On class groups of free products

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Introduction

If R is a commutative ring, let $S(R)$ be the category of supplemented R algebras, where R is central in each object Λ of $S(R)$. Thus, if $\varepsilon_\Lambda\colon \Lambda \to R$ is the augmentation, $\eta_\Lambda\colon R \to \Lambda$ the unit, so $\varepsilon_\Lambda \eta_\Lambda = 1$, then a morphism $f\colon \Lambda \to \Gamma$ of $S(R)$ is a ring homomorphism satisfying $\varepsilon_\Gamma f = \varepsilon_\Lambda$, $f\eta_\Lambda = \eta_\Gamma$. We identify R with its image under η_Λ, and denote the augmentation ideal $\operatorname{Ker} \varepsilon_\Lambda\colon \Lambda \to R$ by $\bar{\Lambda}$.

If Λ and Γ are objects of $S(R)$, then their coproduct exists and is denoted $\Lambda *_R \Gamma$. This is just the free product of Λ and Γ described in [3]. $K_0(\Lambda)$ denotes the Grothendieck group of finitely generated projective left Λ modules, and $\overline{K_0}(\Lambda) = \operatorname{Ker} \varepsilon_{\Lambda,*}\colon K_0(\Lambda) \to K_0(R)$. We shall prove

THEOREM 1. *Suppose that R is regular and $\bar{\Lambda} \otimes_R \bar{\Gamma}$ is a flat R module. Then the inclusions $\Lambda \to \Lambda *_R \Gamma$ and $\Gamma \to \Lambda *_R \Gamma$ induce a direct sum decomposition*

$$\overline{K_0}(\Lambda *_R \Gamma) = \overline{K_0}(\Lambda) \oplus \overline{K_0}(\Gamma) .$$

Equivalently,

$$\begin{aligned} K_0(\Lambda *_R \Gamma) = K_0(R) \oplus \operatorname{Ker} \varepsilon_{\Lambda,*}&\colon K_0(\Lambda) \longrightarrow K_0(R) \\ \operatorname{Ker} \varepsilon_{\Gamma,*}&\colon K_0(\Gamma) \longrightarrow K_0(R) , \end{aligned}$$

where the decomposition is induced by inclusions.

COROLLARY. *If Λ_i, $1 \leqq i \leqq n$, are objects of $S(R)$, where R is regular and, for each index i, $\bar{\Lambda}_i$ is a flat R module, then*

$$\overline{K_0}(\Lambda_1 *_R \Lambda_2 *_R \cdots *_R \Lambda_n) = \overline{K_0}(\Lambda_1) \oplus \overline{K_0}(\Lambda_2) \oplus \cdots \oplus \overline{K_0}(\Lambda_n) .$$

The corollary follows from Theorem 1 by induction on n using properties of flat modules.

If A is a ring, and $A[x]$ is the polynomial ring on an indeterminate x, let $U(A)$ be the subgroup of $K_1(A[x])$ generated by invertible matrices of the form $1 + x\nu$, ν a matrix with entries in A. We shall give a Grothendieck group type definition of $U(A)$ and prove

THEOREM 2. *If R is regular and Λ and Γ are objects of $S(R)$ such that $\bar{\Lambda} \otimes_R \bar{\Gamma}$ is a flat R module, then $U(\Lambda *_R \Gamma) = U(\Lambda) \oplus U(\Gamma)$ where the decom-*

position is induced by inclusions $\Lambda \to \Lambda *_R \Gamma$ *and* $\Gamma \to \Lambda *_R \Gamma$.

Theorems 1 and 2 should be considered in conjunction with the theorem of Stallings [3], that

$$\overline{K_1}(\Lambda *_R \Gamma) = \overline{K_1}(\Lambda) \oplus \overline{K_1}(\Gamma)$$

under the same hypotheses as our Theorem 1. Here $\overline{K_1}(\Lambda) = \operatorname{Ker} \varepsilon_{\Lambda,*}: K_1(\Lambda) \to K_1(R)$, and $K_1(\Lambda)$ is the commutator quotient group of GL (Λ)[1]. In fact our results will be deduced from Stallings theorem, with the aid of a theorem of Bass, Heller, and Swan [2], and some algebraic tricks.

