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Hasse–Witt Invariants for (α, u)-Reflexive Forms and Automorphisms. I: Algebraic K_2-Valued Hasse–Witt Invariants*

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IN MEMORIAM: RALPH HARTZLER FOX (1913–1973)

"We knew not the knots he knew."

0. INTRODUCTION

The classical invariants of (nonsingular) symmetric bilinear forms over a field are rank, discriminant, signatures, and Hasse–Witt invariants. In the theory of (α, u)-reflexive forms [9], an algebraic K_0-valued rank and an algebraic K_1-valued discriminant come into play, leading to various algebraic L-theories L^p, L^h, L^s, as well as to certain periodicity phenomena relating them [10].

In this paper, we generalize for (α, u)-reflexive forms the Hasse–Witt invariants to an algebraic K_2-valued invariant. A corresponding invariant for automorphisms (isometries) of such forms is also defined This is all accomplished in Section 3, and the relation of the invariants defined with the Hasse–Witt invariant in the case of a field is given in Section 4.

The first section motivates and defines involutions $T_{\alpha,u}$ on the even order general linear groups $\mathrm{GL}(2n, R)$ and Steinberg groups $\mathrm{St}(2n; R)$, compatible with the natural homomorphisms $\mathrm{St}(2n; R) \to \mathrm{GL}(2n; R)$, where (R, α, u) is a ring with antistructure As $n \to \infty$, $T_{\alpha,u}$ induces the right involution on K_iR, $i = 1, 2$ Then (α, u)-reflexive forms and automorphisms of even rank and their discriminants are described in terms of $T_{\alpha,u}$, in Section 2. In Sections 2 and 3, there arise "differentials" $H^*(Z_2; K_iR) \to H^*(Z_2; K_{i+1}R)$, $i = 0, 1$, which are carefully described; also, some general computational structure of these differentials is given.

For algebraic L-theory (and for topological applications), K_1R and K_2R are not the right value groups for discriminants and Hasse–Witt invariants of reflexive forms Hence, in Section 5, we describe Steinberg-type groups

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$\mathrm{St}^V(n; R)$ for suitable subgroups $V \subseteq R^\cdot$, the group of units of R, and an exact sequence

$$0 \to K_2{}^V R \to \mathrm{St}^V(R) \to \mathrm{GL}(R) \to K_1{}^V R \to 0,$$

so that $K_1{}^V R$-valued discriminants and $K_2{}^V$-valued Hasse–Witt invariants may be defined with all the right properties. The special choice $R = \mathbb{Z}\pi$, $V = \pm\pi$ gives $K_1{}^V R = Wh(\pi)$, $K_2{}^V R = Wh_2(\pi)$ and the "correct" theory for surgery of manifolds.

In the second paper of this series, we shall give an algebraic L-theory application and interpretation of the ideas set forth here. In particular, an L-theory L^{st} will be described, which fits into an exact triangle

$$\begin{array}{ccc} L_*^{st} & \longrightarrow & L_*{}^s \\ & \nwarrow \quad \swarrow & \\ & H^*(Z_2\,;K_2) & \end{array}$$

From the standpoint of periodicity, the periodicity sequences of Wall [10] may be extended somewhat farther to the left. Also, the direct connection with the unitary K_2 of Sharpe [7] may be established.

Subsequently, computational results and topological applications will be given, including the relation of L^{st} to pseudo-isotopy.

A final word on our restriction of all discussion of forms in this paper to reflexive forms (as opposed to Hermitian or quadratic forms) is in order. This is because the theory of reflexive forms is, in a precise sense, the "fixed point theory" of an involution on a theory of finitely generated projective modules (cf. [3, 8]); moreover, the algebraic K-theory valued invariants we treat arise from this equivariant algebraic K-theoretic situation.

1. Algebraic Preliminaries

Let R be an associative ring with unity. An *antistructure* (α, u) on R consists of an antiautomorphism α of R together with a unit $u \in R$ such that $\alpha(u)\,u = 1$ and $\alpha^2(r) = uru^{-1}$ for every $r \in R$. There is the contravariant duality functor $D_\alpha\colon \mathcal{M}_R \to \mathcal{M}_R$ given by $D_\alpha M = \mathrm{Hom}_R(M, R)$ for $M \in \mathrm{Obj}\,\mathcal{M}_R$, with the conjugate right R-module structure determined by α,

$$(fr)(x) = \alpha(r) f(x), \qquad r \in R, \quad x \in M, \quad f \in \mathrm{Hom}_R(M, R),$$

and $D_\alpha h = \mathrm{Hom}_R(h, R)$ for $h \in \mathrm{Mor}\,\mathcal{M}_R$.

Lemma 1.1. *The formula $(\eta_{\alpha,u}M)(x)(f) = \alpha(f(x))\,u$ defines a natural transformation $\eta_{\alpha,u}\colon 1_{\mathcal{M}_R} \to D_\alpha{}^2$.*

Proof. First, we check that $(\eta_{\alpha,u}M)(x) \in D_\alpha{}^2M$:

$$(\eta_{\alpha,u}M)(x)(fr) = \alpha((fr)(x))\, u = \alpha(\alpha(r) f(x))\, u = \alpha(f(x))\, \alpha^2(r)\, u = \alpha(f(x))\, ur = (\eta_{\alpha,u}M)(x)(f)\, r.$$

Next, we check that $\eta_{\alpha,u}M \in \mathrm{Hom}_R(M, D_\alpha{}^2M)$:

$$(\eta_{\alpha,u}M)(xr)(f) = \alpha(f(xr))\, u = \alpha(f(x)\, r)\, u = \alpha(r)\, \alpha(f(x))\, u = \alpha(r)(\eta_{\alpha,u}M)(x)(f) = ((\eta_{\alpha,u}M)(x)\, r)(f).$$

Hence, $\eta_{\alpha,u}M$ is well defined for every $M \in \mathrm{Obj}\, \mathcal{M}_R$. That it is a natural transformation follows easily. ∎

PROPOSITION 1.2. *The natural transformation $\eta_{\alpha,u}$ defines a self-adjunction of D_α; that is, $(D_\alpha\eta_{\alpha,u})(\eta_{\alpha,u}D_\alpha) = 1_{D_\alpha}$, and the associated homomorphisms*

$$t_{\alpha,u} = t_{\alpha,u}(M, N)\colon \mathrm{Hom}_R(N, D_\alpha M) \to \mathrm{Hom}_R(M, D_\alpha N)$$

defined by $t_{\alpha,u}(M, N)(f) = (D_\alpha f)(\eta_{\alpha,u}M)$ are natural isomorphisms and satisfy $t^2_{\alpha,u} = 1$, i.e.,

$$t_{\alpha,u}(N, M)\, t_{\alpha,u}(M, N) = 1_{\mathrm{Hom}_R(N, D_\alpha M)}.$$

Proof. It suffices to show that $(D_\alpha\eta_{\alpha,u})(\eta_{\alpha,u}D_\alpha) = 1_{D_\alpha}$, the other parts then being routine (cf. Eilenberg and Moore [2], for example). Let $f \in \mathrm{Hom}_R(M, R)$, $g \in \mathrm{Hom}_R(D_\alpha M, R)$, and $h \in \mathrm{Hom}_R(D_\alpha{}^2M, R)$; then, $(\eta_{\alpha,u}D_\alpha M)(f)(g) = \alpha(g(f))\, u$ and

$$(D_\alpha\eta_{\alpha,u}M)(h)(x) = h((\eta_{\alpha,u}M)(x)).$$

Hence, we have the composition

$$(D_\alpha\eta_{\alpha,u}M)((\eta_{\alpha,u}D_\alpha M)(f))(x) = (\eta_{\alpha,u}D_\alpha M)(f)((\eta_{\alpha,u}M)(x)) = \alpha((\eta_{\alpha,u}M)(x)(f))\, u = \alpha(\alpha(f(x))\, u)\, u = u^{-1}\alpha^2 f(x)\, u = f(x) \quad ∎$$

Note. Adjoint and self-adjoint contravariant functors and the reflexive structures determined by the latter are discussed in greater detail in [3].

It is helpful to have an alternate description of $D_\alpha{}^2M$ in the cases where $\eta_{\alpha,u}M$ is an isomorphism. If $\phi\colon R \to S$ is a ring homomorphism then there is the *base change* functor $J_\phi\colon \mathcal{M}_S \to \mathcal{M}_R$ given by $J_\phi M = M$, $J_\phi f = f$ for $M \in \mathrm{Obj}\, \mathcal{M}_S$, $f \in \mathrm{Mor}\, \mathcal{M}_S$, where $J_\phi M$ has the induced R-module structure $x \cdot_\phi r = x\phi(r)$ for $x \in M$, $r \in R$. The following is trivial.

LEMMA 1.3. *The formula $(j_{\alpha,u}M)(x) = xu^{-1}$ defines a natural equivalence of functors $j_{\alpha,u}\colon 1_{\mathcal{M}_R} \to J_{\alpha^2}$.* ∎

It follows that the natural transformation $\theta_{\alpha,u} = \eta_{\alpha,u}j^{-1}_{\alpha,u}\colon J_{\alpha^2} \to D_\alpha{}^2$ has the especially simple form

$$(\theta_{\alpha,u}M)(x)(f) = \alpha(f(xu))\, u = \alpha^{-1}(f(x)).$$

Thus, for example, $(\theta_{\alpha,u}M)(x)(f) = 0$ or 1 according as $f(x) = 0$ or 1 In particular, let F be the free right R-module with finite basis $e_1, \ldots, e_n$; then the *dual basis* of $D_\alpha F$ to $e_1, \ldots, e_n$ is the basis $e_1^*, \ldots, e_n^*$ of $D_\alpha F$ determined by $e_i^*(e_j) = \delta_i^j$ (Kronecker delta). Similarly, $D_\alpha^2 F$ has the basis $e_1^{**}, \ldots, e_n^{**}$ dual to $e_1^*, \ldots, e_n^*$; then, if $J_{\alpha^2}F$ has the basis $e_1, \ldots, e_n$, we have shown the following.

LEMMA 1.4. *For* $i = 1, \ldots, n$, $(\theta_{\alpha,u}F)(e_i) = e_i^{**}$. *Moreover,* $\theta_{\alpha,u}M\colon J_{\alpha^2}M \to D_\alpha^2 M$ *is an isomorphism whenever* $\eta_{\alpha,u}M$ *is; in particular,* $\theta_{\alpha,u}P$ *is an isomorphism for every finitely generated projective R-module P.* ∎

Let $\mathscr{P}_R \subset \mathscr{M}_R$ be the subcategory of finitely generated projective R-modules *and isomorphisms* of such. The restriction of D_α defines a contravariant functor on $\mathscr{P}_R$ by (1 4), however, since the morphisms of $\mathscr{P}_R$ are isomorphisms, it is more convenient to consider the covariant functor $T_\alpha\colon \mathscr{P}_R \to \mathscr{P}_R$ given by $T_\alpha P = D_\alpha P$, $T_\alpha f = D_\alpha f^{-1}$ for $P \in \operatorname{Obj} \mathscr{P}_R$, $f \in \operatorname{Mor} \mathscr{P}_R$. We have $T_\alpha^2 = D_\alpha^2$ on $\mathscr{P}_R$, and J_{α^2} restricts to a functor on $\mathscr{P}_R$. Hence, by (1.4) we have the natural equivalences

$$\eta_{\alpha,u}\colon 1_{\mathscr{P}_R} \cong T_\alpha^2, \qquad \theta_{\alpha,u}\colon J_{\alpha^2} \cong T_\alpha^2$$

of functors on $\mathscr{P}_R$. As before, we also have the natural involutions

$$t_{\alpha,u}(P, Q)\colon \mathscr{P}_R(Q, T_\alpha P) \cong \mathscr{P}_R(P, T_\alpha Q).$$

Since T_α is a product preserving functor and $\eta_{\alpha,u}$ is a natural equivalence of functors on $\mathscr{P}_R$, it follows that T_α induces an involution, also denoted T_α, on K_*R, the Quillen–Segal algebraic K-theory of R. Although T_α^2 is the identity on K_*R, it is not the case that, for K_1R or K_2R, T_α is induced from an involution on $\mathrm{GL}(P, R)$ or on $\mathrm{St}(n, R)$. However, for isomorphisms between projectives of the form $P \oplus T_\alpha P$, we can do considerably better.

