) with

$$C = \mathcal{C}(N : C(X)^{1-*} \rightarrow C(X))_{*+1}.$$

The instant surgery obstruction (45) is the nonsingular quadratic form

$$I(C, \psi) = \left(\text{coker} \left(\begin{pmatrix} -2 \\ N^t \\ 1 + 2XN^t \end{pmatrix} : A^r \rightarrow A^r \oplus A^r \oplus A^r \right), \begin{pmatrix} \frac{1}{4}(N + N^t - 2NXN^t) & 1 & N \\ 0 & -2X & 2 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

such that there is defined an isomorphism

$$\begin{pmatrix} 1 & -4X & 2 \\ N^t & 1 - 2N^tX & N^t \end{pmatrix} : I(C, \psi) \rightarrow \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^tXN) & 1 - 2NX \\ 0 & -2X \end{pmatrix} \right).$$

(iii) *The even multiple skew-suspension isomorphisms of the symmetric Q -groups*

$$\bar{S}^{2i} : Q^{n-4i}(C(X)_{*+2i}) \xrightarrow{\cong} Q^n(C(X)); \{\phi_s \mid s \geq 0\} \mapsto \{\phi_s \mid s \geq 0\} \quad (i \in \mathbb{Z})$$

are defined also for the hyperquadratic, quadratic and twisted quadratic Q -groups. $\square$

2.6. The universal chain bundle

For any A -module chain complexes B, C the additive group $H_0(\text{Hom}_A(C, B))$ consists of the chain homotopy classes of A -module chain maps $f : C \rightarrow B$. For a chain ε -bundle (B, β) there is thus defined a morphism

$$H_0(\text{Hom}_A(C, B)) \rightarrow \widehat{Q}^0(C^{0-*}, \varepsilon); (f : C \rightarrow B) \mapsto \widehat{f}^*(\beta).$$

Proposition 55 (Weiss [21]). (i) *For every ring with involution A and $\varepsilon = \pm 1$ there exists a universal chain ε -bundle $(B^{A, \varepsilon}, \beta^{A, \varepsilon})$ over A such that for any finite f.g. projective A -module chain complex C the morphism*

$$H_0(\text{Hom}_A(C, B^{A, \varepsilon})) \rightarrow \widehat{Q}^0(C^{0-*}, \varepsilon); (f : C \rightarrow B^{A, \varepsilon}) \mapsto \widehat{f}^*(\beta^{A, \varepsilon})$$

is an isomorphism. Thus every chain ε -bundle (C, γ) is classified by a chain ε -bundle map

$$(f, \chi) : (C, \gamma) \rightarrow (B^{A, \varepsilon}, \beta^{A, \varepsilon}).$$

(ii) The universal chain ε -bundle $(B^{A,\varepsilon}, \beta^{A,\varepsilon})$ is characterized (uniquely up to equivalence) by the property that its Wu classes are A -module isomorphisms

$$\widehat{v}_k(\beta^{A,\varepsilon}) : H_k(B^{A,\varepsilon}) \xrightarrow{\cong} \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon) \quad (k \in \mathbb{Z}).$$

(iii) An n -dimensional (ε -symmetric, ε -quadratic) Poincaré pair over A has a canonical universal ε -bundle $(B^{A,\varepsilon}, \beta^{A,\varepsilon})$ -structure.

(iv) The 4-periodic $(B^{A,\varepsilon}, \beta^{A,\varepsilon})$ -structure L -groups are the 4-periodic versions of the ε -symmetric and ε -hyperquadratic L -groups of A :

$$L\langle B^{A,\varepsilon}, \beta^{A,\varepsilon} \rangle^{n+4*}(A, \varepsilon) = L^{n+4*}(A, \varepsilon),$$

$$\widehat{L}\langle B^{A,\varepsilon}, \beta^{A,\varepsilon} \rangle^{n+4*}(A, \varepsilon) = \widehat{L}^{n+4*}(A, \varepsilon).$$

(v) The twisted ε -quadratic Q -groups of $(B^{A,\varepsilon}, \beta^{A,\varepsilon})$ fit into an exact sequence

$$\cdots \rightarrow L_n(A, \varepsilon) \xrightarrow{1+T_\varepsilon} L^{n+4*}(A, \varepsilon) \rightarrow Q_n(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \xrightarrow{\partial} L_{n-1}(A, \varepsilon) \rightarrow \cdots,$$

with

$$\partial : Q_n(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \rightarrow L_{n-1}(A, \varepsilon); (\phi, \theta) \mapsto (C, \psi)$$

given by the construction of Proposition 42(ii), with

$$C = C(\phi_0 : (B^{A,\varepsilon})^{n-*} \rightarrow B^{A,\varepsilon})_{*+1} \text{ etc.}$$

For $\varepsilon = 1$ write

$$(B^{A,1}, \beta^{A,1}) = (B^A, \beta^A)$$

and note that

$$(B^{A,-1}, \beta^{A,-1}) = (B^A, \beta^A)_{*-1}.$$

In general, the chain A -modules $B^{A,\varepsilon}$ are not finitely generated, although $B^{A,\varepsilon}$ is a direct limit of f.g. free A -module chain complexes. In our applications the involution on A will satisfy the following conditions:

Proposition 56 (Connolly and Ranicki [10, Section 2.6]). *Let A be a ring with an even involution such that $\widehat{H}^0(\mathbb{Z}_2; A)$ has a one-dimensional f.g. projective A -module resolution*

$$0 \rightarrow C_1 \xrightarrow{d} C_0 \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow 0.$$

Let $(C, \gamma) = \mathcal{C}(d, \chi)$ be the cone of a chain bundle map $(d, \chi) : (C_1, 0) \rightarrow (C_0, \delta)$ with

$$\widehat{v}_0(\delta) = x : C_0 \rightarrow \widehat{H}^0(\mathbb{Z}_2; A)$$

and set

$$(B^A(i), \beta^A(i)) = (C, \gamma)_{*+2i} \quad (i \in \mathbb{Z}).$$

(i) *The chain bundle over A*

$$(B^A, \beta^A) = \bigoplus_i (B^A(i), \beta^A(i))$$

is universal.

(ii) *The twisted quadratic Q -groups of (B^A, β^A) are given by*

$$Q_n(B^A, \beta^A) = \begin{cases} Q_0(C, \gamma) & \text{if } n \equiv 0 \pmod{4}, \\ \ker(J_\gamma : Q^1(C) \rightarrow \widehat{Q}^1(C)) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ Q_{-1}(C, \gamma) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The inclusion $(B^A(2j), \beta^A(2j)) \rightarrow (B^A, \beta^A)$ is a chain bundle map which induces isomorphisms

$$Q_n(B^A, \beta^A) \cong \begin{cases} Q_n(B^A(2j), \beta^A(2j)) & \text{if } n = 4j, 4j - 1, \\ \ker(J_{\beta^A(2j)} : Q^n(B^A(2j)) \rightarrow \widehat{Q}^n(B^A(2j))) & \text{if } n = 4j + 1. \end{cases}$$

Proof. (i) The Wu classes of the chain bundle $(C, \gamma)_{*+2i}$ are isomorphisms

$$\widehat{v}_k(\gamma) : H_k(C_{*+2i}) \xrightarrow{\cong} \widehat{H}^k(\mathbb{Z}_2; A)$$

for $k = 2i, 2i + 1$.

(ii) See [10] for the detailed analysis of the exact sequence of 38(ii)

$$\begin{aligned} \cdots \rightarrow \sum_{i=-\infty}^{\infty} Q_n(B^A(i), \beta^A(i)) \rightarrow Q_n(B^A, \beta^A) \rightarrow \sum_{i < j} H_n(B^A(i) \otimes_A B^A(j)) \\ \rightarrow \sum_{i=-\infty}^{\infty} Q_{n-1}(B^A(i), \beta^A(i)) \rightarrow \cdots \end{aligned}$$

□

As in the introduction:

Definition 57. A ring with involution A is r -even for some $r \geq 1$ if

- (i) A is commutative, with the identity involution,
- (ii) $2 \in A$ is a nonzero divisor,
- (iii) $\widehat{H}^0(\mathbb{Z}_2; A)$ is a f.g. free A_2 -module of rank r with a basis $\{x_1 = 1, x_2, \dots, x_r\}$.

Example 58. $\mathbb{Z}$ is 1-even.

Proposition 59. If A is 1-even the polynomial extension $A[x]$ is 2-even, with $A[x]_2 = A_2[x]$ and $\{1, x\}$ an $A_2[x]$ -module basis of $\widehat{H}^0(\mathbb{Z}_2; A[x])$.

Proof. For any $a = \sum_{i=0}^{\infty} a_i x^i \in A[x]$:

$$\begin{aligned} a^2 &= \sum_{i=0}^{\infty} (a_i)^2 x^{2i} + 2 \sum_{0 \leq i < j < \infty} a_i a_j x^{i+j} \\ &= \sum_{i=0}^{\infty} a_i x^{2i} \in A_2[x]. \end{aligned}$$

The $A_2[x]$ -module morphism

$$A_2[x] \oplus A_2[x] \rightarrow \widehat{H}^0(\mathbb{Z}_2; A[x]); \quad (p, q) \mapsto p^2 + q^2 x$$

is thus an isomorphism, with inverse

$$\widehat{H}^0(\mathbb{Z}_2; A[x]) \xrightarrow{\cong} A_2[x] \oplus A_2[x]; \quad a = \sum_{i=0}^{\infty} a_i x^i \mapsto \left(\sum_{j=0}^{\infty} a_{2j} x^j, \sum_{j=0}^{\infty} a_{2j+1} x^j \right). \quad \square$$

Proposition 59 is the special case $k = 1$ of a general result: if A is 1-even and $t_1, t_2, \dots, t_k$ are commuting indeterminates over A then the polynomial ring

$A[t_1, t_2, \dots, t_k]$ is 2^k -even with

$$\{x_1 = 1, x_2, x_3, \dots, x_{2^k}\} = \{(t_1)^{i_1} (t_2)^{i_2} \cdots (t_k)^{i_k} \mid i_j = 0 \text{ or } 1, 1 \leq j \leq k\}$$

an $A_2[t_1, t_2, \dots, t_k]$ -module basis of $\widehat{H}^0(\mathbb{Z}_2; A[t_1, t_2, \dots, t_k])$.

We can now prove Theorem C.

Theorem 60. *Let A be an r -even ring with involution.*

(i) *The A -module morphism*

$$x : A^r \rightarrow \widehat{H}^0(\mathbb{Z}_2; A); (a_1, a_2, \dots, a_r) \mapsto \sum_{i=1}^r (a_i)^2 x_i$$

fits into a one-dimensional f.g. free A -module resolution of $\widehat{H}^0(\mathbb{Z}_2; A)$,

$$0 \rightarrow C_1 = A^r \xrightarrow{2} C_0 = A^r \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \rightarrow 0.$$

The symmetric and hyperquadratic L -groups of A are 4-periodic

$$L^n(A) = L^{n+4}(A), \quad \widehat{L}^n(A) = \widehat{L}^{n+4}(A).$$

(ii) *Let $(C(X), \gamma(X))$ be the chain bundle over A given by the construction of (53) for*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_r \end{pmatrix} \in \text{Sym}_r(A),$$

with $C(X) = \mathcal{C}(2 : A^r \rightarrow A^r)$. The chain bundle over A defined by

$$(B^A, \beta^A) = \bigoplus_i (C(X), \gamma(X))_{*+2i} = \bigoplus_i (B^A(i), \beta^A(i))$$

is universal. The hyperquadratic L -groups of A are given by

$$\widehat{L}^n(A) = Q_n(B^A, \beta^A)$$

$$= \begin{cases} Q_0(C(X), \gamma(X)) = \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A) + \{2(N + N^t) - N^t X N \mid N \in M_r(A)\}} & \text{if } n = 0, \\ \text{im}(N_{\gamma(X)} : Q_1(C(X), \gamma(X)) \rightarrow Q^1(C(X))) = \ker(J_{\gamma(X)} : Q^1(C(X)) \rightarrow \widehat{Q}^1(C(X))) \\ = \frac{\{N \in M_r(A) \mid N + N^t \in 2\text{Sym}_r(A), \frac{1}{2}(N + N^t) - N^t X N \in \text{Quad}_r(A)\}}{2M_r(A)} & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ Q_{-1}(C(X), \gamma(X)) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + \{L - LXL \mid L \in \text{Sym}_r(A)\}} & \text{if } n = 3, \end{cases}$$

with

$$\begin{aligned} \partial : \widehat{L}^0(A) &\rightarrow L_{-1}(A); \quad M \mapsto \left(H_{-1}(A^r); A^r, \text{im} \left(\begin{pmatrix} 1 - XM \\ M \end{pmatrix} : A^r \rightarrow A^r \oplus (A^r)^* \right) \right), \\ \partial : \widehat{L}^1(A) &\rightarrow L_0(A); \quad N \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^t X N) & 1 - 2NX \\ 0 & -2X \end{pmatrix} \right), \\ \partial : \widehat{L}^3(A) &\rightarrow L_2(A); \quad M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix} \right). \end{aligned}$$

Proof. Combine Proposition 30, Theorem 54 and Proposition 56, noting that the direct summand

$$\widehat{Q}^2(C(X)) = \text{Sym}_r(A)/\text{Quad}_r(A) \subseteq Q_1(C(X), \gamma(X))$$

is precisely the image of $H_2(C(X) \otimes C(X)_{*+2}) = \widehat{Q}^2(C(X))$ under the first map in the exact sequence

$$\begin{aligned} H_2(C(X) \otimes_A C(X)_{*+2}) &\rightarrow Q_1(C(X), \gamma(X)) \oplus Q_1(C(X)_{*+2}, \gamma(X)_{*+2}) \\ &\rightarrow Q_1(C(X) \oplus C(X)_{*+2}, \gamma(X) \oplus \gamma(X)_{*+2}) \rightarrow H_1(C(X) \otimes_A C(X)_{*+2}) = 0 \end{aligned}$$

of Proposition 38(ii), with $Q_1(C(X)_{*+2}, \gamma(X)_{*+2}) = 0$, so that

$$Q_1(C(X) \oplus C(X)_{*+2}, \gamma(X) \oplus \gamma(X)_{*+2}) = \ker(J_{\gamma(X)} : Q^1(C(X)) \rightarrow \widehat{Q}^1(C(X))). \quad \square$$

We can now prove Theorem A.

Corollary 61. Let A be a 1-even ring with $\psi^2 = 1$.

(i) The universal chain bundle (B^A, β^A) over A is given by

$$B^A : \dots \longrightarrow B_{2k+2}^A = A \xrightarrow{0} B_{2k+1}^A = A \xrightarrow{2} B_{2k}^A = A \xrightarrow{0} B_{2k-1}^A = A \longrightarrow \dots,$$

$$(\beta^A)_{-4k} = 1 : B_{2k}^A = A \rightarrow (B^A)^{2k} = A \quad (k \in \mathbb{Z}).$$

(ii) The hyperquadratic L -groups of A are given by

$$\widehat{L}^n(A) = Q_n(B^A, \beta^A) = \begin{cases} A_8 & \text{if } n \equiv 0 \pmod{4}, \\ A_2 & \text{if } n \equiv 1, 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \end{cases}$$

with

$$\begin{aligned} \partial : \widehat{L}^0(A) = A_8 &\rightarrow L_{-1}(A); \quad a \mapsto \left(H_{-1}(A); A, \operatorname{im} \left(\begin{pmatrix} 1 & -a \\ & a \end{pmatrix} : A \rightarrow A \oplus A \right) \right), \\ \partial : \widehat{L}^1(A) = A_2 &\rightarrow L_0(A); \quad a \mapsto \left(A \oplus A, \begin{pmatrix} a(1-a)/2 & 1-2a \\ 0 & -2 \end{pmatrix} \right), \\ \partial : \widehat{L}^3(A) = A_2 &\rightarrow L_2(A); \quad a \mapsto \left(A \oplus A, \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

(iii) The map $L^0(A) \rightarrow \widehat{L}^0(A)$ sends the Witt class $(K, \lambda) \in L^0(A)$ of a nonsingular symmetric form (K, λ) over A to

$$[K, \lambda] = \lambda(v, v) \in \widehat{L}^0(A) = A_8$$

for any $v \in K$ such that

$$\lambda(x, x) = \lambda(x, v) \in A_2 \quad (x \in K).$$

Proof. (i)+(ii) The A -module morphism

$$\widehat{v}_0(\beta^A) : H_0(B^A) = A_2 \rightarrow \widehat{H}^0(\mathbb{Z}_2; A); \quad a \mapsto a^2$$

is an isomorphism. Apply Theorem 60 with $r = 1$, $x_1 = 1$.

