CHAPTER 5

The Atiyah–Singer Index Theorem*

Peter B. Gilkey

Mathematics Department, University of Oregon, Eugene, OR 97403, USA

E-mail: gilkey@math.uoregon.edu

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0. Introduction

Here is a brief outline to the paper. In Section 1, we review some basic facts concerning Clifford algebras and spin structures. In Section 2, we discuss the spectral theory of self-adjoint elliptic partial differential operators and give the Hodge decomposition theorem. In Section 3, we define the classical elliptic complexes: de Rham, signature, spin, spin^c, Yang–Mills, and Dolbeault; these elliptic complexes are all of Dirac type. In Section 4, we define the various characteristic classes for vector bundles that we shall need: Chern forms, Pontrjagin forms, Chern character, Euler form, Hirzebruch L polynomial, $\widehat{A}$ genus, and Todd polynomial. In Section 5, we discuss the characteristic classes for principal bundles. In Section 6, we give the Atiyah–Singer index theorem; the Chern–Gauss–Bonnet formula, the Hirzebruch signature formula, and the Riemann–Roch formula are special cases of the index theorem. We also discuss the equivariant index theorem and the index theorem for manifolds with boundary. We have given a short bibliography at the end of this article and refer to the extensive bibliography on the index theorem prepared Dr. Herbert Schröder which is contained in [12] for a more complete list of references.

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1. Clifford algebras and spin structures

We refer to Atiyah, Bott, and Shapiro [1], Hitchin [15], and Husemoller [16] for further details concerning the material of this section. Give $\mathbb{R}^m$ the usual inner product. The *exterior algebra* $\Lambda(\mathbb{R}^m)$ is the universal unital real algebra generated by $\mathbb{R}^m$ subject to the relations

$$v \wedge w + w \wedge v = 0.$$

Similarly, the *Clifford algebra* $C(\mathbb{R}^m)$ is the universal unital real algebra generated by $\mathbb{R}^m$ subject to the relations:

$$v * w + w * v = -2(v, w).$$

Let $\{e_i\}$ be the usual orthonormal basis for $\mathbb{R}^m$. If I is a collection of indices $I = \{1 \leqslant i_1 < \cdots < i_p \leqslant m\}$, let $|I| = p$, let

$$e_I^{\Lambda} := e_{i_1} \wedge \cdots \wedge e_{i_p} \quad \text{and let} \quad e_I^{C} := e_{i_1} * \cdots * e_{i_p}.$$

Both $\Lambda(\mathbb{R}^m)$ and $\mathcal{C}(\mathbb{R}^m)$ inherit natural innerproducts. The $\{e_I^\Lambda\}$ are an orthonormal basis for $\Lambda(\mathbb{R}^m)$ and the $\{e_I^{\mathcal{C}}\}$ are an orthonormal basis for $\mathcal{C}(\mathbb{R}^m)$. As the defining relation for $\Lambda(\mathbb{R}^m)$ is homogeneous, $\Lambda(\mathbb{R}^m)$ is a $\mathbb{Z}$ graded algebra where

$$\Lambda^p(\mathbb{R}^m) := \operatorname{span}\{e_I^\Lambda\}_{|I|=p}.$$

As the defining relation for $\mathcal{C}(\mathbb{R}^m)$ is $\mathbb{Z}_2$ graded, $\mathcal{C}(\mathbb{R}^m)$ is a $\mathbb{Z}_2$ graded algebra where the grading into even and odd is given by

$$\mathcal{C}^e(\mathbb{R}^m) := \operatorname{span}\{e_I^{\mathcal{C}}\}_{|I|=\text{even}} \quad \text{and} \quad \mathcal{C}^o(\mathbb{R}^m) := \operatorname{span}\{e_I^{\mathcal{C}}\}_{|I|=\text{odd}}.$$

Let

$$\operatorname{ext}(v)\omega := v \wedge \omega$$

be exterior multiplication and let $\operatorname{int}(v)$ be the dual (interior multiplication); it is defined by the identity $(\operatorname{ext}(v)\omega_1, \omega_2) = (\omega_1, \operatorname{int}(v)\omega_2)$. Suppose that v is a unit vector. Then we can choose an orthonormal basis $\{e_i\}$ so that $v = e_1$. Relative to such an adapted orthonormal basis we have:

$$\operatorname{ext}(v)(e_I^\Lambda) := \begin{cases} 0 & \text{if } i_1 = 1, \\ e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 > 1, \end{cases}$$

$$\operatorname{int}(v)(e_I^\Lambda) := \begin{cases} e_{i_2} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 > 1. \end{cases}$$

In other words, in this adapted orthonormal basis, exterior multiplication by v adds the index '1' to I while interior multiplication by v removes the index '1' from I. If c is a linear map from $\mathbb{R}^m$ to a unital algebra $\mathcal{A}$ such that $c(v)^2 = -|v|^2 1_{\mathcal{A}}$, we polarize to see that

$$c(v)c(w) + c(w)c(v) = -2(v, w)1_{\mathcal{A}}.$$

As c preserves the defining relation, c extends to a representation of $\mathcal{C}(\mathbb{R}^m)$. Let

$$c(v) := \operatorname{ext}(v) - \operatorname{int}(v).$$

Then $c(v)^2\omega = -|v|^2\omega$ so c extends to a unital algebra morphism from $\mathcal{C}$ to the algebra of endomorphisms of $\Lambda(\mathbb{R}^m)$. Again, relative to an adapted orthonormal basis where $v = e_1$, we have

$$c(v)(e_I^\Lambda) := \begin{cases} -e_{i_2} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 = 1, \\ e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 > 1. \end{cases}$$

The map $v \to c(v)1$ extends to a natural additive unitary isomorphism from $\mathcal{C}(\mathbb{R}^m)$ to $\Lambda(\mathbb{R}^m)$ which sends $e_I^{\mathcal{C}}$ to e_I^Λ; this map is *not* an algebra isomorphism.

