Ghys, Étienne; Ranicki, Andrew

Signatures in algebra, topology and dynamics. (English) Zbl 06701552

Ghys, Étienne (ed.) et al., Six papers on signatures, braids and Seifert surfaces. Rio de Janeiro: Sociedade Brasileira de Matemática (SBM) (ISBN 978-85-8337-103-8/pbk). Ensaios Matemáticos 30, 1-173 (2016).

This remarkable – and enjoyable – paper begins with the modest observation that the classification of conic sections into ellipse, parabola, hyperbola familiar from high-school geometry, formalized in Sylvester's law of inertia from 1852, immediately suggests the notion of rank and signature for quadratic forms. This sets the tone for the remaining (large number of) pages: the simple idea of quadratic forms as almost the first non-trivial polynomial object one can study giving rise to the concept of 'signature', which is then placed in a historical context and studied deeply. Goethe's observation *"Die Mathematiker sind eine Art Franzosen; redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anders"* comes to mind, as this fascinating text – while remaining definitely 'about' signatures – wanders far and wide across different parts of mathematics. As the authors argue, there is much that can be said about the pervasive presence and influence of the theory of quadratic forms across many areas of mathematics, and this substantial survey presents 'some *morceaux choisis*' as a general introduction, with a focus on algebra, topology and dynamics and the intention of helping readers engage with other papers on braids and Seifert surfaces in the same collection.

After a brief introduction, the main body of the survey covers – *inter alia* – the following main themes: 1) The basic machinery of quadratic forms on $\mathbb{R}^n$. Sturm's theorem counting real zeros of a polynomial is introduced in familiar guise as a number of sign variations, then reformulated following Sylvester as the signature of a symmetric matrix, and finally as an equality in an associated Witt group. 2) A first visit to the topological part of the story, explaining how the homology of a manifold may be equipped with intersection forms, bringing signatures into the picture as invariants of manifolds. 3) The Maslov class, and an explanation of how it naturally arises in physics, topology, number theory, dynamical systems, geometry, and so on. 4) Hamiltonian dynamical systems and an explanation of how signatures arise for them, the invariants of Calabi and Ruelle and an introduction to the Conley-Zehnder index. 5) Number theory, the extraordinarily pervasive presence of the authors' 'favourite group' $\mathrm{SL}_2(\mathbb{R})$, in this case as it relates to low-dimensional topology and signatures. Torus bundles and how one constructs a 3-manifold from an element of $\mathrm{SL}_2(\mathbb{R})$, lens spaces, and Dedekind sums. 6) An appendix by the second author introduces the algebraic L-theory required for applications to the topology of manifolds, in particular the construction of algebraic cobordism groups.

Throughout, the emphasis is on insight mathematical, historical, and motivational rather than proofs. Along with many entertaining asides and observations, there is an extensive bibliography for a reader who wishes to go further or dig deeper. Both the subject matter and the effort put in by the expert authors to explain concepts while referring to proofs elsewhere make this an attractive and useful contribution.

For the entire collection see [Zbl 06646412].

Reviewer: Thomas B. Ward (Leeds)

MSC:

11E81 Algebraic theory of quadratic forms