(ii) The computation of $\widehat{L}^*(A) = Q_*(B^A, \beta^A)$ is given by Theorem 60, using the fact that $a - a^2 \in 2A$ ($a \in A$) for a 1-even A with $\psi^2 = 1$. The explicit descriptions of ∂ are special cases of the formulae in Theorem 54(ii).

(iii) As in Example 48 regard (K, λ) as a zero-dimensional symmetric Poincaré complex (D, ϕ) with

$$\phi_0 = \varepsilon \lambda^{-1} : D^0 = K \rightarrow D^0 = K^*.$$

The Spivak normal chain bundle $\gamma = \lambda^{-1} \in \widehat{Q}^0(D^{0-*})$ is classified by the chain bundle map $(v, 0) : (D, \gamma) \rightarrow (B^A, \beta^A)$ with

$$g : D_0 = K^* \rightarrow \widehat{H}^0(\mathbb{Z}_2; A); \quad x \mapsto \lambda^{-1}(x, x) = x(v).$$

The algebraic normal invariant $(\phi, 0) \in Q_0(D, \gamma)$ has image

$$g_*(\phi, 0) = \lambda(v, v) \in Q_0(B^A, \beta^A) = A_8. \quad \square$$

Example 62. For $R = \mathbb{Z}$,

$$\widehat{L}^n(\mathbb{Z}) = Q_n(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) = \begin{cases} \mathbb{Z}_8 & \text{if } n \equiv 0 \pmod{4}, \\ \mathbb{Z}_2 & \text{if } n \equiv 1, 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

as recalled (from [15]) in the Introduction.

3. The generalized Arf invariant for forms

A nonsingular ε -quadratic form (K, ψ) over A corresponds to a zero-dimensional ε -quadratic Poincaré complex over A . The zero-dimensional ε -quadratic L -group $L_0(A, \varepsilon)$ is the Witt group of nonsingular ε -quadratic forms, and similarly for $L^0(A, \varepsilon)$ and ε -symmetric forms. In this section we define the ‘generalized Arf invariant’

$$(K, \psi; L) \in Q_1(B^{A, \varepsilon}, \beta^{A, \varepsilon}) = \widehat{L}^{4*+1}(A, \varepsilon)$$

for a nonsingular ε -quadratic form (K, ψ) over A with a lagrangian L for the ε -symmetric form $(K, \psi + \varepsilon\psi^*)$, so that

$$\begin{aligned} (K, \psi) &= \partial(K, \psi; L) \in \ker(1 + T : L_0(A, \varepsilon) \rightarrow L^{4*}(A, \varepsilon)) \\ &= \text{im}(\partial : Q_1(B^{A, \varepsilon}, \beta^{A, \varepsilon}, \varepsilon) \rightarrow L_0(A, \varepsilon)). \end{aligned}$$

3.1. Forms and formations

Given a f.g. projective A -module K and the inclusion $j : L \rightarrow K$ of a direct summand, let $f : C \rightarrow D$ be the chain map defined by

$$\begin{aligned} C : \cdots \rightarrow 0 \rightarrow C_k = K^* \rightarrow 0 \rightarrow \cdots, \\ D : \cdots \rightarrow 0 \rightarrow D_k = L^* \rightarrow 0 \rightarrow \cdots, \\ f = j^* : C_k = K^* \rightarrow D_k = L^*. \end{aligned}$$

The symmetric Q -group

$$Q^{2k}(C) = H^0(\mathbb{Z}_2; S(K), (-1)^k T) = \{\phi \in S(K) \mid \phi^* = (-1)^k \phi\}$$

is the additive group of $(-1)^k$ -symmetric pairings on K , and

$$f^{\%} = S(j) : Q^{2k}(C) \rightarrow Q^{2k}(D); \quad \phi \mapsto f\phi f^* = j^* \phi j = \phi|_L$$

sends such a pairing to its restriction to L . A $2k$ -dimensional symmetric (Poincaré) complex $(C, \phi \in Q^{2k}(C))$ is the same as a (nonsingular) $(-1)^k$ -symmetric form (K, ϕ) . The relative symmetric Q -group of f :

$$\begin{aligned} Q^{2k+1}(f) &= \ker(f^{\%} : Q^{2k}(C) \rightarrow Q^{2k}(D)) \\ &= \{\phi \in S(K) \mid \phi^* = (-1)^k \phi \in S(K), \phi|_L = 0 \in S(L)\}, \end{aligned}$$

consists of the $(-1)^k$ -symmetric pairings on K which restrict to 0 on L . The submodule $L \subset K$ is a lagrangian for (K, ϕ) if and only if ϕ restricts to 0 on L and

$$L^{\perp} = \{x \in K \mid \phi(x)(L) = \{0\} \subset A\} = L,$$

if and only if $(f : C \rightarrow D, (0, \phi) \in Q^{2k+1}(f))$ defines a $(2k+1)$ -dimensional symmetric Poincaré pair, with an exact sequence

$$0 \longrightarrow D^k = L \xrightarrow{f^*=j} C^k = K \xrightarrow{f\phi=j^*\phi} D_k = L^* \longrightarrow 0.$$

Similarly for the quadratic case, with

$$\begin{aligned} Q_{2k}(C) &= H_0(\mathbb{Z}_2; S(K), (-1)^k T), \\ Q_{2k+1}(f) &= \frac{\{(\psi, \chi) \in S(K) \oplus S(L) \mid f^* \psi f = \chi + (-1)^{k+1} \chi^* \in S(L)\}}{\{(\theta + (-1)^{k+1} \theta^*, f\theta f^* + v + (-1)^k v^*) \mid \theta \in S(K), v \in S(L)\}}. \end{aligned}$$

A quadratic structure $\psi \in Q_{2k}(C)$ determines and is determined by the pair (λ, μ) with $\lambda = \psi + (-1)^k \psi^* \in Q^{2k}(C)$ and

$$\mu : K \rightarrow H_0(\mathbb{Z}_2; A, (-1)^k); \quad x \mapsto \psi(x)(x).$$

A $(2k+1)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ is a nonsingular $(-1)^k$ -quadratic form (K, ψ) together with a lagrangian $L \subset K$ for the nonsingular $(-1)^k$ -symmetric form $(K, \psi + (-1)^k \psi^*)$.

Lemma 63. *Let (K, ψ) be a nonsingular $(-1)^k$ -quadratic form over A , and let $L \subset K$ be a lagrangian for $(K, \psi + (-1)^k \psi^*)$. There exists a direct complement for $L \subset K$ which is also a lagrangian for $(K, \psi + (-1)^k \psi^*)$.*

Proof. Choosing a direct complement $L' \subset K$ to $L \subset K$ write

$$\psi = \begin{pmatrix} \mu & \lambda \\ 0 & v' \end{pmatrix} : K = L \oplus L' \rightarrow K^* = L^* \oplus (L')^*$$

with $\lambda : L' \rightarrow L^*$ an isomorphism and

$$\mu + (-1)^k \mu^* = 0 : L \rightarrow L^*.$$

In general $v' + (-1)^k (v')^* \neq 0 : L^* \rightarrow L$, but if the direct complement L' is replaced by

$$L'' = \{(-(\lambda^{-1})^*(v')^*(x), x) \in L \oplus L' \mid x \in L'\} \subset K$$

and the isomorphism

$$\lambda'' : L'' \rightarrow L^*; \quad (-(\lambda^{-1})^*(v')^*(x), x) \mapsto \lambda(x)$$

is used as an identification then

$$\psi = \begin{pmatrix} \mu & 1 \\ 0 & v \end{pmatrix} : K = L \oplus L^* \rightarrow K^* = L^* \oplus L,$$

with $v = (v')^* \mu v' : L^* \rightarrow L$ such that

$$sv + (-1)^k v^* = 0 : L^* \rightarrow L.$$

Thus $L'' = L^* \subset K$ is a direct complement for L which is a lagrangian for $(K, \psi + (-1)^k \psi^*)$, with

$$\psi + (-1)^k \psi^* = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} : K = L \oplus L^* \rightarrow K^* = L^* \oplus L. \quad \square$$

A lagrangian L for the $(-1)^k$ -symmetrization $(K, \psi + (-1)^k \psi^*)$ is a lagrangian for the $(-1)^k$ -quadratic form (K, ψ) if and only if $\psi|_L = \mu$ is a $(-1)^{k+1}$ -symmetrization, i.e.

$$\mu = \theta + (-1)^{k+1} \theta^* : L \rightarrow L^*$$

for some $\theta \in S(L)$, in which case the inclusion $j : (L, 0) \rightarrow (K, \psi)$ extends to an isomorphism of $(-1)^k$ -quadratic forms

$$\begin{pmatrix} 1 & -v^* \\ 0 & 1 \end{pmatrix} : H_{(-1)^k}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \xrightarrow{\cong} (K, \psi),$$

with $v = \psi|_{L^*}$. The $2k$ -dimensional quadratic L -group $L_{2k}(A)$ is the Witt group of stable isomorphism classes of nonsingular $(-1)^k$ -quadratic forms over A , such that

$$(K, \psi) = (K', \psi') \in L_{2k}(A) \text{ if and only if there exists an isomorphism} \\ (K, \psi) \oplus H_{(-1)^k}(L) \cong (K', \psi') \oplus H_{(-1)^k}(L').$$

Proposition 64. *Given a $(-1)^k$ -quadratic form (L, μ) over A such that*

$$\mu + (-1)^k \mu^* = 0 : L \rightarrow L^*,$$

let (B, β) be the chain bundle over A given by

$$B : \cdots \rightarrow 0 \rightarrow B_{k+1} = L \rightarrow 0 \rightarrow \cdots, \\ \beta = \mu \in \widehat{Q}^0(B^{0-*}) = \text{Hom}_A(L, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) = \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1}T).$$

(i) *The $(2k+1)$ -dimensional twisted quadratic Q -group of (B, β) :*

$$Q_{2k+1}(B, \beta) = \frac{\{v \in S(L^*) \mid v + (-1)^k v^* = 0\}}{\{\phi - \phi \mu \phi^* - (\theta + (-1)^{k+1} \theta^*) \mid \phi^* = (-1)^{k+1} \phi, \theta \in S(L^*)\}} \\ = \text{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T) \rightarrow \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T))$$

classifies nonsingular $(-1)^k$ -quadratic forms (K, ψ) over A for which there exists a lagrangian L for $(K, \psi + (-1)^k \psi^)$ such that*

$$\psi|_L = \mu \in \text{im}(\widehat{H}^1(\mathbb{Z}_2; S(L), (-1)^k T) \rightarrow H_0(\mathbb{Z}_2; S(L), (-1)^k T)) \\ = \ker(1 + (-1)^k T : H_0(\mathbb{Z}_2; S(L), (-1)^k T) \rightarrow H^0(\mathbb{Z}_2; S(L), (-1)^k T)).$$

Specifically, for any $(-1)^k$ -quadratic form (L^, v) such that*

$$v + (-1)^k v^* = 0 : L^* \rightarrow L,$$

the nonsingular $(-1)^k$ -quadratic form (K, ψ) defined by

$$\psi = \begin{pmatrix} \mu & 1 \\ 0 & v \end{pmatrix} : K = L \oplus L^* \rightarrow K^* = L^* \oplus L$$

is such that L is a lagrangian of $(K, \psi + (-1)^k \psi^*)$, and

$$\partial : Q_{2k+1}(B, \beta) \rightarrow L_{2k}(A); \quad v \mapsto (K, \psi).$$

(ii) The algebraic normal invariant of a $(2k+1)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ concentrated in degree k with

$$C_k = K^*, \quad D_k = L^*,$$

$$f\psi_0 f^* = \mu \in \ker(1 + (-1)^k T : H_0(\mathbb{Z}_2; S(L), (-1)^k T) \rightarrow H^0(\mathbb{Z}_2; S(L), (-1)^k T))$$

is given by

$$(\phi, \theta) = v \in Q_{2k+1}(C(f), \gamma) = Q_{2k+1}(B, \beta),$$

with

$$\widehat{v}_{k+1}(\gamma) = \widehat{v}_{k+1}(\beta) : L = H_{k+1}(f) = H_{k+1}(B) \rightarrow \widehat{H}^{k+1}(\mathbb{Z}_2; A); \quad x \mapsto \mu(x)(x)$$

and $v = \psi|_{L^*}$ the restriction of ψ to any lagrangian $L^* \subset K$ of $(K, \psi + (-1)^k \psi^*)$ complementary to L .

Proof. (i) Given $(-1)^{k+1}$ -symmetric forms (L^*, v) , (L^*, ϕ) and $\theta \in S(L^*)$ replacing v by

$$v' = v + \phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) : L^* \rightarrow L$$

results in a $(-1)^k$ -quadratic form (K, ψ') such that there is defined an isomorphism

$$\begin{pmatrix} 1 & \phi^* \\ 0 & 1 \end{pmatrix} : (K, \psi') \rightarrow (K, \psi)$$

which is the identity on L .

(ii) This is the translation of Proposition 42(iii) into the language of forms and lagrangians. $\square$

More generally:

Proposition 65. Given $(-1)^k$ -quadratic forms (L, μ) , (L^*, ν) over A such that

$$\mu + (-1)^k \mu^* = 0 : L \rightarrow L^*, \quad \nu + (-1)^k \nu^* = 0 : L^* \rightarrow L$$

define a nonsingular $(-1)^k$ -quadratic form

$$(K, \psi) = \left(L \oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \right),$$

such that L and L^* are complementary lagrangians of the nonsingular $(-1)^k$ -symmetric form

$$(K, \psi + (-1)^k \psi^*) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} \right)$$

and let $(f : C \rightarrow D, (\delta\phi, \psi))$ be the $(2k+1)$ -dimensional (symmetric, quadratic) Poincaré pair concentrated in degree k defined by

$$f = \begin{pmatrix} 1 & 0 \end{pmatrix} : C_k = K^* = L^* \oplus L \rightarrow D_k = L^*, \quad \delta\phi = 0,$$

with $\mathcal{C}(f) \simeq L_{*-k-1}$.