If $m \geqslant 3$, then the fundamental group of the special orthogonal group $SO(m)$ is $\mathbb{Z}_2$; we use the Clifford algebra to describe the universal cover. Let $\mathrm{Pin}(m)$ be the set of all elements $\omega \in C(\mathbb{R}^m)$ which can be written in the form $\omega = v_1 * \cdots * v_k$ for some k where the v_i are unit vectors in $\mathbb{R}^m$ and let $\mathrm{Spin}(m)$ be the subset of elements where k can be taken to be even;

$$\mathrm{Spin}(m) = \mathrm{Pin}(m) \cap C^e(\mathbb{R}^m).$$

Then $\mathrm{Pin}(m)$ and $\mathrm{Spin}(m)$ are smooth manifolds which are given the structure of a Lie group by Clifford multiplication. The Lie group $\mathrm{Pin}(m)$ has two arc components; $\mathrm{Spin}(m)$ is the connected component of the identity in $\mathrm{Pin}(m)$. Let

$$\chi(\omega) = (-1)^k \quad \text{for } \omega = v_1 * \cdots * v_k \in \mathrm{Pin}(m)$$

define a representation of $\mathrm{Pin}(m)$ onto $\mathbb{Z}_2$ whose kernel is $\mathrm{Spin}(m)$. Let

$$\rho(\omega) : x \mapsto \chi(\omega)\omega * x * \omega^{-1}$$

define representations $\rho : \mathrm{Spin}(m) \to SO(m)$ and $\rho : \mathrm{Pin}(m) \to O(m)$. If v is a unit vector, then $\rho(v)$ is reflection in the hyperplane $v^\perp$; relative to an adapted orthonormal basis where $v = e_1$, we have

$$\rho(v)e_1 = -e_1 \quad \text{and} \quad \rho(v)e_i = e_i \quad \text{for } i > 1.$$

The representation ρ defines a short exact sequence

$$\mathbb{Z}_2 \to \mathrm{Spin}(m) \to SO(m)$$

and exhibits $\mathrm{Spin}(m)$ as the universal cover of $SO(m)$ if $m \geqslant 3$. The orthogonal group $O(m)$ is not connected; it has two components. There are two distinct universal covers of $O(m)$ distinguished by the induced multiplication on the set of arc components. One universal cover $\mathrm{Pin}(m)$ is as defined above; the other is defined by taking the opposite sign in the definition of the Clifford algebra. We omit details as this will not play a role in our discussion.

Let $U(n)$ be the unitary group. We complexify to define:

$$C_c(\mathbb{R}^m) := C(\mathbb{R}^m) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and}$$

$$\mathrm{Spin}^c(m) := \mathrm{Spin}(m) \otimes_{\mathbb{Z}_2} U(1) \subset C_c(\mathbb{R}^m).$$

Since we identify $(-g) \otimes (-\lambda)$ with $g \otimes \lambda$ in $\mathrm{Spin}^c(m)$, the map $\sigma : g \otimes \lambda \mapsto \lambda^2$ defines a representation from $\mathrm{Spin}^c(m)$ to $U(1)$. Since $\rho(-g) = \rho(g)$, we may extend ρ to a representation from $\mathrm{Spin}^c(m)$ to $SO(m)$ by defining $\rho(g \otimes \lambda) = \rho(g)$. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, let $\mathbb{RP}^n := S^n/\mathbb{Z}_2$ be real projective space, and let $SU(n)$ be the special unitary group. We have that

$$SO(3) = \mathbb{RP}^3, \quad \mathrm{Spin}(3) = S^3 = SU(2),$$

$$\mathrm{Spin}^c(3) = U(2) \quad \text{and} \quad \mathrm{Spin}(4) = S^3 \times S^3.$$

Let $\varepsilon(2n) = \varepsilon(2n+1) := (\sqrt{-1})^n$. We define the normalized orientation by:

$$\mathrm{orn}_m := \varepsilon(m) e_1 * \cdots * e_m \in \mathcal{C}_c(\mathbb{R}^m); \quad \mathrm{orn}_m^2 = -1.$$

Let $M_\nu(\mathbb{C})$ be the algebra of $\nu \times \nu$ complex matrices. Then $\mathcal{C}_c(\mathbb{R}^{2n}) = M_{2^n}(\mathbb{C})$. This isomorphism defines an irreducible representation $\mathcal{S}$ of $\mathcal{C}_c(\mathbb{R}^{2n})$ of dimension 2^n which is called the *spin representation.* Every complex representation of $\mathcal{C}(\mathbb{R}^{2n})$ or equivalently of $\mathcal{C}_c(\mathbb{R}^{2n})$ is isomorphic to the direct sum of copies of $\mathcal{S}$. The normalized orientation orn_{2n} anti-commutes with elements of $\mathcal{C}_c^o(\mathbb{R}^{2n})$ and commutes with elements of $\mathcal{C}_c^e(\mathbb{R}^{2n})$. The restriction of $\mathcal{S}$ to $\mathcal{C}_c^e(2n)$ is no longer irreducible; it decomposes into two representations $\mathcal{S}^\pm$ called the *half spin* representations; $\mathcal{S}^\pm(\mathrm{orn}_{2n}) = \pm 1$ on $\mathcal{S}^\pm$. Clifford multiplication defines a natural map intertwining the representations $\rho \otimes \mathcal{S}^\pm$ and $\mathcal{S}^\mp$ of $\mathrm{Spin}(2n)$ and of $\mathrm{Spin}^c(2n)$:

$$c : \mathbb{R}^{2n} \otimes \mathcal{S}^\pm \to \mathcal{S}^\mp.$$

For example, let $\{u, v\}$ be an orthonormal basis for $\mathbb{R}^2$. Then $\mathcal{C}_c(\mathbb{R}^2) = \mathcal{S}_1 \oplus \mathcal{S}_2$ decomposes as the sum of 2 copies of $\mathcal{S}$ where

$$\mathcal{S}_1 := \mathrm{Span}\{u + \sqrt{-1} \cdot v, 1 - \sqrt{-1} \cdot u * v\}$$

and

$$\mathcal{S}_2 := \mathrm{Span}\{u - \sqrt{-1} \cdot v, 1 + \sqrt{-1} \cdot u * v\}.$$

