LOCALIZATIONS IN ASSOCIATIVE RINGS

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INTRODUCTION

In this article we develop the methods of [1]. In the first section we describe the construction of a localization of an arbitrary ring R with respect to an arbitrary set Σ of (rectangular) matrices over the ring R, analogous to Ore's construction, and which essentially coincides with it in the case when Σ is the set of the elements in the ring satisfying the left or right Ore condition (Theorem 2). We prove a criterion for the equality of two elements in the ring $R\Sigma^{-1}$, analogous to the criterion for the equality of two fractions in Ore's construction. This criterion allows us to find a fairly simple condition, written in the form of a system of quasiidentities, which is necessary and sufficient for the potential invertibility of the set Σ (Corollary 3).

In the second section, we introduce the idea of an n-independent set Σ, and we prove that for such a set, the ring $R\Sigma^{-1}$ is an n-FI-ring (Theorem 6). As an application, we give an answer to Bergman's problem [2] on the connection between the dependence parameters on the rings R and $R\Sigma^{-1}$, where Σ is the set of all complete quadratic matrices of a given fixed order $l > 0$. It turns out that if R is an n-FI-ring, then $R\Sigma^{-1}$ is an $(n - 2l)$-FI-ring. In particular, if $\Sigma = R \setminus \{0\}$ is the set of all nonzero elements of the n-FI-ring R, then the ring $R\Sigma^{-1}$ is an $(n - 2)$-FI-ring.

All the rings we consider have a unity, which is preserved by homomorphisms. If R is an arbitrary ring, then we denote by ${}^mR^n$ the set of matrices over the ring R with m rows and n columns. If one of the indices m, n is equal to zero, then by definition we assume that the set ${}^mR^n$ consists of a unique empty zero matrix. We denote by the symbols ${}^m0^n$ and 1_n, respectively, the zero matrix in ${}^mR^n$, and the unit matrix of order n. If in writing ${}^mR^n$ one of the indices is omitted, then this means that its value is either arbitrary or can be found uniquely from the context. This also refers to the notation ${}^m0^n$, 1_n and other similar notation. We denote by $\mathbf{N}$ the set of nonnegative integers.

On the set $\mathrm{Mat}(R) = \bigcup_{m,n\in\mathbf{N}} {}^mR^n$ we define the partial operations of addition

$$\begin{cases} {}^mR^n \times {}^mR^n \to {}^mR^n \\ (a, b) \to a + b \end{cases} \quad m, n \in \mathbf{N},$$

and multiplication

$$\begin{cases} {}^mR^n \times {}^nR^k \to {}^mR^k \\ (a, b) \to ab \end{cases} \quad m, n, k \in \mathbf{N},$$

which turn this set into a preadditive category, whose morphisms are matrices and whose objects are the natural numbers, and, moreover, the number 0 is the zero object, i.e., $1_0 = {}^00^0$.

As well as the category Mat (R), it is useful to consider the category Mat^2 (R), whose elements (morphisms) are matrices with a fixed expansion into cells, where some cells may be empty. The operations of addition (or multiplication) of two matrices in Mat^2 (R) are defined if and only if the expansions into cells are compatible, i.e., the operations may be carried out cellwise. The result of the operation is the ordinary sum (or product) of the matrices, whose cellular expansion is naturally defined by the expansions of the terms being added (multiplied). All computations with cellular matrices will be performed in Mat^2 (R), unless we specify otherwise.

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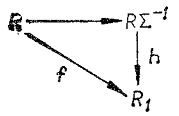
A DESCRIPTION OF THE CONSTRUCTION, AND ITS FUNDAMENTAL PROPERTIES

Let R be an arbitrary ring, and $\Sigma \subseteq \mathrm{Mat}(R)$ an arbitrary set of matrices. We recall that the homomorphism of rings $f:R \to R_1$ is called Σ-inverting if the images of the matrices in the set Σ are invertible over the ring R_1. Amongst the Σ-inverting homomorphisms there is a universal one, which we call a localization of the ring R with respect to Σ, and we write thus:

$$u_\Sigma : R \to R\Sigma^{-1}.$$

This homomorphism is characterized by the fact that for any Σ-inverting homomorphism $f:R \to R_1$, there exists a homomorphism $h:R\Sigma^{-1} \to R_1$, such that we have the following commutative diagram:



We call the set Σ of rectangular matrices over the ring R multiplicative, if it satisfies the following conditions:

1. The empty (0×0)-matrix 1_0 belongs to Σ.

2. If $a, b \in \Sigma$, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in \Sigma$ for any matrix c of suitable dimensions.

For any set Σ, we can easily construct a least multiplicative set $\overline{\Sigma}$ containing Σ. We call the set $\overline{\Sigma}$ the multiplicative closure of the set Σ. Localizations of the ring R with respect to Σ and $\overline{\Sigma}$ are naturally isomorphic, and therefore, without loss of generality, we need consider only multiplicative sets.

In this section we give a construction for a localization of the ring R with respect to the multiplicative set Σ. In the future it will be convenient to construct, not the ring $R\Sigma^{-1}$ itself, but the whole set $\mathrm{Mat}\,(R\Sigma^{-1})$ of rectangular matrices over this ring, together with the partial operations of addition and multiplication, defined on this set. We first construct some set $M(R, \Sigma)$ with partial operations $\oplus$, $\odot$, a mapping $\varepsilon:\mathrm{Mat}\,(R) \to M(R, \Sigma)$ preserving the operations, and an equivalence relation Θ_Σ, stable with respect to the above operations. We will then establish a correspondence between equivalence classes and matrices over some ring, so that the mapping ε, restricted to (1×1)-matrices, defines a universal Σ-inverting homomorphism. The relation θ_Σ, which is first defined nonconstructively, is then studied, as a result of which we obtain a criterion for the equality of two elements of the ring $R\Sigma^{-1}$ in terms of the ring R. The final results on this construction are formulated below in Theorem 2 and its three corollaries.

We say that we are given an $\mathcal{M}$*-object* if we are given:

1. A nonempty set M.
2. Some expansion $M = \bigcup\limits_{m,n \in \mathbf{N}} {}^mM^n$ into pairwise nonintersecting subsets.
3. A family of mappings

$$\begin{cases} {}^mM^n \times {}^mM^n \to {}^mM^n \\ (a, b) \to a \oplus b. \end{cases} \quad m, n \in \mathbf{N},$$

4. A family of mappings

$$\begin{cases} {}^mM^n \times {}^nM^k \to {}^mM^k \\ (a, a) \to a \odot b \end{cases} \quad m, n, k \in \mathbf{N}.$$

A fundamental example for us of an $\mathcal{M}$-object is the set of all rectangular matrices over the ring R, with the partial operations +, · defined on it. Analogously, any preadditive category, whose objects are natural numbers, may be considered as an $\mathcal{M}$-object. Below we define the $\mathcal{M}$-object $M(R, \Sigma)$ and an equivalence relation θ_Σ on it, so that the equivalence classes correspond to matrices over $R\Sigma^{-1}$.

If M_1 and M_2 are two $\mathcal{M}$-objects, then an $\mathcal{M}$*-morphism* from M_1 to M_2 is a family of mappings

$$f = \{{}^m f^n : {}^m M_1^n \to {}^m M_2^n \mid m, n \in \mathbf{N}\}$$

such that $f(a \oplus b) = f(a) \oplus f(b)$, $f(c \odot d) = f(c) \odot f(d)$ every time the values $a \oplus b$, $c \oplus d$ are defined.

Any homomorphism of rings $f: R \to R_1$ induces an $\mathcal{M}$-morphism $\mathrm{Mat}(f): \mathrm{Mat}(R) \to \mathrm{Mat}(R_1)$.

An equivalence relation θ on the $\mathcal{M}$-object M is called a *congruence* if the following conditions are satisfied:

1. $(a\theta b,\ a \in {}^m M^n) \Rightarrow b \in {}^m M^n$.
2. $(a\theta b,\ c\theta d) \Rightarrow (a \oplus c)\theta(b \oplus d)$, if $a \oplus c$ is defined.
3. $(a\theta b,\ c\theta d) \Rightarrow (a \odot c)\theta(b \odot d)$, if $a \odot c$ is defined.