(i) The Spivak normal bundle of $(f : C \rightarrow D, (\delta\phi, \psi))$ is given by

$$\gamma = \mu \in \widehat{Q}^0(\mathcal{C}(f)^{0-*}) = \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1}T)$$

and

$$\begin{aligned} Q_{2k+1}(\mathcal{C}(f), \gamma) &= \frac{\{\lambda \in S(L^*) \mid \lambda + (-1)^k \lambda^* = 0\}}{\{\phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) \mid \phi^* = (-1)^{k+1}\phi, \theta \in S(L^*)\}} \\ &= \text{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T) \\ &\rightarrow \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T)). \end{aligned}$$

The algebraic normal invariant of $(f : C \rightarrow D, (\delta\phi, \psi))$ is

$$(\phi, \theta) = \nu \in Q_{2k+1}(\mathcal{C}(f), \gamma).$$

(ii) Let (B, β) be a chain bundle concentrated in degree $k+1$

$$\begin{aligned} B : \cdots \rightarrow 0 \rightarrow B_{k+1} \rightarrow 0 \rightarrow \cdots, \\ \beta \in \widehat{Q}^0(B^{0-*}) = \text{Hom}_A(B_{k+1}, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) = \widehat{H}^0(\mathbb{Z}_2; S(B_{k+1}), (-1)^{k+1}T), \end{aligned}$$

so that

$$\begin{aligned} Q_{2k+1}(B, \beta) &= \frac{\{\lambda \in S(B^{k+1}) \mid \lambda + (-1)^k \lambda^* = 0\}}{\{\phi - \phi \beta \phi^* - (\theta + (-1)^{k+1} \theta^*) \mid \phi^* = (-1)^{k+1} \phi, \theta \in S(B^{k+1})\}} \\ &= \text{coker}(J_\beta : H^0(\mathbb{Z}_2; S(B^{k+1}), (-1)^{k+1} T) \\ &\rightarrow \widehat{H}^0(\mathbb{Z}_2; S(B^{k+1}), (-1)^{k+1} T)). \end{aligned}$$

A (B, β) -structure on $(f : C \rightarrow D, (\delta\phi, \psi))$ is given by a chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \rightarrow (B, \beta)$, corresponding to an A -module morphism $g : L \rightarrow B_{k+1}$ such that

$$g^* \beta g = \mu \in \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1} T),$$

with

$$(g, \chi)_\% : Q_{2k+1}(\mathcal{C}(f), \gamma) \rightarrow Q_{2k+1}(B, \beta); \lambda \mapsto g \lambda g^*.$$

The 4-periodic (B, β) -structure cobordism class is thus given by

$$\begin{aligned} (K, \psi; L) &= (f : C \rightarrow D, (\delta\phi, \psi)) = (g, \chi)_\%(\phi, \theta) = g v g^* \\ &\in \widehat{L}\langle B, \beta \rangle^{4*+2k+1}(A) = Q_{2k+1}(B, \beta), \end{aligned}$$

with

$$\begin{aligned} (K, \psi) &= \left(B_{k+1} \oplus B^{k+1}, \begin{pmatrix} \beta & 1 \\ 0 & g v g^* \end{pmatrix} \right) \\ &\in \text{im}(\partial : Q_{2k+1}(B, \beta) \rightarrow L_{2k}(A)) = \ker(L_{2k}(A) \rightarrow L\langle B, \beta \rangle^{4*+2k}(A)). \end{aligned}$$

3.2. The generalized Arf invariant

Definition 66. The generalized Arf invariant of a nonsingular $(-1)^k$ -quadratic form (K, ψ) over A together with a lagrangian $L \subset K$ for the $(-1)^k$ -symmetric form $(K, \psi + (-1)^k \psi^*)$ is the image

$$(K, \psi; L) = (g, \chi)_\%(\phi, \theta) \in \widehat{L}^{4*+2k+1}(A) = Q_{2k+1}(B^A, \beta^A)$$

of the algebraic normal invariant $(\phi, \theta) \in Q_{2k+1}(\mathcal{C}(f), \gamma)$ (43) of the corresponding $(2k+1)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi) \in Q_{2k+1}^{2k+1}(f))$

$$\begin{aligned} (\phi, \theta) &= v \in Q_{2k+1}(\mathcal{C}(f), \gamma) \\ &= \text{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1} T) \rightarrow \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1} T)) \end{aligned}$$

under the morphism $(g, \chi)_\%$ induced by the classifying chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \rightarrow (B^A, \beta^A)$. As in 64 $v = \psi|_{L^*}$ is the restriction of ψ to a lagrangian $L^* \subset K$ of $(K, \psi + (-1)^k \psi^*)$ complementary to L .

A nonsingular $(-1)^k$ -symmetric formation $(K, \phi; L, L')$ is a nonsingular $(-1)^k$ -symmetric form (K, ϕ) together with two lagrangians L, L' . This type of formation is essentially the same as a $(2k+1)$ -dimensional symmetric Poincaré complex concentrated in degrees $k, k+1$, and represents an element of $L^{4*+2k+1}(A)$.

Proposition 67. (i) *The generalized Arf invariant is such that*

$$(K, \psi; L) = 0 \in \mathcal{Q}_{2k+1}(B^A, \beta^A) = \widehat{L}^{4*+2k+1}(A)$$

if and only if there exists an isomorphism of $(-1)^k$ -quadratic forms

$$(K, \psi) \oplus H_{(-1)^k}(L') \cong H_{(-1)^k}(L'')$$

such that

$$((K, \psi + (-1)^k \psi^*) \oplus (1+T)H_{(-1)^k}(L'); L \oplus L', L'') = 0 \in L^{4*+2k+1}(A).$$

(ii) *If (K, ψ) is a nonsingular $(-1)^k$ -quadratic form over A and $L, L' \subset K$ are lagrangians for $(K, \psi + (-1)^k \psi^*)$ then*

$$\begin{aligned} (K, \psi; L) - (K, \psi; L') &= (K, \psi + (-1)^k \psi^*; L, L') \\ &\in \text{im}(L^{4*+2k+1}(A) \rightarrow \widehat{L}^{4*+2k+1}(A)) = \ker(\widehat{L}^{4*+2k+1}(A) \rightarrow L_{2k}(A)). \end{aligned}$$

Proof. This is the translation of the isomorphism $\mathcal{Q}_{2k+1}(B^A, \beta^A) \cong \widehat{L}^{4*+2k+1}(A)$ given by 46 into the language of forms and formations. $\square$

Example 68. Let A be a field, so that each $\widehat{H}^n(\mathbb{Z}_2; A)$ is a free A -module, and the universal chain bundle over A can be taken to be

$$B^A = \widehat{H}^*(\mathbb{Z}_2; A) : \cdots \longrightarrow B_n^A = \widehat{H}^n(\mathbb{Z}_2; A) \xrightarrow{0} B_{n-1}^A = \widehat{H}^{n-1}(\mathbb{Z}_2; A) \xrightarrow{0} \cdots$$

If A is a perfect field of characteristic 2 with the identity involution squaring defines an A -module isomorphism

$$A \xrightarrow{\cong} \widehat{H}^n(\mathbb{Z}_2; A); \quad a \mapsto a^2.$$

Every nonsingular $(-1)^k$ -quadratic form over A is isomorphic to one of the type

$$(K, \psi) = \left(L \oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \right),$$

with $L = A^\ell$ f.g. free and

$$\mu = (-1)^{k+1} \mu^* : L \rightarrow L^*, \quad \nu = (-1)^{k+1} \nu^* : L^* \rightarrow L.$$

For $j = 1, 2, \dots, \ell$ let

$$\begin{aligned} e_j &= (0, \dots, 0, 1, 0, \dots, 0) \in L, \quad g_j = \mu(e_j)(e_j) \in A, \\ e_j^* &= (0, \dots, 0, 1, 0, \dots, 0) \in L^*, \quad h_j = \nu(e_j^*)(e_j^*) \in A. \end{aligned}$$

The generalized Arf invariant in this case was identified in [18, §11] with the original invariant of Arf [1]

$$(K, \psi; L) = \sum_{j=1}^{\ell} g_j h_j \in Q_{2k+1}(B^A, \beta^A) = A/\{c + c^2 \mid c \in A\}.$$

For $k = 0$ we have:

Proposition 69. *Suppose that the involution on A is even. If (K, ψ) is a nonsingular quadratic form over A and L is a lagrangian of $(K, \psi + \psi^*)$ then L is a lagrangian of (K, ψ) , the Witt class is*

$$(K, \psi) = 0 \in L_0(A),$$

the algebraic normal invariant is

$$(\phi, \theta) = 0 \in Q_1(\mathcal{C}(f), \gamma) = 0$$

and the generalized Arf invariant is

$$(K, \psi; L) = (g, \chi)_{\%}(\phi, \theta) = 0 \in \widehat{L}^{4*+1}(A) = Q_1(B^A, \beta^A).$$

Proof. By hypothesis $\widehat{H}^1(\mathbb{Z}_2; A) = 0$, and $L = A^\ell$, so that by Proposition 64(i)

$$Q_1(\mathcal{C}(f), \gamma) = \widehat{H}^0(\mathbb{Z}_2; S(L^*), -T) = \bigoplus_{\ell} \widehat{H}^1(\mathbb{Z}_2; A) = 0. \quad \square$$

For $k = 1$ we have:

Theorem 70. Let A be an r -even ring with A_2 -module basis $\{x_1 = 1, x_2, \dots, x_r\} \subset \widehat{H}^0(\mathbb{Z}_2; A)$, and let

$$X = \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_r \end{pmatrix} \in \text{Sym}_r(A)$$

so that by Theorem 60

$$Q_3(B^A, \beta^A) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A) + \{L - LXL \mid L \in \text{Sym}_r(A)\}}.$$

(i) Given $M \in \text{Sym}_r(A)$ define the nonsingular (-1) -quadratic form over A

$$(K_M, \psi_M) = \left(A^r \oplus (A^r)^*, \begin{pmatrix} X & 1 \\ 0 & M \end{pmatrix} \right)$$

such that $L_M = A^r \subset K_M$ is a lagrangian of $(K_M, \psi_M - \psi_M^*)$. The function

$$Q_3(B^A, \beta^A) \rightarrow \widehat{L}^{4*+3}(A); \quad M \mapsto (K_M, \psi_M; L_M)$$

is an isomorphism, with inverse given by the generalized Arf invariant.

(ii) Let (K, ψ) be a nonsingular (-1) -quadratic form over A of the type

$$(K, \psi) = \left(L \oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \right),$$

with

$$\mu - \mu^* = 0 : L \rightarrow L^*, \quad \nu - \nu^* = 0 : L^* \rightarrow L$$

and let $g : L \rightarrow A^r$, $h : L^* \rightarrow A^r$ be A -module morphisms such that

$$\mu = g^* X g \in \widehat{H}^0(\mathbb{Z}_2; S(L), T), \quad \nu = h^* X h \in \widehat{H}^0(\mathbb{Z}_2; S(L^*), T).$$

The generalized Arf invariant of $(K, \psi; L)$ is

$$(K, \psi; L) = g \nu g^* = g h^* X h g^* \in Q_3(B^A, \beta^A).$$

If $L = A^\ell$ then

$$g = (g_{ij}) : L = A^\ell \rightarrow A^r, \quad h = (h_{ij}) : L^* = A^\ell \rightarrow A^r,$$

with the coefficients $g_{ij}, h_{ij} \in A$ such that

$$\mu(e_j)(e_j) = \sum_{i=1}^r (g_{ij})^2 x_i, \quad \nu(e_j^*)(e_j^*) = \sum_{i=1}^r (h_{ij})^2 x_i \in \widehat{H}^0(\mathbb{Z}_2; A)$$

$$(e_j = (0, \dots, 0, 1, 0, \dots, 0) \in L = A^\ell, \quad e_j^* = (0, \dots, 0, 1, 0, \dots, 0) \in L^* = A^\ell)$$

and

$$(K, \psi; L) = gh^* X hg^* = \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 \\ 0 & 0 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_r \end{pmatrix} \in Q_3(B^A, \beta^A),$$

with

$$c_i = \sum_{k=1}^r \left(\sum_{j=1}^{\ell} g_{ij} h_{kj} \right)^2 x_k \in \widehat{H}^0(\mathbb{Z}_2; A).$$

(iii) For any $M = (m_{ij}) \in \text{Sym}_r(A)$ let $h = (h_{ij}) \in M_r(A)$ be such that

$$m_{jj} = \sum_{i=1}^r (h_{ij})^2 x_i \in \widehat{H}^0(\mathbb{Z}_2; A) \quad (1 \leq j \leq r),$$

so that

$$M = \begin{pmatrix} m_{11} & 0 & 0 & \dots & 0 \\ 0 & m_{22} & 0 & \dots & 0 \\ 0 & 0 & m_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m_{rr} \end{pmatrix} = h^* X h \in \widehat{H}^0(\mathbb{Z}_2; M_r(A), T) = \frac{\text{Sym}_r(A)}{\text{Quad}_r(A)}$$

and the generalized Arf invariant of the triple $(K_M, \psi_M; L_M)$ in (i) is

$$(K_M, \psi_M; L_M) = h^* Xh = M \in Q_3(B^A, \beta^A)$$

(with $g = (\delta_{ij})$ here).

Proof. (i) The isomorphism $Q_3(B^A, \beta^A) \rightarrow \widehat{L}^3(A); M \mapsto (K_M, \psi_M; L_M)$ is given by Proposition 46.

(ii) As in Definition 66 let $(\phi, \theta) \in Q_3(\mathcal{C}(f), \gamma)$ be the algebraic normal invariant of the three-dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ concentrated in degree 1, with

$$f = \begin{pmatrix} 1 & 0 \end{pmatrix} : C_1 = K^* = L^* \oplus L \rightarrow D_1 = L^*, \quad \delta\phi = 0.$$

The A -module morphism

$$\widehat{v}_2(\gamma) : H_2(\mathcal{C}(f)) = H^1(D) = L \rightarrow \widehat{H}^0(\mathbb{Z}_2; A); \quad y \mapsto \mu(y)(y)$$

is induced by the A -module chain map

$$g : \mathcal{C}(f) \simeq L_{*-2} \rightarrow B^A(1)$$

and

$$(g, 0) : (\mathcal{C}(f), \gamma) \rightarrow (B^A(1), \beta^A(1)) \rightarrow (B^A, \beta^A)$$

is a classifying chain bundle map. The induced morphism

$$\begin{aligned} (g, 0)_\% : Q_3(\mathcal{C}(f), \gamma) &= \text{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), T) \rightarrow \widehat{H}^0(\mathbb{Z}_2; S(L^*), T)) \\ &\rightarrow Q_3(B^A, \beta^A) = \text{coker}(J_X : H^0(\mathbb{Z}_2; M_r(A), T) \rightarrow \widehat{H}^0(\mathbb{Z}_2; M_r(A), T)); \quad \sigma \mapsto g\sigma g^* \end{aligned}$$

sends the algebraic normal invariant

$$(\phi, \theta) = v = h^* Xh \in Q_3(\mathcal{C}(f), \gamma)$$

to the generalized Arf invariant

$$(g, 0)_\%(\phi, \theta) = gh^* Xhg^* \in Q_3(B^A, \beta^A).$$

(iii) By construction. $\square$

In particular, the generalized Arf invariant for $A = \mathbb{Z}_2$ is just the classical Arf invariant.

4. The generalized Arf invariant for linking forms

An ε -quadratic formation $(Q, \psi; F, G)$ over A corresponds to a one-dimensional ε -quadratic Poincaré complex. The one-dimensional ε -quadratic L -group $L_1(A, \varepsilon)$ is the Witt group of ε -quadratic formations, or equivalently the cobordism group of one-dimensional ε -quadratic Poincaré complexes over A . We could define a generalized Arf invariant $\alpha \in Q_2(B^A, \beta^A, \varepsilon)$ for any formation with a null-cobordism of the one-dimensional ε -symmetric Poincaré complex, so that

$$\begin{aligned} (Q, \psi; F, G) &= \partial(\alpha) \in \ker(1 + T_\varepsilon : L_1(A, \varepsilon) \rightarrow L^{4*+1}(A, \varepsilon)) \\ &= \text{im}(\partial : Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \rightarrow L_1(A, \varepsilon)). \end{aligned}$$

However, we do not need quite such a generalized Arf invariant here. For our application to UNil, it suffices to work with a localization $S^{-1}A$ of A and to only consider a formation $(Q, \psi; F, G)$ such that

$$F \cap G = \{0\}, \quad S^{-1}(Q/(F + G)) = 0,$$

which corresponds to a $(-\varepsilon)$ -quadratic linking form (T, λ, μ) over (A, S) with

$$T = Q/(F + G), \quad \lambda : T \times T \rightarrow S^{-1}A/A.$$

Given a lagrangian $U \subset T$ for the $(-\varepsilon)$ -symmetric linking form (T, λ) we define in this section a ‘linking Arf invariant’

$$(T, \lambda, \mu; U) \in Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) = \widehat{L}^{4*+2}(A, \varepsilon)$$

such that

$$\begin{aligned} (Q, \psi; F, G) &= \partial(T, \lambda, \mu; U) \in \ker(1 + T_\varepsilon : L_1(A, \varepsilon) \rightarrow L^{4*+1}(A, \varepsilon)) \\ &= \text{im}(\partial : Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \rightarrow L_1(A, \varepsilon)). \end{aligned}$$

4.1. Linking forms and formations

Given a ring with involution A and a multiplicative subset $S \subset A$ of central nonzero divisors such that $\overline{S} = S$ let $S^{-1}A$ be the localized ring with involution obtained from A by inverting S . We refer to [16] for the localization exact sequences in ε -symmetric and ε -quadratic algebraic L -theory

$$\begin{aligned} \cdots \rightarrow L^n(A, \varepsilon) &\rightarrow L^n_I(S^{-1}A, \varepsilon) \rightarrow L^n(A, S, \varepsilon) \rightarrow L^{n-1}(A, \varepsilon) \rightarrow \cdots, \\ \cdots \rightarrow L_n(A, \varepsilon) &\rightarrow L_n^I(S^{-1}A, \varepsilon) \rightarrow L_n(A, S, \varepsilon) \rightarrow L_{n-1}(A, \varepsilon) \rightarrow \cdots. \end{aligned}$$

with $I = \text{im}(\tilde{K}_0(A) \rightarrow \tilde{K}_0(S^{-1}A))$, $L^n(A, S, \varepsilon)$ the cobordism group of $(n-1)$ -dimensional ε -symmetric Poincaré complexes (C, ϕ) over A such that $H_*(S^{-1}C) = 0$, and similarly for $L_n(A, S, \varepsilon)$. An (A, S) -module is an A -module T with a one-dimensional f.g. projective A -module resolution

$$0 \rightarrow P \xrightarrow{d} Q \rightarrow T \rightarrow 0$$

such that $S^{-1}d : S^{-1}P \rightarrow S^{-1}Q$ is an $S^{-1}A$ -module isomorphism. In particular,

$$S^{-1}T = 0.$$

The dual (A, S) -module is defined by

$$\begin{aligned} T^\wedge &= \text{Ext}_A^1(T, A) = \text{Hom}_A(T, S^{-1}A/A) \\ &= \text{coker}(d^* : Q^* \rightarrow P^*), \end{aligned}$$

with

$$A \times T^\wedge \rightarrow T^\wedge; (a, f) \mapsto (x \mapsto f(x)\bar{a}).$$

For any (A, S) -modules T, U there is defined a duality isomorphism

$$\text{Hom}_A(T, U) \rightarrow \text{Hom}_A(U^\wedge, T^\wedge); f \mapsto f^\wedge,$$

with

$$f^\wedge : U^\wedge \rightarrow T^\wedge; g \mapsto (x \mapsto g(f(x))).$$

An element $\lambda \in \text{Hom}_A(T, T^\wedge)$ can be regarded as a sesquilinear linking pairing

$$\lambda : T \times T \rightarrow S^{-1}A/A; (x, y) \mapsto \lambda(x, y) = \lambda(x)(y),$$

with

$$\begin{aligned} \lambda(x, ay + bz) &= a\lambda(x, y) + b\lambda(x, z), \\ \lambda(ay + bz, x) &= \lambda(y, x)\bar{a} + \lambda(z, x)\bar{b}, \\ \tilde{\lambda}^\wedge(x, y) &= \overline{\lambda(y, x)} \in S^{-1}A/A \quad (a, b \in A, x, y, z \in T). \end{aligned}$$

Definition 71. Let $\varepsilon = \pm 1$.