We may identify $\mathcal{S}^+ = \mathcal{S}_1^+$ with the span of $u + \sqrt{-1}v$ and $\mathcal{S}^- = \mathcal{S}_1^-$ with the span of $1 - \sqrt{-1}u * v$. Similarly, let $\{u, v, w, x\}$ be an orthonormal basis for $\mathbb{R}^4$. Then $\mathcal{C}_c(\mathbb{R}^4) = \widehat{\mathcal{S}}_1 \oplus \widehat{\mathcal{S}}_2 \oplus \widehat{\mathcal{S}}_3 \oplus \widehat{\mathcal{S}}_4$ decomposes as the sum of 4 copies of $\mathcal{S}$; these representation spaces can be taken to be the tensor product of the representation spaces in dimension 2. Thus

$$\widehat{\mathcal{S}}_1 := \mathcal{S}_1 \otimes \mathcal{S}_1, \qquad \widehat{\mathcal{S}}_2 := \mathcal{S}_2 \otimes \mathcal{S}_1, \qquad \widehat{\mathcal{S}}_3 := \mathcal{S}_1 \otimes \mathcal{S}_2, \qquad \widehat{\mathcal{S}}_4 := \mathcal{S}_2 \otimes \mathcal{S}_2,$$

or equivalently:

$$\begin{aligned}
\widehat{\mathcal{S}}_1 := \mathrm{Span}\big\{&(u + \sqrt{-1} \cdot v)(w + \sqrt{-1} \cdot x), (u + \sqrt{-1} \cdot v)(1 - \sqrt{-1} \cdot w * x), \\
&(1 - \sqrt{-1} \cdot u * v)(w + \sqrt{-1} \cdot x), (1 - \sqrt{-1} \cdot u * v)(1 - \sqrt{-1} \cdot w * x)\big\}, \\
\widehat{\mathcal{S}}_2 := \mathrm{Span}\big\{&(u - \sqrt{-1} \cdot v)(w + \sqrt{-1} \cdot x), (u - \sqrt{-1} \cdot v)(1 - \sqrt{-1} \cdot w * x), \\
&(1 + \sqrt{-1} \cdot u * v)(w + \sqrt{-1} \cdot x), (1 + \sqrt{-1} \cdot u * v)(1 - \sqrt{-1} \cdot w * x)\big\}, \\
\widehat{\mathcal{S}}_3 := \mathrm{Span}\big\{&(u + \sqrt{-1} \cdot v)(w - \sqrt{-1} \cdot x), (u + \sqrt{-1} \cdot v)(1 + \sqrt{-1} \cdot w * x), \\
&(1 - \sqrt{-1} \cdot u * v)(w - \sqrt{-1} \cdot x), (1 - \sqrt{-1} \cdot u * v)(1 + \sqrt{-1} \cdot w * x)\big\},
\end{aligned}$$

$$\widehat{\mathcal{S}}_4 := \operatorname{Span}\big\{(u-\sqrt{-1}\cdot v)(w-\sqrt{-1}\cdot x), (u-\sqrt{-1}\cdot v)(1+\sqrt{-1}\cdot w*x), (1+\sqrt{-1}\cdot u*v)(w-\sqrt{-1}\cdot x), (1+\sqrt{-1}\cdot u*v)(1+\sqrt{-1}\cdot w*x)\big\}.$$

Let

$$c_1 := \begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}, \quad c_2 := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad c_3 := \begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},$$

$$\widehat{c}_1 := \sqrt{-1}\cdot c_1\otimes c_1, \quad \widehat{c}_2 := \sqrt{-1}\cdot c_1\otimes c_2, \quad \widehat{c}_3 := \sqrt{-1}\cdot c_1\otimes c_3,$$

$$\widehat{c}_4 := c_2\otimes 1.$$

We can also define the spin representations for $m=2$ or $m=4$ by taking

$$c(a_1u+a_2v) := a_1c_1+a_2c_2 \quad \text{if } m=2,$$

$$c(a_1u+a_2v+a_3w+a_4x) := a_1\widehat{c}_1+a_2\widehat{c}_2+a_3\widehat{c}_3+a_4\widehat{c}_4 \quad \text{if } m=4.$$

The *Stiefel–Whitney classes* are $\mathbb{Z}_2$ characteristic classes of a real vector bundle. They are characterized by the properties:

(a) If $\dim(V)=r$, then $w(V)=1+w_1(V)+\cdots+w_r(V)$ for $w_i\in H^i(M;\mathbb{Z}_2)$.

(b) If $f: M_1\to M_2$, then $f^*(w(V))=w(f^*V)$.

(c) We have $w(V\oplus W)=w(V)w(W)$, i.e.

$$w_k(V\oplus W)=\sum_{i+j=k} w_i(V)w_j(W).$$

(d) If L is the Möbius line bundle over $\mathbb{RP}^1=S^1$, then $w_1(L)\neq 0$.