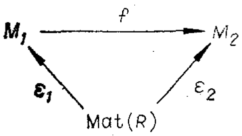
If $f: M_1 \to M_2$ is an $\mathcal{M}$-morphism, then the relation

$$\theta_f : a\theta_f b \Leftrightarrow f(a) = f(b)$$

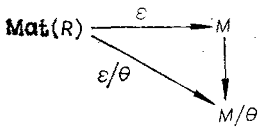
is clearly a congruence, which we call the kernel congruence of the morphism f.

If θ is a congruence on the $\mathcal{M}$-object M, then on the factor-set M/θ we clearly have defined the structure of an $\mathcal{M}$-object.

Let R be an arbitrary ring. We call an *R-object* any $\mathcal{M}$-object together with a fixed $\mathcal{M}$-morphism $\varepsilon: \mathrm{Mat}(R) \to M$. If (M_1, ε_1), (M_2, ε_2) are two R-objects, then we call an *R-morphism* any $\mathcal{M}$-morphism $f: M_1 \to M_2$ for which we have the following commutative diagram:



We call the congruence θ on the R-object (M, ε) a ring congruence if the factor-object M/θ is a preadditive category (i.e., the operations $\oplus$, $\odot$ are associative, there exist neutral elements, $\langle {}^m M^n/\theta;\ \oplus\rangle$ are Abelian groups, and the distributive identities hold) and the mapping ε/θ in the commutative diagram



is a morphism of preadditive categories (i.e., takes zero elements to zero elements and unit elements to unit elements).

In particular, this definition requires that the natural number 0 is a zero object in M/θ.

If θ is a ring congruence on the R-object (M, ε), then it is easily proved that the factor-object $(M/\theta, \varepsilon/\theta)$ is naturally isomorphic to the R-object

$$\mathrm{Mat}({}^1\varepsilon^1/\theta) : \mathrm{Mat}(R) \to \mathrm{Mat}({}^1 M^1/\theta).$$

We construct the isomorphism as follows:

Let ${}^m E_{ij}^n \in {}^m R^n$ be the matrix whose (i, j)-th element is one and the remaining element are zeros. Moreover, let ${}^m e_{ij}^n$ be the image of the matrix ${}^m E_{ij}^n$ for the mapping ε/θ. We associate with each element $a \in {}^m M^n/\theta$ a matrix of order $m \times n$ over the ring ${}^1 M^1/\theta$, whose (i, j)-th element is ${}^1 e_{1i}^m \cdot a \cdot {}^n e_{j1}^1$. It is easily verified that this correspondence is an isomorphism of R-objects.

We call the ring congruence θ on the R-object (M, ε) (*universal* and) Σ*-inverting*, if the homomorphism of rings ${}^1\varepsilon^1/\theta: R \to {}^1 M^1/\theta$ is a (universal) Σ-inverting homomorphism. Here $\Sigma \subseteq \mathrm{Mat}(R)$ is some multiplicative set of matrices.

We shall now construct an object (M, Σ) with the following properties:

(a) For any Σ-inverting homomorphism $f: R \to R_1$, there exists a mapping $h_f: M(R, \Sigma) \to \mathrm{Mat}(R_1)$ which is an R-morphism [to the R-object $\mathrm{Mat}(f): \mathrm{Mat}(R) \to \mathrm{Mat}(R_1)$].

(b) For any matrix $a \in \Sigma \cap {}^mR^n$ there exists an element $a^* \in {}^nM^m(R, \Sigma)$ such that $h_f(a^*) = f(a)^{-1}$ for any Σ-inverting homomorphism f.

(c) $M(R, \Sigma)$ is generated as an R-object by the set $\{a^* | a \in \Sigma\}$.

On such an object, any kernel congruence θ_{h_f} is Σ-inverting, the intersection of all such congruences

$$\theta_\Sigma = \bigcap_f \theta_{h_f}$$

is universal and Σ-inverting, and the homomorphism ${}^1\varepsilon^1/\theta_\Sigma : R \to {}^1M^1(R, \Sigma)/\theta_\Sigma$ is a localization of R with respect to Σ.

Construction of the R-object $M(R, \Sigma)$. Denote by $M(R)$ the set of rectangular matrices with a fixed expansion into cells, of the form

$$a = \left(\begin{array}{c|c} a' & \tilde{a} \\ \hline a^0 & 'a \end{array}\right), \tag{1}$$

i.e., the quadruple of matrices $(a', \tilde{a}, a^0, 'a)$, whose dimensions allow us to form the matrix (1). We shall think of the expansion (1) as being distinct from other expansions of the matrix a which we shall encounter. Thus, on the set $M(R)$ there are defined four mappings

$$a \to a',\ a \to \tilde{a},\ a \to a^0,\ a \to 'a.$$

Some of the above four cells of the matrix $a \in M(R)$ may be empty. In this case we shall use abbreviated notation, i.e., $(\overline{a^0}|)$, $(\underline{a'|\tilde{a}})$, and so on.

Set

$${}^mM^n(R) \rightleftharpoons \{a \in M(R) \mid \tilde{a} \in {}^mR^n\},\quad a \oplus b \rightleftharpoons \left(\begin{array}{cc|c} a' & b' & \tilde{a} + \tilde{b} \\ \hline a^0 & 0 & 'a \\ 0 & b^0 & 'b \end{array}\right),$$

$$a \odot b \rightleftharpoons \left(\begin{array}{cc|c} a' & \tilde{a}b' & \tilde{a}\tilde{b} \\ \hline a^0 & 'ab' & 'a\tilde{b} \\ 0 & b^0 & 'b \end{array}\right),\ \varepsilon(a) \rightleftharpoons (\underline{|a}).$$

It is easily verified that we have thus defined an R-object $(M(R), \varepsilon)$ and that the mapping $a \to \tilde{a}$ is an R-morphism $M(R) \to \mathrm{Mat}(R)$.

For the matrix $a \in {}^mR^n$, set

$$a^* \rightleftharpoons \left(\begin{array}{c|c} -1 & 0 \\ \hline a & 1 \end{array}\right) \in {}^nM^m(R).$$

If $f: R \to R_1$ is a homomorphism of rings, then the induced mapping $M(f): M(R) \to M(R_1)$ may be assumed to be an R-morphism in view of the following commutative diagram:

$$\begin{array}{ccc} M(R) & \xrightarrow{M(f)} & M(R_1) \\ \uparrow{\scriptstyle\varepsilon} & & \uparrow{\scriptstyle\varepsilon} \\ \mathrm{Mat}(R) & \xrightarrow{\mathrm{Mat}(f)} & \mathrm{Mat}(R_1). \end{array}$$

We also note that the mapping $M(f)$ preserves the operation $a \to a^*$.

If Σ is a multiplicative set of matrices, then the set

$$M(R, \Sigma) \rightleftharpoons \{a \in M(R) \mid a^0 \in \Sigma\}$$

is closed with respect to the operations $\oplus$, $\odot$, and therefore on it there is induced an R-object structure, which we also denote by $M(R, \Sigma)$, assuming that the mapping ε is fixed.

Denote by $\hat{a}$ $(a \in M(R))$ the matrix

$$\begin{pmatrix} 1 & a' & \tilde{a} \\ 0 & a^0 & 'a \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, if $p \in {}^mM^0(R)$, $q \in {}^0M^n(R)$, then

$$\hat{p} = \begin{pmatrix} 1_m & p' \\ 0 & p^0 \end{pmatrix}, \quad \hat{q} = \begin{pmatrix} q^0 & 'q \\ 0 & -1_n \end{pmatrix}.$$

If $a \in {}^mM^n(R)$ and the matrix a^0 is invertible, then the matrix $\hat{a}$ is also invertible and there exists a unique matrix $\omega(a) \in {}^mM^n(R)$ such that

$$\hat{a}^{-1} = \widehat{\omega(a)}.$$

The following identities can be verified immediately:

$$\begin{gathered} \omega(a \oplus b) = \omega(a) \odot \omega(b), \\ \omega(a \odot b) = \omega(a) \odot \omega(b), \\ \omega(\underline{|a}) = (\underline{|a}), \\ \omega(a^*) = \left(\frac{a^{-1}}{a^{-1}}\middle|\frac{a^{-1}}{a^{-1}}\right). \end{gathered} \tag{2}$$

Each of these identities is true whenever the value of its left- or right-hand side is defined. In particular, if the set Σ consists of invertible matrices and is closed with respect to inversion, then identities (2) hold for any matrices in the set $M(R, \Sigma)$. The first three identities mean that the mapping ω is, in this case, an automorphism of the R-object $M(R, \Sigma)$.