Let $H_\alpha\colon \mathscr{P}_R \to \mathscr{P}_R$ be the *hyperbolic module functor*, given by $H_\alpha P = P \oplus T_\alpha P$, $H_\alpha f = f \oplus T_\alpha f$ for $P \in \operatorname{Obj} \mathscr{P}_R$, $f \in \operatorname{Mor} \mathscr{P}_R$ Then $\bar{H}_\alpha \mathscr{P}_R$ denotes the category with objects $H_\alpha P$ for $P \in \operatorname{Obj} \mathscr{P}_R$ and with morphisms from $H_\alpha P$ to $H_\alpha Q$ *all* the morphisms in $\mathscr{P}_R$ from $H_\alpha P$ to $H_\alpha Q$ Since $T_\alpha H_\alpha P \cong H_\alpha T_\alpha P$ naturally in $P \in \operatorname{Obj} \mathscr{P}_R$, the functor $T_\alpha\colon \bar{H}_\alpha \mathscr{P}_R \to \bar{H}_\alpha \mathscr{P}_R$ is defined, and $\eta_{\alpha,u}\colon 1_{\bar{H}_\alpha \mathscr{P}_R} \cong T_\alpha^2$.

For $P \in \operatorname{Obj} \mathscr{P}_R$, there is the *hyperbolic form*

$$\psi_{\alpha,u}P = \begin{pmatrix} 0 & 1_{T_\alpha P} \\ \eta_{\alpha,u}P & 0 \end{pmatrix} H_\alpha P \to H_\alpha T_\alpha P$$

in $\bar{H}_\alpha \mathscr{P}_R$. Let $T_{\alpha,u}\colon \bar{H}_\alpha \mathscr{P}_R \to \bar{H}_\alpha \mathscr{P}_R$ be the functor given by $T_{\alpha,u} H_\alpha P = H_\alpha P$, $T_{\alpha,u} f = (\psi_{\alpha,u}Q)^{-1}(T_\alpha f)(\psi_{\alpha,u}P)$ for $P \in \operatorname{Obj} \mathscr{P}_R$, $f \in \mathscr{P}_R(H_\alpha P, H_\alpha Q)$. Then the following is immediate.

LEMMA 1 5. $\psi_{\alpha,u}$ *is a natural equivalence of functors on* $\bar{H}_\alpha \mathscr{P}_R$, $\psi_{\alpha,u}\colon T_{\alpha,u} \cong T_\alpha$. ∎

Noting that $\psi_{\alpha,u}T_\alpha = T_\alpha\psi_{\alpha,u}$ and that $(T_\alpha\psi_{\alpha,u})(\psi_{\alpha,u}) = \eta_{\alpha,u}\colon 1_{\bar{H}_\alpha\mathscr{P}_R} \cong T_\alpha^{\ 2}$, we have, not only that $T_{\alpha,u}$ fixes the objects of $\bar{H}_\alpha\mathscr{P}_R$, but also that $T^2_{\alpha,u} = 1_{\bar{H}_\alpha\mathscr{P}_R}$.

Since the canonical functor $\bar{H}_\alpha\mathscr{P}_R \to \mathscr{P}_R$ is cofinal, we have

$$\begin{aligned} K_i\bar{H}_\alpha\mathscr{P}_R &\cong (1 + T_\alpha)\, K_0R \qquad &&\text{if} \quad i = 0, \\ &\cong K_iR &&\text{if} \quad i > 0. \end{aligned}$$

Furthermore, $T_{\alpha,u} = T_\alpha$ on K_iR, $i > 0$, and $T^2_{\alpha,u} = 1$ on $\mathrm{GL}(H_\alpha P; R)$ for $P \in \mathrm{Obj}\,\mathscr{P}_R$. We shall see that the same holds for the Steinberg group $\mathrm{St}(H_\alpha F; R)$ for $F \in \mathrm{Obj}\,\mathscr{P}_R$ a free module with specified basis $e_1, \ldots, e_n$. For then $H_\alpha F$ has the basis $e_1, \ldots, e_n, e_1{}^*, \ldots, e_n{}^*$, and in matrix notation,

$$T_{\alpha,u}M = \begin{pmatrix} 0 & I_nu^{-1} \\ I_n & 0 \end{pmatrix} M^{-\alpha} \begin{pmatrix} 0 & I_n \\ I_nu & 0 \end{pmatrix}$$

in $\mathrm{GL}(2n; R) \cong \mathrm{GL}(H_\alpha F; R)$, where M^α denotes conjugate transpose of M by α, $M^{-\alpha}$ the inverse of M^α, and I_n is the identity of $\mathrm{GL}(n; R)$.

PROPOSITION 1.6. *If e^r_{ij} is an elementary matrix in $E(2n; R) \subset \mathrm{GL}(2n; R) \cong \mathrm{GL}(H_\alpha F; R)$, then $T_{\alpha,u}e^r_{ij} = e^s_{pq}$ for suitable unique p, q, s (depending upon notational convention only).*

Proof. We simply give below the result of applying $T_{\alpha,u}$ to the elementary matrix e^r_{ij}. The proposition follows from the same kind of arguments given in Milnor [5, 9.2, 9.4], except that we are working in $E(2n; R)$. Let the basis element e_i correspond to the integer i, and let the basis element $e_i{}^*$ correspond to the integer $-i$, where $i = 1, \ldots, n$. Then

$$\begin{aligned} T_{\alpha,u}e^r_{ij} &= e^{-\alpha(r)}_{-j,-i} \qquad && i, j \text{ both} > 0, \\ &= e^{-u^{-1}\alpha(r)}_{-j,-i} && i > 0, \ j < 0, \\ &= e^{-\alpha(r)u}_{-j,-i} && i < 0, \ j > 0, \\ &= e^{-\alpha^{-1}(r)}_{-j,-i} && i, j \text{ both} < 0. \quad \blacksquare \end{aligned}$$

For convenience, we record how $T_{\alpha,u}$ acts on $E(2n; R)$ in blockwise notation (subscripts are to be ordered $1, \ldots, n, -1, \ldots, -n$).

$$T_{\alpha,u}\begin{pmatrix} I_n & E \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & -u^{-1}E^\alpha \\ 0 & I_n \end{pmatrix},$$

$$T_{\alpha,u}\begin{pmatrix} I_n & 0 \\ E & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -E^\alpha u & I_n \end{pmatrix},$$

$$T_{\alpha,u}\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} E_2^{-\alpha^{-1}} & 0 \\ 0 & E_1^{-\alpha} \end{pmatrix}$$

Also, for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2n; R)$, we have

$$T_{\alpha,u}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^{\alpha^{-1}} & u^{-1}B^{\alpha} \\ C^{\alpha}u & A^{\alpha} \end{pmatrix}$$

Using the Steinberg relations for $E(2n; R)$, the following is easily checked (cf. Milnor [5, Chap 5, 10.4]).

COROLLARY 1.7. *The involution $T_{\alpha,u}$ on $E(2n; R)$ lifts to an involution $T_{\alpha,u}$ on* St$(2n; R)$ *defined by the permutation of generators*

$$\begin{aligned} T_{\alpha,u}x_{ij}^{r} &= x_{-j,-i}^{-\alpha(r)} && i, j \text{ both } > 0, \\ &= x_{-j,-i}^{-u^{-1}\alpha(r)} && i > 0, \quad j < 0, \\ &= x_{-j,-i}^{-\alpha(r)u} && i < 0, \quad j > 0, \\ &= x_{-j,-i}^{-\alpha^{-1}(r)} && i, j \text{ both } < 0. \quad \blacksquare \end{aligned}$$

Letting $n \to \infty$ gives the corresponding result for $\mathrm{St}(R) =$ "St$(2\infty; R)$." Thus we have a commutative diagram with exact rows and vertical morphisms all involutions (here, $\mathrm{GL}(R) =$ "GL$(2\infty; R)$," also)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2R & \longrightarrow & \mathrm{St}(R) & \xrightarrow{\phi} & \mathrm{GL}(R) & \longrightarrow & K_1R & \longrightarrow & 0 \\ & & {\scriptstyle T_{\alpha,u}=T_\alpha}\downarrow & & {\scriptstyle T_\alpha}\downarrow & & \downarrow{\scriptstyle T_\alpha} & & \downarrow{\scriptstyle T_{\alpha,u}=T_\alpha} & & \\ 0 & \longrightarrow & K_2R & \longrightarrow & \mathrm{St}(R) & \xrightarrow{\phi} & \mathrm{GL}(R) & \longrightarrow & K_1R & \longrightarrow & 0. \end{array}$$

2. REFLEXIVE FORMS ON HYPERBOLIC MODULES

Recall that a (nonsingular) (α, u)*-reflexive form* on $P \in \mathrm{Obj}\, \mathscr{P}_R$ is an element $g \in \mathscr{P}_R(P, T_\alpha P)$ such that $\eta_{\alpha,u}P = (T_\alpha g)\, g$ (in other words, $t_{\alpha,u}(P, T_\alpha P)(g) = (D_\alpha g)(\eta_{\alpha,u}P) = g$) A basic observation in our approach to reflexive forms is the following.

PROPOSITION 2.1. *If $P \in \mathrm{Obj}\, \mathscr{P}_R$, then the (nonsingular) (α, u)-reflexive forms g on $H_\alpha P$ are in one-to-one correspondence with the elements $g' \in \mathrm{GL}(H_\alpha P; R)$ such that $1_{H_\alpha P} = (T_{\alpha,u}g')\, g'$, according to the rule $g = (\psi_{\alpha,u}P)\, g'$.*

Proof. Since $T_\alpha\psi_{\alpha,u}P = \psi_{\alpha,u}T_\alpha P$ and $(\psi_{\alpha,u}T_\alpha P)(\psi_{\alpha,u}P) = \eta_{\alpha,u}H_\alpha P$, we have

$$\begin{aligned} T_{\alpha,u}((\psi_{\alpha,u}P)^{-1} g) &= (\psi_{\alpha,u}P)^{-1}\,(T_\alpha\psi_{\alpha,u}P)^{-1}\,(T_\alpha g)(\psi_{\alpha,u}P) \\ &= (\eta_{\alpha,u}P)^{-1}\,(T_\alpha g)(\psi_{\alpha,u}P) \end{aligned}$$

Hence, if $g = (\psi_{\alpha,u}P)\, g'$, then $(T_{\alpha,u}g')\, g' = 1_{H_\alpha P}$ if and only if $1_{H_\alpha P} = (\eta_{\alpha,u}P)^{-1}\,(T_\alpha g)\, g$, i e., if and only if g is (α, u)-reflexive. $\blacksquare$

Of course, the hyperbolic form $\psi_{\alpha,u}P$ is itself an (α, u)-reflexive form on $H_\alpha P$, and it corresponds in Proposition 2.1 to $1_{H_\alpha P}$ The following is clear.