For example, we have $H^*(\mathbb{RP}^n;\mathbb{Z}_2)=\mathbb{Z}_2[x_1]/(x_1^{n+1}=0)$ is a truncated polynomial algebra; here $x_1=w_1(L)\in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is the first Stiefel–Whitney class of the classifying real line bundle over $\mathbb{RP}^n$. We have

$$w\big(T\mathbb{RP}^n\big)=(1+w_1)^{n+1}.$$

More generally, let $Gr_p(m)$ be the Grassmannian of unoriented p-dimensional planes in $\mathbb{R}^m$; $\mathbb{RP}^m=Gr_1(m+1)$. Let E be the classifying p plane bundle over $Gr_p(m)$ and let $E^\perp$ be the complementary bundle;

$$E := \big\{(\pi,\xi)\in Gr_p(m)\times\mathbb{R}^m:\ \xi\in\pi\big\} \quad\text{and}$$

$$E^\perp := \big\{(\pi,\xi)\in Gr_p(m)\times\mathbb{R}^m:\ \xi\perp\pi\big\}.$$

Let $w := w(E)$ and let $\overline{w} := w(E^\perp)$. Since $E\oplus E^\perp$ is the trivial m plane bundle over $Gr_p(m)$, we have $w\overline{w}=1$. We use this relation to solve for $\overline{w}$ in terms of w; $\overline{w}_1=w_1$,

$\overline{w}_2 = w_1^2 + w_2$, etc. For dimensional reasons, we have $\overline{w}_i = 0$ for $i > \dim(E^\perp) = m - i$. This is the only relation imposed. We have, see Borel [5],

$$H^*(Gr_p(m); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_p]/\overline{w}_i = 0 \quad \text{for } i > m - p.$$

The integral *Chern classes* are $\mathbb{Z}$ characteristic classes of a complex vector bundle. They are characterized by the properties:

(a) If $\dim(V) = r$, then $c(V) = 1 + c_1(V) + \cdots + c_r(V)$ for $c_i \in H^i(M; \mathbb{Z})$.
(b) If $f : M_1 \to M_2$, then $f^*(c(V)) = c(f^*V)$.
(c) We have $c(V \oplus W) = c(V)c(W)$, i.e.

$$c_k(V \oplus W) = \sum_{i+j=k} c_i(V)c_j(W).$$

(d) If L is the classifying line bundle over $\mathbb{CP}^1 = S^2$, then $c_1(L)[\mathbb{CP}^1] = -1$.

We have $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[x_2]/(x_2^{n+1} = 0)$ is a truncated polynomial algebra; here $x_2 = c_1(L) \in H^2(\mathbb{CP}^n; Z)$ is the first Chern class of the classifying complex line bundle over $\mathbb{CP}^n$. Let $T^{1,0}$ be the holomorphic tangent bundle of $\mathbb{CP}^n$; $T^{1,0}$ is isomorphic to $\Lambda^{0,1}\mathbb{CP}^n$. We have

$$c(T^{1,0}) = (1 + x_2)^{n+1}.$$

We can complexify a real vector bundle V to define an associated complex vector bundle. The integral *Pontrjagin classes* are defined in terms of the Chern classes:

$$p_i(V) := (-1)^i c_{2i}(V_{\mathbb{R}}\mathbb{C}) \in H^{4i}(M; \mathbb{Z}).$$

For example,

$$p(TS^m) = 1 \quad \text{and} \quad p(T\mathbb{CP}^n) = (1 + x_2^2)^{n+1}.$$

Fix a fiber metric on a real vector bundle V and let e_α be local orthonormal frames for V over contractable coordinate neighborhoods $\mathcal{O}_\alpha$. We may express $e_\alpha = \phi_{\alpha\beta}e_\beta$ where $\phi_{\alpha\beta}$ maps the overlap $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ to the orthogonal group $O(r)$. These satisfy the *cocycle condition*: $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$. We say that V is *orientable* if we can reduce the structure group to $SO(r)$; this means that we can choose the e_α so $\det(\phi_{\alpha\beta}) = 1$; V is orientable if and only if $w_1(V) = 0$. We say that V admits a *spin structure* if V is orientable and if we can define lifts $\widetilde{\phi}_{\alpha\beta}$ to $\text{Spin}(r)$ preserving the cocycle condition; similarly, we say that V admits a *spin^c structure* if V is orientable and if we can define lifts $\widetilde{\phi}_{\alpha\beta}$ to $\text{Spin}^c(r)$ preserving the cocycle condition. Let s be a spin^c structure on V. We use σ to define an associated complex line bundle $\sigma(s)$ over M with transition functions $\sigma(\widetilde{\phi}_{\alpha\beta}) \in U(1)$.

By choosing a fiber metric for a line bundle L, we can reduce the structure group to $O(1) = \pm 1$ in the real setting or to $U(1) = S^1$ in the complex setting. Let $\varepsilon_{\alpha\beta}$ be the transition functions of L. If s is a spin structure on V and if L is real or if s is a spin^c structure on V and if L is complex, we twist the structure s by L to define a new structure

s_L with lifts $\widetilde{\phi}_{\alpha\beta}\varepsilon_{\alpha\beta}$. Let $\mathrm{Vect}^1_{\mathbb{R}}(M)$ and $\mathrm{Vect}^1_{\mathbb{C}}(M)$ be the set of isomorphism classes of real and complex line bundles over M. We use the map $s \mapsto s_L$ to parametrize inequivalent spin and spin^c structures on V by $\mathrm{Vect}^1_{\mathbb{R}}(M)$ and $\mathrm{Vect}^1_{\mathbb{C}}(M)$. We use tensor product to make $\mathrm{Vect}^1_{\mathbb{R}}(M)$ and $\mathrm{Vect}^1_{\mathbb{C}}(M)$ into Abelian groups. The first Stiefel–Whitney class is a group isomorphism from $\mathrm{Vect}^1_{\mathbb{R}}(M)$ to $H^1(M;\mathbb{Z}_2)$ which provides a natural equivalence between these two functors. Similarly, the integral first Chern class is a group isomorphism from $\mathrm{Vect}^1_{\mathbb{C}}(M)$ to $H^2(M;\mathbb{Z})$ which provides a natural equivalence between these other two functors. Thus inequivalent spin and spin^c structures on V are parametrized by $H^1(M;\mathbb{Z}_2)$ and $H^2(M;\mathbb{Z})$; there exist inequivalent spin structures if and only if $H^1(M;\mathbb{Z}_2)\neq 0$ and there exist inequivalent spin^c structures if and only if $H^2(M;\mathbb{Z})\neq 0$. Note that the complex line bundle $\sigma(s_L)$ associated to the twisted spin^c structure is the complex line bundle $\sigma(s)$ twisted by L^2, i.e. $\sigma(s_L)=\sigma(s)\otimes L^2$.