(i) An ε -symmetric linking form over (A, S) (T, λ) is an (A, S) -module T together with $\lambda \in \text{Hom}_A(T, T^\wedge)$ such that $\widehat{\lambda} = \varepsilon\lambda$, so that

$$\overline{\lambda(x, y)} = \varepsilon\lambda(y, x) \in S^{-1}A/A \quad (x, y \in T).$$

The linking form is *nonsingular* if $\lambda : T \rightarrow T^\wedge$ is an isomorphism. A *lagrangian* for (T, λ) is an (A, S) -submodule $U \subset T$ such that the sequence

$$0 \longrightarrow U \xrightarrow{j} T \xrightarrow{j^\wedge \lambda} U^\wedge \longrightarrow 0$$

is exact with $j \in \text{Hom}_A(U, T)$ the inclusion. Thus λ restricts to 0 on U and

$$U^\perp = \{x \in T \mid \lambda(x)(U) = \{0\} \subset S^{-1}A/A\} = U.$$

(ii) A (nonsingular) ε -quadratic linking form over (A, S) (T, λ, μ) is a (nonsingular) ε -symmetric linking form (T, λ) together with a function

$$\mu : T \rightarrow Q_\varepsilon(A, S) = \frac{\{b \in S^{-1}A \mid \varepsilon\bar{b} = b\}}{\{a + \varepsilon\bar{a} \mid a \in A\}}$$

such that

$$\begin{aligned} \mu(ax) &= a\mu(x)\bar{a}, \\ \mu(x + y) &= \mu(x) + \mu(y) + \lambda(x, y) + \lambda(y, x) \in Q_\varepsilon(A, S), \\ \mu(x) &= \lambda(x, x) \in \text{im}(Q_\varepsilon(A, S) \rightarrow S^{-1}A/A) \quad (x, y \in T, a \in A). \end{aligned}$$

A *lagrangian* $U \subset T$ for (T, λ, μ) is a lagrangian for (T, λ) such that $\mu|_U = 0$.

We refer to [16, 3.5] for the development of the theory of ε -symmetric and ε -quadratic linking formations over (A, S) .

From now on, we shall only be concerned with A, S which satisfy:

Hypothesis 72. A, S are such that

$$\widehat{H}^*(\mathbb{Z}_2; S^{-1}A) = 0.$$

Example 73. Hypothesis 72 is satisfied if $1/2 \in S^{-1}A$, e.g. if A is even and

$$S = (2)^\infty = \{2^i \mid i \geq 0\} \subset A, \quad S^{-1}A = A[1/2].$$

Proposition 74. (i) For $n = 2$ (resp. 1) the relative group $L^n(A, S, \varepsilon)$ in the ε -symmetric L -theory localization exact sequence

$$\cdots \rightarrow L^n(A, \varepsilon) \rightarrow L^n_I(S^{-1}A, \varepsilon) \rightarrow L^n(A, S, \varepsilon) \rightarrow L^{n-1}(A, \varepsilon) \rightarrow \cdots$$

is the Witt group of nonsingular $(-\varepsilon)$ -symmetric linking forms (resp. ε -symmetric linking formations) over (A, S) , with $I = \text{im}(\tilde{K}_0(A) \rightarrow \tilde{K}_0(S^{-1}A))$. The skew-suspension maps

$$\bar{S} : L^n(A, S, \varepsilon) \rightarrow L^{n+2}(A, S, -\varepsilon) \quad (n \geq 1)$$

are isomorphisms if and only if the skew-suspension maps

$$\bar{S} : L^n(A, \varepsilon) \rightarrow L^{n+2}(A, -\varepsilon) \quad (n \geq 0)$$

are isomorphisms.

(ii) The relative group $L_n(A, S, \varepsilon)$ for $n = 2k$ (resp. $2k + 1$) in the ε -quadratic L -theory localization exact sequence

$$\cdots \rightarrow L_n(A, \varepsilon) \rightarrow L_n^I(S^{-1}A, \varepsilon) \rightarrow L_n(A, S, \varepsilon) \rightarrow L_{n-1}(A, \varepsilon) \rightarrow \cdots$$

is the Witt group of nonsingular $(-1)^k \varepsilon$ -quadratic linking forms (resp. formations) over (A, S) .

(iii) The 4-periodic ε -symmetric and ε -quadratic localization exact sequences interleave in a commutative braid of exact sequences

$$\begin{array}{ccccc}
 & & \partial & & \\
 & \searrow & & \searrow & \\
 Q_{n+1}(B^A, \beta^A, \epsilon) & & L_n(A, \epsilon) & & L_n^I(S^{-1}A, \epsilon) \\
 & \searrow^{\partial^S} & \nearrow & \searrow & \nearrow \\
 & L_{n+1}(A, S, \epsilon) & & L^{n+4*}(A, \epsilon) & \\
 & \nearrow & \searrow & \nearrow & \searrow \\
 L_{n+1}^I(S^{-1}A, \epsilon) & & L^{n+4*+1}(A, S, \epsilon) & & Q_n(B^A, \beta^A, \epsilon) \\
 & \nearrow & \searrow & \nearrow & \\
 & & & &
 \end{array}$$

Proof. (i)+(ii) See [16, §3].

(iii) For A, S satisfying Hypothesis 72 the ε -symmetrization maps for the L -groups of $S^{-1}A$ are isomorphisms

$$1 + T_\varepsilon : L_n^I(S^{-1}A, \varepsilon) \xrightarrow{\cong} L_n^n(S^{-1}A, \varepsilon). \quad \square$$

Definition 75. (i) An ε -quadratic S -formation $(Q, \psi; F, G)$ over A is an ε -quadratic formation such that

$$S^{-1}F \oplus S^{-1}G = S^{-1}Q,$$

or equivalently such that $Q/(F + G)$ is an (A, S) -module.

(ii) A *stable isomorphism* of ε -quadratic S -formations over A

$$[f] : (Q_1, \psi_1; F_1, G_1) \rightarrow (Q_2, \psi_2; F_2, G_2)$$

is an isomorphism of the type

$$f : (Q_1, \psi_1; F_1, G_1) \oplus (N_1, \nu_1; H_1, K_1) \rightarrow (Q_2, \psi_2; F_2, G_2) \oplus (N_2, \nu_2; H_2, K_2),$$

with $N_1 = H_1 \oplus K_1$, $N_2 = H_2 \oplus K_2$.

Proposition 76. (i) A $(-\varepsilon)$ -quadratic S -formation $(Q, \psi; F, G)$ over A determines a nonsingular ε -quadratic linking form (T, λ, μ) over (A, S) , with

$$T = Q/(F + G),$$

$$\lambda : T \times T \rightarrow S^{-1}A/A; (x, y) \mapsto (\psi - \varepsilon\psi^*)(x)(z)/s,$$

$$\mu : T \rightarrow Q_\varepsilon(A, S); y \mapsto (\psi - \varepsilon\psi^*)(x)(z)/s - \psi(y)(y)$$

$$(x, y \in Q, z \in G, s \in S, sy - z \in F).$$

(ii) The isomorphism classes of nonsingular ε -quadratic linking forms over A are in one-one correspondence with the stable isomorphism classes of $(-\varepsilon)$ -quadratic S -formations over A .

Proof. See Proposition 3.4.3 of [16]. $\square$

For any $S^{-1}A$ -contractible f.g. projective A -module chain complexes concentrated in degrees $k, k + 1$

$$C : \cdots \rightarrow 0 \rightarrow C_{k+1} \rightarrow C_k \rightarrow 0 \rightarrow \cdots,$$

$$D : \cdots \rightarrow 0 \rightarrow D_{k+1} \rightarrow D_k \rightarrow 0 \rightarrow \cdots$$

there are natural identifications

$$\begin{aligned} H^{k+1}(C) &= H_k(C)^\wedge, \quad H_k(C) = H^{k+1}(C)^\wedge, \\ H^{k+1}(D) &= H_k(D)^\wedge, \quad H_k(D) = H^{k+1}(D)^\wedge, \\ H_0(\operatorname{Hom}_A(C, D)) &= \operatorname{Hom}_A(H_k(C), H_k(D)) = \operatorname{Tor}_1^A(H^{k+1}(C), H_k(D)), \\ H_1(\operatorname{Hom}_A(C, D)) &= H^{k+1}(C) \otimes_A H_k(D) = \operatorname{Ext}_A^1(H_k(C), H_k(D)), \\ H_{2k}(C \otimes_A D) &= H_k(C) \otimes_A H_k(D) = \operatorname{Ext}_A^1(H^{k+1}(C), H_k(D)), \\ H_{2k+1}(C \otimes_A D) &= \operatorname{Hom}_A(H^{k+1}(C), H_k(D)) = \operatorname{Tor}_1^A(H_k(C), H_k(D)). \end{aligned}$$

In particular, an element $\lambda \in H_{2k+1}(C \otimes_A D)$ is a sesquilinear linking pairing

$$\lambda : H^{k+1}(C) \times H^{k+1}(D) \rightarrow S^{-1}A/A.$$

An element $\phi \in H_{2k}(C \otimes_A D)$ is a chain homotopy class of chain maps $\phi : C^{2k-*} \rightarrow D$, classifying the extension

$$0 \rightarrow H_k(D) \rightarrow H_k(\phi) \rightarrow H^{k+1}(C) \rightarrow 0.$$

Proposition 77. *Given an (A, S) -module T let*

$$B : \cdots \rightarrow 0 \rightarrow B_{k+1} \xrightarrow{d} B_k \rightarrow 0 \rightarrow \cdots$$

be a f.g. projective A -module chain complex concentrated in degrees $k, k+1$ such that $H^{k+1}(B) = T$, $H^k(B) = 0$, so that $H_k(B) = T^\wedge$, $H_{k+1}(B) = 0$. The Q -groups in the exact sequence

$$\begin{array}{ccccccc} Q^{2k+2}(B) = 0 & \longrightarrow & \widehat{Q}^{2k+2}(B) & \xrightarrow{H} & Q_{2k+1}(B) & \xrightarrow{1+T} & Q^{2k+1}(B) \\ & & \xrightarrow{J} & & \widehat{Q}^{2k+1}(B) & & \end{array}$$

have the following interpretation in terms of T .

(i) *The symmetric Q -group*

$$Q^{2k+1}(B) = H^0(\mathbb{Z}_2; \operatorname{Hom}_A(T, T^\wedge), (-1)^{k+1})$$

is the additive group of $(-1)^{k+1}$ -symmetric linking pairings λ on T , with $\phi \in Q^{2k+1}(B)$ corresponding to

$$\lambda : T \times T \rightarrow S^{-1}A/A; \quad (x, y) \mapsto \phi_0(d^*)^{-1}(x)(y) \quad (x, y \in B^{k+1}).$$

(ii) *The quadratic Q -group*

$$Q_{2k+1}(B)$$

$$= \frac{\{(\psi_0, \psi_1) \in \text{Hom}_A(B^k, B_{k+1}) \oplus S(B^k) \mid d\psi_0 = \psi_1 + (-1)^{k+1}\psi_1^* \in S(B^k)\}}{\{((\chi_0 + (-1)^{k+1}\chi_0^*)d^*, d\chi_0d^* + \chi_1 + (-1)^k\chi_1^*) \mid (\chi_0, \chi_1) \in S(B^{k+1}) \oplus S(B^k)\}}$$

is the additive group of $(-1)^{k+1}$ -quadratic linking structures (λ, μ) on T . The element $\psi = (\psi_0, \psi_1) \in Q_{2k+1}(B)$ corresponds to

$$\begin{aligned} \lambda : T \times T &\rightarrow S^{-1}A/A; \quad (x, y) \mapsto \psi_0(d^*)^{-1}(x)(y) \quad (x, y \in B^{k+1}), \\ \mu : T &\rightarrow Q_{(-1)^{k+1}}(A, S); \quad x \mapsto \psi_0(d^*)^{-1}(x)(x). \end{aligned}$$

(iii) *The hyperquadratic Q -groups of B*

$$\widehat{Q}^n(B) = H_n(\widehat{d}^{\%} : \widehat{W}^{\%} B_{k+1} \rightarrow \widehat{W}^{\%} B_k)$$

are such that

$$\begin{aligned} \widehat{Q}^{2k}(B) &= \frac{\{(\delta, \chi) \in S(B^{k+1}) \oplus S(B^k) \mid \delta^* = (-1)^{k+1}\delta, d\delta d^* = \chi + (-1)^{k+1}\chi^*\}}{\{(\mu + (-1)^{k+1}\mu^*, d\mu d^* + \nu + (-1)^k\nu^*) \mid (\mu, \nu) \in S(B^{k+1}) \oplus S(B^k)\}}, \\ \widehat{Q}^{2k+1}(B) &= \frac{\{(\delta, \chi) \in S(B^{k+1}) \oplus S(B^k) \mid \delta^* = (-1)^k\delta, d\delta d^* = \chi + (-1)^k\chi^*\}}{\{(\mu + (-1)^k\mu^*, d\mu d^* + \nu + (-1)^{k+1}\nu^*) \mid (\mu, \nu) \in S(B^{k+1}) \oplus S(B^k)\}}, \end{aligned}$$

with universal coefficient exact sequences

$$\begin{aligned} 0 \rightarrow T^{\wedge} \otimes_A \widehat{H}^k(\mathbb{Z}_2; A) \rightarrow \widehat{Q}^{2k}(B) &\xrightarrow{\widehat{v}_{k+1}} \text{Hom}_A(T, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) \rightarrow 0, \\ 0 \rightarrow T^{\wedge} \otimes_A \widehat{H}^{k+1}(\mathbb{Z}_2; A) \rightarrow \widehat{Q}^{2k+1}(B) &\xrightarrow{\widehat{v}_k} \text{Hom}_A(T, \widehat{H}^k(\mathbb{Z}_2; A)) \rightarrow 0. \end{aligned}$$

Let $f : C \rightarrow D$ be a chain map of $S^{-1}A$ -contractible A -module chain complexes concentrated in degrees $k, k+1$, inducing the A -module morphism

$$f^* = j : U = H^{k+1}(D) \rightarrow T = H^{k+1}(C).$$

By Proposition 77(i) a $(2k+1)$ -dimensional symmetric Poincaré complex (C, ϕ) is essentially the same as a nonsingular $(-1)^{k+1}$ -symmetric linking form (T, λ) , and a $(2k+2)$ -dimensional symmetric Poincaré pair $(f : C \rightarrow D, (\delta\phi, \phi))$ is essentially the same as a lagrangian U for (T, λ) , with $j = f^* : U \rightarrow T$ the inclusion. Similarly, a $(2k+1)$ -dimensional quadratic Poincaré complex (C, ψ) is essentially the same as

a nonsingular $(-1)^{k+1}$ -quadratic linking form (T, λ, μ) , and a $(2k+2)$ -dimensional quadratic Poincaré pair $(f : C \rightarrow D, (\delta\psi, \psi))$ is essentially the same as a lagrangian $U \subset T$ for (T, λ, μ) . A $(2k+2)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi))$ is a nonsingular $(-1)^{k+1}$ -quadratic linking form (T, λ, μ) together with a lagrangian $U \subset T$ for the nonsingular $(-1)^{k+1}$ -symmetric linking form (T, λ) .