Set $\tilde{\omega}(a) \rightleftharpoons \widetilde{\omega(a)}$. It follows from the definition of ω that $\tilde{\omega} = (a) = \tilde{a} - a'(a^0)^{-1} \cdot {}'a$.

Now let $f: R \to R_1$ be some Σ-inverting homomorphism. Denote by Σ_1 the set $f(\Sigma) \cup f(\Sigma)^{-1}$ consisting of the images of the matrices in Σ and the inverses of these images. The R-morphism h_f in the commutative diagram

$$\begin{array}{ccc} M(R,\Sigma) & \xrightarrow{M(f)} & M(R_1,\Sigma_1) \\ & \searrow^{h_f} & \downarrow \tilde{\omega} \\ & & \mathrm{Mat}(R_1) \end{array}$$

is Σ-inverting, in view of the fourth identity in (2). Now set

$$\theta_\Sigma \rightleftharpoons \bigcap_f \theta_{h_f},$$

where the intersection is taken over all Σ-inverting homomorphisms.

Proposition 1. The relation θ_Σ is a universal Σ-inverting congruence.

Proof. Of the conditions (a)-(c) formulated above, it remains to prove only (c), which is satisfied in view of the identity

$$\left(\frac{a'}{a^0}\middle|\frac{\tilde{a}}{'a}\right) = (\underline{|\tilde{a}}) \oplus ((\underline{|-a'}) \odot a^* \odot (\underline{|'a}).$$

We can now give an explicit description of the congruence θ_Σ.

The proposition is proved.

The set Σ of matrices over the ring R is called *saturated* if it contains all the matrices $a \in \mathrm{Mat}(R)$ for which the matrices $u_\Sigma(a)$ are invertible over the ring $R\Sigma^{-1}$. (Here u_Σ is a universal Σ-inverting homomorphism.) For any set Σ, there exists a unique saturated set $\bar{\Sigma} \supseteq \Sigma$ such that the universal Σ- and Σ'-inverting homomorphisms coincide. Moreover, a universal Σ-inverting congruence θ_Σ on the set $M(R, \Sigma)$ is, clearly, the restriction to this set of the congruence $\theta_{\bar{\Sigma}}$ [defined on the set $M(R, \bar{\Sigma})$]. The set $\bar{\Sigma}$ is called the saturation of the set Σ.

We call the matrix $c \in M(R, \Sigma)$ Σ-*incomplete* if it can be written in the form

$$\left(\frac{c'}{c^0}\middle|\frac{\tilde{c}}{'c}\right) = \left(\frac{a'}{a^0}\middle|\right) \cdot (\overline{b^0|'b}), \quad a^0, b^0 \in \bar{\Sigma},$$

where $\bar{\Sigma}$ is the saturation of the set Σ.

LEMMA 1. Any Σ-incomplete matrix represents the zero matrix over the ring $R\Sigma^{-1}$.

Proof. We may assume that the set Σ is saturated. It is sufficient to prove that $h_f(c) = 0$ for any Σ-inverting homomorphism f. Therefore, we may assume that the set Σ consists of invertible matrices and is closed with respect to inversion. Under these assumptions, we show by direct verification that

$$\omega(c) = \left(\begin{array}{c|c} a_1' & 0 \\ \hline b_1^0 a_1^0 & 'b_1 \end{array}\right),$$

where $a_1 = \omega(a)$, $b_1 = \omega(b)$. Thus, $\overline{\omega}(c) = 0$. The lemma is proved.

For matrices $a, b \in M(R)$, set

$$\theta b \rightleftharpoons (\underline{|-1}) \odot b, \quad a \ominus b \rightleftharpoons a \oplus (\ominus b).$$

LEMMA 2. The following equations in the category $\mathrm{Mat}^2(R)$ are equivalent:

$$\begin{pmatrix} a' & \tilde{a} \\ a^0 & 'a \end{pmatrix} \cdot \begin{pmatrix} q^0 & 'q \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & p' \\ 0 & p^0 \end{pmatrix} \cdot \begin{pmatrix} b' & \tilde{b} \\ b^0 & 'b \end{pmatrix} = 0$$
$$\begin{pmatrix} a' & p' \\ a^0 & p^0 \end{pmatrix} \cdot \begin{pmatrix} q^0 & 'q \\ b^0 & 'b \end{pmatrix} = \begin{pmatrix} -b' & \tilde{a} - \tilde{b} \\ 0 & 'a \end{pmatrix}, \tag{3}$$
$$\left(\begin{array}{c|c} a' & \tilde{a} \\ \hline a^0 & 'a \end{array}\right) \odot \left(\begin{array}{c|c} b' & \tilde{b} \\ \hline b^0 & 'b \end{array}\right) - \left(\begin{array}{cc|} a' & p' \\ \hline a^0 & p^0 \\ 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc|c} \overline{1} & q^0 & 'q \\ 0 & b^0 & 'b \end{array}\right).$$

Proof. Each of the above three inequalities is equivalent to the following equations:

$$a' \cdot q^0 + p' \cdot b^0 = -b', \quad a' \cdot 'q + p' \cdot 'b = \tilde{a} - \tilde{b},$$
$$a^0 \cdot q^0 + p^0 \cdot b^0 = 0, \quad a^0 \cdot 'q + p^0 \cdot 'b = 'a.$$

This can be verified directly. The lemma is proved.

COROLLARY. If $a, b \in M(R, \Sigma)$, $p \in M^0(R)$, $q \in {}^0M(R)$ and in the category $\mathrm{Mat}^2(R)$ we have the equation

$$a\hat{q} + \hat{p}b = 0, \tag{4}$$

i.e., the equation

$$\begin{pmatrix} a' & \tilde{a} \\ a^0 & 'a \end{pmatrix} \begin{pmatrix} q^0 & 'q \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & p' \\ 0 & p^0 \end{pmatrix} \begin{pmatrix} b' & \tilde{b} \\ b^0 & 'b \end{pmatrix} = 0,$$

then $a \theta_\Sigma b$.

This statement follows immediately from Lemmas 1 and 2.

Fix some multiplicative set Σ of matrices over the ring R.

If the conditions of the corollary are satisfied, then we say that the matrix a is connected to b from the left, and that b is connected to a from the right, and we write this fact as $a \to b$.

LEMMA 3. The relation $\to$ is reflexive and transitive; if $a \to b$ and $c \to d$, then $a \oplus c \to b \oplus d$, $a \odot c \to b \odot d$ whenever the corresponding operations are defined.

Proof. The equation $a \cdot (-1) + 1 \cdot a = 0$ proves reflexivity. If $a\hat{q} + \hat{p}b = 0$, $b\hat{v} = \hat{u}c = 0$ are relations of the form of (4), then the relation $a \cdot (-\widehat{qv}) + (\widehat{pu}) \cdot c = 0$ is also of the form of (4). Therefore, $a \to c$.

Let $a\hat{q} + \hat{p}b = 0$, $c\hat{v} + \hat{u}d = 0$. If the value of $a \oplus c$ is defined, then

$$(a \oplus c) \cdot (\widehat{q \oplus v}) + (\widehat{p \oplus v}) \cdot (b \oplus d) = 0.$$

If, moreover, $a \odot c$ is defined, then

$$(a \odot c)\cdot \widehat{\left(q \odot \left(\begin{array}{c|c} d' & \tilde{d} \\ \hline v^0 & {}'v \end{array}\right)\right)} + \widehat{\left(\left(\begin{array}{c|c} p' & \tilde{a} \\ \hline p^0 & {}'a \end{array}\right) \odot u\right)}\cdot (b \odot d) = 0.$$

Both these equations can be verified directly. The lemma is proved.

To perform concrete computations in a universal Σ-inverting ring, we need the following.