1. Applications to group rings

If R is a commutative ring and G is a group, then the group ring $R[G]$ is considered a supplemented R algebra by the augmentation $g \to 1$, $g \in G$. If H is a group, $G * H$ is the free product of groups, and we have the relation

$$R[G * H] = R[G] *_R R[H] .$$

Thus, Theorem 1 implies

THEOREM 1.1. *If R is regular, then*

$$\overline{K_0}(R[G * H]) = \overline{K_0}(R[G]) \oplus \overline{K_0}(R[H]) .$$

If G is a free group of finite rank n, then

$$G = T_1 * T_2 * \cdots * T_n ,$$

where each T_i is an infinite cyclic group. Thus if R is regular

$$\overline{K_0}(R[G]) = \oplus_{i=1}^{n} \overline{K_0}(R[T_i]) .$$

But if R is regular, then $R[T_i]$ is regular, and the theorem of Grothendieck [2] states that $K_0(R) \to K_0(R[T_i])$, induced by inclusion $R \to R[T_i]$, is an isomorphism. Hence, $\overline{K_0}(R[T_i]) = 0$.

THEOREM 1.2. *If R is regular and G is free, then the map*

$$K_0(R) \longrightarrow K_0(R[G])$$

induced by inclusion $R \to R[G]$ is an isomorphism.

PROOF. The preceding discussion establishes this result if G is free of finite rank. The general case is reduced to the case of finite rank by observing that a matrix over $R[G]$ involves entries which are sums of words involving only a finite number of free generations of G.

COROLLARY. *If T is free abelian and G is free, then*

$$K_0(\mathbf{Z}[T \times G]) \cong \mathbf{Z} .$$

PROOF. As above, one reduces to the case with T free abelian of finite rank and G free of finite rank. Then $\mathbf{Z}[T]$ is regular and

$$\mathbf{Z}[T \times G] = \mathbf{Z}[T][G] .$$

Hence $K_0(\mathbf{Z}[T \times G]) \cong K_0(\mathbf{Z}[T])$. By successive applications of the theorem of Grothendieck, one deduces that

$$K_0(\mathbf{Z}[T]) \cong K_0(\mathbf{Z}) \cong \mathbf{Z} .$$

Remark. In his extension of Novikov's splitting lemma, Wall [4] assumes that M is a closed manifold with free abelian fundamental group. The essential restriction on the fundamental group G of M needed for the proof of the splitting lemma is that $K_0(\mathbf{Z}[T \times G]) \cong \mathbf{Z}$ for free abelian groups T. As a consequence of the last corollary, the splitting lemma applies to manifolds with fundamental group $T \times G$, G free and T free abelian, in particular to manifolds with free fundamental group.

2. The functor $U(A)$

If A is a ring we construct a category $\mathfrak{L}(A)$ as follows. An object of $\mathfrak{L}(A)$ will be a pair (P, ν) where P is a finitely generated projective left A module, and ν is a nilpotent endomorphism of P. A morphism $(P, \nu) \xrightarrow{f} (P', \nu')$ is a homomorphism of left A modules $f\colon P \to P'$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\nu} & P \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ P' & \xrightarrow{\nu'} & P' . \end{array}$$

The diagram of morphisms

$$(P', \nu') \xrightarrow{f} (P, \nu) \xrightarrow{g} (P'', \nu'')$$

is a short exact sequence if the corresponding diagram

$$0 \longrightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \longrightarrow 0$$

is a short exact sequence of A-modules. We can form the Grothendieck group $K_0(\mathfrak{L}(A))$. It is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian groups. $K_0(\mathfrak{L}(A))$ can be described as the quotient group of the free abelian group generated by isomorphism classes of objects (P, ν) by the subgroup generated by all $(P, \nu) - (P', \nu') - (P'', \nu'')$, where

$$(P', \nu) \xrightarrow{f} (P, \nu) \xrightarrow{g} (P'', \nu'')$$

is a short exact sequence.

Definition. $L(A)$ is the quotient of $K_0(\mathfrak{L}(A))$ by the subgroup generated by classes of $(P, 0)$. It is a covariant functor from the category of rings to the category of abelian groups.