LEMMA 2 2 *Let $P, Q \in \operatorname{Obj} \mathscr{P}_R$, and let $g = (\psi_{\alpha,u}P)g'$, $h = (\psi_{\alpha,u}Q)h'$ be (α, u)-reflexive forms on P, Q, respectively; if $\tau: H_\alpha P \oplus H_\alpha Q \cong H_\alpha(P \oplus Q)$ is the canonical isomorphism which interchanges the middle two summands, then*

$$g \boxplus h = (T_\alpha \tau)(g \oplus h)\tau^{-1} = (\psi_{\alpha,u}(P \oplus Q))(g' \boxplus h')$$

is an (α, u)-reflexive form on $H_\alpha(P \oplus Q)$, and $g' \boxplus h' = \tau(g' \oplus h')\tau^{-1}$ ▮

Thus, stabilizing the (α, u)-reflexive form $g = (\psi_{\alpha,u}P)g'$ on $H_\alpha P$ by orthogonal direct sum with the hyperbolic form $\psi_{\alpha\ u}Q$ on $H_\alpha Q$ corresponds in Proposition 2.1 to stabilizing $g' \in \mathrm{GL}(H_\alpha P, R)$ by direct sum with $1_{H_\alpha Q}$ The process of stabilizing g or g' this way will always be followed by τ_*, viz ,

$$\mathrm{GL}(H_\alpha P; R) \hookrightarrow \mathrm{GL}(H_\alpha P \oplus H_\alpha Q; R) \underset{\tau_*}{\cong} \mathrm{GL}(H_\alpha(P \oplus Q); R),$$

as with the direct sum operations $\boxplus$. This is basically in agreement with notation adopted in the previous section for free modules F and for $\mathrm{GL}(H_\alpha F, R)$.

Having identified $g' \in \mathrm{GL}(H_\alpha P; R)$ such that $1_{H_\alpha P} = (T_{\alpha,u}g')g'$ with the (α, u)-reflexive form $(\psi_{\alpha,u}P)g'$, we should point out that an element $f \in \mathrm{GL}(H_\alpha P; R)$ satisfies $T_{\alpha,u}f = f$ if and only if f is an *automorphism* of the hyperbolic form $\psi_{\alpha,u}P$. More generally, we have the following.

LEMMA 2.3. *If $g = (\psi_{\alpha,u}P)g'$, $h = (\psi_{\alpha,u}Q)h'$ are (α, u)-reflexive forms on $P, Q \in \operatorname{Obj} \mathscr{P}_R$, respectively, then $f \in \mathscr{P}_R(H_\alpha P, H_\alpha Q)$ is an isomorphism of (α, u)-reflexive forms from g to h if and only if $T_{\alpha,u}f = h'fg'^{-1}$.*

Proof. Expanding $T_{\alpha,u}f = (\psi_{\alpha,u}Q)^{-1}(T_\alpha f)(\psi_{\alpha,u}P)$, we have $T_{\alpha,u}f = h'fg'^{-1}$ if and only if $(D_\alpha f)hf = g$, which is just the condition for f to be an isomorphism from g to h ▮

The *discriminant with respect to $P \in \operatorname{Obj} \mathscr{P}_R$* of an (α, u)-reflexive form g on $H_\alpha P$ is defined to be the class of $(\psi_{\alpha,u}P)^{-1}g \in \mathrm{GL}(H_\alpha P; R)$ in K_1R,

$$\operatorname{disc}_P g = [(\psi_{\alpha,u}P)^{-1}g] \in K_1R.$$

Similarly, if $e \in \mathscr{P}_R(P, Q)$ and if $f \in \mathscr{P}_R(H_\alpha P, H_\alpha Q)$ is an isomorphism of (α, u)-reflexive forms from g to h, the *discriminant with respect to e* of f is defined to be the class of $(H_\alpha e)^{-1}f \in \mathrm{GL}(H_\alpha P; R)$ in K_1R,

$$\operatorname{disc}_e f = [(H_\alpha e)^{-1}f] \in K_1R.$$

If $P = Q$, $e = 1_P$, and $g = h$, then $\operatorname{disc}_1 f$ is simply the *determinant* of the automorphism f of g. From Proposition 2 1, it is clear that

$$(1 + T_\alpha)\operatorname{disc}_P g = 0;$$

on the other hand, since $T_{\alpha,u}H_\alpha = H_\alpha$, we have from Lemma 2.3 that

$$(1 - T_\alpha)\, \mathrm{disc}_e f = \mathrm{disc}_P\, g - \mathrm{disc}_Q\, h,$$

which vanishes if g and h have the same relative discriminants, e.g., when f is an automorphism. From a somewhat different viewpoint, Lemma 2.3 tells us that, up to stable isomorphism of (g, P), $\mathrm{disc}_P\, g$ is well-defined modulo $(1 - T_\alpha)\, K_1R$. Similarly, up to stable choice of e, $\mathrm{disc}_e f$ is well-defined modulo $(1 + T_\alpha)\, K_1R$.

Now, if $f \in \mathscr{P}_R(H_\alpha P, H_\alpha Q)$ is an isomorphism from g to h of (α, u)-reflexive forms, then, in the absence of any $e \in \mathscr{P}_R(P, Q)$, we have from Lemma 2 3 that

$$(\psi_{\alpha,u}P)^{-1} g = (T_{\alpha,u}f)^{-1} ff^{-1}(\psi_{\alpha,u}Q)^{-1} hf,$$

and since conjugation does not alter a class modulo commutators,

$$\mathrm{disc}_P\, g = \mathrm{disc}_Q\, h + [(T_{\alpha,u}f)^{-1} f] \in K_1R.$$

The element $[(T_{\alpha,u}f)^{-1} f] \in K_1R$ is of particular interest, as explained by the following, whose proof is omitted.

LEMMA 2 4. *Let $x \in Z^-(Z_2 ; K_0R)$, then there are $P, Q \in \mathrm{Obj}\, \mathscr{P}_R$ such that $H_\alpha P \cong H_\alpha Q$ and $x = [P] - [Q]$. If $f \in \mathscr{P}_R(H_\alpha P, H_\alpha Q)$, then, modulo $(1 - T_\alpha)\, K_1R$, the element $d^-_{\alpha,u}x = [(T_{\alpha,u}f)^{-1} f]$ is well defined. The resulting function*

$$d^-_{\alpha,u}\colon Z^-(Z_2 ; K_0R) \to K_1R/(1 - T_\alpha)\, K_1R$$

is a homomorphism which vanishes on $(1 - T_\alpha)\, K_0R$ and satisfies $(1 + T_\alpha)\, d^-_{\alpha,u} = 0$, and so induces a homomorphism

$$d^-_{\alpha,u}\colon H^-(Z_2 ; K_0R) \to H^-(Z_2 ; K_1R). \quad \blacksquare$$

Note. $Z^\pm(Z_2 ;\ ) = \mathrm{Ker}(1 \mp T_\alpha)$; $H^\pm(Z_2 ;\ ) = Z^\pm(Z_2 ;\ )/\mathrm{Im}(1 \pm T_\alpha)$.

COROLLARY 2.5. *If g is a nonsingular (α, u)-reflexive form on $H_\alpha P$, $P \in \mathrm{Obj}\, \mathscr{P}_R$, then the (global) discriminant of g*

$$\mathrm{disc}\, g = [\mathrm{disc}_P\, g] \in \mathrm{Coker}[d^-_{\alpha,u}\colon H^-(Z_2 ; K_0R) \to H^-(Z_2 ; K_1R)]$$

is well defined on the stable isomorphism class of g (as opposed to the stable isomorphism class of the pair (g, P)). $\blacksquare$

Suppose now that $P \in \mathrm{Obj}\, \mathscr{P}_R$ satisfies $P \cong T_\alpha P$; then, for $g \in \mathscr{P}_R(P, T_\alpha P)$, there is defined the *hyperbolic automorphism* of $\psi_{\alpha,u}P$

$$\psi_{\alpha,u}g = \begin{pmatrix} 0 & t_{\alpha,u}g^{-1} \\ g & 0 \end{pmatrix} \in \mathrm{GL}(H_\alpha P; R)$$

As complement to Lemma 2.4, we have the following, which is easily verified

LEMMA 2.6 *Let* $x \in Z^+(Z_2 ; K_0R)$; *then there are* $P, Q \in \operatorname{Obj} \mathscr{P}_R$ *such that* $P \cong T_\alpha P$, $Q \cong T_\alpha Q$, *and* $x = [P] - [Q]$. *If* $g \in \mathscr{P}_R(P, T_\alpha P)$, $h \in \mathscr{P}_R(Q, T_\alpha Q)$, *then, modulo* $(1 + T_\alpha) K_1R$, *the element*

$$d^+_{\alpha,u}x = [\psi_{\alpha,u}g] - [\psi_{\alpha,u}h] \in K_1R/(1 + T_\alpha) K_1R$$

is well defined. The resulting function

$$d^+_{\alpha,u}: Z^+(Z_2 ; K_0R) \to K_1R/(1 + T_\alpha) K_1R$$

is a homomorphism which vanishes on $(1 + T_\alpha) K_0R$ *and satisfies* $(1 - T_\alpha) d^+_{\alpha,u} = 0$, *and so induces a homomorphism*

$$d^+_{\alpha,u}: H^+(Z_2 ; K_0R) \to H^+(Z_2 ; K_1R). \quad ▮$$

Hence, modulo both the stable choice of e and stable hyperbolic automorphisms (or modulo hyperbolic stabilization), the discriminant of an automorphism of an (α, u)-hyperbolic form is well defined as an element of

$$\operatorname{Coker}[d^+_{\alpha,u}: H^+(Z_2 ; K_0R) \to H^+(Z_2 ; K_1R)].$$

If stabilization is via $H_\alpha F$, F free, then topological applications require that the determinant or relative discriminant only be considered modulo $[\psi_{\alpha,u}(F \to T_\alpha F)]$, where $F \to T_\alpha F$ is the dual basis map which sends e_i to e_i^*.

At any rate, we have defined $(\pm)$-*discriminant* homomorphisms $d^\pm_{\alpha,u}$: $H^\pm(Z_2 ; K_0R) \to H^\pm(Z_2 ; K_1R)$, which are of some interest in algebraic K-theory. Presumably, they may be identified with a row of differentials in the E_2-term of the equivariant algebraic K-theory spectral sequence due to Vance [8].

Recall that if (α, u) is an antistructure on R, then (α, v) is an antistructure on R if and only if $v = \epsilon u$, $\epsilon \in Z = Z(R)$, and $\alpha(\epsilon)\epsilon = 1$. For example, there are always the cases $\epsilon = \pm 1$.

PROPOSITION 2.7. *Let* $\epsilon \in Z = Z(R)$ *be a unit satisfying* $\alpha(\epsilon)\epsilon = 1$. *If* $P \in \operatorname{Obj} \mathscr{P}_R$ *supports an* $(\alpha, \epsilon u)$-*reflexive form, then, in the pairing* $K_1Z \otimes K_0R \to K_1R$,

$$d^+_{\alpha,u}[P] = [[-\epsilon][P]] \in H^+(Z_2 , K_1R).$$

In particular, if P *supports an* $(\alpha, -u)$-*reflexive form, then* $d^+_{\alpha,u}[P] = 0$.

Proof. Let $g \in \mathscr{P}_R(P, T_\alpha P)$ satisfy $t_{\alpha,\epsilon u}g = g$; since $t_{\alpha,\epsilon u} = \epsilon t_{\alpha,u}$, we have

$$\psi_{\alpha,u}g = \begin{pmatrix} 0 & t_{\alpha,u}g^{-1} \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon g^{-1} \\ g & 0 \end{pmatrix} \in \mathrm{GL}(H_\alpha P, R)$$

Using $1_P \oplus g: P \oplus P \xrightarrow{\cong} H_\alpha P$, we have

$$(1 \oplus g)^{-1} (\psi_{\alpha,u}g)(1 \oplus g) = \begin{pmatrix} 0 & \epsilon 1_P \\ 1_P & 0 \end{pmatrix} \in \mathrm{GL}(P \oplus P; R),$$

which gives the desired result under the identifications $R = Z \otimes_Z R$, $P = Z \otimes_Z P$. ∎

In the same vein, we have the following (proof omitted).