LEMMA 4. Consider the following pairs of matrices:

$$1.\quad \left(\begin{array}{ccc|c} c_1' & c_2' & c_3' & \tilde{c} \\ \hline c_{11} & c_{12} & c_{13} & {}'c_1 \\ 0 & c_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & {}'c_3 \end{array}\right), \left(\begin{array}{cc|c} c_1' & c_3' & \tilde{c} \\ \hline c_{11} & c_{13} & {}'c_1 \\ c_{31} & c_{33} & {}'c_3 \end{array}\right),$$

$$2.\quad \left(\begin{array}{cc|c} c_1' & c_3' & \tilde{c} \\ \hline c_{11} & c_{13} & {}'c_1 \\ c_{31} & c_{33} & {}'c_3 \end{array}\right), \left(\begin{array}{ccc|c} c_1' & 0 & c_3' & \tilde{c} \\ \hline c_{11} & 0 & c_{13} & {}'c_1 \\ c_{21} & c_{22} & c_{23} & {}'c_2 \\ c_{31} & 0 & c_{33} & {}'c_3 \end{array}\right),$$

$$3.\quad \left(\begin{array}{c|c} c' & \tilde{c} \\ \hline c^0 & {}'c \end{array}\right), \left(\begin{array}{c|c} c' & \tilde{c} \\ \hline c^0 & {}'c \end{array}\right)\cdot\left(\begin{array}{c|c} v^0 & {}'v \\ \hline 0 & 1 \end{array}\right),$$

$$4.\quad \left(\begin{array}{c|c} 1 & u' \\ \hline 0 & u^0 \end{array}\right)\cdot\left(\begin{array}{c|c} c' & \tilde{c} \\ \hline c^0 & {}'c \end{array}\right), \left(\begin{array}{c|c} c' & \tilde{c} \\ \hline c^\circ & {}'c \end{array}\right).$$

If in some of these pairs both matrices belong to the set $M(R, \Sigma)$, then the first of them is connected to the second from the left.

Proof. The relation of connectivity is easily written out in each case. Let a be the first matrix of a pair, and b the second. In the first case we must take

$$p = \left(\begin{array}{cc|} 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right), \quad q = \overline{\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right)};$$

in the second,

$$p = \left(\begin{array}{ccc|} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad q = \overline{\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)};$$

in the third, $p = \left(\begin{array}{c|} 0 \\ \hline 1 \end{array}\right)$, $q = \overline{(-v^0 \mid -{}'v)}$; and in the fourth, $p = u$, $q = \overline{(-1 \mid 0)}$. The lemma is proved.

Denote by the symbol $\sim$ the equivalence relation generated by the relation $\rightarrow$. In the conditions of Lemma 4, replacing the first matrix of any pair by the second does not change the equivalence class. In the first case we call this substitution reduction by a trivial row; in the second, insertion of a trivial column; and in the third, triangular transformation of columns. The inverse substitution is called, in the first case, insertion of a trivial row; in the second, reduction by a trivial column; and in the fourth, triangular transformation of rows.

LEMMA 5. The relation $\sim$ coincides with the universal Σ-inverting congruence θ_Σ.

Proof. The relation $\sim$ is a congruence, by Lemma 3. It is sufficient to prove that $M/\sim$ is a preadditive category with zero morphisms $(|{}^m0^n)/\sim$ and unit morphisms $(|1_n)/\sim$, and that $(|a) \odot a^* \sim (|1)$, $a^* \odot (|a) \sim (|1)$ for any matrix $a \in \Sigma$.

The operations $\oplus$, $\odot$ are already associative on the set $M(R)$ and have neutral elements $(|{}^m0^n)$ and $(|1_n)$, respectively. The equations

$$\begin{pmatrix} a' & 0 \\ a^\circ & 0 \end{pmatrix}\cdot \widehat{\overline{(-1\ 1 \mid 0)}} + \widehat{\left(\begin{array}{cc|} 0 & 0 \\ \hline 1 & -1 \end{array}\right)}\cdot (a \ominus a) = 0,$$

$$\begin{pmatrix} a' & 0 \\ a^\circ & 0 \end{pmatrix} \cdot \widehat{(|0)} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (0) = 0$$

prove the equivalence $a \ominus a \sim (|0)$, i.e., the sets ${}^mM^n(R, \Sigma)$ are groups with respect to the operation $\oplus$. The commutativity of these groups follows from the obvious identity $\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b)$. Distributivity follows from the equations

$$((a \odot c) \oplus (b \odot c)) \cdot \hat{q} + \hat{p} \cdot ((a \oplus b) \odot c) = 0,$$
$$(a \odot (b \oplus c)) \cdot \hat{v} + \hat{u} \cdot ((a \odot b) \oplus (a \odot c)) = 0,$$

where

$$p = \left(\begin{array}{ccc|} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \quad q = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array} \right), \quad u = \left(\begin{array}{cccc|} 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

$$v = \left(\begin{array}{ccccc|c} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right).$$

Finally, for any matrix $a \in \Sigma$, the equations

$$(1) \cdot (0 \;\; -1) + \widehat{(1|)} \cdot ((\underline{|a}) \odot a^*) = 0,$$
$$(a^* \odot (\underline{|a})) \cdot \widehat{(|1)} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1) = 0$$

show that the relation $\sim$ is a Σ-inverting congruence. Since $\sim \subseteq \theta_\Sigma$ in view of the corollary of Lemma 2, then the statement of Lemma 5 now follows from Proposition 1. The lemma is proved.

The equivalence relation $\sim$ on the set M(R, Σ) can be considered in the usual way as some categroy $\mathscr{E}$, whose objects are the elements of the set M(R, Σ), and whose morphisms are pairs (a, b) such that $a \sim b$.

In view of its reflexivity and transitivity, the relation $\to$ defines a subcategory, which we denote by $\mathscr{R}$.

We introduce three subcategories of $\mathscr{R}$ with the same set of objects as in $\mathscr{R}$.

The subcategory $\mathscr{R}_e$ consists, by definition, of pairs of the form

$$\left(\begin{array}{cc|c} a' & a_2 & \tilde{a} \\ \hline a^0 & a_{12} & 'a \\ 0 & a_{22} & 0 \end{array} \right), \quad \left(\begin{array}{c|c} a' & \tilde{a} \\ \hline a^0 & 'a \end{array} \right),$$

where $a^0, a_{22} \in \Sigma$; the subcategory $\mathscr{R}_m$ consists of pairs of the form

$$\left(\begin{array}{c|c} a' & \tilde{a} \\ \hline a^0 & 'a \end{array} \right), \quad \left(\begin{array}{cc|c} 0 & a' & \tilde{a} \\ \hline a_{11} & a_{12} & 'a_1 \\ 0 & a^0 & 'a \end{array} \right),$$

where $a^0, a_{11} \in \Sigma$, and the subcategory $\mathscr{R}_i$ consists of pairs of the form (a, b), which may be included in a relation of the form (4), such that the matrices p^0 and q^0 are invertible over the ring R.

It follows from Lemmas 3 and 4 that the above subsets are really subcategories of $\mathscr{R}$, and, moreover, the subcategory $\mathscr{R}_i$ is closed with respect to inversion of morphisms (in the category $\mathscr{E}$).

For sets $\mathscr{A}$ and $\mathscr{B}$ of morphisms in the category $\mathscr{E}$, we use the normal notation

$$\mathscr{A} \cdot \mathscr{B} = \{\gamma \in \mathscr{E} \mid \exists \alpha \in \mathscr{A}, \exists \beta \in \mathscr{B} \; (\alpha \cdot \beta = \gamma)\},$$
$$\mathscr{A}^{-1} = \{\alpha^{-1} \mid \alpha \in \mathscr{A}\} = \{(b, a) \mid (a, b) \in \mathscr{A}\}.$$

LEMMA 6.

(a) $\mathcal{R}_m^{-1}\cdot\mathcal{R}\subseteq\mathcal{R}\cdot\mathcal{R}_m^{-1}$,

(b) $\mathcal{R}\cdot\mathcal{R}_e^{-1}\subseteq\mathcal{R}_e^{-1}\cdot\mathcal{R}$,

(c) $\mathcal{R}=\mathcal{R}_m\cdot\mathcal{R}_i\cdot\mathcal{R}_e$,

(d) $\mathscr{E}=\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}$.

Proof. (a) The pair (c, b) belongs to the set $\mathcal{R}_m^{-1}\cdot\mathcal{R}$ if and only if

$$c=\left(\begin{array}{cc|c}0 & a' & \tilde{a}\\ \hline a_{11} & a_{12} & 'a_1\\ 0 & a^0 & 'a\end{array}\right),\quad a_{11},\ a^0\in\Sigma$$

and we have a relation $a\hat{q}+\hat{p}b=0$ of the form (4). Set

$$p_1\rightleftharpoons\left(\begin{array}{cc|}0 & p'\\ \hline 1 & 0\\ 0 & p^0\end{array}\right),\quad q_1\rightleftharpoons\left(\begin{array}{cc|c}\hline -1 & 0 & 0\\ 0 & q^0 & 'q\end{array}\right)$$

and define the row $(b_{11}b_{12}'b_1)$ by the equation

$$(a_{11}a_{12}'a_1)\cdot\hat{q}_1+(b_{11}b_{12}'b_1)=0. \tag{5}$$

(This equation defines the dimension of the unit matrix in q_1.) Then for a suitable dimension of the unit matrix in p_1, we have the equation

$$c\hat{q}_1+\hat{p}_1\cdot\begin{pmatrix}0 & b' & \tilde{b}\\ b_{11} & b_{12} & 'b_1\\ 0 & b^0 & 'b\end{pmatrix}=0,$$

where $b_{11}\in\mathscr{E}$ by (5). Thus, $(c,b)\in\mathcal{R}\cdot\mathcal{R}_m^{-1}$.