A real vector V is orientable if and only if $w_1(V)=0$. It admits a spin structure if and only if $w_1(V)=0$ and $w_2(V)=0$. It admits a spin^c structure if and only if $w_1(V)=0$ and if $w_2(V)$ can be lifted from $H^2(M;\mathbb{Z}_2)$ to $H^2(M;\mathbb{Z})$. If V_1 admits a spin structure, then $V\oplus V_1$ admits a spin structure if and only if V admits a spin structure. If V_1 admits a spin^c structure, then $V\oplus V_1$ admits a spin^c structure if and only if V admits a spin^c structure. If V is the underlying real vector bundle of a complex vector bundle W, then V admits a natural orientation and spin^c structure.

We say that a manifold M is spin or spin^c if the tangent bundle TM has a spin or spin^c structure. The sphere S^m is spin for any m. Note that $\mathbb{RP}^1=S^1$. Let $m>1$. Real projective space $\mathbb{RP}^m$ is orientable if and only if m is odd, spin^c if and only if m is odd, and spin if and only if $m\equiv 3 \bmod 4$. We have

$$H^1(\mathbb{RP}^m;\mathbb{Z}_2)=\mathbb{Z}_2 \quad\text{and}\quad H^2(\mathbb{RP}^m;\mathbb{Z})=\mathbb{Z}_2.$$

Let $j\geqslant 1$. There are 2 inequivalent spin structures on $\mathbb{RP}^{4j-1}$ and 2 inequivalent spin^c structures on $\mathbb{RP}^{4j+1}$. Complex projective space $\mathbb{CP}^m$ always admits a spin^c structure. It admits a spin structure if and only if m is odd. The spin structure is unique; the spin^c structure is not. If $m=2$ and if M is orientable, then M admits a spin structure. The group of n-th roots of unity acts by complex multiplication on the unit sphere S^{2k-1} in $\mathbb{C}^k$. For $k\geqslant 2$, the *lens space* $L(k;n)$ is the quotient $S^{2k-1}/\mathbb{Z}_n$. If k is odd and if n is even, $L(k;n)$ does not admit a spin structure; $L(k,n)$ admits a spin structure if n is odd or if k and n are both even. The spin structure is unique if n is odd; there are two spin structures if n is even. The lens space $L(k;n)$ always admits a spin^c structure and there are n inequivalent spin^c structures. The product of spin manifolds is spin; the product of spin^c manifolds is spin^c. The connected sum of spin manifolds is spin; the connected sum of spin^c manifolds is spin^c.

If M is an even dimensional spin manifold, let the *spin bundle* $\mathcal{S}(M)$ be the bundle defined by the spinor representation $\mathcal{S}$. The Levi-Civita connection lifts to a connection called the *spin connection* on $\mathcal{S}(M)$. Clifford multiplication defines a representation c of the Clifford algebra of the tangent bundle on the spin bundle $\mathcal{S}(M)$. Let $\vec{e}:=\{e_i\}$ be a local orthonormal frame for the tangent bundle. The frame $\vec{e}$ defines two local frames $\pm\vec{s}$ for $\mathcal{S}(M)$; this sign ambiguity plays no role in the local theory and reflects the fact that we have two lifts from the principal SO bundle to the principal Spin bundle. Let Γ_{ijk} be the

Christoffel symbols of the Levi-Civita connection. Then the connection 1-form of the spin connection is an endomorphism valued 1 form which is given by

$$\frac{1}{4}\sum_{ijk}\Gamma_{ijk}e_i\otimes c(e_j)c(e_k).$$

2. Spectral theory

We refer to Gilkey [12] and to Seeley [19] for further details concerning the material of this section. Let M be a compact Riemannian manifold without boundary. Let $x=(x^1,\dots,x^m)$ be a system of local coordinates on M. Let $\partial_i^x=\partial/\partial x^i$. If $\alpha=(\alpha_1,\dots,\alpha_m)$ is a multi-index, let

$$\partial_\alpha^x:=(\partial_1^x)^{\alpha_1}\cdots(\partial_m^x)^{\alpha_m},\quad \xi^\alpha:=\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}\quad\text{and}\quad |\alpha|:=\alpha_1+\cdots+\alpha_m.$$

Let V and W be smooth complex vector bundles over M and let D mapping $C^\infty(V)$ to $C^\infty(W)$ be a partial differential operator of order n. Choose local frames for V and W to decompose

$$D=\sum_{|\alpha|\leqslant n}a_\alpha(x)\partial_\alpha^x,$$

where the a_α are linear maps from V to W. We define the *leading symbol* of D by replacing differentiation with multiplication:

$$\sigma_L(D)(x,\xi):=\sum_{|\alpha|=n}a_\alpha(x)\xi^\alpha.$$

If we identify ξ with the cotangent vector $\xi:=\sum_i\xi_i\,dx^i$, then $\sigma_L(D)(x,\xi)$ is an invariantly defined map which is homogeneous of degree n from the cotangent bundle T^*M to the bundle of endomorphisms from V to W. We have

$$\sigma_L(D^*)=(-1)^n\sigma_L(D)^*\quad\text{and}\quad\sigma_L(D_1\circ D_2)=\sigma_L(D_1)\circ\sigma_L(D_2).$$

The leading symbol is sometimes defined with factors of $\sqrt{-1}$ to make formulas involving the Fourier transform and the adjoint more elegant; we delete these factors in the interests of simplicity.