PROPOSITION 2.8. *Let* $Z = Z(R)$ *and* $P \in \mathrm{Obj}\, \mathscr{P}_Z$ *support an* $(\alpha|_Z, \epsilon)$-*reflexive form; then, in the pairing* $K_1R \otimes K_0Z \to K_1R$,

$$d^+_{\alpha,u}[R \otimes_Z P] = [[-\epsilon u][P]] \in \bar{\bar{H}}(Z_2, K_1R).$$

In particular, $d^+_{\alpha,u}[R] = [-u]$. ∎

As for $d^-_{\alpha,u}$, we have the following, using Lemma 2.3.

PROPOSITION 2.9. *If* $P, Q \in \mathrm{Obj}\, \mathscr{P}_R$ *and the hyperbolic forms* $\psi_{\alpha,u}P$, $\psi_{\alpha,u}Q$ *are isomorphic, then* $d^-_{\alpha,u}([P] - [Q]) = 0$ ∎

3. Reflexive Forms with Vanishing Discriminant

Let g, h be (α, u)-reflexive forms on $H_\alpha P$, $H_\alpha Q$, respectively, where $P, Q \in \mathrm{Obj}\, \mathscr{P}_R$; then, by Lemma 2 2, the relative discriminant is additive:

$$\mathrm{disc}_{P\oplus Q}\, g \boxplus h = \mathrm{disc}_P\, g + \mathrm{disc}_Q\, h.$$

Thus, for example, we may fix g, P and then choose Q such that $P \oplus Q = F$ is free, say with basis $e_1, \ldots, e_n$; of course, $\mathrm{disc}_F\, g + \psi_{\alpha,u}Q = \mathrm{disc}_P\, g$. If this vanishes, and if n is sufficiently large, then, with respect to the canonical basis $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$ of $H_\alpha F$, the matrix of

$$(\psi_{\alpha,u}F)^{-1}(g \boxplus \psi_{\alpha,u}Q) \in \mathrm{GL}(H_\alpha F; R)$$

is an element $\hat{g} \in E(2n; R) \subseteq \mathrm{GL}(2n; R)$ satisfying $(T_{\alpha,u}\hat{g})\hat{g} = 1$ by Proposition 2.1.

Now let $\tilde{g} \in \mathrm{St}(2n; R)$ be an element such that $\phi\tilde{g} = \hat{g}$, where $\phi\colon \mathrm{St}(m; R) \to E(m; R)$ denotes the canonical epimorphism. Since $\phi T_{\alpha,u} = T_{\alpha,u}\phi$ by Corollary 1 7, we have

$$(T_{\alpha,u}\tilde{g})\tilde{g} \in C_{2n}R = \mathrm{Ker}[\phi\colon \mathrm{St}(2n; R) \to E(2n; R)];$$

hence, there is defined the element $\tilde{G}_{\alpha,u}\tilde{g} = [(T_{\alpha,u}\tilde{g})\tilde{g}] \in K_2R$.

PROPOSITION 3 1. *If* $\hat{g} \in E(2n; R)$ *satisfies* $(T_{\alpha,u}\hat{g})\hat{g} = 1$, *and if* $\tilde{g} \in \mathrm{St}(2n; R)$ *satisfies* $\phi\tilde{g} = \hat{g}$, *then*

$$\tilde{G}_{\alpha,u}\tilde{g} \in Z^+(Z_2; K_2R).$$

If $\tilde{g}' \in \mathrm{St}(2n; R)$ *also satisfies* $\phi\tilde{g}' = \hat{g}$, *then*

$$\tilde{G}_{\alpha,u}\tilde{g}' - \tilde{G}_{\alpha,u}\tilde{g} = (1 + T_\alpha)[\tilde{g}'\tilde{g}^{-1}] \in (1 + T_\alpha) K_2R;$$

hence, $\hat{G}_{\alpha,u}\hat{g} = [\tilde{G}_{\alpha,u}\tilde{g}] \in H^+(Z_2 ; K_2R)$ *is well defined.*

Proof. That $\tilde{G}_{\alpha,u}\tilde{g} = T_\alpha\tilde{G}_{\alpha,u}\tilde{g}$ follows from the fact that $T_{\alpha,u}^2 = 1$ on $\mathrm{St}(2n; R)$ by Corollary 1.7. The second assertion follows from the computation

$$\begin{aligned}[(T_{\alpha,u}\tilde{g}')\tilde{g}'] - [(T_{\alpha,u}g)g] &= [(T_{\alpha,u}\tilde{g}')\tilde{g}'\tilde{g}^{-1}(T_{\alpha,u}\tilde{g})^{-1}] \\ &= [\tilde{g}'\tilde{g}^{-1}] + [T_{\alpha,u}(\tilde{g}'\tilde{g}^{-1})]. \quad \blacksquare\end{aligned}$$

If $P_1, P_2 \in \mathrm{Obj}\,\mathscr{P}_R$, $e \in \mathscr{P}_R(P_1, P_2)$, and if $f \in \mathscr{P}_R(H_\alpha P_1, H_\alpha P_2)$ is an isomorphism from g_1 to g_2 of (α, u)-reflexive forms, then we may choose $Q \in \mathrm{Obj}\,\mathscr{P}_R$ such that $P_i \oplus Q = F_i$ is free, say with basis $e_{i1}, \ldots, e_{in}$, $i = 1, 2$. Then $f \boxplus 1_{H_\alpha Q} \in \mathscr{P}_R(H_\alpha F_1, H_\alpha F_2)$ is an isomorphism from $g_1 \boxplus \psi_{\alpha,u}Q$ to $g_2 \boxplus \psi_{\alpha,u}Q$, and $\mathrm{disc}_{e\oplus 1_Q} f \boxplus 1_{H_\alpha Q} = \mathrm{disc}_e f$ (a special case of additivity of the relative discriminant of isomorphisms of (α, u)-reflexive forms) With respect to the appropriate canonical bases determined by the e_{1j} and the e_{2j}, let $\hat{e}$, $\hat{f}$, $\hat{g}_i$ be the matrices of

$$e \oplus 1_Q, H_\alpha(e \oplus 1_Q)^{-1} (f \boxplus 1_{H_\alpha Q}), (\psi_{\alpha,u}F_i)^{-1} (g_i \boxplus \psi_{\alpha,u}Q),$$

respectively, $i = 1, 2$; also, set $\hat{h}_2 = (H_\alpha\hat{e})\,\hat{g}_2(H_\alpha\hat{e})^{-1}$, where $H_\alpha\hat{e} = \hat{e} \oplus \hat{e}^{-\alpha}$ Since $T_{\alpha,u}H_\alpha\hat{e} = H_\alpha\hat{e}$, we have $(T_{\alpha,u}\hat{f})\hat{g}_1 = \hat{h}_2\hat{f}$ by Lemma 2 3 and $(T_{\alpha,u}\hat{g}_i)\hat{g}_i = 1$, $(T_{\alpha,u}\hat{h}_2)\,\hat{h}_2 = 1$ by Lemma 2 2. If $\mathrm{disc}_e f$ and $\mathrm{disc}_{P_i} g_i$ all vanish, and if n is sufficiently large, then $\hat{f}$, $\hat{g}_i$, $\hat{h}_2 \in E(2n; R)$, $i = 1, 2$.

Now let $\tilde{f}, \tilde{g}_1, \tilde{h}_2 \in \mathrm{St}(2n; R)$ be elements such that $\phi\tilde{f} = \hat{f}$, $\phi\tilde{g}_1 = \hat{g}_1$, $\phi\tilde{h}_2 = \hat{h}_2$; then $(T_{\alpha,u}\tilde{f})\tilde{g}_1\tilde{f}^{-1}\tilde{h}_2^{-1} \in C_{2n}R$, and hence there is defined the element

$$\tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}_1, \tilde{h}_2) = [(T_{\alpha,u}\tilde{f})\,\tilde{g}_1\tilde{f}^{-1}\tilde{h}_2^{-1}] \in K_2R.$$

PROPOSITION 3 2. *If* $\hat{f}, \hat{g}, \hat{h} \in E(2n, R)$ *satisfy* $(T_{\alpha,u}\hat{g})\hat{g} = 1$, $(T_{\alpha,u}\hat{h})\,\hat{h} = 1$, *and* $(T_{\alpha,u}\hat{f})\hat{g} = \hat{h}\hat{f}$, *and if* $\tilde{f}, \tilde{g}, \tilde{h} \in \mathrm{St}(2n; R)$ *satisfy* $\phi\tilde{f} = \hat{f}$, $\phi\tilde{g} = \hat{g}$, $\phi\tilde{h} = \hat{h}$, *then*

$$(1 + T_\alpha)\,\tilde{G}_{\alpha,u}(\tilde{f}, \tilde{g}, \tilde{h}) = \tilde{G}_{\alpha,u}\tilde{g} - \tilde{G}_{\alpha,u}\tilde{h}.$$

If $\tilde{f}' \in \mathrm{St}(2n; R)$ *also satisfies* $\phi\tilde{f}' = \hat{f}$, *then*

$$\tilde{G}_{\alpha,u}(\tilde{f}'; \tilde{g}, \tilde{h}) - \tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}, \tilde{h}) = (1 - T_\alpha)[\tilde{f}\tilde{f}'^{-1}] \in (1 - T_\alpha)\,K_2R;$$

hence, $\hat{G}_{\alpha,u}(\hat{f}; \hat{g}, \hat{h}) = [\tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}, \tilde{h})] \in K_2R/(1 - T_\alpha)\,K_2R$ *is well defined.*

Proof. Permuting some of the terms cyclically in the following computation, we have

$$\begin{aligned}(1 + T_\alpha)\,\tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}, \tilde{h}) &= [\tilde{h}^{-1}(T_{\alpha,u}\tilde{f})\,\tilde{g}\tilde{f}^{-1}] + T_\alpha[(T_{\alpha,u}\tilde{f})\,\tilde{g}\tilde{f}^{-1}\tilde{h}^{-1}] \\ &= [\tilde{h}^{-1}(T_{\alpha,u}\tilde{f})\,\tilde{g}\tilde{f}^{-1}(T_{\alpha,u}^2\tilde{f})(T_{\alpha,u}\tilde{g})(T_{\alpha,u}\tilde{f})^{-1}\,(T_{\alpha,u}\tilde{h})^{-1}] \\ &= [\tilde{g}(T_{\alpha,u}\tilde{g})] + [\tilde{h}^{-1}(T_{\alpha,u}\tilde{h})^{-1}] = \tilde{G}_{\alpha,u}\tilde{g} - \tilde{G}_{\alpha,u}\tilde{h}.\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}\tilde{G}_{\alpha,u}(\tilde{f}'; \tilde{g}, \tilde{h}) - \tilde{G}_{\alpha,u}(\tilde{f}, \tilde{g}, \tilde{h}) &= [(T_{\alpha,u}\tilde{f}')\tilde{g}\tilde{f}'^{-1}\tilde{h}^{-1}\tilde{h}\tilde{f}\tilde{g}^{-1}(T_{\alpha,u}\tilde{f})^{-1}] \\ &= [\tilde{f}'^{-1}\tilde{f}] + [T_{\alpha,u}(\tilde{f}'\tilde{f}^{-1})] \\ &= (1 - T_\alpha)[\tilde{f}\tilde{f}'^{-1}]. \quad \blacksquare\end{aligned}$$

Thus, if $\tilde{G}_{\alpha,u}\tilde{g} = \tilde{G}_{\alpha,u}\tilde{h}$, then $\tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}, \tilde{h}) \in Z^-(Z_2 ; K_2R)$ and $\hat{G}_{\alpha,u}(\hat{f}; \tilde{g}, \tilde{h}) \in H^-(Z_2 : K_2R)$. This is the case, for example, when $\tilde{g} = \tilde{h}$ (and hence $\hat{g} = \hat{h}$); moreover, in this situation, we have the following, whose proof is trivial.