Statement (b) is true in view of the symmetry between rows and columns in all the definitions. Its proof is analogous to the proof of statement (a), and therefore we omit it.

We prove (c). Let $a\hat{q}+\hat{p}b=0$ be a relation of the form (4). Set

$$c\rightleftharpoons\left(\begin{array}{cc|c}0 & a' & \tilde{a}\\ \hline b^0 & 0 & 'b\\ 0 & a^0 & 'a\end{array}\right),\quad d\rightleftharpoons\left(\begin{array}{cc|c}b' & a' & \tilde{b}\\ \hline b^0 & 0 & 'b\\ 0 & a^0 & 0\end{array}\right),\quad p_1=\left(\begin{array}{cc|}p' & 0\\ \hline 1 & 0\\ p^0 & 1\end{array}\right),\quad q_1\rightleftharpoons\left(\begin{array}{cc|c}\hline -1 & 0 & 0\\ q^0 & -1 & 'q\end{array}\right).$$

By direct verification we have $c\hat{q}_1+\hat{p}_1d=0$, and therefore $(a,\ c)\in\mathcal{R}_m$, $(c,\ d)\in\mathcal{R}_i$, $(d,\ b)\in\mathcal{R}_e$. Equation (c) is proved.

To prove (d), it is sufficient to establish that the set $\mathscr{E}_1\rightleftharpoons\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}$ satisfies the conditions $\mathscr{E}_1\cdot\mathscr{E}_1\subseteq\mathscr{E}_1$, $\mathscr{E}_1^{-1}\subseteq\mathscr{E}_1$. We use the statements (a)-(c), which we have already proved. We have

$$\begin{aligned}\mathscr{E}_1\cdot\mathscr{E}_1&=\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}\cdot\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}\subseteq\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}\\ &=\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_e^{-1}\cdot\mathcal{R}_i\cdot\mathcal{R}_m^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}\subseteq\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}=\mathscr{E}_1,\\ \mathscr{E}_1^{-1}&=\mathcal{R}_m\cdot\mathcal{R}_e^{-1}\cdot\mathcal{R}_i\cdot\mathcal{R}_m^{-1}\cdot\mathcal{R}_e\subseteq\mathcal{R}\cdot\mathcal{R}_e^{-1}\cdot\mathcal{R}_i\cdot\mathcal{R}_m^{-1}\cdot\mathcal{R}\subseteq\mathcal{R}_e^{-1}\cdot\mathcal{R}\cdot\mathcal{R}_m^{-1}=\mathscr{E}_1.\end{aligned}$$

The lemma is proved.

COROLLARY. The matrices $a,\ b\in M(R,\ \Sigma)$ are equivalent if and only if they can be included in a relation of the form

$$\begin{pmatrix}a' & * & * & *\\ a^0 & * & * & *\\ 0 & \Sigma & * & *\end{pmatrix}\cdot\begin{pmatrix}* & * & *\\ * & * & *\\ \Sigma & * & *\\ 0 & b^0 & 'b\end{pmatrix}=\begin{pmatrix}0 & -b' & \tilde{a}-\tilde{b}\\ 0 & 0 & 'a\\ 0 & 0 & 0\end{pmatrix}, \tag{6}$$

where the symbol Σ denotes certain matrices in Σ, and the symbol $*$ denotes certain matrices of suitable dimension.

Proof. This statement is a reformulation of statement (d) of Lemma 6, also using Lemma 2.

We now formulate the fundamental theorem on the construction of a localization.

THEOREM 2. Let R be an arbitrary ring, and Σ a multiplicative set of matrices. We define on the set $M(R, \Sigma)$ a relation $\sim$, setting $a \sim b$ if the matrices a and b may be included in a relation of the form (6). Then

1. The relation $\sim$ is a congruence with respect to the operations $\oplus$, $\odot$.
2. There exists a one-to-one correspondence between congruence classes and matrices over a universal Σ-inverting ring $R\Sigma^{-1}$, and, moreover, each element $a \in {}^mM^n(R, \Sigma)$ represents some matrix $[a]$, with m rows and n columns.
3. The elements $a \oplus b$, $c \odot d$ represent the matrices $[a] + [b]$ and $[c]\cdot[d]$, respectively.
4. For any matrix $a \in \Sigma$, the element a^* represents the matrix inverse of $[|a]$.

All the statements of Theorem 2 are already proved. We note that the criterion thus obtained for the equality of two elements in a universal Σ-inverting ring is analogous to the criterion for the equality of two fractions in Ore's construction. Ore's theorem can easily be obtained from Theorem 2.

COROLLARY 1. Under the conditions of Theorem 2, the following statements on the matrix $a \in M(R, \Sigma)$ are equivalent:

(a) The matrix a represents the zero matrix over the ring $R\Sigma^{-1}$.

(b) There exists a relation of the form

$$\begin{pmatrix} a' & * & * \\ a^0 & * & * \\ 0 & \Sigma & * \end{pmatrix} \cdot \begin{pmatrix} * & * \\ * & * \\ \Sigma & * \end{pmatrix} = \begin{pmatrix} 0 & \tilde{a} \\ 0 & 'a \\ 0 & 0 \end{pmatrix}.$$

(c) There exists a relation of the form

$$\begin{pmatrix} * & * & * \\ \Sigma & * & * \end{pmatrix} \cdot \begin{pmatrix} * & * & * \\ \Sigma & * & * \\ 0 & a^0 & 'a \end{pmatrix} = \begin{pmatrix} 0 & a' & \tilde{a} \\ 0 & 0 & 0 \end{pmatrix}.$$

COROLLARY 2. Under the conditions of Theorem 2, the kernel of the universal Σ-inverting homomorphism $u_\Sigma : R \to R\Sigma^{-1}$ consists precisely of those elements $r \in R$ which can be included in a relation of the form

$$\begin{pmatrix} a' & \tilde{a} \\ a^0 & 'a \end{pmatrix} \cdot \begin{pmatrix} b' & \tilde{b} \\ b^0 & 'b \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \quad a^0, b^0 \in \Sigma.$$

COROLLARY 3. The multiplicative set Σ is potentially invertible if and only if all quasiidentities of the following form are satisfied in the ring R:

$$\begin{pmatrix} a' & \tilde{a} \\ a_0 & 'a \end{pmatrix} \cdot \begin{pmatrix} b' & \tilde{b} \\ b^0 & 'b \end{pmatrix} - \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \Rightarrow c = 0,$$

where $a^0, b^0 \in \Sigma$, $c \in {}^1R^1$ and the remaining matrices have suitable dimensions.

Let Σ be a multiplicative set of matrices over a ring. We shall now describe the divisors of zero and the invertible elements in the category Mat $(R\Sigma^{-1})$.

LEMMA 7. Let $a, b, c \in M(R, \Sigma)$, $(c, a \odot b) \in \mathcal{R}_e$. Then there exist matrices $a_1, b_1 \in M(R, \Sigma)$ such that

$$(a_1, a) \in \mathcal{R}_e, \ (b_1, b) \in \mathcal{R}_e, \ a_1 \odot b_1 \to c.$$

Proof. We have

$$c = \left(\begin{array}{ccc|c} a' & \tilde{a}b' & c_1 & \tilde{a}\tilde{b} \\ \hline a^0 & 'ab' & c_2 & '\tilde{ab} \\ 0 & b^0 & c_3 & 'b \\ 0 & 0 & c_0 & 0 \end{array}\right), \quad c^0 \in \Sigma.$$

Set

$$a_1 \rightleftharpoons \left(\begin{array}{cc|c} a' & c_1 & \widetilde{a} \\ \hline a^0 & c_2 & {}'a \\ 0 & c^0 & 0 \end{array}\right), \quad b_1 \rightleftharpoons \left(\begin{array}{cc|c} b' & 0 & \widetilde{b} \\ \hline b^0 & c_3 & {}'b \\ 0 & c^0 & 0 \end{array}\right),$$

$$p \rightleftharpoons \left(\begin{array}{ccc|c} 0 & 0 & 0 & \\ \hline 1 & 0 & 0 & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right), \quad q \rightleftharpoons \left(\begin{array}{ccc|c} \hline -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

For suitable dimensions of the unit matrices in p and q, we have the relation $(a_1 \odot b_1) \cdot \hat{q} + \hat{p} \cdot c = 0$. The lemma is proved.