Proposition 78. *Let U be an (A, S) -module together with an A -module morphism $\mu_1 : U \rightarrow \widehat{H}^{k+1}(\mathbb{Z}_2; A)$, defining a $(-1)^{k+1}$ -quadratic linking form (U, λ_1, μ_1) over (A, S) with $\lambda_1 = 0$.*

(i) *There exists a map of chain bundles $(d, \chi) : (B_{k+2}, 0) \rightarrow (B_{k+1}, \delta)$ concentrated in degree $k+1$ such that the cone chain bundle $(B, \beta) = \mathcal{C}(d, \chi)$ has*

$$\begin{aligned} H_{k+1}(B) &= U, \quad H^{k+2}(B) = U^\wedge, \quad H_{k+2}(B) = H^{k+1}(B) = 0, \\ \beta = [\delta] &= \mu_1 \in \widehat{Q}^0(B^{0-*}) = \text{Hom}_A(U, \widehat{H}^{k+1}(\mathbb{Z}_2; A)). \end{aligned}$$

(ii) *The $(2k+2)$ -dimensional twisted quadratic Q -group of (B, β) as in (i)*

$$\begin{aligned} Q_{2k+2}(B, \beta) &= \frac{\{(\phi, \theta) \in S(B^{k+1}) \oplus S(B^{k+1}) \mid \phi^* = (-1)^{k+1}\phi, \phi - \phi\delta\phi^* = \theta + (-1)^{k+1}\theta^*\}}{\{(d, \chi)_\% (v) + (0, \eta + (-1)^k\eta^*) \mid v \in S(B^{k+2}), \eta \in S(B^{k+1})\}} \\ ((d, \chi)_\% (v) &= (d(v + (-1)^{k+1}v^*)d^*, dvd^* - d(v + (-1)^{k+1}v^*)\chi(v^* + (-1)^{k+1}v)d^*)) \end{aligned}$$

is the additive group of isomorphism classes of extensions of U to a nonsingular $(-1)^{k+1}$ -quadratic linking form (T, λ, μ) over (A, S) such that $U \subset T$ is a lagrangian of the $(-1)^{k+1}$ -symmetric linking form (T, λ) and

$$\beta = \mu|_U : H_{k+1}(B) = U \rightarrow \widehat{H}^{k+1}(\mathbb{Z}_2; A) = \ker(Q_{(-1)^{k+1}}(A, S) \rightarrow S^{-1}A/A).$$

(iii) *An element $(\phi, \theta) \in Q_{2k+2}(B, \beta)$ is the algebraic normal invariant (43) of the $(2k+2)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi)) \in Q_{2k+2}^{2k+2}(f)$ with*

$$\begin{aligned} d_C &= \begin{pmatrix} d & \phi \\ 0 & d^* \end{pmatrix} : C_{k+1} = B_{k+2} \oplus B^{k+1} \rightarrow C_k = B_{k+1} \oplus B^{k+2}, \\ f &= \text{projection} : C \rightarrow D = B^{2k+2-*} \end{aligned}$$

constructed as in Proposition 42(ii), corresponding to the quadruple $(T, \lambda, \mu; U)$ given by

$$j = f^* : U = H^{k+1}(D) = H_{k+1}(B) \rightarrow T = H^{k+1}(C).$$

The A -module extension

$$0 \rightarrow U \rightarrow T \rightarrow U^\wedge \rightarrow 0$$

is classified by

$$[\phi] \in H_{2k+2}(B \otimes_A B) = U \otimes_A U = \text{Ext}_A^1(U^\wedge, U).$$

(iv) The $(-1)^{k+1}$ -quadratic linking form (T, λ, μ) in (iii) corresponds to the $(-1)^k$ -quadratic S -formation $(Q, \psi; F, G)$ with

$$(Q, \psi) = H_{(-1)^k}(F), \quad F = B_{k+2} \oplus B^{k+1},$$

$$G = \text{im} \left(\begin{pmatrix} 1 & 0 \\ -\delta d & 1 - \delta \phi \\ 0 & (-1)^{k+1} d^* \\ d & \phi \end{pmatrix} : B_{k+2} \oplus B^{k+1} \rightarrow B_{k+2} \oplus B^{k+1} \oplus B^{k+2} \oplus B_{k+1} \right)$$

$$\subset F \oplus F^*$$

such that

$$F \cap G = \{0\}, \quad Q/(F + G) = H^{k+1}(C) = T.$$

The inclusion $U \rightarrow T$ is resolved by

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_{k+2} & \xrightarrow{d} & B_{k+1} & \longrightarrow & U \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & B_{k+2} \oplus B^{k+1} & \xrightarrow{\begin{pmatrix} 0 & (-1)^{k+1} d^* \\ d & \phi \end{pmatrix}} & B^{k+2} \oplus B_{k+1} & \longrightarrow & T \longrightarrow 0 \end{array}$$

(v) If the involution on A is even and $k = -1$ then

$$Q_0(B, \beta) = \frac{\{\phi \in \text{Sym}(B^0) \mid \phi - \phi \delta \phi \in \text{Quad}(B^0)\}}{\{d \sigma d^* \mid \sigma \in \text{Quad}(B^1)\}}.$$

An extension of $U = \text{coker}(d : B_1 \rightarrow B_0)$ to a nonsingular quadratic linking form (T, λ, μ) over (A, S) with $\mu|_U = \mu_1$ and $U \subset T$ a lagrangian of (T, λ) is classified by $\phi \in Q_0(B, \beta)$ such that $\lambda : T \rightarrow T^\sim$ is resolved by

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_1 \oplus B^0 & \xrightarrow{\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}} & B^1 \oplus B_0 & \longrightarrow & T \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} 1 & 0 \\ -\delta d & 1 - \delta \phi \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix} & & \downarrow \lambda \\ 0 & \longrightarrow & B_1 \oplus B^0 & \xrightarrow{\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}} & B^1 \oplus B_0 & \longrightarrow & T^\sim \longrightarrow 0 \end{array}$$

and

$$\begin{aligned} T &= \text{coker} \left(\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix} : B_1 \oplus B^0 \rightarrow B^1 \oplus B_0 \right), \\ \lambda : T \times T &\rightarrow S^{-1}A/A; \\ ((x_1, x_0), (y_1, y_0)) &\mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(y_1) + d^{-1}(x_1)(y_0) + (d^*)^{-1}(x_0)(y_1), \\ \mu : T &\rightarrow Q_{+1}(A, S); \\ (x_1, x_0) &\mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(x_1) + d^{-1}(x_1)(x_0) + (d^*)^{-1}(x_0)(x_1) - \delta(x_0)(x_0), \\ (x_0, y_0 \in B_0, \quad x_1, y_1 \in B^1). \end{aligned}$$

4.2. The linking Arf invariant

Definition 79. The *linking Arf invariant* of a nonsingular $(-1)^{k+1}$ -quadratic linking form (T, λ, μ) over (A, S) together with a lagrangian $U \subset T$ for (T, λ) is the image

$$(T, \lambda, \mu; U) = (g, \chi)_\%(\phi, \theta) \in \widehat{L}^{4*+2k+2}(A) = Q_{2k+2}(B^A, \beta^A)$$

of the algebraic normal invariant $(\phi, \theta) \in Q_{2k+2}(\mathcal{C}(f), \gamma)$ (43) of the corresponding $(2k+2)$ -dimensional (symmetric, quadratic) Poincaré pair $(f : C \rightarrow D, (\delta\phi, \psi) \in Q_{2k+2}^{2k+2}(f))$ concentrated in degrees $k, k+1$ with

$$f^* = j : H^{k+1}(D) = U \rightarrow H^{k+1}(C) = T$$

and $(g, \chi)_\%$ induced by the classifying chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \rightarrow (B^A, \beta^A)$.

The chain bundle $(\mathcal{C}(f), \gamma)$ in 79 is (up to equivalence) of the type (B, β) considered in Proposition 78(i): the algebraic normal invariant $(\phi, \theta) \in Q_{2k+2}(B, \beta)$ classifies the extension of (U, β) to a lagrangian of a $(-1)^{k+1}$ -symmetric linking form (T, λ) with

a $(-1)^{k+1}$ -quadratic function μ on T such that $\mu|_U = \beta$. The linking Arf invariant $(T, \lambda, \mu; U) \in Q_{2k+2}(B^A, \beta^A)$ gives the Witt class of $(T, \lambda, \mu; U)$. The boundary map

$$\partial : Q_{2k+2}(B^A, \beta^A) \rightarrow L_{2k+1}(A); (T, \lambda, \mu; U) \mapsto (Q, \psi; F, G)$$

sends the linking Arf invariant to the Witt class of the $(-1)^k$ -quadratic formation $(Q, \psi; F, G)$ constructed in 78(iv).

Theorem 80. *Let A be an r -even ring with A_2 -module basis $\{x_1 = 1, x_2, \dots, x_r\} \subset \widehat{H}^0(\mathbb{Z}_2; A)$, and let*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_r \end{pmatrix} \in \text{Sym}_r(A),$$

so that by Theorem 60

$$Q_{2k}(B^A, \beta^A) = \begin{cases} \frac{\{M \in \text{Sym}_r(A) \mid M - MXM \in \text{Quad}_r(A)\}}{4\text{Quad}_r(A) + \{2(N + N^t) - N^t X N \mid N \in M_r(A)\}} & \text{if } k = 0, \\ 0 & \text{if } k = 1. \end{cases}$$

(i) *Let*

$$S = (2)^\infty \subset A,$$

so that

$$S^{-1}A = A[1/2]$$

and $\widehat{H}^0(\mathbb{Z}_2; A)$ is an (A, S) -module. The hyperquadratic L -group $\widehat{L}^0(A)$ fits into the exact sequence

$$\dots \rightarrow L^1(A, S) \rightarrow \widehat{L}^0(A) \rightarrow L_0(A, S) \rightarrow L^0(A, S) \rightarrow \dots$$

The linking Arf invariant of a nonsingular quadratic linking form (T, λ, μ) over (A, S) with a lagrangian $U \subset T$ for (T, λ) is the Witt class

$$(T, \lambda, \mu; U) \in Q_0(B^A, \beta^A) = \widehat{L}^{4*}(A).$$

(ii) Given $M \in \text{Sym}_r(A)$ such that $M - MXM \in \text{Quad}_r(A)$ let (T_M, λ_M, μ_M) be the nonsingular quadratic linking form over (A, S) corresponding to the (-1) -quadratic S -formation over A (76)

$$(Q_M, \psi_M; F_M, G_M)$$

$$= \left(H_-(A^{2r}); A^{2r}, \text{im} \left(\left(\begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix} \right) : A^{2r} \rightarrow A^{2r} \oplus A^{2r} \right) \right)$$

and let

$$U_M = (A_2)^r \subset T_M = Q_M / (F_M + G_M) = \text{coker}(G_M \rightarrow F_M^*)$$

be the lagrangian for the nonsingular symmetric linking form (T_M, λ_M) over (A, S) with the inclusion $U_M \rightarrow T_M$ resolved by

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^r & \xrightarrow{2I} & A^r & \longrightarrow & U_M \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & A^r \oplus A^r & \xrightarrow{\begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix}} & A^r \oplus A^r & \longrightarrow & T_M \longrightarrow 0 \end{array}$$

The function

$$Q_0(B^A, \beta^A) \rightarrow \widehat{L}^{4*}(A); \quad M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

is an isomorphism, with inverse given by the linking Arf invariant.

(iii) Let (T, λ, μ) be a nonsingular quadratic linking form over (A, S) together with a lagrangian $U \subset T$ for (T, λ) . For any f.g. projective A -module resolution of U

$$0 \rightarrow B_1 \xrightarrow{d} B_0 \rightarrow U \rightarrow 0$$

let

$$\delta \in \text{Sym}(B_0), \quad \phi \in \text{Sym}(B^0), \quad \beta = [\delta] = \mu|_U \in \widehat{Q}^0(B^{0-*})$$

$$= \text{Hom}_A(U, \widehat{H}^0(\mathbb{Z}_2; A))$$

be as in Proposition 78(i) and (v), so that

$$d^*\delta d \in \text{Quad}(B_1), \quad \phi - \phi\delta\phi \in \text{Quad}(B^0)$$

and

$$\phi \in Q_0(B, \beta) = \frac{\ker(J_\delta : \text{Sym}(B^0) \rightarrow \text{Sym}(B^0)/\text{Quad}(B^0))}{\text{im}((d^*)^\% : \text{Quad}(B^1) \rightarrow \text{Sym}(B^0))}$$

classifies $(T, \lambda, \mu; U)$. Lift $\beta : U \rightarrow \widehat{H}^0(\mathbb{Z}_2; A)$ to an A -module morphism $g : B_0 \rightarrow A^r$ such that

$$gd(B_1) \subseteq 2A^r, \quad \delta = g^*Xg \in \widehat{H}^0(\mathbb{Z}_2; S(B^0), T) = \text{Sym}(B^0)/\text{Quad}(B^0).$$

The linking Arf invariant is

$$(T, \lambda, \mu; U) = g\phi g^* \in Q_0(B^A, \beta^A).$$

(iv) For any $M = (m_{ij}) \in \text{Sym}_r(A)$ with $m_{ij} \in 2A$

$$M - MXM = 2(M/2 - 2(M/2)X(M/2)) \in \text{Quad}_r(A)$$

and so M represents an element $M \in Q_0(B^A, \beta^A)$. The invertible matrix

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} \in M_{2r}(A)$$

is such that

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & 2I \\ 2I & vM \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix} \begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix}$$

so that $(Q_M, \psi_M; F_M, G_M)$ is isomorphic to the (-1) -quadratic S -formation

$$(Q'_M, \psi'_M; F'_M, G'_M) = \left(H_-(A^{2r}); A^{2r}, \text{im} \left(\begin{pmatrix} \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix} \\ \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} \end{pmatrix} : A^{2r} \rightarrow A^{2r} \oplus A^{2r} \right) \right),$$

corresponding to the nonsingular quadratic linking form over (A, S)

$$(T'_M, \lambda'_M, \mu'_M) = \left((A_2)^r \oplus (A_2)^r, \begin{pmatrix} -M/4 & I/2 \\ I/2 & 0 \end{pmatrix}, \begin{pmatrix} -M/4 \\ -X \end{pmatrix} \right),$$

with $2T'_M = 0$, and $U'_M = 0 \oplus (A_2)^r \subset T'_M$ a lagrangian for the symmetric linking form (T'_M, λ'_M) . The linking Arf invariant of $(T'_M, \lambda'_M, \mu'_M; U'_M)$ is

$$(T'_M, \lambda'_M, \mu'_M; U'_M) = M \in Q_0(B^A, \beta^A).$$

Proof. (i) $\widehat{H}^0(\mathbb{Z}_2; A)$ has an $S^{-1}A$ -contractible f.g. free A -module resolution

$$0 \longrightarrow A^r \xrightarrow{2} A^r \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \longrightarrow 0.$$

The exact sequence for $\widehat{L}^0(A)$ is given by the exact sequence of Proposition 74(iii)

$$\dots \rightarrow L^{4*+1}(A, S) \rightarrow Q_0(B^A, \beta^A) \rightarrow L_0(A, S) \rightarrow L^{4*}(A, S) \rightarrow \dots$$

and the isomorphism $Q_0(B^A, \beta^A) \cong \widehat{L}^{4*}(A)$.