Proposition 3.3. *Let $\hat{f}, \hat{g} \in E(2n; R)$ satisfy $(T_{\alpha,u}\hat{g})\hat{g} = 1$ and $(T_{\alpha,u}\hat{f})\hat{g} = \hat{g}\hat{f}$; if $\tilde{f}, \tilde{g} \in \mathrm{St}(2n; R)$ are elements such that $\phi\tilde{f} = \hat{f}$, $\phi\tilde{g} = \hat{g}$, then the elements*

$$\tilde{G}_{\alpha,u}(\tilde{f}; \hat{g}) = \tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}, \tilde{g}) \in Z^-(Z_2 ; K_2R),$$
$$\hat{G}_{\alpha,u}(\hat{f}; \hat{g}) = \hat{G}_{\alpha,u}(\hat{f}; \tilde{g}, \tilde{g}) \in H^-(Z_2 ; K_2R)$$

are well defined, independent of the choice of $\tilde{g} \in \mathrm{St}(2n; R)$ such that $\phi\tilde{g} = \hat{g}$. ∎

Suppose that, in the situation preceding (Proposition 3.2), we are given $\tilde{f}, \tilde{g}_i \in \mathrm{St}(2n; R)$ such that $\phi\tilde{f} = \hat{f}$, $\phi\tilde{g}_i = \hat{g}_i$, $i = 1, 2$. Since inner automorphism of $\mathrm{GL}(R)$ by an element $\hat{x} \in \mathrm{GL}(R)$, say $\hat{y} \mapsto \hat{y}^{\hat{x}} = \hat{x}^{-1}\hat{y}\hat{x}$, induces an automorphism of $E(R)$, then covering this automorphism of $E(R)$ is a unique automorphism, $\tilde{y} \mapsto \tilde{y}^{\hat{x}}$, of $\mathrm{St}(R)$. In our case, we have $(T_{\alpha,u}\hat{f})\hat{g}_1 = \hat{g}_2^{-H_\alpha \hat{e}^{-1}}\hat{f}$, and so we have the elements

$$\tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}_1, \tilde{g}_2 ; \hat{e}) = \tilde{G}_{\alpha,u}(\tilde{f}; \tilde{g}_1, \tilde{g}_2^{-H_\alpha \hat{e}^{-1}}) \in K_2R,$$
$$\hat{G}_{\alpha,u}(\hat{f}; \tilde{g}_1, \tilde{g}_2 ; \hat{e}) = \hat{G}_{\alpha,u}(\hat{f}; \tilde{g}_1, \tilde{g}_2^{-H_\alpha \hat{e}^{-1}}) \in K_2R/(1 - T_\alpha)\, K_2R.$$

Thus, for all practical purposes, we may take $\hat{e} = 1$

Suppose now that $\hat{f}, \hat{g}, \hat{h} \in \mathrm{GL}(2n, R)$ satisfy $(T_{\alpha,u}\hat{g})\hat{g} = 1$, $(T_{\alpha,u}\hat{h})\, \hat{h} = 1$, and $(T_{\alpha,u}\hat{f})\hat{g} = \hat{h}\hat{f}$. If, as elements of $\mathrm{GL}(R)$, we have $\hat{g}, \hat{h} \in E(R)$, then $\hat{h}^{\hat{f}} \in E(R)$ and hence also $\hat{f}^{-1}(T_{\alpha,u}\hat{f}) = \hat{h}^{\hat{f}}\hat{g}^{-1} \in E(R)$. Let $\tilde{g}, \tilde{h}, \tilde{y} \in \mathrm{St}(R)$ satisfy $\phi\tilde{g} = \hat{g}$, $\phi\tilde{h} = \hat{h}$, $\phi\tilde{y} = \hat{f}^{-1}(T_{\alpha,u}\hat{f})$, and $\tilde{y}\tilde{g} = \tilde{h}^{\hat{f}}$. Then, since K_1R acts trivially on K_2R, we have

$$\begin{aligned}\tilde{G}_{\alpha,u}\tilde{h} &= [(T_{\alpha,u}\tilde{h})\, \tilde{h}] = [(T_{\alpha,u}\tilde{h})^{\hat{f}}\, \tilde{h}^{\hat{f}}] = [(T_{\alpha,u}\tilde{h})^{\hat{f}}\, \tilde{y}\tilde{g}] = [\tilde{y}(T_{\alpha,u}\tilde{h})^{\hat{f}(\phi\tilde{y})}\, \tilde{g}] \\ &= [\tilde{y}(T_{\alpha,u}\tilde{h})^{T_{\alpha,u}\hat{f}}\, \tilde{g}] = [\tilde{y}(T_{\alpha,u}(\tilde{h}^{\hat{f}}))\, \tilde{g}] = [\tilde{y}(T_{\alpha,u}(\tilde{y}\tilde{g}))\, \tilde{g}] \\ &= [\tilde{y}(T_{\alpha,u}\tilde{y})] + \tilde{G}_{\alpha,u}\tilde{g}.\end{aligned}$$

As for the element $[\tilde{y}(T_{\alpha,u}\tilde{y})] \in K_2R$, we have the following analog of Lemma 2.4.

Proposition 3.4. *Let $x \in Z^+(Z_2 ; K_1R)$; then, if $\hat{f} \in \mathrm{GL}(R)$ represents x, $(T_{\alpha,u}\hat{f})^{-1}\, \hat{f} \in E(R)$. If $\tilde{y} \in \mathrm{St}(R)$ satisfies $\phi\tilde{y} = (T_{\alpha,u}\hat{f})^{-1}\hat{f}$, then, modulo*

$(1 + T_\alpha) K_2R$, *the element* $d^+_{\alpha,u}x = [(T_{\alpha,u}\tilde{y})\tilde{y}]$ *is well defined. The resulting function*

$$d^+_{\alpha,u}: Z^+(Z_2 ; K_1R) \to K_2R/(1 + T_\alpha) K_2R$$

is a homomorphism which vanishes on $(1 + T_\alpha) K_1R$ *and satisfies* $(1 - T_\alpha) d^+_{\alpha,u} = 0$, *and so induces a homomorphism*

$$d^+_{\alpha,u}: H^+(Z_2 ; K_1R) \to H^+(Z_2 ; K_2R)$$

Proof. Clearly, modulo $(1 + T_\alpha) K_2R$, $[(T_{\alpha,u}\tilde{y})\tilde{y}]$ depends on $\hat{f}$ and not on the choice of $\tilde{y} \in \mathrm{St}(R)$ such that $\phi\tilde{y} = (T_{\alpha,u}\hat{f})^{-1}\hat{f}$. Suppose $\hat{f}' = \hat{e}\hat{f}$, where $\hat{e} \in E(R)$, and $\tilde{y}' \in \mathrm{St}(R)$ satisfies $\phi\tilde{y}' = (T_{\alpha,u}\hat{f}')^{-1}\hat{f}'$, if $\tilde{e} \in \mathrm{St}(R)$ satisfies $\phi\tilde{e} = \hat{e}$, then $\tilde{y} = \tilde{y}'(\tilde{e}^{-1}(T_{\alpha,u}\tilde{e}))^{\hat{f}}$ satisfies

$$\phi\tilde{y} = (T_{\alpha,u}\hat{f}')^{-1}\hat{f}'\hat{f}^{-1}\hat{e}^{-1}(T_{\alpha,u}\hat{e})\hat{f} = (T_{\alpha,u}\hat{f})^{-1}\hat{f}$$

Hence, we have

$$\begin{aligned}[(T_{\alpha,u}\tilde{y})\tilde{y}] &= [(T_{\alpha,u}(\tilde{y}'(\tilde{e}^{-1}(T_{\alpha,u}\tilde{e}))^{\hat{f}})\tilde{y}'(\tilde{e}^{-1}(T_{\alpha,u}\tilde{e}))^{\hat{f}}] \\ &= [(T_{\alpha,u}\tilde{y}')((T_{\alpha,u}\tilde{e})^{-1}\tilde{e})^{T_{\alpha,u}\hat{f}}\tilde{y}'(\tilde{e}^{-1}(T_{\alpha,u}\tilde{e}))^{\hat{f}}] \\ &= [(T_{\alpha,u}\tilde{y}')\tilde{y}'((T_{\alpha,u}\tilde{e})^{-1}\tilde{e})^{\hat{f}}(\tilde{e}^{-1}(T_{\alpha,u}\tilde{e}))^{\hat{f}}] \\ &= [(T_{\alpha,u}\tilde{y}')\tilde{y}'],\end{aligned}$$

since $(T_{\alpha,u}\hat{f})(\phi\tilde{y}') = \hat{f}$, and thus $d^+_{\alpha,u}x$ is well defined To see that $d^+_{\alpha,u}$ vanishes on $(1 + T_\alpha) K_1R$, let $\hat{e} \in \mathrm{GL}(n, R)$ represent $x \in K_1R$; then $\hat{f} = H_\alpha\hat{e} \in \mathrm{GL}(2n; R)$ represent $(1 + T_\alpha)x$, and $(T_{\alpha,u}\hat{f})^{-1}\hat{f} = 1$, so that $d^+_{\alpha,u}(1 + T_\alpha)x = 0$. The rest follows easily ∎

The exposition above has been arranged so that the following is now a routine verification.

COROLLARY 3.5. *If g is an (α, u)-reflexive form on $H_\alpha P$, $P \in \mathrm{Obj}\,\mathscr{P}_R$, and if* $\mathrm{disc}\, g = 0$, *then, for $\hat{g} \in E(2n; R)$ the matrix of $(\psi_{\alpha,u}F)^{-1}(g \boxplus \psi_{\alpha,u}Q)$ as before, the element*

$$G_{\alpha,u}g = [\hat{G}_{\alpha,u}\hat{g}] \in \mathrm{Coker}[d^+_{\alpha,u}: H^+(Z_2 ; K_1R) \to H^+(Z_2 ; K_2R)]$$

is well defined on the stable isomorphism class of g (as opposed to the class of (g, P) up to stable isomorphisms with vanishing discriminant). ∎

Continuing the analogy with Section 2, suppose that $\hat{e} \in \mathrm{GL}(n, R)$ satisfies $H_\alpha\hat{e} \in E(2n, R)$; since $T_{\alpha,u}H_\alpha\hat{e} = H_\alpha\hat{e}$, if $\tilde{y} \in \mathrm{St}(2n; R)$ satisfies $\phi\tilde{y} = H_\alpha\hat{e}$, then $(T_{\alpha,u}\tilde{y})\tilde{y}^{-1} \in C_{2n}R$.