We suppose $V=W$ for the moment. We assume V is equipped with a fiber metric and define the L^2 inner product by integration. We say that D is *elliptic* if $\sigma_L(D)(x,\xi)$ is invertible for $\xi\neq 0$. We say that D is *self-adjoint* if

$$(D\phi,\psi)_{L^2(V)}=(\phi,D\psi)_{L^2(V)}\quad\text{for all smooth }\phi,\psi.$$

Let D be self-adjoint and elliptic. There exists a complete orthonormal basis $\{\phi_\nu\}$ for $L^2(V)$ where the ϕ_ν are smooth sections to V with $D\phi_\nu=\lambda_\nu\phi_\nu$; the collection $\{\phi_\nu,\lambda_\nu\}$ is called a *discrete spectral resolution* of D. We have $\dim\ker(D) < \infty$ and $\lim_{\nu\mapsto\infty}|\lambda_\nu| = \infty$. Order the eigenvalues so $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$. Then there exists $\delta > 0$ and $\varepsilon > 0$ so that $|\lambda_\nu| \geqslant \varepsilon\nu^\delta$ if ν is sufficiently large. We have a decomposition

$$C^\infty(V) = \ker(D) \oplus \mathrm{range}(D)$$

which is orthogonal in $L^2(V)$. If $\phi \in L^2(V)$, we may expand $\phi = \sum_\nu a_\nu\phi_\nu$ in a generalized Fourier series; ϕ is smooth if and only if $\lim_{\nu\to\infty}\nu^k a_\nu = 0$ for any k, i.e. the Fourier coefficients decay rapidly.

For example, let $D = -\partial_\theta^2$ on the circle. The corresponding spectral resolution is $\{e^{\sqrt{-1}n\theta}, n^2\}_{n\in\mathbb{Z}}$. The eigenfunctions are smooth and the eigenvalues grow quadratically. More generally, let D be the Laplacian on the sphere S^m. If we introduce polar coordinates (r,θ) on $\mathbb{R}^{m+1}$ for $r \in [0,\infty)$ and $\theta \in S^m$, we can express the Euclidean Laplacian D_e in the form:

$$D_e = -\partial_r^2 - mr^{-1}\partial_r + r^{-2}D.$$

Let $H(m+1, j)$ be the vector space of all harmonic polynomials which are homogeneous of order j in $m+1$ variables. We can use the above equation to see that the restriction of $f \in H(m+1, j)$ to S^m is an eigenfunction for D corresponding to the eigenvalue $j(j+m-1)$. In fact, all eigenfunctions of the Laplacian on S^m arise in this way and $\{j(j+m-1), H(m+1,j)\}_{j\geqslant 0}$ is the discrete spectral resolution of the scalar Laplacian on the sphere S^m. The eigenvalues $j(j+m-1)$ appear with multiplicity

$$\dim H(m+1, j) = \binom{m+j}{m} - \binom{m+j-2}{m}.$$

Thus if we order the eigenvalues in increasing order and repeat each eigenvalue according to multiplicity, we have λ_ν grows like $\nu^{2/m}$. We refer to [12, Section 4.2] for further details.

Let $\{\phi_\nu, \lambda_\nu\}$ be the discrete spectral resolution of a self-adjoint elliptic second-order partial differential operator D. If we assume that the leading symbol of D is negative definite, then there are only a finite number of negative eigenvalues. The estimate $\lambda_\nu \geqslant \nu^\varepsilon$ for large ν shows that the *heat trace*

$$h(t, D) := \mathrm{tr}_{L^2} e^{-tD} = \sum_\nu e^{-t\lambda_\nu}$$

is analytic for $t > 0$. There exists an asymptotic series as $t \downarrow 0$ of the form

$$h(t, D) \approx \sum_{n\geqslant 0} a_n(D) t^{(n-m)/2},$$

where $a_n(D) = \int_M a_n(x, D)$ are locally computable invariants. The $a_n(x, D)$ are invariant polynomials in the jets of the total symbol of D with coefficients which are smooth functions of the leading symbol of D. For example, if $D = \delta d$ is the scalar Laplacian, the leading symbol of D is given by the metric tensor; D is a self-adjoint elliptic partial differential operator. Let $R_{ijkl;\dots}$ denote the components of the covariant derivatives of the curvature tensor relative to a local orthonormal frame for the tangent bundle of M. We adopt the *Einstein convention* and sum over repeated indices. Let $\rho_{ij} := R_{ikkj}$ be the Ricci tensor and $\mathcal{R} := \rho_{ii}$ be the scalar curvature. Then, see [12, Section 4.1] we have:

$$\begin{aligned}
a_0(D) &= (4\pi)^{-m/2}\int_M 1, \qquad a_2(D) = (4\pi)^{-m/2}\int_M \frac{1}{6}\mathcal{R},\\
a_4(D) &= (4\pi)^{-m/2}\int_M \left\{\frac{5}{360}\mathcal{R}^2 - \frac{1}{180}|\rho|^2 + \frac{1}{180}|R|^2\right\},\\
a_6(D) &= (4\pi)^{-m/2}\int_M \left\{\frac{17}{7!}\mathcal{R}_{;k}\mathcal{R}_{;k} - \frac{2}{7!}\rho_{ij;k}\rho_{ij;k} - \frac{4}{7!}\rho_{jk;n}\rho_{jn;k}\right.\\
&\quad + \frac{9}{7!}R_{ijkl;n}R_{ijkl;n} + \frac{28}{7!}\mathcal{R}\mathcal{R}_{;nn} - \frac{8}{7!}\rho_{jk}\rho_{jk;nn} + \frac{24}{7!}\rho_{jk}\rho_{jn;kn}\\
&\quad + \frac{12}{7!}R_{ijk\ell}R_{ijk\ell;nn} + \frac{35}{9\cdot 7!}\mathcal{R}^3 - \frac{14}{7!}\mathcal{R}\rho^2 + \frac{14}{3\cdot 7!}\mathcal{R}|R|^2\\
&\quad - \frac{208}{9\cdot 7!}\rho_{jk}\rho_{jn}\rho_{kn} - \frac{64}{3\cdot 7!}\rho_{ij}\rho_{kl}R_{ikjl} - \frac{16}{3\cdot 7!}\rho_{jk}R_{jn\ell i}R_{kn\ell i}\\
&\quad \left. - \frac{44}{9\cdot 7!}R_{ijkn}R_{ij\ell p}R_{kn\ell p} - \frac{80}{9\cdot 7!}R_{ijkn}R_{i\ell kp}R_{j\ell np}\right\}.
\end{aligned}$$

There are similar formulas for the Laplacian on p forms we will discuss presently; note that the covariant derivatives of the curvature tensor enter nontrivially into these formulas.

Exterior differentiation d is a first order partial differential operator from the space of smooth p forms $C^\infty(\Lambda^p M)$ to the space of smooth $p+1$ forms $C^\infty(\Lambda^{p+1} M)$. We have $d^2 = 0$. The *de Rham* theorem provides a natural isomorphism between the cohomology groups $H^p(M;\mathbb{C})$ which are defined topologically and the *de Rham cohomology groups*

$$H^p(M;\mathbb{C}) := \ker(d_p)/\mathrm{range}(d_{p-1})$$

which are defined geometrically. We have

$$d\Big(\sum_I f_I\,dx^I\Big) = \sum_{I,i}\frac{\partial f_I}{\partial x^i}\,dx^i\wedge dx^I, \quad \text{so } \sigma_L(d)(x,\xi) = \mathrm{ext}(\xi).$$

The leading symbol of interior differentiation δ is the dual $-\mathrm{int}$. The leading symbol of $d+\delta$ is is given by Clifford multiplication $c := \mathrm{ext} - \mathrm{int}$. The Laplacian

$$\Delta_p := d_{p-1}\delta_{p-1} + \delta_p d_p = (d+\delta)^2_p$$

on $C^\infty(\Lambda^p M)$ is a self-adjoint second order partial differential operator with scalar leading symbol. Let $ds^2 = \sum_{ij} g_{ij}\, dx^i \circ dx^j$ be the metric tensor and let I be the identity map on Λ^p. We have

$$\sigma_L \Delta_p(x,\xi) = c(\xi)^2 = -|\xi|^2 I = -\sum_{ij} g^{ij}\xi_i\xi_j I.$$

Thus Δ_p is elliptic. We have $\ker\Delta_p = \ker d_p \cap \ker\delta_{p-1}$. We may decompose

$$\begin{aligned} C^\infty(\Lambda^p M) &= \ker(\Delta_p) \oplus \mathrm{range}(\Delta_p) \\ &= \ker(\Delta_p) \oplus \mathrm{range}(d_{p-1}) \oplus \mathrm{range}(\delta_p); \end{aligned}$$

these are orthogonal direct sum decompositions in $L^2(\Lambda^p M)$. If $\omega \in \ker\Delta_p$, then $d_p\omega = 0$. This yields the *Hodge decomposition* theorem: the map $\omega \to [\omega]$ is an isomorphism

$$\ker\Delta_p = H^p(M;\mathbb{C})$$

from the space of harmonic p forms to the de Rham cohomology groups. Let ∇ be the Levi-Civita connection and let $\{e_i\}$ be a local orthonormal frame for the tangent bundle TM. Then:

$$d = \mathrm{ext}\circ\nabla = \sum_i \mathrm{ext}(e^i)\nabla_{e_i} \quad \text{and} \quad \delta = -\mathrm{int}\circ\nabla = -\sum_i \mathrm{int}(e^i)\nabla_{e_i}.$$

Let orn be the normalized orientation on an orientable manifold M. Then

$$c(\mathrm{orn})^2 = 1,\ c(\xi)c(\mathrm{orn}) = (-1)^{m-1}c(\mathrm{orn})c(\xi), \quad \nabla\mathrm{orn} = 0,$$

$$(d+\delta)c(\mathrm{orn}) = (-1)^{m-1}c(\mathrm{orn})(d+\delta) \quad \text{and} \quad c(\mathrm{orn})\Delta_p = \Delta_{m-p}c(\mathrm{orn}).$$

Consequently $c(\mathrm{orn})$ intertwines $\ker(\Delta_p)$ with $\ker(\Delta_{m-p})$ and defines an isomorphism called *Poincaré duality*

$$H^p(M;\mathbb{C}) = \ker(\Delta_p) = \ker\Delta_{m-p} = H^{m-p}(M;\mathbb{C}).$$

Let *dvol* be the oriented volume form; the image of orn in $\Lambda^m M \otimes_{\mathbb{R}} \mathbb{C}$ is $\varepsilon(m)dvol$. This is a harmonic form which generates $H^m(M;\mathbb{C}) = \mathbb{C}$. The *Hodge* $\star$ operator is an isometry from $\Lambda^p M$ to $\Lambda^{m-p} M$ characterized by the property

$$(\star\phi_p, \psi_{m-p})dvol = \phi_p \wedge \psi_{m-p}.$$

There is a universal fourth root of unity $\varepsilon(m,p)$ so that

$$\star\phi_p = \varepsilon(m,p)c(\mathrm{orn})\phi_p;$$

thus $c(\mathrm{orn})$ is essentially just the Hodge operator. We shall use $c(\mathrm{orn})$ rather than $\star$ to simplify the sign conventions.