(ii) The isomorphism

$$Q_0(B^A, \beta^A) \rightarrow \widehat{L}^{4*}(A); \quad M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

is given by Proposition 46.

(iii) Combine (ii) and Proposition 78.

(iv) By construction. $\square$

5. Application to UNil

5.1. Background

The topological context for the unitary nilpotent L -groups UNil_* is the following. Let N^n be a closed connected manifold together with a decomposition into n -dimensional connected submanifolds $N_-, N_+ \subset N$ such that

$$N = N_- \cup N_+$$

and

$$N_\cap = N_- \cap N_+ = \partial N_- = \partial N_+ \subset N$$

is a connected $(n - 1)$ -manifold with $\pi_1(N_\cap) \rightarrow \pi_1(N_\pm)$ injective. Then

$$\pi_1(N) = \pi_1(N_-) *_{\pi_1(N_\cap)} \pi_1(N_+),$$

with $\pi_1(N_\pm) \rightarrow \pi_1(N)$ injective. Let M be an n -manifold. A homotopy equivalence $f : M \rightarrow N$ is called *splittable along $N_\cap$* if it is homotopic to a map f' , transverse regular to $N_\cap$ (whence $f'^{-1}(N_\cap)$ is an $(n - 1)$ -dimensional submanifold of M), and whose restriction $f'^{-1}(N_\cap) \rightarrow N_\cap$, and a fortiori also $f'^{-1}(N_\pm) \rightarrow N_\pm$, is a homotopy equivalence.

We ask the following question: given a simple homotopy equivalence $f : M \rightarrow N$, when is M h -cobordant to a manifold M' such that the induced homotopy equivalence $f' : M' \rightarrow N$ is splittable along $N_\cap$? The answer is given by Cappell [5,6]: the problem has a positive solution if and only if a Whitehead torsion obstruction

$$\overline{\Phi}(\tau(f)) \in \widehat{H}^n(\mathbb{Z}_2; \ker(\widetilde{K}_0(A) \rightarrow \widetilde{K}_0(B_+) \oplus \widetilde{K}_0(B_-)))$$

(which is 0 if f is simple) and an algebraic L -theory obstruction

$$\chi^h(f) \in \text{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

vanish, where

$$A = \mathbb{Z}[\pi_1(N_\cap)], \quad B_\pm = \mathbb{Z}[\pi_1(N_\pm)], \quad \mathcal{N}_\pm = B_\pm - A.$$

The groups $\text{UNil}_*(A; \mathcal{N}_-, \mathcal{N}_+)$ are 4-periodic and 2-primary, and vanish if the inclusions $\pi_1(N_\cap) \hookrightarrow \pi_1(N_\pm)$ are square root closed. The groups $\text{UNil}_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ arising from the expression of the infinite dihedral group as a free product

$$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$$

are of particular interest. Cappell [3] showed that

$$\text{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \text{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}_2 - \{1\}], \mathbb{Z}[\mathbb{Z}_2 - \{1\}])$$

contains $(\mathbb{Z}_2)^\infty$, and deduced that there is a manifold homotopy equivalent to the connected sum $\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ which does not have a compatible connected sum decomposition. With

$$B = \mathbb{Z}[\pi_1(N)] = B_1 *_A B_2,$$

the map

$$\text{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow L_{n+1}(B)$$

given by sending the splitting obstruction $\chi^h(f)$ to the surgery obstruction of an $(n+1)$ -dimensional normal map between f and a split homotopy equivalence, is a split monomorphism, and

$$L_{n+1}(B) = L_{n+1}^K(A \rightarrow B_+ \cup B_-) \oplus \text{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

with $K = \ker(\tilde{K}_0(A) \rightarrow \tilde{K}_0(B_+) \oplus \tilde{K}_0(B_-))$. Farrell [11] established a factorization of this map as

$$\text{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow \text{UNil}_{n+1}(B; B, B) \longrightarrow L_{n+1}(B).$$

Thus the groups $\text{UNil}_n(A; A, A)$ for any ring A with involution acquire special importance, and we shall use the usual abbreviation

$$\text{UNil}_n(A) = \text{UNil}_n(A; A, A).$$

Cappell [3–5] proved that $\text{UNil}_{4k}(\mathbb{Z}) = 0$ and that $\text{UNil}_{4k+2}(\mathbb{Z})$ is infinitely generated. Farrell [11] showed that for any ring A , $4\text{UNil}_*(A) = 0$. Connolly and Koźniewski [9] obtained $\text{UNil}_{4k+2}(\mathbb{Z}) = \bigoplus_1^\infty \mathbb{Z}_2$.

For any ring with involution A let NL_* denote the L -theoretic analogues of the nilpotent K -groups

$$NK_*(A) = \ker(K_*(A[x]) \rightarrow K_*(A)),$$

that is

$$NL_*(A) = \ker(L_*(A[x]) \rightarrow L_*(A)),$$

where $A[x] \rightarrow A$ is the augmentation map $x \mapsto 0$. Ranicki [16, 7.6] used the geometric interpretation of $\text{UNil}_*(A)$ to identify $NL_*(A) = \text{UNil}_*(A)$ in the case when $A = \mathbb{Z}[\pi]$ is the integral group ring of a finitely presented group π . The following was obtained by pure algebra:

Proposition 81 (Connolly and Ranicki [10]). *For any ring with involution A*

$$\text{UNil}_*(A) \cong NL_*(A).$$

It was further shown in [10] that $\text{UNil}_1(\mathbb{Z}) = 0$ and $\text{UNil}_3(\mathbb{Z})$ was computed up to extensions, thus showing it to be infinitely generated.

Connolly and Davis [8] related $\text{UNil}_3(\mathbb{Z})$ to quadratic linking forms over $\mathbb{Z}[x]$ and computed the Grothendieck group of the latter. By Proposition 81

$$\text{UNil}_3(\mathbb{Z}) \cong \ker(L_3(\mathbb{Z}[x]) \rightarrow L_3(\mathbb{Z})) = L_3(\mathbb{Z}[x])$$

using the classical fact $L_3(\mathbb{Z}) = 0$. From a diagram chase one gets

$$L_3(\mathbb{Z}[x]) \cong \ker(L_0(\mathbb{Z}[x], (2)^\infty) \rightarrow L_0(\mathbb{Z}, (2)^\infty)).$$

By definition, $L_0(\mathbb{Z}[x], (2)^\infty)$ is the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$, with $2^n T = 0$ for some $n \geq 1$. Let $\mathcal{L}(\mathbb{Z}[x], 2)$ be a similar Witt group, the difference being that the underlying module T is required to satisfy $2T = 0$. The main results of [8] are

$$L_0(\mathbb{Z}[x], (2)^\infty) \cong \mathcal{L}(\mathbb{Z}[x], 2)$$

and

$$\mathcal{L}(\mathbb{Z}[x], 2) \cong \frac{x\mathbb{Z}_4[x]}{\{2(p^2 - p) \mid p \in x\mathbb{Z}_4[x]\}} \oplus \mathbb{Z}_2[x].$$

By definition, a ring A is *one-dimensional* if it is hereditary and noetherian, or equivalently if every submodule of a f.g. projective A -module is f.g. projective. In particular, a Dedekind ring A is one-dimensional. The symmetric and hyperquadratic L -groups of a one-dimensional A are 4-periodic

$$L^n(A) = L^{n+4}(A), \quad \widehat{L}^n(A) = \widehat{L}^{n+4}(A).$$

Proposition 82 (Connolly and Ranicki [10]). *For any one-dimensional ring A with involution*

$$Q_{n+1}(B^{A[x]}, \beta^{A[x]}) = Q_{n+1}(B^A, \beta^A) \oplus \text{UNil}_n(A) \quad (n \in \mathbb{Z}).$$

Proof. For any ring with involution A the inclusion $A \rightarrow A[x]$ and the augmentation $A[x] \rightarrow A; x \mapsto 0$ determine a functorial splitting of the exact sequence

$$\cdots \rightarrow L_n(A[x]) \rightarrow L^n(A[x]) \rightarrow \widehat{L}^n(A[x]) \rightarrow L_{n-1}(A[x]) \rightarrow \cdots$$

as a direct sum of the exact sequences

$$\begin{aligned} \cdots \rightarrow L_n(A) \rightarrow L^n(A) \rightarrow \widehat{L}^n(A) \rightarrow L_{n-1}(A) \rightarrow \cdots, \\ \cdots \rightarrow NL_n(A) \rightarrow NL^n(A) \rightarrow N\widehat{L}^n(A) \rightarrow NL_{n-1}(A) \rightarrow \cdots. \end{aligned}$$

with $\widehat{L}^{n+4*}(A) = Q_n(B^A, \beta^A)$. It is proved in [10] that for a one-dimensional A

$$L^n(A[x]) = L^n(A), \quad NL^n(A) = 0, \quad N\widehat{L}^{n+1}(A) = NL_n(A) = \text{UNil}_n(A). \quad \square$$

Example 83. Proposition 82 applies to $A = \mathbb{Z}$, so that

$$Q_{n+1}(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = Q_{n+1}(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \oplus \text{UNil}_n(\mathbb{Z})$$

with $Q_*(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) = \widehat{L}^*(\mathbb{Z})$ as given by Example 62.

5.2. The computation of $Q_*(B^{A[x]}, \beta^{A[x]})$ for 1-even A with $\psi^2 = 1$

We shall now compute the groups

$$\widehat{L}^n(A[x]) = Q_n(B^{A[x]}, \beta^{A[x]}) \quad (n \pmod{4})$$

for a 1-even ring A with $\psi^2 = 1$. The special case $A = \mathbb{Z}$ computes

$$\widehat{L}^n(\mathbb{Z}[x]) = Q_n(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \widehat{L}^n(\mathbb{Z}) \oplus \text{UNil}_{n-1}(\mathbb{Z}).$$

Proposition 84. The universal chain bundle over $A[x]$ is given by

$$(B^{A[x]}, \beta^{A[x]}) = \bigoplus_{i=-\infty}^{\infty} (C(X), \gamma(X))_{*+2i},$$

with $(C(X), \gamma(X))$ the chain bundle over $A[x]$ given by the construction of (53) for

$$X = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \text{Sym}_2(A[x]).$$

The twisted quadratic Q -groups of $(B^{A[x]}, \beta^{A[x]})$ are

$$Q_n(B^{A[x]}, \beta^{A[x]}) = \begin{cases} Q_0(C(X), \gamma(X)) = \frac{\{M \in \text{Sym}_2(A[x]) \mid M - MXM \in \text{Quad}_2(A[x])\}}{4\text{Quad}_2(A[x]) + \{2(N + N^t) - N^t X N \mid N \in M_2(A[x])\}} & \text{if } n = 0, \\ \text{im}(N_{\gamma(X)} : Q_1(C(X), \gamma(X)) \rightarrow Q^1(C(X))) = \ker(J_{\gamma(X)} : Q^1(C(X)) \rightarrow \widehat{Q}^1(C(X))) \\ = \frac{\{N \in M_2(A[x]) \mid N + N^t \in 2\text{Sym}_2(A[x]), \frac{1}{2}(N + N^t) - N^t X N \in \text{Quad}_2(A[x])\}}{2M_2(A[x])} & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ Q_{-1}(C(X), \gamma(X)) = \frac{\text{Sym}_2(A[x])}{\text{Quad}_2(A[x]) + \{L - LXL \mid L \in \text{Sym}_2(A[x])\}} & \text{if } n = 3. \end{cases}$$

Proof. A special case of Theorem 60, noting that by Proposition 59 $A[x]$ is 2-even, with $\{1, x\}$ an $A_2[x]$ -module basis for $\widehat{H}^0(\mathbb{Z}_2; A[x])$. $\square$

Our strategy for computing $Q_*(B^{A[x]}, \beta^{A[x]})$ will be to first compute $Q_*(C(1), \gamma(1))$, $Q_*(C(x), \gamma(x))$ and then to compute $Q_*(C(X), \gamma(X))$ for

$$(C(X), \gamma(X)) = (C(1), \gamma(1)) \oplus (C(x), \gamma(x))$$

using the exact sequence given by Proposition 38(ii)

$$\begin{aligned} \cdots \rightarrow H_{n+1}(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_n(C(1), \gamma(1)) \oplus Q_n(C(x), \gamma(x)) \\ \rightarrow Q_n(C(X), \gamma(X)) \rightarrow H_n(C(1) \otimes_{A[x]} C(x)) \rightarrow \cdots \end{aligned}$$

The connecting maps ∂ have components

$$\begin{aligned} \partial(1) : H_{n+1}(C(1) \otimes_{A[x]} C(x)) &\rightarrow \widehat{Q}^{n+1}(C(1)) \rightarrow Q_n(C(1), \gamma(1)) \\ (f(1) : C(x)^{n+1-*} \rightarrow C(1)) &\mapsto (0, \widehat{f(1)}^{\%}(S^{n+1}\gamma(x))), \\ \partial(x) : H_{n+1}(C(1) \otimes_{A[x]} C(x)) &\rightarrow \widehat{Q}^{n+1}(C(x)) \rightarrow Q_n(C(x), \gamma(x)) \\ (f(x) : C(1)^{n+1-*} \rightarrow C(x)) &\mapsto (0, \widehat{f(x)}^{\%}(S^{n+1}\gamma(1))). \end{aligned}$$

Proposition 85. (i) *The twisted quadratic Q -groups*

$$Q_n(C(1), \gamma(1)) = \begin{cases} \frac{A[x]}{2A[x] + \{a - a^2 \mid a \in A[x]\}} & \text{if } n = -1, \\ \frac{\{a \in A[x] \mid a - a^2 \in 2A[x]\}}{8A[x] + \{4b - 4b^2 \mid b \in A[x]\}} & \text{if } n = 0, \\ \frac{\{a \in A[x] \mid a - a^2 \in 2A[x]\}}{2A[x]} & \text{if } n = 1, \end{cases}$$

(as given by Theorem 54) are such that

$$Q_n(C(1), \gamma(1)) \cong \begin{cases} A_2[x] & \text{if } n = -1, \\ A_8 \oplus A_4[x] \oplus A_2[x] & \text{if } n = 0, \\ A_2 & \text{if } n = 1, \end{cases}$$

with isomorphisms

$$\begin{aligned} f_{-1}(1) : \mathcal{Q}_{-1}(C(1), \gamma(1)) &\rightarrow A_2[x]; \quad \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j} \right) x^{i+1}, \\ f_0(1) : \mathcal{Q}_0(C(1), \gamma(1)) &\rightarrow A_8 \oplus A_4[x] \oplus A_2[x]; \\ \sum_{i=0}^{\infty} a_i x^i &\mapsto \left(a_0, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j/2} \right) x^i, \sum_{k=0}^{\infty} (a_{2k+2/2}) x^k \right), \\ f_1(1) : \mathcal{Q}_1(C(1), \gamma(1)) &\rightarrow A_2; \quad a = \sum_{i=0}^{\infty} a_i x^i \mapsto a_0. \end{aligned}$$

The connecting map components $\partial(1)$ are given by

$$\begin{aligned} \partial(1) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] &\rightarrow \mathcal{Q}_0(C(1), \gamma(1)); \quad c \mapsto (0, 2c, 0), \\ \partial(1) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] &\rightarrow \mathcal{Q}_{-1}(C(1), \gamma(1)); \quad c \mapsto cx. \end{aligned}$$

(ii) The twisted quadratic $\mathcal{Q}$ -groups

$$\mathcal{Q}_n(C(x), \gamma(x)) = \begin{cases} \frac{A[x]}{2A[x] + \{a - a^2x \mid a \in A[x]\}} & \text{if } n = -1, \\ \frac{\{a \in A[x] \mid a - a^2x \in 2A[x]\}}{8A[x] + \{4b - 4b^2x \mid b \in A[x]\}} & \text{if } n = 0, \\ \frac{\{a \in A[x] \mid a - a^2x \in 2A[x]\}}{2A[x]} & \text{if } n = 1 \end{cases}$$

(as given by Theorem 54) are such that

$$\mathcal{Q}_n(C(x), \gamma(x)) \cong \begin{cases} A_2[x] & \text{if } n = -1, \\ A_4[x] \oplus A_2[x] & \text{if } n = 0, \\ 0 & \text{if } n = 1, \end{cases}$$

with isomorphisms

$$\begin{aligned} f_{-1}(x) : \mathcal{Q}_{-1}(C(x), \gamma(x)) &\rightarrow A_2[x]; \quad a = \sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j-1} \right) x^i, \\ f_0(x) : \mathcal{Q}_0(C(x), \gamma(x)) &\rightarrow A_4[x] \oplus A_2[x]; \\ \sum_{i=0}^{\infty} a_i x^i &\mapsto \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j-1/2} \right) x^i, \sum_{k=0}^{\infty} (a_{2k+1/2}) x^k \right). \end{aligned}$$

The connecting map components $\partial(x)$ are given by

$$\partial(x) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \rightarrow Q_0(C(x), \gamma(x)); \quad c \mapsto (2c, 0),$$

$$\partial(x) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \rightarrow Q_{-1}(C(x), \gamma(x)); \quad c \mapsto c.$$

Proof. (i) We start with $Q_1(C(1), \gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that $a(x) - a(x)^2 \in 2A[x]$ if and only if

$$a_{2i+1}, \quad a_{2i+2} - (a_{i+1})^2 \in 2A \quad (i \geq 0),$$

if and only if $a_k \in 2A$ for all $k \geq 1$, so that $f_1(1)$ is an isomorphism.