PROPOSITION 3 6. *Let $x \in Z^-(Z_2 , K_1R)$, then x is represented by $\hat{e} \in \mathrm{GL}(n; R)$ such that $H_\alpha\hat{e} \in E(2n; R)$ for some n. If $\tilde{y} \in \mathrm{St}(2n; R)$ satisfies $\phi\tilde{y} = H_\alpha\hat{e}$, then,* *modulo* $(1 - T_\alpha) K_2R$, *the element* $d^-_{\alpha,u}x = [(T_{\alpha,u}\tilde{y})\tilde{y}^{-1}]$ *is well defined. The resulting function*

$$d^-_{\alpha,u}\colon Z^-(Z_2\,; K_1R) \to K_2R/(1 - T_\alpha) K_2R$$

is a homomorphism which vanishes on $(1 - T_\alpha) K_1R$ *and satisfies* $(1 + T_\alpha) d^-_{\alpha,u} = 0$, *and so induces a homomorphism*

$$d^-_{\alpha,u}\colon H^-(Z_2\,; K_1R) \to H^-(Z_2\,, K_2R).$$

Proof. Exercise. ∎

There are various ways of relating Proposition 3.6 with $\hat{G}_{\alpha,u}(\hat{f}; \hat{g})$ and the like Perhaps the most obvious comes from the fact that, for $\hat{e} \in \mathrm{GL}(n; R)$, $H_\alpha\hat{e}$ is the matrix of an automorphism of $\psi_{\alpha,u}F$ with respect to a basis of the form $e_1, \ldots, e_n$, $e_1^*, \ldots, e_n^*$. Thus, if $[\hat{e}] \in Z^-(Z_2\,; K_1R)$, then $d^-_{\alpha,u}[\hat{e}] = \hat{G}_{\alpha,u}(H_\alpha\hat{e}; 1) \in H^-(Z_2\,; K_2R)$. The following result for $d^-_{\alpha,u}$ is quite basic

THEOREM 3.7 *Let* $\epsilon \in Z = Z(R)$ *be a unit satisfying* $\alpha(\epsilon)\,\epsilon = 1$ *If* $\hat{g} \in \mathrm{GL}(2n; R)$ *satisfies* $(T_{\alpha,\epsilon u}\hat{g})\hat{g} = 1$, *then, in the pairing* $K_1Z \otimes K_1R \to K_2R$,

$$d^-_{\alpha,u}[\hat{g}] = [[-\epsilon][\hat{g}]] \in H^-(Z_2\,; K_2R)$$

In particular, if g *is an* $(\alpha, -u)$*-reflexive form on* $H_\alpha P$, $P \in \mathrm{Obj}\,\mathscr{P}_R$, *then* $d^-_{\alpha,u}[\mathrm{disc}_P\, g] = 0$ ($\mathrm{disc}_P\, g$ *is taken in the* $(\alpha, -u)$ *antistructure*).

Proof. From the fact that $(T_{\alpha,\epsilon u}\hat{g})\hat{g} = 1$, we have $T_\alpha\hat{g} = \hat{v}\hat{g}^{-1}\hat{v}^{-1}$, where

$$\hat{v} = \begin{pmatrix} 0 & 1 \\ \epsilon u & 0 \end{pmatrix} \in \mathrm{GL}(2n; R).$$

Now, following the notational scheme of Proposition 1.6, let block elementary matrices be denoted

$$e^{\hat{a}}_{+-} = \begin{pmatrix} 1 & \hat{a} \\ 0 & 1 \end{pmatrix}, \qquad e^{\hat{a}}_{-+} = \begin{pmatrix} 1 & 0 \\ \hat{a} & 1 \end{pmatrix} \in E(2m; R) \qquad \text{for} \quad \hat{a} \in M_mR,$$

and let $x^{\hat{a}}_{\pm\mp} \in \mathrm{St}(2m; R)$ denote the block Steinberg generator such that $\phi x^{\hat{a}}_{\pm\mp} = e^{\hat{a}}_{\pm\mp}$. In other words, if $e^{\hat{a}}_{\pm\mp}$ is expressed as the canonical product of m^2 ordinary elementary matrices in $E(2m; R)$, then $x^{\hat{a}}_{\pm\mp}$ is expressed as the corresponding product of m^2 ordinary Steinberg generators in $\mathrm{St}(2m, R)$. For $\hat{h} \in \mathrm{GL}(m; R)$, let

$$w_{\pm\mp}(\hat{h}) = x^{\hat{h}}_{\pm\mp} x^{-\hat{h}^{-1}}_{\mp\pm} x^{\hat{h}}_{\pm\mp}\,,$$

$$h_{\pm\mp}(\hat{h}) = w_{\pm\mp}(\hat{h})\, w_{\pm\mp}(-1)$$

in St($2m$; R) be the block analogs of the elements $w_{ij}(u)$, $h_{ij}(u)$ in Milnor [5, Sect. 9]; then

$$\phi w_{\pm\mp}(\hat{h}) = \begin{pmatrix} 0 & \pm\hat{h}^{\pm 1} \\ \mp\hat{h}^{\mp 1} & 0 \end{pmatrix}, \qquad \phi h_{\pm\mp}(\hat{h}) = \begin{pmatrix} \hat{h}^{\pm 1} & 0 \\ 0 & \hat{h}^{\mp 1} \end{pmatrix}.$$

Hence, with $m = 2n$, we have

$$\begin{aligned} H_\alpha \hat{g} &= \begin{pmatrix} 1 & 0 \\ 0 & \hat{v} \end{pmatrix} \begin{pmatrix} \hat{g} & 0 \\ 0 & \hat{g}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{v}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{v} \end{pmatrix} \phi h_{+-}(\hat{g}) \begin{pmatrix} 1 & 0 \\ 0 & \hat{v}^{-1} \end{pmatrix} \\ &= \phi w_{+-}(\hat{g}\hat{v}^{-1})\, \phi w_{+-}(-\hat{v}^{-1}) = \phi h_{+-}(\hat{g}\hat{v}^{-1})\, \phi h_{+-}(\hat{v}^{-1})^{-1}, \end{aligned}$$

much as in [5, Corollary 9.4]. Now, using Corollary 1.7, we have

$$\left.\begin{aligned} T_{\alpha,u} w_{+-}(\hat{h}) &= w_{+-}(-u^{-1}\hat{h}^{\alpha}) \\ T_{\alpha,u} w_{+-}(\hat{h}) &= w_{+-}(-\hat{h}^{\alpha} u) \end{aligned}\right\} \in \mathrm{St}(2m; R)$$

for $\hat{h} \in \mathrm{GL}(m, R)$. Hence, with $\tilde{y} = w_{+-}(\hat{g}\hat{v}^{-1})\, w_{+-}(-\hat{v}^{-1})$, we have

$$\begin{aligned} T_{\alpha,u}\tilde{y} &= w_{+-}(-u^{-1}\hat{v}^{-\alpha}\hat{g}^{\alpha})\, w_{+-}(u^{-1}\hat{v}^{-\alpha}) = w_{+-}(-\epsilon\hat{v}^{-1}\hat{g}^{\alpha})\, w_{+-}(\epsilon\hat{v}^{-1}) \\ &= w_{+-}(-\epsilon\hat{g}\hat{v}^{-1})\, w_{+-}(\epsilon\hat{v}^{-1}), \end{aligned}$$

since $\hat{v}^{-\alpha} = \hat{v}$, $u^{-1}\hat{v} = \epsilon\hat{v}^{-1}$, and $\hat{v}^{-1}\hat{g}^{\alpha} = \hat{g}\hat{v}^{-1}$ ($T_\alpha \hat{g} = \hat{g}^{-\alpha}$) Thus we have

$$\begin{aligned} (T_{\alpha,u}\tilde{y})\tilde{y}^{-1} &= w_{+-}(-\epsilon\hat{g}\hat{v}^{-1})\, w_{+-}(\epsilon\hat{v}^{-1})\, w_{+-}(\hat{v}^{-1})\, w_{+-}(-\hat{g}\hat{v}^{-1}) \\ &= h_{+-}(-\epsilon\hat{g}\hat{v}^{-1})\, h_{+-}(-\epsilon\hat{v}^{-1})^{-1}\, h_{+-}(\hat{v}^{-1})\, h_{+-}(\hat{g}\hat{v}^{-1})^{-1}, \end{aligned}$$

since $w_{\pm\mp}(\hat{h})\, w_{\pm\mp}(-\hat{h}) = 1$ and $w_{\pm\mp}(\hat{h}_1)\, w_{\pm\mp}(\hat{h}_2) = h_{\pm\mp}(\hat{h}_1)\, h_{\pm\mp}(-\hat{h}_2)^{-1}$, as in [5, Sect. 9] Letting $n \to \infty$ and continuing our calculation, we have, upon inserting $h_{+-}(-\epsilon)\, h_{+-}(-\epsilon)^{-1}$ between $h_{+-}(\hat{v}^{-1})$ and $h_{+-}(\hat{g}\hat{v}^{-1})^{-1}$,

$$\begin{aligned} [(T_{\alpha,u}\tilde{y})\tilde{y}^{-1}] &= -[-\epsilon][\hat{v}^{-1}] + [h_{+-}(-\epsilon\hat{g}\hat{v}^{-1})\, h_{+-}(-\epsilon)^{-1})\, h_{+-}(\hat{g}\hat{v}^{-1})^{-1}] \\ &= [-\epsilon][\hat{v}] + [-\epsilon][\hat{g}\hat{v}^{-1}] = [-\epsilon][\hat{g}], \end{aligned}$$

by mild extension of [5, Lemmas 8.2, 9.4]. The theorem follows. ∎

For the appropriate analog of Proposition 2.8, we offer the following without proof.

PROPOSITION 3.8. *Let $\epsilon \in Z = Z(R)$ be a unit satisfying $\alpha(\epsilon)\,\epsilon = 1$. If $\hat{g} \in \mathrm{GL}(n; Z)$ satisfies $(T_{\alpha,\epsilon}\hat{g})\hat{g} = 1$, then, in the pairing $K_1R \otimes K_1Z \to K_2R$,*

$$d^-_{\alpha,u}[R \otimes_Z \hat{g}] = [[-\epsilon u][\hat{g}]] \in H^-(Z_2\,; K_2R).$$

In particular, $d^-_{\alpha,u}[-1] = [[-u][-1]]$. ∎

A vanishing result for $d^+_{\alpha,u}$ is the following.

THEOREM 3.9. *If $\hat{f} \in \mathrm{GL}(2n; R)$ is the matrix with respect to $e_1, \ldots, e_n$, $e_1^*, \ldots, e_n^*$ of an automorphism of an (α, u)-reflexive form g with vanishing relative discriminant on $H_\alpha F$, then $d^+_{\alpha,u}[\hat{f}] = 0$.*

Proof. Let $\hat{g} \in \mathrm{GL}(2n; R)$ be the matrix of $(\psi_{\alpha,u}F)^{-1}g$; then $\hat{g} \in E(R)$, and $(T_{\alpha,u}\hat{f})\hat{g} = \hat{g}\hat{f}$, by Lemma 2 3. Thus $(T_{\alpha,u}\hat{f})^{-1}\hat{f} = \hat{g}\hat{g}^{-\hat{f}}$. Let $\tilde{g} \in \mathrm{St}(R)$ satisfy $\phi\tilde{g} = \hat{g}$ and set $\tilde{y} = \tilde{g}\tilde{g}^{-\hat{f}}$. Then, noting that $\hat{g} = (T_{\alpha,u}\hat{g})^{-1}$ near the end, we have

$$
\begin{aligned}
d^+_{\alpha,u}[\hat{f}] = [(T_{\alpha,u}\tilde{y})\,\tilde{y}] &= [(T_{\alpha,u}\tilde{g})(T_{\alpha,u}\tilde{g})^{-T_{\alpha,u}\hat{f}}\tilde{g}\tilde{g}^{-\hat{f}}] \\
&= [(T_{\alpha,u}\tilde{g})\,\tilde{g}(T_{\alpha,u}\tilde{g})^{-(T_{\alpha,u}\hat{f})\hat{g}}\,\tilde{g}^{-\hat{f}}] \\
&= \hat{G}_{\alpha,u}\hat{g} + [(T_{\alpha,u}\tilde{g})^{-\hat{g}\hat{f}}\,\tilde{g}^{-\hat{f}}] \\
&= \hat{G}_{\alpha,u}\hat{g} + [(T_{\alpha,u}\tilde{g}^{\hat{g}^{-1}})^{-\hat{f}}\,\tilde{g}^{-\hat{f}}] \\
&= \hat{G}_{\alpha,u}\hat{g} + [(T_{\alpha,u}\tilde{g})^{-1}\,\tilde{g}^{-1}] = 0. \quad \blacksquare
\end{aligned}
$$

Now we can show how the $d^\pm_{\alpha,u}$ of this and the preceding section are differentials.