If Λ is a ring, recall [11] that an element of $K_1(\Lambda)$ is represented by a pair (Q, α) where Q is a finitely generated Λ module and α is an automorphism of Q. For a ring A, we associate to the object (P, ν) in $\mathfrak{L}(A)$ the pair

$$\left(A[x] \otimes_A P, 1 + (x \otimes \nu)\right) .$$

Here $A[x]$ is the polynomial ring on an indeterminate x, and $A \to A[x]$ is the inclusion. Since ν is nilpotent, $1 + x \otimes \nu$ is an automorphism. This association is additive and defines a homomorphism φ: $L(A) \to K_1(A[x])$.

Definition. $U(A)$ is the image of φ in $K_1(A[x])$.

We consider $A[x]$ as a supplemented A algebra by the augmentation $x \mapsto 0$. Clearly the composition

$$L(A) \xrightarrow{\varphi} K_1(A[x]) \longrightarrow K_1(A)$$

is zero.

LEMMA 2.1. *The relative group $K_1(A[x], (x))$ is the kernel of the map*

$$K_1(A[x]) \longrightarrow K_1(A) .$$

PROOF. The easiest way to see this is to invoke the functor K_2[5] and the exact sequence

$$K_2(A[x]) \longrightarrow K_2(A) \longrightarrow K_1(A[x], (x)) \longrightarrow K_1(A[x]) \longrightarrow K_1(A) .$$

Since $A[x] \to A$ has right inverse, $K_i(A[x]) \to K_i(A)$ is surjective, $(i = 1,2)$ whence the sequence

$$0 \longrightarrow K_1(A[x], (x)) \longrightarrow K_1(A[x]) \longrightarrow K_1(A) \longrightarrow 0$$

is split exact. Alternatively, a direct argument may be given in terms of matrices.

Thus φ: $L(A) \to K_1(A[x])$ actually has its image on $K_1(A[x], (x))$.

LEMMA 2.2. φ: $L(A) \to K_1(A[x], (x))$ *is surjective.*

PROOF. An element α of $K_1(A[x], (x))$ is represented by an invertible matrix M congruent to 1 modulo the ideal (x). We may then apply the linearization trick of Higman [6], multiplying M on right and left by elementary matrices, each congruent to 1 modulo (x), to get a matrix representing α of the form $1 + x\nu$, where ν is a matrix with entries in A. Since $1 + x\nu$ is invertible, it follows that ν is nilpotent [2]. Let $1 + x\nu$ act on $A[x]^n$. Then the pair $(A^n, \nu) \in \mathfrak{L}(A)$ represents a class $\beta \in L(A)$ such that $\varphi(\beta) = \alpha$.

COROLLARY 2.3. $U(A) = K_1(A[x], (x)) \subset K_1(A[x])$.

According to Bass, the map $\varphi\colon L(A) \to U(A)$ is an isomorphism. This result will not be used in this paper, but it provides a convenient way of describing $K_1(A[x], (x))$.

3. Passing from $S(R)$ to $S(R[T])$ and to $S(R[x])$

Let Λ be an object of $S(R)$. We let T be an infinite cyclic group, and x denote a polynominal indeterminate. Then $\Lambda[T]$(respectively $\Lambda[x]$) may be considered as a supplemented $R[T]$(respectively $R[x]$) algebra. For $\Lambda[T] = \Lambda \otimes_R R[T](\Lambda[x] = \Lambda \otimes_R[x])$ and the augmentation is in each case $\varepsilon_\Lambda \otimes 1$. If Γ is also an object of $S(R)$, the coproduct of $\Lambda[T]$ and $\Gamma[T]$ exists in $S(R[T])$, and is the free product $\Lambda[T] *_{R[T]} \Gamma[T]$. Similarly the coproduct of $\Lambda[x]$ and $\Gamma[x]$ exists in $S(R[x])$ and is the free product $\Lambda[x] *_{R[x]} \Gamma[x]$.

PROPOSITION 3.1. $\Lambda[T] *_{R[T]} \Gamma[T] = (\Lambda *_R \Gamma)[T]$ *as objects of* $S(R[T])$, *and* $\Lambda[x] *_{R[x]} \Gamma[x] = (\Lambda *_R \Gamma)[x]$ *as objects of* $S(R[x])$.