Next, we consider $Q_{-1}(C(1), \gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a(x) \in 2A[x] + \{b(x) - b(x)^2 \mid b(x) \in A[x]\}$$

if and only if there exist $b_1, b_2, \dots \in A$ such that

$$a_0 = 0, \quad a_1 = b_1, \quad a_2 = b_2 - b_1, \quad a_3 = b_3, \quad a_4 = b_4 - b_2, \quad \dots \in A_2,$$

if and only if

$$a_0 = \sum_{j=0}^{\infty} a_{(2i+1)2^j} = 0 \in A_2 \quad (i \geq 0)$$

(with $b_{(2i+1)2^j} = \sum_{k=0}^j a_{(2i+1)2^k} \in A_2$ for any $i, j \geq 0$). Thus $f_{-1}(1)$ is well-defined and injective. The morphism $f_{-1}(1)$ is surjective, since

$$\sum_{i=0}^{\infty} c_i x^i = f_{-1}(1) \left(c_0 + \sum_{i=0}^{\infty} c_{i+1} x^{2i+1} \right) \in A_2[x] \quad (c_i \in A).$$

The map $\widehat{Q}^1(C(1)) \rightarrow Q_0(C(1), \gamma(1))$ is given by

$$\begin{aligned} \widehat{Q}^1(C(1)) &= A_2[x] \rightarrow Q_0(C(1), \gamma(1)) = A_8 \oplus A_4[x] \oplus A_2[x], \\ a &= \sum_{i=0}^{\infty} a_i x^i \mapsto \left(4a_0, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j} \right) x^i, 0 \right). \end{aligned}$$

If $a = c^2x$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\left(4a_0, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} 2a_{(2i+1)2j} \right) x^i \right) = (0, 2c) \in A_8 \oplus A_4[x],$$

so that the composite

$$\begin{aligned} \partial(1) : H_1(C(1) \otimes_{A[x]} C(x)) &= A_2[x] \rightarrow \widehat{Q}^1(C(1)) \rightarrow Q_0(C(1), \gamma(1)) \\ &= A_8 \oplus A_4[x] \oplus \mathbb{Z}_2[x] \end{aligned}$$

is given by $c \mapsto (0, 2c, 0)$.

Next, we consider $Q_0(C(1), \gamma(1))$. A polynomial $a(x) \in A \oplus 2xA[x]$ is such that

$$a(x) \in 8A[x] + \{4(b(x) - b(x)^2) \mid b(x) \in A[x]\},$$

if and only if there exist $b_1, b_2, \dots \in A$ such that

$$a_0 = 0, \quad a_1 = 4b_1, \quad a_2 = 4(b_2 - b_1), \quad a_3 = 4b_3, \quad a_4 = 4(b_4 - b_2), \dots \in A_8,$$

if and only if

$$a_1 = a_2 = a_3 = a_4 = \dots = 0 \in A_4,$$

$$a_0 = \sum_{j=0}^{\infty} a_{(2i+1)2j} = 0 \in A_8 \quad (i \geq 0).$$

Thus $f_0(1)$ is well-defined and injective. The morphism $f_0(1)$ is surjective, since

$$\begin{aligned} \left(a_0, \sum_{i=0}^{\infty} b_i x^i, \sum_{i=0}^{\infty} c_i x^i \right) &= f_0(1) \left(a_0 + 2 \sum_{i=0}^{\infty} b_i x^{2i+1} + 2 \sum_{i=0}^{\infty} c_i x^{2i+2} \right) \\ &\in A_8 \oplus A_4[x] \oplus A_2[x] \quad (a_0, b_i, c_i \in A). \end{aligned}$$

The map $\widehat{Q}^0(C(1)) \rightarrow Q_{-1}(C(1), \gamma(1))$ is given by

$$\widehat{Q}^0(C(1)) = A_2[x] \rightarrow Q_{-1}(C(1), \gamma(1)) = A_2[x],$$

$$a = \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j} \right) x^{i+1}.$$

If $a = c^2x$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$a_0 + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j} \right) x^{i+1} = cx \in A_2[x],$$

so that the composite

$$\partial(1) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \rightarrow \widehat{Q}^0(C(1)) \rightarrow Q_{-1}(C(1), \gamma(1)) = A_2[x]$$

is given by $c \mapsto cx$.

(ii) We start with $Q_1(C(x), \gamma(x))$. For any polynomial $a = \sum_{i=0}^{\infty} a_i x^i \in A[x]$

$$a - a^2x = \sum_{i=0}^{\infty} a_i x^i - \sum_{i=0}^{\infty} a_i x^{2i+1} \in A_2[x].$$

Now $a - a^2x \in 2A[x]$ if and only if the coefficients $a_0, a_1, \dots \in A$ are such that

$$a_0 = a_1 - a_0 = a_2 = a_3 - a_1 = \dots = 0 \in A_2,$$

if and only if

$$a_0 = a_1 = a_2 = a_3 = \dots = 0 \in A_2.$$

It follows that $Q_1(C(x), \gamma(x)) = 0$.

Next, $Q_{-1}(C(x), \gamma(x))$. A polynomial $a = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a \in 2A[x] + \{b - b^2x \mid v \in A[x]\},$$

if and only if there exist $b_0, b_1, \dots \in A$ such that

$$a_0 = b_0, \quad a_1 = b_1 - b_0, \quad a_2 = b_2, \quad a_3 = b_3 - b_1, \quad a_4 = b_4, \quad \dots \in A_2,$$

if and only if

$$\sum_{j=0}^{\infty} a_{(2i+1)2j-1} = 0 \in A_2 \quad (i \geq 0).$$

Thus $f_{-1}(x)$ is well-defined and injective. The morphism $f_{-1}(x)$ is surjective, since

$$\begin{aligned} \sum_{i=0}^{\infty} c_i x^i &= f_{-1}(x) \left(\sum_{i=0}^{\infty} c_i x^{2i} \right) \\ &\in A_2[x] \quad (c_i \in A). \end{aligned}$$

The map $\widehat{Q}^0(C(x)) \rightarrow Q_{-1}(C(x), \gamma(x))$ is given by

$$\begin{aligned} \widehat{Q}^0(C(x)) &= A_2[x] \rightarrow Q_{-1}(C(x), \gamma(x)) = A_2[x]; \\ b &= \sum_{i=0}^{\infty} b_i x^i \mapsto \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1} \right) x^i. \end{aligned}$$

If $b = c^2$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1} \right) x^i = c \in A_2[x],$$

so that the composite

$$\partial(x) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \rightarrow \widehat{Q}^0(C(x)) \rightarrow Q_{-1}(C(x), \gamma(x)) = A_2[x]$$

is just the identity $c \mapsto c$.

Next, $Q_0(C(x), \gamma(x))$. For any $a \in A[x]$

$$a \in 8A[x] + \{4(b - b^2x) \mid b \in A[x]\},$$

if and only there exist $b_0, b_1, \dots \in A$ such that

$$a_0 = 4b_0, \quad a_1 = 4(b_1 - b_0), \quad a_2 = 4b_2, \quad a_3 = 4(b_3 - b_1), \quad \dots \in A_8,$$

if and only if

$$a_0 = a_1 = a_2 = a_3 = \dots = 0 \in A_4,$$

$$\sum_{j=0}^{\infty} a_{(2i+1)2^j-1} = 0 \in A_8 \quad (i \geq 0).$$

Thus $f_0(x)$ is well-defined and injective. The morphism $f_0(x)$ is surjective, since

$$\left(\sum_{i=0}^{\infty} c_i x^i, \sum_{i=0}^{\infty} d_i x^i \right) = f_0(x) \left(\sum_{i=0}^{\infty} 2c_i x^{2i} + \sum_{i=0}^{\infty} 2d_i x^{2i+1} \right) \\ \in A_4[x] \oplus A_2[x] \quad (c_i, d_i \in A).$$

The map $\widehat{Q}^1(C(x)) \rightarrow Q_0(C(x), \gamma(x))$ is given by

$$\widehat{Q}^1(C(x)) = A_2[x] \rightarrow Q_0(C(x), \gamma(x)) = A_4[x] \oplus A_2[x]; \\ b = \sum_{i=0}^{\infty} b_i x^i \mapsto \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} 2b_{(2i+1)2^j-1} \right) x^i, 0 \right).$$

If $b = c^2$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j} x^i \right) = 2c \in A_4[x],$$

so that the composite

$$\partial(x) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \rightarrow \widehat{Q}^1(C(x)) \rightarrow Q_0(C(x), \gamma(x)) \\ = A_4[x] \oplus A_2[x]$$

is given by $c \mapsto (2c, 0)$.

We can now prove Theorem B:

Theorem 86. *The hyperquadratic L-groups of $A[x]$ for a 1-even A with $\psi^2 = 1$ are given by*

$$\widehat{L}^n(A[x]) = Q_n(B^{A[x]}, \beta^{A[x]}) = \begin{cases} A_8 \oplus A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 0 \pmod{4}, \\ A_2 & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ A_2[x] & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(i) For $n = 0$

$$Q_0(B^{A[x]}, \beta^{A[x]}) = \frac{\{M \in \text{Sym}_2(A[x]) \mid M - MXM \in \text{Quad}_2(A[x])\}}{4\text{Quad}_2(A[x]) + \{2(N + N^t) - 4N^t XN \mid N \in M_2(A[x])\}}.$$

An element $M \in Q_0(B^{A[x]}, \beta^{A[x]})$ is represented by a matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \text{Sym}_2(A[x]) \quad \left(a = \sum_{i=0}^{\infty} a_i x^i, c = \sum_{i=0}^{\infty} c_i x^i \in A[x] \right),$$

with $a - a_0, b, c \in 2A[x]$. The isomorphism

$$Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} \widehat{L}^0(A[x]) = \widehat{L}^1(A[x], (2)^\infty); \quad M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

sends M to the Witt class of the nonsingular quadratic linking form (T_M, λ_M, μ_M) over $(A[x], (2)^\infty)$ with a lagrangian $U_M \subset T_M$ for (T_M, λ_M) corresponding to the (-1) -quadratic $(2)^\infty$ -formation over $A[x]$

$$\partial(M) = \left(H_-(A[x]^4); A[x]^4, \text{im} \left(\begin{pmatrix} I & 0 \\ -2X & I - XM \\ 0 & 2I \\ 2I & M \end{pmatrix} : A[x]^4 \rightarrow A[x]^4 \oplus A[x]^4 \right) \right)$$

(80), with

$$\partial : Q_0(B^{A[x]}, \beta^{A[x]}) = \widehat{L}^0(A[x]) \rightarrow L_{-1}(A[x]); \quad M \mapsto \partial(M).$$

The inverse isomorphism is defined by the linking Arf invariant (79). Writing

$$2\Delta : A_2[x] \rightarrow A_4[x] \oplus A_4[x]; \quad d \mapsto (2d, 2d),$$

there are defined isomorphisms

$$Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} A_8 \oplus \text{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x];$$

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \quad (c' = c - b^2)$$

$$\mapsto \left(a_0, \left[\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j/2} \right) x^i, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c'_{(2i+1)2j-1/2} \right) x^i \right], \sum_{k=0}^{\infty} (a_{2k+2}/2) x^k, \sum_{k=0}^{\infty} (c'_{2k+1}/2) x^k \right),$$

$$\text{coker}(2\Delta) \xrightarrow{\cong} A_4[x] \oplus A_2[x]; \quad [d, e] \mapsto (d - e, d).$$

In particular $M \in Q_0(B^{A[x]}, \beta^{A[x]})$ can be represented by a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix}$.

(ii) For $n = 1$:

$$\begin{aligned} Q_1(B^{A[x]}, \beta^{A[x]}) \\ = \frac{\{N \in M_2(A[x]) \mid N + N^t \in 2\text{Sym}_2(A[x]), \frac{1}{2}(N + N^t) - N^t X N \in \text{Quad}_2(A[x])\}}{2M_2(A[x])} \end{aligned}$$

and there is defined an isomorphism

$$Q_1(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} Q_1(B^A, \beta^A) = A_2; \quad N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a_0,$$

with

$$\begin{aligned} \partial : Q_1(B^{A[x]}, \beta^{A[x]}) &= \widehat{L}^1(A[x]) = A_2 \rightarrow L_0(A[x]); \\ a_0 &\mapsto A[x] \otimes_A \left(A \oplus A, \begin{pmatrix} a_0(a_0 - 1)/2 & 1 - 2a_0 \\ 0 & -2 \end{pmatrix} \right). \end{aligned}$$

(iii) For $n = 2$:

$$Q_2(B^{A[x]}, \beta^{A[x]}) = 0.$$

(iv) For $n = 3$:

$$Q_3(B^{A[x]}, \beta^{A[x]}) = \frac{\text{Sym}_2(A[x])}{\text{Quad}_2(A[x]) + \{M - MXM \mid M \in \text{Sym}_2(A[x])\}}.$$

There is defined an isomorphism

$$\begin{aligned} Q_3(B^{A[x]}, \beta^{A[x]}) &\xrightarrow{\cong} A_2[x]; \\ M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &= \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} \mapsto d_0 + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} d_{(2i+1)2j} \right) x^{i+1} \\ (a' = a - b^2x, \quad c' = c - b^2 \in A[x], \quad d = a' + c'x = a + cx \in A_2[x]). \end{aligned}$$

The isomorphism

$$Q_3(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} \widehat{L}^3(A[x]); \quad M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto (K_M, \psi_M; L_M)$$

sends M to the Witt class of the nonsingular (-1) -quadratic form over $A[x]$

$$(K_M, \psi_M) = \left(A[x]^2 \oplus A[x]^2, \begin{pmatrix} X & 1 \\ 0 & M \end{pmatrix} \right),$$

with a lagrangian $L_M = A[x]^2 \oplus 0 \subset K_M$ for $(K_M, \psi_M - \psi_M^*)$ (70), and

$$\partial : Q_3(B^{A[x]}, \beta^{A[x]}) = \widehat{L}^3(A[x]) \rightarrow L_2(A[x]); \quad M \mapsto (K_M, \psi_M).$$

In particular $M \in Q_3(B^{A[x]}, \beta^{A[x]})$ can be represented by a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix}$. The inverse isomorphism is defined by the generalized Arf invariant (66).

Proof. Proposition 84 expresses $Q_n(B^{A[x]}, \beta^{A[x]})$ in terms of 2×2 matrices. We deal with the four cases separately.