THEOREM 3.10. $d^\pm_{\alpha,u}d^\pm_{\alpha,\pm u} = 0$.

Proof. If $g \in \mathscr{P}_R(P, T_\alpha P)$, then $T_{\alpha,u}\psi_{\alpha,u}g = \psi_{\alpha,u}g$, so that $(d^+_{\alpha,u})^2 = 0$ by Theorem 3.9. To show that $d^-_{\alpha,u}d^-_{\alpha,-u} = 0$, let $x \in Z^-(Z_2 : K_0R)$; clearly, in the representation $x = [P] - [Q]$ of Lemma 2.4, we may take $P = F$ to be free, say with basis $e_1, \ldots, e_n$. Now let $f \in \mathscr{P}_R(H_\alpha F, H_\alpha Q)$ and let $\hat{g} \in \mathrm{GL}(2n; R)$ be the matrix of $(T_{\alpha,-u}f)^{-1}f \in \mathrm{GL}(H_\alpha F; R)$ with respect to $e_1, \ldots, e_n$, $e_1^*, \ldots, e_n^*$. Then $(T_{\alpha,-u}\hat{g})\hat{g} = 1$, so $d^-_{\alpha,u}[\hat{g}] = 0$ by Theorem 3.7 Since $[\hat{g}] = d^-_{\alpha,-u}x$, it follows that $d^-_{\alpha,u}d^-_{\alpha,-u} = 0$. $\blacksquare$

Let $d^{p,q}_{\alpha,u}\colon H^p(Z_2\,; K_{-q}R) \to H^{p+2}(Z_2\,; K_{-q+1}R)$ for $p \geqslant 0$, $q = 0, -1$, be defined by

$$
\begin{aligned}
d^{p,0}_{\alpha\,u} &= d^+_{\alpha,u} && \text{for } p \equiv 0 \\
&= d^-_{\alpha,u} && \phantom{\text{for } p} \equiv 1 \qquad (\text{mod } 4), \\
&= d^+_{\alpha,-u} && \phantom{\text{for } p} \equiv 2 \\
&= d^-_{\alpha,-u} && \phantom{\text{for } p} \equiv 3
\end{aligned}
$$

$$
\begin{aligned}
d^{p,-1}_{\alpha\,u} &= d^+_{\alpha,-u} && \text{for } p \equiv 0 \\
&= d^-_{\alpha,u} && \phantom{\text{for } p} \equiv 1 \qquad (\text{mod } 4). \\
&= d^+_{\alpha,u} && \phantom{\text{for } p} \equiv 2 \\
&= d^-_{\alpha,-u} && \phantom{\text{for } p} \equiv 3
\end{aligned}
$$

Then these should be differentials in the E_2-term of the spectral sequence of Vance [8] for equivariant algebraic K-theory:

$$E_2^{p,q} = H^p(Z_2\,;K_{-q}R) \underset{p}{\Rightarrow} KR^*_{Z_2}(S^\infty, A),$$

$(E_r^{p,q} = E_r^{p,q}(R, \alpha, u), KR^*_{Z_2} = K(R, \alpha, u)^*_{Z_2})$.

4. The Case of a Commutative Ring

In this section, R will be a commutative ring; hence, α is an automorphism of period 1 or 2, and u is a unit such that $\alpha(u)\,u = 1$.

Proposition 4.1. *Let $v \in R$ be a unit such that $T_\alpha v = v^{\pm 1}$ (i.e., $\alpha(v)\,v^{\pm 1} = 1$); then $[v] \in Z^\pm(Z_2\,; K_1R)$ and*

$$d^\pm_{\alpha,u}[v] = [[\pm u][v]] \in H^\pm(Z_2\,; K_2R).$$

Proof. If $T_\alpha v = v$, then $(T_{\alpha,u}(v \oplus 1))^{-1}(v \oplus 1) = v \oplus v^{-1} = \phi h_{1,-1}(v)$; on the other hand, if $T_\alpha v = v^{-1}$, then $H_\alpha v = v \oplus v^{-1} = \phi h_{1,-1}(v)$. In either case, let $\tilde{y} = h_{1,-1}(v) \in \mathrm{St}(2, R)$; then, using Corollary 1.7, we have

$$\begin{aligned}[(T_{\alpha,u}\tilde{y})^{\pm 1}\tilde{y}] &= [(T_{\alpha,u}h_{1,-1}(v))^{\pm 1}\,h_{1,-1}(v)]\\ &= [(h_{1,-1}(-u^{-1}v^{\mp 1})\,h_{1,-1}(-u^{-1})^{-1})^{\pm 1}\,h_{1,-1}(v)]\\ &= -[\pm u^{-1}][v] = [\pm u][v]\end{aligned}$$

in K_2R. Hence, $d^\pm_{\alpha,u}[v] = [[\pm u][v]]$. ∎

Corollary 4.2. *If $H^\pm(Z_2\,; SK_1R) = 0$, then $d^\pm_{\alpha,\pm 1}\colon H^\pm(Z_2\,; K_1R) \to H^\pm(Z_2\,; K_2R)$ vanishes* ∎

This is the case, for example, if R is a Dedekind domain or the group ring of a finite abelian group with either cyclic or elementary abelian Sylow 2-subgroup. We shall be particularly interested in the case of a field.

The following calculation leads directly to the main result of this section.

Lemma 4.3. *Let $a, b \in R$ be units, and let*

$$\hat{g} = \begin{pmatrix} \bigcirc & & a^{-1}b^{-1} & 0 \\ & & 0 & 1 \\ a & 0 & & \\ 0 & b & & \bigcirc \end{pmatrix} \in \mathrm{GL}(4; R).$$

Then $\hat{g} \in E(4; R)$, $(T_{id,1}\hat{g})\hat{g} = 1$, and $\hat{G}_{id,1}\hat{g} = [[-a][-b]] \in H^+(Z_2\,; K_2R)$.

Proof. We have the factorization

$$\hat{g} = \begin{pmatrix} \bigcirc & & -a^{-1} & 0 \\ & & 0 & -b^{-1} \\ a & 0 & & \\ 0 & b & \bigcirc & \end{pmatrix} \begin{pmatrix} 1 & 0 & & \bigcirc \\ 0 & 1 & & \\ & & -b^{-1} & 0 \\ \bigcirc & & 0 & -b \end{pmatrix}$$

$$= \phi(w_{-1,1}(a)\, w_{-2,2}(b)\, h_{-2,-1}(-b)) \in E(4; R).$$

Setting $\tilde{g} = w_{-1,1}(a)\, w_{-2,2}(b)\, h_{-2,-1}(-b) \in \mathrm{St}(4; R)$ and using Corollary 1.7, we have $T_{id,1}\tilde{g} = w_{-1,1}(-a)\, w_{-2,2}(-b)\, h_{2,1}(-b^{-1})$ Now using [5, Lemmas 8.2, 9.6, Corollary 9.4], we have

$$\begin{aligned} \tilde{G}_{id\,1}\tilde{g} &= [w_{-1,1}(-a)\, w_{-2,2}(-b)\, h_{2,1}(-b^{-1})\, w_{-1,1}(a)\, w_{-2,2}(b)\, h_{-2,-1}(-b)] \\ &= [h_{-2,-1}(-b^{-2}a^{-1})\, h_{-2,-1}(b^{-1}a^{-1})^{-1}\, h_{-2,-1}(-b)] \\ &= -[-b][-b^{-2}a^{-1}] = -[-b][-a^{-1}] = [-a^{-1}][-b] \end{aligned}$$

in $Z^{+}(Z_2\,; K_2R)$. Thus $\hat{G}_{id,1}\hat{g} = [[-a][-b]]$. ∎

Now we can identify our invariant $G_{id,1}$ in the case $R = E$ is a field. Note that, by Corollary 4.2, $G_{id,1}$ takes values in $H^{+}(Z_2\,; K_2E) \cong K_2E/2K_2E$, the isomorphism following directly from Matsumoto's theorem [5, Theorem 11.1].

THEOREM 4 4. *Let E be a field, g an $(id, 1)$-reflexive (i e., symmetric bilinear) form of even rank with vanishing discriminant* disc g; *then, for every symbol $\varphi\colon E^{\cdot} \times E \to \{\pm 1\}$, the Hasse–Witt invariant $h_\varphi g$ equals $\bar{\varphi}G_{id,1}g$, where $\bar{\varphi}$ is the unique homomorphism such that the following diagram is commutative.*

$$\begin{array}{ccccc} K_1E \times K_1E & \longrightarrow & K_2E & \longrightarrow & K_2E/2K_2E \\ \cong \big\downarrow \det\times\det & & & & \big\downarrow \bar{\varphi} \\ E^{\cdot} \times E^{\cdot} & & \xrightarrow{\quad\varphi\quad} & & \{\pm 1\} \end{array}$$

(h_φ is defined in Milnor and Husemoller [6, p. 80]).

Proof. Clearly, disc g lives in $H^{-}(Z_2\,; K_1E) = E^{\cdot}/E^{\cdot 2}$ and coincides with the usual discriminant for forms of even rank. Thus, in the Witt ring $W(E)$, g represents an element of I^2, $I \subseteq W(E)$ being the fundamental ideal generated by forms of even rank Since both h_φ and $G_{id,1}$ are homomorphisms of I^2, it suffices to show that $h_\varphi g = \bar{\varphi}G_{id,1}g$ for a set of generators for I^2. But such a set is given by the forms

$$g = \langle a\rangle + \langle b\rangle + \langle a^{-1}b^{-1}\rangle + \langle 1\rangle, \qquad a, b \in E^{\cdot}.$$

Now, Lemma 4 3 computes $G_{id,1}g = \hat{G}_{id,1}\hat{g} = [[-a][-b]]$; hence, we must have $h_\varphi g = \bar{\varphi}[[-a][-b]] = \varphi(-a, -b)$. This is indeed the case, as is easily checked from the definition of h. ▮

It follows from Theorem 4.4 that $G_{id,1}: I^2/I^3 \cong K_2E/2K_2E$ is the universal Hasse–Witt invariant in the case of a field E. For this reason, the invariants $\tilde{G}_{\alpha,u}$, $\hat{G}_{\alpha,u}$, $G_{\alpha,u}$ of the preceding section are all called generalized *Hasse–Witt invariants.*

5. Extensions of the Hasse–Witt Invariants

We begin by defining a Steinberg-like group $\mathrm{St}^V(n; R)$, where $V \subseteq R^\cdot$ is a subgroup of the group $R^\cdot$ of units of the ring R, $n \geqslant 1$. In terms of generators and relations, $\mathrm{St}^V(n; R)$ is presented as follows

Generators. (1) x_{ij}^r, $r \in R$, $1 \leqslant i, j \leqslant n$, $i \neq j$,

(2) $y(v_1, \ldots, v_n)$, $v_1, \ldots, v_n \in V$.