Analogously, we prove the following dual statement:

<u>LEMMA 7'.</u> Let $a,\ b,\ c \in M(R,\ \Sigma),\ (a \odot b,\ c) \in \mathscr{R}_m$. Then there exist matrices $a_1,\ b_1 \in M(R,\ \Sigma)$ such that

$$(a,\ a_1) \in \mathscr{R}_m,\ (b,\ b_1) \in \mathscr{R}_m,\ c \to a_1 \odot b_1.$$

<u>Proposition 3.</u> Let Σ be a multiplicative set of matrices over the ring R. Then for any relation

$$u \cdot v = [\,\lfloor r \rfloor],\quad u, v \in \operatorname{Mat}(R\Sigma^{-1}),\quad r \in \operatorname{Mat}(R)$$

there exist matrices $a,\ b \in M(R,\ \Sigma)$ such that [a] = u, [b] = v, and we have a relation of the form

$$\begin{pmatrix} a' & \widetilde{a} & * \\ a^0 & {}'a & * \end{pmatrix} \cdot \begin{pmatrix} * & * \\ b' & \widetilde{b} \\ b^0 & {}'b \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}. \tag{7}$$

<u>Proof.</u> Let $u = [a_2],\ -v = [b_2],\ a_2,\ b_2 \in M(R,\ \Sigma)$. By Lemma 6, there exist matrices $c, d \in$ M(R, Σ) such that

$$(c, a_2 \odot b_2) \in \mathscr{R}_e, \quad c \to d, \quad ((\lfloor -r), d) \in \mathscr{R}_m.$$

By Lemma 7, there exist matrices $a_1,\ b_1 \in M(R,\ \Sigma)$ such that

$$(a_1,\ a_2) \in \mathscr{R}_e,\ (b_1,\ b_2) \in \mathscr{R}_e,\ a_1 \odot b_1 \to c.$$

By Lemma 3 we have $a_1 \odot b_1 \to d$. The matrix d is of the form $\left(\begin{array}{c|c} 0 & -r \\ \hline d^0 & {}'d \end{array}\right)$. Consider the relation connecting the matrices $a_1 \odot b_1$ and d:

$$\begin{pmatrix} a_1' & \widetilde{a}_1 b_1' & \widetilde{a}_1 \widetilde{b}_1 \\ a_1^0 & {}'a_1 b_1' & {}'a_1 \widetilde{b}_1 \\ 0 & b_1^0 & {}'b_1 \end{pmatrix} \cdot \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & p_1 \\ 0 & p_2 \\ 0 & p_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & -r \\ d^0 & {}'d \end{pmatrix} = 0.$$

From this relation we immediately obtain the following relations:

$$\begin{pmatrix} a_1' & \widetilde{a}_1 & p_1 \\ a_1^0 & {}'a_1 & p_2 \end{pmatrix} \cdot \begin{pmatrix} q_{11} & q_{12} \\ b_1' q_{21} & b_1' q_{22} - \widetilde{b} \\ d^0 & {}'d \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} b_1' & \widetilde{b}_1 \\ b_1^0 & {}'b_1 \end{pmatrix} \cdot \begin{pmatrix} q_{21} & q_{22} \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & p_3 \end{pmatrix} \cdot \begin{pmatrix} -b_1' q_{21} & -b_1' q_{22} + \widetilde{b} \\ d^0 & {}'d \end{pmatrix} = 0. \tag{9}$$

Set

$$a \rightleftharpoons a_1, \quad b \rightleftharpoons \left(\begin{array}{c|c} b_1' q_{21} & b_1' q_{22} - \widetilde{b} \\ \hline d^0 & {}'d \end{array}\right).$$

By (9) we have $b \sim (\ominus b_1) \sim (\ominus b_2)$. Thus, [$a$] = u, [b] = v and relation (8) is of the required form (7). The proposition is proved.

2. INDEPENDENT SETS OF MATRICES

Let Σ be some (not necessarily multiplicative) set of matrices over a fixed ring R. We call a relation of Σ-dependence any relation $a \cdot b = 0$ of the form

$$\left(\frac{a'}{a^0}\middle|\frac{x_1}{x_2}\middle|\frac{u_1}{u_2}\right)\cdot\left(\begin{array}{c|c} v_1 & v_2 \\ \hline y_1 & y_2 \\ \hline b^0 & 'b \end{array}\right)=0, \quad a^0, b^0 \in \dot{\Sigma}. \tag{10}$$

Speaking more formally, we say we are given a relation of Σ-dependence if we are given 12 matrices [appearing in relation (10)] satisfying the four relations

$$a' \cdot v_1 + x_1 \cdot y_1 + u_1 \cdot b^0 = 0, \quad a' \cdot v_2 + x_1 \cdot y_2 + u_1 \cdot 'b = 0.$$
$$a^0 \cdot v_1 + x_2 \cdot y_1 + u_2 \cdot b^0 = 0, \quad a^0 \cdot v_2 + x_2 \cdot y_2 + u_2 \cdot 'b = 0,$$

where the dimensions of the matrices allow us to constitute the matrices a and b in relation (10) so that $a \cdot b = 0$.

Some of the 12 matrices in (10) may be empty, and in this case we use the reduced notation, in the same way as in Sec. 1.

The number of columns of the matrix $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, which is equal to the number of rows in the matrix $y = (y_1, y_2)$, is called the length of the relation (10).

If relation (10) is of the form

$$\left(\begin{array}{c|cc|c} a' & x_{11} & 0 & 0 \\ \hline a^0 & & & 0 \end{array}\right)\cdot\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ y_{21} & y_{22} \\ \hline b^0 & 'b \end{array}\right)=0,$$

then it is called trivial.

If for relation (10) there exists a pair of mutually inverse matrices

$$\alpha=\left(\begin{array}{c|cc|c} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \hline \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \hline \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \end{array}\right), \beta=\left(\begin{array}{c|c|c} \beta_{11} & \beta_{12} & \beta_{13} \\ \hline \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \hline \beta_{41} & \beta_{42} & \beta_{43} \end{array}\right) \tag{11}$$

such that the relation $a\alpha \cdot \beta b = 0$ is a trivial relation of Σ-dependence of the same length as the original relation $a \cdot b = 0$, then we say that the pair (α, β) *trivializes* (more precisely, Σ-*trivializes*) the relation (10).

Expansion in the matrices α, β is performed so that multiplication is cellwise, and the chosen expansions are compatible. For example,

$$\alpha\beta=\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad a\alpha\beta = a.$$

Denote by Σ_0 the matrix $\Sigma \cup \{1_0\}$, obtained by adding the empty (0×0)-matrix. We call the set Σ n-independent if any relation of Σ-dependence of length $\leqslant n$ can be Σ_0-trivialized.

We call sets which are 0-independent, independent sets. We note that independent sets consist of nondivisors of zero [since relations of the form $(\overline{a^0}\,\|)\cdot(|\,\underline{\underline{v}})=0$, $(\|\underline{u})\cdot(\overline{\overline{b_0}}\,|)=0$ are trivializable].

LEMMA 8. Let Σ be an independent multiplicative set of matrices over the ring R, $a \in M(R, \Sigma)$. Then the following statements are equivalent:

(a) The matrix a represents the zero matrix over the ring $R\Sigma^{-1}$.

(b) There exists a relation of the form

$$\begin{pmatrix} a' & \tilde{a} \\ a_0 & 'a \end{pmatrix} = \begin{pmatrix} b' \\ b_0 \end{pmatrix} \cdot (c^0\ 'c), \tag{12}$$

in which $b^0 \in \Sigma$.

(c) There exists a relation of the form (12) in which $c^0 \in \Sigma$.