PROOF. One observes that $(\Lambda *_R \Gamma)[T]$, equipped with the inclusions $\Lambda[T] \to (\Lambda *_R \Gamma)[T]$ and $\Gamma[T] \to (\Lambda *_R \Gamma)[T]$, is a coproduct of $\Lambda[T]$ and $\Gamma[T]$ in $S(R[T])$; similarly for $(\Lambda *_R \Gamma)[x]$.

4. Proofs of Theorems 1 and 2

The theorem of Stallings states (although not in Stallings notation) that

$$(*)\qquad \begin{aligned} K_1(\Lambda *_R \Gamma) = K_1(R) &\oplus \operatorname{Ker} \varepsilon_{\Lambda,*}\colon K_1(\Lambda) \longrightarrow K_1(R) \\ &\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}\colon K_1(\Gamma) \longrightarrow K_1(R) \end{aligned}$$

with the decomposition induced by inclusions, provided the hypotheses of Theorem 1 are satisfied.

Let T be an infinite cyclic group. Then $R[T]$ is regular if R is regular. Also $\overline{\Lambda[T]} \otimes_{R[T]} \overline{\Gamma[T]} = \bar{\Lambda} \otimes_R \bar{\Gamma} \otimes_R (R[T])$ is a flat $R[T]$ module. Thus the theorem of Stallings applies to $(\Lambda *_R \Gamma)[T]$:

$$(**)\qquad \begin{aligned} K_1((\Lambda *_R \Gamma)[T]) = K_1(R[T]) &\oplus \operatorname{Ker} \varepsilon_{\Lambda[T],*}\colon K_1(\Lambda[T]) \longrightarrow K_1(R[T]) \\ &\oplus \operatorname{Ker} \varepsilon_{\Gamma[T],*}\colon K_1(\Gamma[T]) \longrightarrow K_1(R[T]) . \end{aligned}$$

We proceed to compute left and right sides of $(**)$.

If A is any ring, Theorem 2′ of [2] gives a canonical decomposition

$$K_1(A[T]) = K_0(A) \oplus K_1(A) \oplus V_A ,$$

where V_A is generated by classes of unipotent matrices of the form $1 + (t^{\pm 1} - 1)\nu$. Here t is a generator of T and ν is a nilpotent matrix over A. Also, if A is regular, then $V_A = 0$.

Thus we may compute the left side of $(**)$:

$$K_1((\Lambda *_R \Gamma)[T]) = K_0(\Lambda *_R \Gamma) \oplus K_1(\Lambda *_R \Gamma) \oplus V_{\Lambda *_R \Gamma}$$
$$(1) \qquad = K_0(\Lambda *_R \Gamma) \oplus K_1(R) \oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K_1(\Lambda) \longrightarrow K_1(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_1(\Gamma) \longrightarrow K_1(R) \oplus V_{\Lambda *_R \Gamma} ,$$

where we have applied $(*)$ to compute $K_1(\Lambda *_R \Gamma)$.

Now $K_1(R[T]) = K_0(R) \oplus K_1(R)$, $K_1(\Lambda[T]) = K_0(\Lambda) \oplus K_1(\Lambda) \oplus V_\Lambda$, $K_1(\Gamma[T])$
$= K_0(\Gamma) \oplus K_1(\Gamma) \oplus V_\Gamma$,

and

$$\operatorname{Ker} \varepsilon_{\Lambda[T],*}: K_1(\Lambda[T]) \longrightarrow K_1(R[T])$$
$$= \operatorname{Ker} \varepsilon_{\Lambda,*}: K_0(\Lambda) \longrightarrow K_0(R) \oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_1(\Lambda) \longrightarrow K_1(R) \oplus V_\Lambda .$$

Similarly for $\operatorname{Ker} \varepsilon_{\Gamma[T],*}$. Here we have used the fact $V_R = 0$.

Thus, the right side of $(**)$ is computed as

$$(2) \qquad K_1((\Lambda *_R \Gamma)[T]) = K_0(R) \oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K_0(\Lambda) \longrightarrow K_0(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_0(\Gamma) \longrightarrow K_0(R) \oplus K_1(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K_1(\Lambda) \longrightarrow K_1(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_1(\Gamma) \longrightarrow K_1(R) \oplus V_\Lambda \oplus V_\Gamma .$$

If we compare the right sides of equations (1) and (2), and examine the composite isomorphism, we see that this isomorphism carries K_0 terms to K_0 terms, K_1 terms to K_1 terms, and V-terms to V-terms. In particular

$$K_0(\Lambda *_R \Gamma) = K_0(R) \oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K^0(\Lambda) \longrightarrow K_0(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_0(\Gamma) \longrightarrow K_0(R) ,$$

where the decomposition is induced by inclusions. The result can be restated

$$\overline{K_0}(\Lambda *_R \Gamma) = \overline{K_0}(\Lambda) \oplus \overline{K_0}(\Gamma) ,$$

which completes the proof of Theorem 1.