(i) Let $n = 0$. Proposition 85 gives an exact sequence

$$0 \rightarrow H_1(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)) \rightarrow Q_0(C(X), \gamma(X)) \rightarrow 0$$

with

$$\begin{aligned} H_1(C(1) \otimes_{A[x]} C(x)) &= A_2[x] \\ \rightarrow Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)) &= (A_8 \oplus A_4[x] \oplus A_2[x]) \oplus (A_4[x] \oplus A_2[x]); \\ x &\mapsto ((0, 2c, 0), (2c, 0), \end{aligned}$$

so that there is defined an isomorphism

$$\text{coker}(\partial) \xrightarrow{\cong} A_8 \oplus \text{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x]; \quad (s, t, u, v, w) \mapsto (s, [t, v], u, w).$$

We shall define an isomorphism $Q_0(C(X), \gamma(X)) \cong \text{coker}(\partial)$ by constructing a splitting map

$$Q_0(C(X), \gamma(X)) \rightarrow Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)).$$

An element in $Q_0(C(X), \gamma(X))$ is represented by a symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \text{Sym}_2(A[x]),$$

such that

$$M - MXM = \begin{pmatrix} a - a^2 - b^2x & b - ab - bcx \\ b - ab - bcx & c - b^2 - c^2x \end{pmatrix} \in \text{Quad}_2(A[x]),$$

so that

$$a - a^2 - b^2x, \quad c - b^2 - c^2x \in 2A[x].$$

Given $a = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ let

$$d = \max\{i \geq 0 \mid a_i \notin 2A\} \quad (= 0 \text{ if } a \in 2A[x])$$

so that $a \in A_2[x]$ has degree $d \geq 0$,

$$(a_d)^2 = a_d \neq 0 \in A_2$$

and $a - a^2 \in A_2[x]$ has degree $2d$. Thus if $b \neq 0 \in A_2[x]$ the degree of $a - a^2 = b^2x \in A_2[x]$ is both even and odd, so $b \in 2A[x]$ and hence also $a - a^2, c - c^2x \in 2A[x]$. It follows from $a(1 - a) = 0 \in A_2[x]$ that $a = 0$ or $1 \in A_2[x]$, so $a - a_0 \in 2A[x]$. Similarly, it follows from $c(1 - cx) = 0 \in A_2[x]$ that $c = 0 \in A_2[x]$, so $c \in 2A[x]$. The matrices defined by

$$N = \begin{pmatrix} 0 & -b/2 \\ 0 & 0 \end{pmatrix} \in M_2(A[x]), \quad M' = \begin{pmatrix} a & 0 \\ 0 & c - b^2 \end{pmatrix} \in \text{Sym}_2(A[x])$$

are such that

$$M + 2(N + N^t) - 4N^t X N = M' \in \text{Sym}_2(A[x])$$

and so $M = M' \in Q_0(C(X), \gamma(X))$. The explicit splitting map is given by

$$Q_0(C(X), \gamma(X)) \rightarrow Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)); \quad M = M' \mapsto (a, c - b^2).$$

The isomorphism

$$Q_0(C(X), \gamma(X)) \xrightarrow{\cong} \text{coker}(\partial); \quad M \mapsto (a, c - b^2)$$

may now be composed with the isomorphisms given in the proof of Proposition 85(i)

$$\begin{aligned} Q_0(C(1), \gamma(1)) &\xrightarrow{\cong} A_8 \oplus A_4[x] \oplus A_2[x]; \\ \sum_{i=0}^{\infty} d_i x^i &\mapsto \left(d_0, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} d_{(2i+1)2^j} / 2 \right) x^i, \sum_{k=0}^{\infty} (d_{2k+2} / 2) x^k \right), \\ Q_0(C(x), \gamma(x)) &\xrightarrow{\cong} A_4[x] \oplus A_2[x]; \\ \sum_{i=0}^{\infty} e_i x^i &\mapsto \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} e_{(2i+1)2^j-1} / 2 \right) x^i, \sum_{k=0}^{\infty} (e_{2k+1} / 2) x^k \right). \end{aligned}$$

(ii) Let $n = 1$. If $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A[x])$ represents an element $N \in Q_1(B^{A[x]}, \beta^{A[x]})$

$$\begin{aligned} N + N^t &= \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \in 2\text{Sym}_2(A[x]), \\ \frac{1}{2}(N + N^t) - N^t X N &= \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} - \begin{pmatrix} a^2 + c^2 x & ab + cd x \\ ab + cd x & b^2 + d^2 x \end{pmatrix} \\ &\in \text{Quad}_2(A[x]) \end{aligned}$$

then

$$b+c, \quad a-a^2-c^2x, \quad d-b^2-d^2x \in 2A[x].$$

If $d \notin 2A[x]$ then the degree of $d-d^2x = b^2 \in A_2[x]$ is both even and odd, so that $d \in 2A[x]$ and hence $b, c \in 2A[x]$. Thus $a-a^2 \in 2A[x]$ and so (as above) $a-a_0 \in 2A[x]$. It follows that

$$Q_1(B^{A[x]}, \beta^{A[x]}) = Q_1(B^A, \beta^A) = A_2.$$

(iii) Let $n = 2$. $Q_2(B^{A[x]}, \beta^{A[x]}) = 0$ by 85.

(iv) Let $n = 3$. Proposition 85 gives an exact sequence

$$0 \rightarrow H_0(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x)) \rightarrow Q_3(C(X), \gamma(X)) \rightarrow 0$$

with

$$\begin{aligned} \partial : H_0(C(1) \otimes_{A[x]} C(x)) &= A_2[x] \rightarrow \\ Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x)) &= A_2[x] \oplus A_2[x]; \quad c \mapsto (cx, c), \end{aligned}$$

so that there is defined an isomorphism

$$\operatorname{coker}(\partial) \xrightarrow{\cong} A_2[x]; (a, b) \mapsto a + bx.$$

We shall define an isomorphism $Q_3(C(X), \gamma(X)) \cong \operatorname{coker}(\partial)$ by constructing a splitting map

$$Q_3(C(X), \gamma(X)) \rightarrow Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x)).$$

For any $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \operatorname{Sym}_2(A[x])$ the matrices

$$L = \begin{pmatrix} 0 & -b \\ -b & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} a - b^2x & 0 \\ 0 & c - b^2 \end{pmatrix} \in \operatorname{Sym}_2(A[x])$$

are such that

$$M' = M + L - LXL \in \operatorname{Sym}_2(A[x])$$

so $M = M' \in Q_3(C(X), \gamma(X))$. The explicit splitting map is given by

$$Q_3(C(X), \gamma(X)) \rightarrow Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x));$$

$$M = M' \mapsto (a - b^2x, c - b^2).$$

The isomorphism

$$Q_3(C(X), \gamma(X)) \xrightarrow{\cong} Q_{-1}(C(1), \gamma(1)); \quad M \mapsto (a - b^2x) + (c - b^2)x = a + cx$$

may now be composed with the isomorphism given in the proof of Proposition 85(ii)

$$Q_{-1}(C(1), \gamma(1)) \xrightarrow{\cong} A_2[x]; \quad d = \sum_{i=0}^{\infty} d_i x^i \mapsto d_0 + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} d_{(2i+1)2j} \right) x^{i+1}. \quad \square$$

Remark 87. (i) Substituting the computation of $Q_*(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ given by Theorem 86 in the formula

$$Q_{n+1}(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = Q_{n+1}(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \oplus \operatorname{UNil}_n(\mathbb{Z})$$

recovers the computations

$$\mathrm{UNil}_n(\mathbb{Z}) = NL_n(\mathbb{Z}) = \begin{cases} 0 & \text{if } n \equiv 0, 1 \pmod{4}, \\ \mathbb{Z}_2[x] & \text{if } n \equiv 2 \pmod{4}, \\ \mathbb{Z}_4[x] \oplus \mathbb{Z}_2[x]^3 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

of Connolly and Ranicki [10] and Connolly and Davis [8].

(ii) The twisted quadratic Q -group

$$Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \mathbb{Z}_8 \oplus L_{-1}(\mathbb{Z}[x]) = \mathbb{Z}_8 \oplus \mathrm{UNil}_3(\mathbb{Z})$$

fits into a commutative braid of exact sequences

$$\begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & \xrightarrow{\quad} & Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) & \xrightarrow{\quad} & L_{-1}(\mathbb{Z}[x]) & \xrightarrow{\quad} & L^{-1}(\mathbb{Z}[1/2][x]) = 0 \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ & L^0(\mathbb{Z}[x]) = \mathbb{Z} & & L_0(\mathbb{Z}[x], (2)^\infty) & & L^{-1}(\mathbb{Z}[x]) = 0 & \\ & \nearrow 8 & \nearrow & \searrow & \nearrow & \nearrow & \\ L_0(\mathbb{Z}[x]) = \mathbb{Z} & & L^0(\mathbb{Z}[1/2][x]) = \mathbb{Z} \oplus \mathbb{Z}_2 & & L^0(\mathbb{Z}[x], (2)^\infty) = \mathbb{Z}_2 & & 0 \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ & & & & & & \end{array}$$

with $L_0(\mathbb{Z}[x], (2)^\infty)$ (resp. $L^0(\mathbb{Z}[x], (2)^\infty)$) the Witt group of nonsingular quadratic (resp. symmetric) linking forms over $(\mathbb{Z}[x], (2)^\infty)$, and

$$L^0(\mathbb{Z}[x], (2)^\infty) \xrightarrow{\cong} \mathbb{Z}_2; (T, \lambda) \mapsto n \text{ if } |\mathbb{Z} \otimes_{\mathbb{Z}[x]} T| = 2^n.$$

The twisted quadratic Q -group $Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ is thus the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$ with $|\mathbb{Z} \otimes_{\mathbb{Z}[x]} T| = 4^m$ for some $m \geq 0$. $Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ can also be regarded as the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$ together with a lagrangian $U \subset T$ for the symmetric linking form (T, λ) . The isomorphism class of any such quadruple $(T, \lambda, \mu; U)$ is an element $\phi \in Q_0(B, \beta)$. The chain bundle β is classified by a chain bundle map

$$(f, \chi) : (B, \beta) \rightarrow (B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$$

and the Witt class is given by the linking Arf invariant

$$(T, \lambda, \mu; U) = (f, \chi)_\%(\phi) \in Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \mathbb{Z}_8 \oplus \mathbb{Z}_4[x] \oplus \mathbb{Z}_2[x]^3.$$

(iii) Here is an explicit procedure obtaining the generalized linking Arf invariant

$$(T, \lambda, \mu; U) \in Q_0(B^{A[x]}, \beta^{A[x]}) = A_8 \oplus A_4[x] \oplus A_2[x]^3$$

for a nonsingular quadratic linking form (T, λ, μ) over $(A[x], (2)^\infty)$ together with a lagrangian $U \subset T$ for the symmetric linking form (T, λ) such that $[U] = 0 \in \tilde{K}_0(A[x])$, for any 1-even ring A with $\psi^2 = 1$.

Use a set of $A[x]$ -module generators $\{g_1, g_2, \dots, g_u\} \subset U$ to obtain a f.g. free $A[x]$ -module resolution

$$0 \rightarrow B_1 \xrightarrow{d} B_0 = A[x]^u \xrightarrow{(g_1, g_2, \dots, g_u)} U \rightarrow 0.$$

Let $(p_i, q_i) \in A_2[x] \oplus A_2[x]$ be the unique elements such that

$$\mu(g_i) = (p_i)^2 + x(q_i)^2 \in \hat{H}^0(\mathbb{Z}_2; A[x]) = A_2[x] \quad (1 \leq i \leq u)$$

and use arbitrary lifts $(p_i, q_i) \in A[x] \oplus A[x]$ to define

$$b_i = (p_i)^2 + x(q_i)^2 \in A[x], \\ p = (p_1, p_2, \dots, p_u), \quad q = (q_1, q_2, \dots, q_u) \in A[x]^u.$$

The diagonal symmetric form on B_0 ,

$$\beta = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_u \end{pmatrix} \in \text{Sym}(B_0)$$

is such that

$$d^* \beta d \in \text{Quad}(B_1) \subset \text{Sym}(B_1)$$

and represents the chain bundle

$$\beta = \mu|_U \in \hat{Q}^0(B^{-*}) = \text{Hom}_A(U, \hat{H}^0(\mathbb{Z}_2; A[x])).$$

The $A[x]$ -module morphisms

$$f_0 = \begin{pmatrix} p \\ q \end{pmatrix} : B_0 = A[x]^u \rightarrow B_0^{A[x]} = A[x] \oplus A[x]; \quad (a_1, a_2, \dots, a_u) \mapsto \sum_{i=1}^u a_i (p_i, q_i), \\ f_1 : B_1 = A[x]^u \rightarrow B_1^{A[x]} = A[x] \oplus A[x]; \quad a = (a_1, a_2, \dots, a_u) \mapsto \frac{f_0 d(a)}{2}$$

define a chain bundle map

$$(f, 0) : (B, \beta) \rightarrow (B^{A[x]}, \beta^{A[x]}),$$

with

$$\beta_0^{A[x]} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} : B_0^{A[x]} = A[x] \oplus A[x] \rightarrow (B_0^{A[x]})^* = A[x] \oplus A[x].$$

The $(2)^\infty$ -torsion dual of U has f.g. free $A[x]$ -module resolution

$$0 \rightarrow B^0 = A[x]^u \xrightarrow{d^*} B^1 \rightarrow U^\wedge \rightarrow 0.$$

Lift a set of $A[x]$ -module generators $\{h_1, h_2, \dots, h_u\} \subset U^\wedge$ to obtain a basis for B^1 , and hence an identification $B^1 = A[x]^u$. Also, lift these generators to elements $\{h_1, h_2, \dots, h_u\} \subset T$, so that $\{g_1, g_2, \dots, g_u, h_1, h_2, \dots, h_u\} \subset T$ is a set of $A[x]$ -module generators such that

$$d^{-1} = (\lambda(g_i, h_j)) \in \frac{\text{Hom}_{A[1/2][x]}(B_0[1/2], B_1[1/2])}{\text{Hom}_{A[x]}(B_0, B_1)}.$$

Lift the symmetric $u \times u$ matrix $(\lambda(h_i, h_j))$ with entries in $A[1/2][x]/A[x]$ to a symmetric form on the f.g. free $A[1/2][x]$ -module $B^1[1/2] = A[1/2][x]^u$:

$$\Lambda = (\lambda_{ij}) \in \text{Sym}(B^1[1/2])$$

such that $\lambda_{ii} \in A[1/2][x]$ has image $\mu(h_i) \in A[1/2][x]/2A[x]$. Let $\phi = (\phi_{ij})$ be the symmetric form on $B^0 = A[x]^u$ defined by

$$\phi = d\Lambda d^* \in \text{Sym}(B^0) \subset \text{Sym}(B^0[1/2]).$$

Then T has a f.g. free $A[x]$ -module resolution

$$0 \rightarrow B_1 \oplus B^0 \xrightarrow{\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}} B^1 \oplus B_0 \xrightarrow{(g_1, \dots, g_u, h_1, \dots, h_u)} T \rightarrow 0$$

and

$$\phi_{ii} - \sum_{j=1}^u (\phi_{ij})^2 b_j \in 2A[x].$$

The symmetric form on $(B_0^{A[x]})^* = A[x] \oplus A[x]$ defined by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = f_0 \phi f_0^* = \begin{pmatrix} \phi(p, p) & \phi(p, q) \\ \phi(q, p) & \phi(q, q) \end{pmatrix} \in \text{Sym}((B_0^{A[x]})^*)$$

$$(p = (p_1, p_2, \dots, p_u), q = (q_1, q_2, \dots, q_u) \in B^0 = A[x]^u)$$

is of the type considered in the proof of Theorem 86(i), with

$$a - a^2 = b^2 x, \quad c - c^2 x = b^2 \in A_2[x], \quad b \in 2A[x].$$

The Witt class is

$$\begin{aligned} (T, \lambda, \mu; U) &= (f, 0)_\%(\phi) \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \in Q_0(B^{A[x]}, \beta^{A[x]}) \quad (c' = c - b^2), \end{aligned}$$

with isomorphisms

$$\begin{aligned} Q_0(B^{A[x]}, \beta^{A[x]}) &\xrightarrow{\cong} A_8 \oplus \text{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x]; \\ \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} &\mapsto \left(a_0, \left[\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2j} / 2 \right) x^i, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c'_{(2i+1)2j-1} / 2 \right) x^i \right], \right. \\ &\quad \left. \sum_{k=0}^{\infty} (a_{2k+2} / 2) x^k, \sum_{k=0}^{\infty} (c'_{2k+1} / 2) x^k \right), \\ \text{coker}(2\Delta) &\xrightarrow{\cong} A_4[x] \oplus A_2[x]; \quad [m, n] \mapsto (m - n, m), \end{aligned}$$

where

$$2\Delta : A_2[x] \rightarrow A_4[x] \oplus A_4[x]; \quad m \mapsto (2m, 2m)$$

as in Theorem 86, and

$$Q_0(B^{A[x]}, \beta^{A[x]}) = A_8 \oplus A_4[x] \oplus A_2[x]^3.$$

For Dedekind A the splitting formula of [10] gives

$$\text{UNil}_3(A) \cong Q_0(B^{A[x]}, \beta^{A[x]}) / A_8 \cong A_4[x] \oplus A_2[x]^3.$$

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