Proof. The implications (b) ⇒ (a) and (c) ⇒ (a) follow from Lemma 1. We prove that (b) follows from (a). By Corollary 1 of Theorem 2, there exists a relation of the form

$$\begin{pmatrix} a' & p_1 & a_{12} \\ a^0 & p_2 & a_{22} \\ 0 & p_3 & a_{32} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ q_1 & q_2 \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{a} \\ 0 & 'a \\ 0 & 0 \end{pmatrix}, \quad p_3, b_{21} \in \Sigma.$$

The relation of Σ-dependence $(p_3 \| a_{32}) \cdot \left(\frac{q_1}{b_{21}}\right) = 0$ is trivializable by the condition, and we may assume that it is trivial, i.e., $a_{32} = 0$, $q_1 = 0$. Then $q_2 = 0$, since p_3 is a nondivisor of zero. Thus, we have the relation

$$\begin{pmatrix} a' & a_{12} \\ a^0 & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{a} \\ 0 & 'a \end{pmatrix}.$$

Let (α, β) be a pair of matrices trivializing the relation

$$\left(\frac{a'}{a^0} \middle\| \frac{a_{12}}{a_{22}}\right) \cdot \left(\frac{b_{11}}{b_{21}}\right) = 0,$$

and let

$$\begin{pmatrix} a' & a_{12} \\ a^0 & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} b' & 0 \\ b_0 & 0 \end{pmatrix}, \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{a} \\ d^0 & 'a \end{pmatrix}.$$

Then

$$\begin{pmatrix} a' & \tilde{a} \\ a^0 & 'a \end{pmatrix} = \begin{pmatrix} b' \\ b^0 \end{pmatrix} \cdot (\beta_{11}\ \tilde{d}),$$

which is what we required.

The implication (a) ⇒ (c) is proved analogously. The lemma is proved.

COROLLARY. Any independent multiplicative set Σ of matrices over an arbitrary ring R is potentially invertible.

Proof. Let $r \in R$ be an element in the kernel of a universal Σ-inverting homomorphism. By Corollary 2 of Theorem 2, there exists a relation of the form

$$\begin{pmatrix} a' & \tilde{a} \\ a^{0\prime} & a \end{pmatrix} \cdot \begin{pmatrix} b' & \tilde{b} \\ b^0 & 'b \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}.$$

We may assume that $'a = 0$, $b' = 0$. In this case $\tilde{a} = 0$, $\tilde{b} = 0$, since a^0 and b^0 are nondivisors of zero. Hence it follows that $r = 0$. The statement is proved.

We note that this statement may be formally deduced from Lemma 8.

Proposition 4. Let Σ be an n-independent multiplicative set of matrices over the ring R. Then the ring $R\Sigma^{-1}$ is an n-FI-ring.

Proof. Let u be a row of length n, and v a column of length n over the ring $R\Sigma^{-1}$ with $u \cdot v = 0$. It is sufficient to prove that there exists a pair of mutually inverse square matrices

$$[\gamma],\quad [\delta] \in \mathrm{Mat}(R\Sigma^{-1}),\quad \gamma,\quad \delta \in {}^nM^n(R, \Sigma)$$

such that either the last element of the row $u \cdot [\gamma]$ or the first element of the column $[\delta] \cdot v$ is zero.

By Proposition 3, there exist matrices $a \in {}^1M^n(R, \Sigma)$, $b \in {}^nM^1(R, \Sigma)$ such that $[a] = u$, $[b] = v$ and we have a relation of the form

$$\left(\begin{array}{c|c|c} a' & \tilde{a} & p_1 \\ \hline a^0 & {}'a & p_2 \end{array}\right) \cdot \left(\begin{array}{c|c} q_1 & q_2 \\ \hline b' & \tilde{b} \\ \hline b^0 & {}'b \end{array}\right) = 0.$$

Let (α, β) be a pair of mutually inverse matrices of the form (11), trivializing this relation. Then for some matrices $c, d \in M(R, \Sigma)$, we have

$$\left(\begin{array}{c|c|c} a' & \tilde{a} & p_1 \\ \hline a^0 & {}'a & p_2 \end{array}\right) \alpha = \left(\begin{array}{c|cc|c} c' & \tilde{c} & 0 & 0 \\ \hline c^0 & {}'c & 0 & 0 \end{array}\right), \quad \beta \left(\begin{array}{c|c} q_1 & q_2 \\ \hline b^1 & \tilde{b} \\ \hline b^0 & {}'b \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ d' & \tilde{d} \\ \hline d^0 & {}'d \end{array}\right).$$

Set

$$\gamma \rightleftharpoons \left(\begin{array}{cc|cc} b' & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \hline b^0 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ 0 & c^0 & {}'c & 0 \end{array}\right), \quad \delta \rightleftharpoons \left(\begin{array}{cc|c} 0 & \beta_{21} & \beta_{22} \\ d' & \beta_{31} & \beta_{32} \\ \hline d^0 & \beta_{41} & \beta_{42} \\ 0 & a^0 & {}'a \end{array}\right).$$

We verify that the matrices $[\gamma]$ and $[\delta]$ are mutually inverse. We have

$$\delta \odot \gamma = \left(\begin{array}{cccc|cc} 0 & \beta_{21} & \beta_{22}b' & \beta_{22}\alpha_{21} & \beta_{22}\alpha_{22} & \beta_{22}\alpha_{23} \\ d' & \beta_{31} & \beta_{32}b' & \beta_{32}\alpha_{21} & \beta_{32}\alpha_{22} & \beta_{32}\alpha_{23} \\ \hline d^0 & \beta_{41} & \beta_{42}b' & \beta_{42}\alpha_{21} & \beta_{42}\alpha_{22} & \beta_{42}\alpha_{23} \\ 0 & a^0 & {}'ab' & {}'a\alpha_{21} & {}'a\alpha_{22} & {}'a\alpha_{23} \\ 0 & 0 & b^0 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ 0 & 0 & 0 & c^0 & {}'c & 0 \end{array}\right).$$

We use Lemma 4. We multiply the matrix we obtain, respectively, on the left and on the right, by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \beta_{23} & 0 \\ 0 & 1 & 0 & 0 & \beta_{33} & 0 \\ 0 & 0 & 1 & 0 & \beta_{43} & 0 \\ 0 & 0 & 0 & 1 & p_1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 0 & & 0 \\ 0 & 1 & q_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As a result, we obtain a matrix

$$\left(\begin{array}{cccc|cc} 0 & \beta_{21} & 0 & 0 & 1 & 0 \\ d' & \beta_{31} & 0 & 0 & 0 & 1 \\ \hline d^0 & \beta_{41} & 0 & 0 & 0 & 0 \\ 0 & a^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ 0 & 0 & 0 & c^0 & {}'c & 0 \end{array}\right),$$

which, by reducing the trivial rows and columns, gives us the matrix $\left(\begin{array}{|cc} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$.

Analogously, the matrix $\gamma \odot \delta$, after triangular transformations and reductions, gives us the unit matrix. Thus, $\delta \odot \gamma \sim (|\underline{1}), \gamma \odot \delta \sim (|\underline{1})$.

We now verify the equivalence

$$a \odot \gamma \sim \left(\begin{array}{c|cc} c' & \tilde{c} & 0 \\ \hline c^0 & {}'c & 0 \end{array}\right), \quad \delta \odot b \sim \left(\begin{array}{c|c} 0 & 0 \\ d' & \tilde{d} \\ \hline d^0 & {}'d \end{array}\right). \tag{13}$$

We have

$$a \odot \gamma = \begin{pmatrix} a' & \tilde{a}b' & \tilde{a}\alpha_{21} & \tilde{a}\alpha_{22} & \tilde{a}\alpha_{23} \\ a^0 & 'ab' & 'a\alpha_{21} & 'a\alpha_{22} & 'a\alpha_{23} \\ 0 & b^0 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ 0 & 0 & c^0 & 'c & 0 \end{pmatrix}.$$

Multiply the matrix thus obtained, respectively, on the left and on the right, by the matrices

$$\begin{pmatrix} 1 & 0 & p_1 & 0 \\ 0 & 1 & p_2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & q_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As a result, we obtain a matrix

$$\begin{pmatrix} a' & 0 & c' & \tilde{c} & 0 \\ a^0 & 0 & 0 & 0 & 0 \\ 0 & b^0 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ 0 & 0 & c^0 & 'c & 0 \end{pmatrix},$$

which after reduction gives us the matrix

$$\begin{pmatrix} c' & \tilde{c} & 0 \\ c^0 & 'c & 0 \end{pmatrix}.$$

The second equivalence in (13) is proved analogously. The proposition is proved.