Relations. (1) $x_{ij}^r x_{ij}^s = x_{ij}^{r+s}$,

(2) $y(u_1, \ldots, u_n)\, y(v_1, \ldots, v_n) = y(u_1v_1, \ldots, u_nv_n)$,

(3) $y(v_1, \ldots, v_n)\, x_{ij}^r\, y(v_1, \ldots, v_n)^{-1} = x_{ij}^{v_i r v_j^{-1}}$,

(4)
$$\begin{aligned}[x_{ij}^r, x_{kl}^s] &= 1 &&\text{if } i \neq 1,\ j \neq k,\\ &= x_{il}^{rs} &&\text{if } i \neq 1,\ j = k,\\ &= x_{kj}^{-sr} &&\text{if } i = 1,\ j \neq k,\end{aligned}$$

(5) $w_{ij}(u)\, x_{ji}^r w_{ij}(-u) = x_{ij}^{-uru}$, where $u \in R^\cdot$ and $w_{ij}(u) = x_{ij}^u x_{ji}^{-u^{-1}} x_{ij}^u$,

(6) $h_{ij}(v) = y(1, \ldots, \underset{i}{v}, \ldots, \underset{j}{v^{-1}}, \ldots, 1)$, $v \in V$, where $h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1)$.

Clearly, for $V = \{1\}$, $\mathrm{St}^V(n; R) = \mathrm{St}(n; R)$.

There are canonical homomorphisms $\psi_V: \mathrm{St}(n; R) \to \mathrm{St}^V(n; R)$ given by $\psi_V x_{ij}^r = x_{ij}^r$ and $\phi_V: \mathrm{St}^V(n; R) \to \mathrm{GL}(n; R)$ given by $\phi_V x_{ij}^r = e_{ij}^r$, $\phi_V y(v_1, \ldots, v_n) = \mathrm{diag}(v_1, \ldots, v_n)$. If $E^V(n; R) = \phi_V \mathrm{St}^V(n; R)$, then $E(n; R)$ is a normal subgroup of $E^V(n; R)$. Let $W_1{}^V(n; R) = E^V(n; R)/E(n; R)$ and $W_2{}^V(n; R) = \mathrm{Ker}[\psi_V: \mathrm{St}(n; R) \to \mathrm{St}^V(n; R)]$. From relation (3), it follows that $\psi_V \mathrm{St}(n; R)$ is a normal subgroup of $\mathrm{St}^V(n; R)$, and so ϕ_V induces a surjection $\bar{\phi}_V$: Coker $\psi_V \to W_1{}^V(n; R)$.

If $\bar{\phi}_V$: Coker $\psi_V \to W_1{}^V(n; R)$ is an isomorphism, then V is called an $\mathrm{St}(n)$-*subgroup* of R; V is called an $\mathrm{St}(\infty)$-*subgroup* of $R^\cdot$ if it is an $\mathrm{St}(n)$-subgroup for almost all n, and V is called an St-*subgroup* of $R^\cdot$ if it is an $\mathrm{St}(n)$-subgroup for all n.

The significance of an St(n)-subgroup $V \subseteq R$ is that it leads immediately to the following commutative diagram with exact rows and columns, where $K_2{}^V(n; R) = \mathrm{Ker}[\phi_V\colon \mathrm{St}^V(n; R) \to E^V(n; R)]$.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_2{}^V(n; R) & \longrightarrow & K_2(n; R) & \longrightarrow & K_2{}^V(n; R) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & W_2{}^V(n; R) & \longrightarrow & \mathrm{St}(n; R) & \xrightarrow{\psi_V} & \mathrm{St}^V(n; R) & \longrightarrow & W_1{}^V(n; R) \longrightarrow 1 \\
 & & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_V} & & \| \\
 & & 1 & \longrightarrow & E(n; R) & \longrightarrow & E^V(n; R) & \longrightarrow & W_1{}^V(n; R) \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Letting $n \to \infty$, where $\mathrm{St}^V(n; R) \hookrightarrow \mathrm{St}^V(n+1; R)$, etc., in the obvious way, we have, for an St(∞)-subgroup $V \subseteq R^{\cdot}$, the following commutative diagram with exact rows and columns, where $K_1{}^V(R) = K_1(R)/W_1{}^V(R)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_2{}^V(R) & \longrightarrow & K_2(R) & \longrightarrow & K_2{}^V(R) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_2{}^V(R) & \longrightarrow & \mathrm{St}(R) & \xrightarrow{\psi_V} & \mathrm{St}^V(R) & \longrightarrow & W_1{}^V(R) \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_V} & & \downarrow \\
 & & 1 & \longrightarrow & E(R) & \longrightarrow & \mathrm{GL}(R) & \longrightarrow & K_1(R) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 1 & & K_1{}^V(R) & = & K_1{}^V(R) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & 0 & & 0
\end{array}$$

It could perhaps be that all subgroups of $R^{\cdot}$ are, say, St(∞)-subgroups. The best we can do is the following, which provides a source of St-subgroups.

PROPOSITION 5.1. *If the kernel of the canonical homomorphism $V \to K_1 R$ is the commutator subgroup of V, then $V \subseteq R^{\cdot}$ is an* St-*subgroup.*

Proof. For $n = 1$, $\bar{\phi}_V = id: V \cong V$ is clearly an isomorphism. Suppose $n > 1$, and let $w \in \mathrm{St}^V(n; R)$ be an element such that $\bar{\phi}_V[w] = 1$ in $W_1{}^V(n; R)$, i.e., such that $\phi_V w \in E(n; R)$. Using relations (1), (2), (3), we may suppose that $w = xy(v_1, \ldots, v_n)$, where x is a word in the x_{ij}^r. Now, using relation (6), we have $y(v_1, \ldots, v_n) = h_{1,2}(v_1)\, y(1, v_1 v_2, v_3, \ldots, v_n)$. Continuing inductively, we have

$$y(v_1, \ldots, v_n) = h_{1,2}(v_1) \cdots h_{n-1,n}(v_1 \cdots v_{n-1})\, y(1, \ldots, 1, v_1 \cdots v_n),$$

and so we may in fact suppose that $w = xy(1, \ldots, 1, v_0)$, where x is a word in the x_{ij}^r. From the assumption that the kernel of $V \to K_1 R$ is the commutator subgroup of V, it follows that v_0 is an element of the commutator subgroup of V. Now, for $u, v \in V$, we have

$$\begin{aligned} y(1, \ldots, 1, [u, v]) &= y(1, \ldots, 1, v^{-1}u^{-1}, uv)\, y(1, \ldots, 1, u, u^{-1})\, y(1, \ldots, 1, v, v^{-1}) \\ &= h_{n,n-1}(uv)\, h_{n,n-1}(u)^{-1}\, h_{n,n-1}(v)^{-1}, \end{aligned}$$

again using relation (6). Hence, w may be expressed as a word in the x_{ij}^r, and so w is in the image of ψ_V. That is, $[w] \in \mathrm{Coker}\, \psi_V$, and V is an $\mathrm{St}(n)$-subgroup of $R^{\cdot}$ for every n ∎

Two important examples of rings and St-subgroups, in virtue of Proposition 5.1, are the following.

EXAMPLE 1. Any subgroup V of the group of units of a commutative ring R is an St-subgroup. The particular choice $V = R^{\cdot}$ gives $W_1^{R^{\cdot}}(n; R) = W_1^{R^{\cdot}}(R) = R^{\cdot}$, and since $SK_1(R) = K_1(R)/R^{\cdot} = K_1^{R^{\cdot}}(R)$, we define

$$SK_2(R) = K_2^{R^{\cdot}}(R).$$

By the work of Dennis and Stein, the ring of integers in $\mathbb{Q}(-17)^{1/2}$ is a ring R with $SK_2 R \neq 0$ [1, Sect. 3].

EXAMPLE 2. Any subgroup $V \subsetneq \pm\pi$ such that V/V' injects monomorphically into $\pm\pi/\pi'$ is an St-subgroup of the group of units of the group ring $\mathbb{Z}\pi$. In particular, we may choose $V = \pm\pi$, and then we have $K_1^{\pm\pi}(\mathbb{Z}\pi) = Wh(\pi)$ and, in the notation of Hatcher [4], $W_2^{\pm\pi}(\mathbb{Z}\pi) = W(\pi) \cap K_2\mathbb{Z}\pi$, and so we have

$$Wh_2(\pi) = K_2^{\pm\pi}(\mathbb{Z}\pi).$$

It would appear that basic objects of topological algebraic K_2-theory should somehow be the groups in the exact sequence

$$0 \to Wh_2(\pi) \to \mathrm{St}(\pi) \to \mathrm{GL}(\mathbb{Z}\pi) \to Wh(\pi) \to 0,$$

as well as the sequence itself, where $\mathrm{St}(\pi) = \mathrm{St}^{\pm\pi}(\mathbb{Z}\pi)$.

Now suppose that R is a ring with antistructure (α, u), and that $V \subseteq R^{\cdot}$ is an St(∞)-subgroup such that $\alpha V = V$. Then $T_\alpha W_i{}^V(R) = W_i{}^V(R)$, $i = 1, 2$, and so T_α induces an involution T_α on $K_i{}^V(R)$, $i = 1, 2$.

If $V \subseteq R^{\cdot}$ is an St($2n$)-subgroup such that $\alpha V = V$, then, with the notational convention of Proposition 1.6–Corollary 1.7, there is an involution $T_{\alpha,u}$ on $\mathrm{St}^V(2n; R)$ given by

$$\begin{aligned} T_{\alpha,u} x_{ij}^r &= x_{-j,-i}^{-\alpha(r)} && i, j \text{ both} > 0, \\ &= x_{-j,-i}^{-u^{-1}\alpha(r)} && i > 0, \quad j < 0, \\ &= x_{-j,-i}^{-\alpha(r)u} && i < 0, \quad j > 0, \\ &= x_{-j,-i}^{-\alpha^{-1}(r)} && i, j \text{ both} < 0, \end{aligned}$$

$$T_{\alpha,u} y(v_1, \ldots, v_{-n}) = y(\alpha^{-1}v_{-1}^{-1}, \ldots, \alpha^{-1}v_{-n}^{-1}, \alpha v_1^{-1}, \ldots, \alpha v_n^{-1}).$$

For an St(∞)-subgroup V with $\alpha V = V$, these involutions are compatible with stabilization and commute with ϕ_V, ψ_V. Hence, in this case the commutative diagram preceding Proposition 5.1 is a commutative diagram of groups-with-involution.

Now, if $V \subseteq R^{\cdot}$ is an St-subgroup (or, with care being taken to stabilize appropriately, an St(∞)-subgroup) such that $\alpha V = V$, then the program of Sections 2 and 3 may be repeated with $K_i{}^V R$ replacing $K_i R$, $i = 1, 2$, $E^V(n; R)$ replacing $E(n; R)$, $\mathrm{St}^V(n; R)$ replacing $\mathrm{St}(n; R)$, etc For example, we obtain relative discriminants disc^V with values in $K_1{}^V R$ and homomorphisms

$$d_{\alpha,u}^{\pm}: H^{\pm}(Z_2; K_0 R) \to H^{\pm}(Z_2; K_1{}^V R).$$

Similarly, Hasse–Witt invariants $\tilde{G}_{\alpha,u}^V$ in $Z^{\pm}(Z_2; K_2{}^V R)$, $\hat{G}_{\alpha,u}^V$ in $H^{\pm}(Z_2; K_2{}^V R)$, and homomorphisms

$$d_{\alpha,u}^{\pm}: H^{\pm}(Z_2; K_1{}^V R) \to H^{\pm}(Z_2; K_2{}^V R)$$

are defined. With the obvious rewording, the results of both sections hold true in this situation.

Algebraic L-theoretic ramifications will be taken up in the next paper of this series, along with topological applications.

Note added in proof. S. C. Geller has shown the sufficient condition provided by Proposition 5.1 is also necessary.

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