We begin now the proof of Theorem 2. The polynomial ring $R[x]$ is regular if R is regular, and $\overline{\Lambda[x]} \otimes_{R[x]} \overline{\Gamma[x]} = \bar{\Lambda} \otimes_R \bar{\Gamma} \otimes_R (R[x])$ is a flat $R[x]$ module. Thus the theorem of Stallings applies to $(\Lambda *_R \Gamma)[x]$.

$$(3) \qquad K_1((\Lambda *_R \Gamma)[x]) = K_1(R[x]) \oplus \operatorname{Ker} \varepsilon_{\Lambda[x],*}: K_1(\Lambda[x]) \longrightarrow K_1(R[x])$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma[x],*}: K_1(\Gamma[x]) \longrightarrow K_1(R[x]) .$$

From the split exact sequence in the proof of 2.1, we deduce that

$$K_1(\Lambda *_R \Gamma[x]) = K_1(\Lambda *_R \Gamma) \oplus U(\Lambda *_R \Gamma)$$
$$(4) \qquad = K_1(R) \oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K_1(\Lambda) \longrightarrow K_1(R)$$
$$\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_1(\Gamma) \longrightarrow K_1(R)$$
$$\oplus U(\Lambda *_R \Gamma) .$$

Now $K_1(R[x]) = K_1(R), K_1(\Lambda[x]) = K_1(\Lambda) \oplus U(\Lambda)$, and $K_1(\Gamma[x]) = K_1(\Gamma) \oplus U(\Gamma)$. Thus

$$\begin{aligned} &\operatorname{Ker} \varepsilon_{\Lambda[x],*}: K_2(\Lambda[x]) \longrightarrow K_1(R[x]) \\ &\quad = \operatorname{Ker} \varepsilon_{\Lambda[x],*}: K_1(\Lambda) \longrightarrow K_1(R) \oplus U(\Lambda)\ , \end{aligned}$$

and similarly for $\operatorname{Ker} \varepsilon_{\Gamma[x],*}$. Thus, the right side of (3) is computed as

$$\begin{aligned} K_1((\Lambda *_R \Gamma)[x]) = K_1(R) &\oplus \operatorname{Ker} \varepsilon_{\Lambda,*}: K_1(\Lambda) \longrightarrow K_1(R) \\ &\oplus \operatorname{Ker} \varepsilon_{\Gamma,*}: K_1(\Gamma) \longrightarrow K_1(R) \\ &\oplus U(\Lambda) \oplus U(\Gamma)\ . \end{aligned} \tag{5}$$

If we compare the right sides of equations (4) and (5), examining the composite isomorphism, we deduce, as in the proof of Theorem 1, that

$$U(\Lambda *_R \Gamma) = U(\Lambda) \oplus U(\Gamma)\ .$$

This complete the proof of Theorem 2.

We remark finally that Bass has shown (unpublished) a remarkable connection between V_A and $U(A)$. If T is an infinite cyclic group generated by t, and x is a polynomial indeterminate, then there are maps

$$\psi_+,\ \psi_-: A[x] \longrightarrow A[T]$$

given by $\psi_+(x) = t - 1$, $\psi_-(x) = t^{-1} - 1$. The compositions, ω_+ and ω_-, of maps

$$L(A) \xrightarrow{\varphi} U(A) \subset K_1(A[x]) \xrightarrow{\psi_{+,*}} K_1(A[T])$$

$$L(A) \xrightarrow{\varphi} U(A) \subset K_1(A[x]) \xrightarrow{\psi_{-,*}} K_1(A[T])$$

and respectively, provide a canonical decomposition of V_A as $\operatorname{Im} \omega_+ \oplus \operatorname{Im} \omega_-$. This fact can also be used, with the proof of Theorem 1, to deduce Theorem 2.

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