LEMMA 9. The multiplicative closure $\overline{\Sigma}$ of the n-independent set Σ is n-independent.

Proof. We may assume that the set Σ contains the empty matrix 1_0. Consider the $\overline{\Sigma}$-dependence relation $a \cdot b = 0$ of the form (10). We must prove that it is trivializable. We do this by induction on the sum of the number of rows of the matrix a^0 and the number of columns of the matrix b^0. If each of the matrices a^0, b^0 belongs to the set Σ, then the relation under consideration is, by the condition, trivializable. Let one of the matrices, e.g., a^0, not belong to the set Σ. Then we may rewrite relation (10) in the form

$$\begin{pmatrix} a_{11} & a_{12} & x_1 & u_1 \\ a_{21} & a_{22} & x_2 & u_2 \\ 0 & a_{32} & x_3 & u_3 \end{pmatrix} \cdot \begin{pmatrix} v_{11} & v_{21} & y_1 & b^0 \\ v_{13} & v_{22} & y_2 & 'b \end{pmatrix}^T = 0, \quad a_{21}, a_{32} \in \overline{\Sigma}.$$

[Here and henceforward, the symbol T denotes the formal, i.e., nonexpanded into cells, transposition of matrices, e.g., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \rightleftarrows \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c, and d are matrices.] The relation

$$\overline{(a_{32} \mid x_3 \mid u_3)} \cdot \begin{pmatrix} v_{21} & y_1 & b^0 \\ v_{22} & y_2 & 'b \end{pmatrix}^T = 0$$

is trivializable, by the inductive hypothesis, and we may assume that it is trivial. In this case, the relation under consideration is of the form

$$\begin{pmatrix} a_{11} & a_{12} & x_{11} & x_{12} & u_1 \\ a_{21} & a_{22} & x_{21} & x_{22} & u_2 \\ 0 & a_{32} & x_{31} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_{11} & 0 & 0 & y_{21} & b^0 \\ v_{12} & 0 & 0 & y_{22} & 'b \end{pmatrix}^T = 0, \tag{14}$$

so that we have the relation

$$\begin{pmatrix} a_{11} & x_{12} & u_1 \\ a_{21} & a_{22} & u_2 \end{pmatrix} \cdot \begin{pmatrix} v_{11} & y_{21} & b^0 \\ v_{12} & y_{22} & 'b \end{pmatrix}^T = 0,$$

which, by the induction hypothesis, is trivialized by some pair (α, β). Inserting the rows and columns of the unit matrices in the suitable places, we obtain a pair which trivializes relation (14).

The lemma is proved.

If there exists at least one n-independent set Σ over the ring R, then all relations of length $\leqslant n$ of the form

$$(\underline{|x|})\cdot(\overline{|y})=0$$

are trivializable, and therefore the ring R is an n-FI-ring. (All the definitions and results concerning n-FI-rings may be found in [3].)

LEMMA 10. The set Σ of matrices over the n-FI-ring R is n-independent if and only if all the following relations are trivializable:

$$(\overline{a|x|})\cdot\left(\left|\frac{v}{\underline{y}}\right.\right)=0,\quad a\in\Sigma,$$

$$(\underline{|x|u})\cdot\left(\left.\frac{\overline{y}}{b}\right|\right)=0,\quad b\in\Sigma,$$

$$(\overline{a|x|u})\cdot(\underline{v|y|b})^{T}=0,\quad a,b\in\Sigma$$

where the above relations have length not greater than n.

Proof. Consider an arbitrary Σ_0-dependence relation $a\cdot b = 0$ of the form (10), with length $\leqslant n$. We may assume that the relation

$$(\overline{a^0|x_2|u_2})\cdot(\underline{v_1|y_1|b^0})^{T}=0$$

is trivial, and the relation under consideration is of the form

$$\left(\frac{a'}{a^0}\left|\frac{x_{11}\ x_{12}}{x_{21}\ 0}\right|\frac{u_1}{0}\right)\cdot\left(\frac{0}{v_2}\left|\frac{0\quad y_{21}}{y_{12}\ y_{22}}\right|\frac{b^0}{'b}\right)^{T}=0.$$

In this case we have the relations

$$(\underline{|x_{12}|u_1})\cdot(\underline{|y_{21}|b^0})^{T}=0,\quad(\overline{a^0|x_{21}|})\cdot(\overline{v_2|y_{12}|})^{T}=0$$

and we may assume that these relations are also trivial. Then the relation under consideration takes the form

$$\left(\frac{a'}{a^0}\left|\frac{x_{11}\ x_{12}\ x_{13}\ 0}{x_{21}\ 0\quad 0\quad 0}\right|\frac{0}{0}\right)\cdot\left(\frac{0}{0}\left|\frac{0\ 0\quad 0\quad y_{41}}{0\ y_{22}\ y_{32}\ y_{42}}\right|\frac{b^0}{'b}\right)^{T}=0$$

and we have the relation $(\underline{|x_{13}|})\cdot(\overline{|y_{32}})=0$, which is trivializable, by the definition of an n-FI-ring. The lemma is proved.

COROLLARY. The set Σ of matrices over an arbitrary ring R is independent if and only if it consists of nondivisors of zero, and all relations of the form

$$(\overline{a\|u})\cdot\left(\left.\frac{v}{\overline{b}}\right|\right)=0,\quad a,b\in\Sigma$$

are Σ-trivializable.

Lemmas 9 and 10 allow us to omit the requirement of multiplicativity in the corollary of Lemma 8 and in Proposition 4. We formulate these results in the form of two theorems.

THEOREM 5. Let Σ be a set of rectangular matrices over the ring R, consisting of nondivisors of zero and satisfying the following condition:

For any relation

$$a\cdot v+u\cdot b=0,\quad a,\ b\in\Sigma$$

there exists a pair of mutually invertible matrices

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

over the ring R, such that

$$a \cdot \alpha_{11} + u \cdot \alpha_{21} \in \Sigma, \quad a \cdot \alpha_{12} + u \cdot \alpha_{22} = 0,$$
$$\beta_{11} \cdot v + \beta_{12} \cdot b = 0, \quad \beta_{21} \cdot v + \beta_{22} \cdot b \in \Sigma.$$

Then the set Σ is potentially invertible.

THEOREM 6. Let Σ be a set of rectangular matrices over the n-FI-ring R, such that any relation

$$a \cdot v + x \cdot y + u \cdot b = 0, \ x \in R^m, \ y \in {}^mR, \ m \leqslant n, \quad a, \ b \in \Sigma \cup \{1_0\} \rightleftharpoons \Sigma_0$$

is Σ_0-trivializable. Then the ring $R\Sigma^{-1}$ is an n-FI-ring.

Let l be a positive integer. If the ring R is a $2l$-FI-ring, then the set Σ_l of all complete square matrices of order l satisfies the condition of Theorem 5. If the ring R is a $(2l + n)$-FI-ring, then the set Σ_l also satisfies the condition of Theorem 6. This follows easily from the definition of a k-FI-ring, and from some of their very simple properties (see, e.g., Theorem 1.1.1 of [3]). Hence we have:

COROLLARY 1. A set of complete square matrices of order l over a $2l$-FI-ring is potentially invertible.

This statement is a generalization of Theorem 2 of [1], where the same thing is proved for $l = 1$.

COROLLARY 2. If Σ is the set of all complete square matrices of order l over the $(2l + n)$-FI-ring R, then the ring $R\Sigma^{-1}$ is an n-FI-ring.

This statement gives an answer to Bergman's problem in [2], p. 77.

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INDICATORS OF ENTIRE HERMITIAN-POSITIVE FUNCTIONS OF FINITE ORDER

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We shall say that an entire Hermitian-positive function (e.H.p.f.) is an entire function $f: \mathbf{C} \to \mathbf{C}$, $f(0) = 1$, whose restriction to the real axis is a Hermitian-positive function. The class of e.H.p.f.'s coincides with the class of entire characteristic functions of probability distributions, i.e., with the class of functions of the form

$$f(z) = \int_{-\infty}^{\infty} e^{izu} P(du), \tag{0}$$

where P is a probability measure on the line, and the integral converges absolutely for all $z \in \mathbf{C}$. The finiteness of the order of the e.H.p.f. f is equivalent [1, p. 54] to the condition

$$\varliminf_{r \to \infty} (\ln r)^{-1} \ln \ln (1/P(\{x : |x| > r\})) > 1.$$

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