It is worth considering a few examples. Let S^n be the standard sphere in $\mathbb{R}^m$. Then $H^j(S^n;\mathbb{C})=0$ for $j\neq 0,n$. We have

$$H^0(S^n;\mathbb{C})=\ker(\Delta_0)=1\cdot\mathbb{C}$$

is generated by the constant function and

$$H^n(S^n;\mathbb{C})=\ker(\Delta_n)=dvol\cdot\mathbb{C}$$

is generated by the volume form on S^n. Let $\mathbb{CP}^n$ denote complex projective space. Then $H^{2j}(\mathbb{CP}^n;\mathbb{C})=\mathbb{C}$ for $0\leqslant j\leqslant n$; $H^\nu(\mathbb{CP}^n;\mathbb{C})=0$ otherwise. Let x_2 be the Kaehler form of the Fubini–Study metric; alternatively, we could take x_2 to be the first Chern form of the hyperplane line bundle. Then x_2^j is a harmonic $2j$ form which generates $H^{2j}(\mathbb{CP}^n;\mathbb{C})=\mathbb{C}$ for $0\leqslant j\leqslant n$. Let $U(n)$ be the unitary group embedded in $M_n(\mathbb{C})$. Let $\sigma:=u^{-1}du$ be the Maurer–Cartan form. Let $\theta_k(u)=\mathrm{tr}(\sigma^{2k-1})$; $\mathrm{tr}(\sigma^{2k})=0$. Let

$$\Theta_I:=\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}\quad\text{for } I:=\{1\leqslant i_1<\cdots<i_p\leqslant n\}.$$

Then the Θ_I are harmonic forms on $U(n)$ which form a basis for the cohomology of the unitary group. A basis for the cohomology of the special unitary group $SU(n)$ is given by the Θ_I where $i_\nu\geqslant 2$ and a basis for the cohomology of the orthogonal group $O(n)$ is given by the Θ_I where all the i_ν are even.

Let $\mathcal{V}:=\{V_0,\dots,V_n\}$ be a finite collection of vector bundles over M and let $d:=\{d_0,\dots,d_{n-1}\}$ be a collection of first order partial differential operators where $d_p:C^\infty(V_p)\to C^\infty(V_{p+1})$. We say that $(\mathcal{V},d)$ is an *elliptic complex* if

(a) $d_p\circ d_{p-1}=0$,

(b) $\ker\sigma_L(d_p)(x,\xi)=\mathrm{range}\,\sigma_L(d_{p-1})(x,\xi)$ for $\xi\neq 0$.

We define the *associated Laplacian* $\Delta:=(d+d^*)^2$; this decomposes in the form $\Delta:=\bigoplus_p\Delta_p$, where the Δ_p are elliptic self-adjoint second order partial differential operators on $C^\infty(V_p)$. The Hodge decomposition theorem generalizes this setting to identify the cohomology groups with the harmonic sections:

$$H^p(\mathcal{V},d):=\frac{\ker(d_p)}{\mathrm{range}(d_{p-1})}=\ker(\Delta_p).$$

The cohomology groups are finite-dimensional. We define the *index* of this elliptic complex by

$$\mathrm{index}(\mathcal{V},d):=\sum_p(-1)^p\dim H^p(\mathcal{V},d)=\sum_p(-1)^p\dim\ker\Delta_p.$$

Let $a_n(\cdot)$ be the constant term in the asymptotic expansion of the heat trace. Define

$$a_m(x,d):=\sum_p(-1)^p a_m(x,\Delta_p).$$

We then have

$$\mathrm{index}(\mathcal{V}, d) = \sum_p (-1)^p \mathrm{tr}_{L^2} e^{-t\Delta_p} = \int_M a_m(x, d).$$

This gives a local formula for the index.

To illustrate this, we recover the Chern–Gauss–Bonnet formula in dimensions $m = 2$ and $m = 4$ using formulas from [12, Section 4.1]. Let Δ_p be the p form valued Laplacian. We have:

$$a_0(\Delta_p) = (4\pi)^{-m/2} \int_M \binom{m}{p}.$$

$$a_2(\Delta_p) = (4\pi)^{-m/2} \frac{1}{6} \int_M \left\{ \binom{m}{p} - 6\binom{m-2}{p-1} \right\} \tau.$$

$$\begin{aligned} a_4(\Delta_p) = (4\pi)^{-m/2} \frac{1}{360} \int_M \Bigg\{ &\left\{ 5\binom{m}{p} - 60\binom{m-2}{p-1} + 180\binom{m-4}{p-2} \right\} \tau^2 \\ &+ \left\{ -2\binom{m}{p} + 180\binom{m-2}{p-1} - 720\binom{m-4}{p-2} \right\} |\rho|^2 \\ &+ \left\{ 2\binom{m}{p} - 30\binom{m-2}{p-1} + 180\binom{m-4}{p-2} \right\} |R|^2 \Bigg\}. \end{aligned}$$

The index of the de Rham complex is the Euler characteristic. Thus

$$\chi(M^2) = \sum_p (-1)^p a_2(\Delta_p) = (4\pi)^{-m/2} \int_M \tau,$$

$$\chi(M^4) = \sum_p (-1)^p a_4(\Delta_p) = (4\pi)^{-m/2} \frac{1}{360} \int_M \left\{ 180\tau^2 - 720|\rho|^2 + 180|R|^2 \right\}.$$

We draw some consequences of the observation that there is a local formula for the index. Let $\{\mathcal{V}, d_\varepsilon\}$ be a smooth 1 parameter family of elliptic complexes. Since $a_m(x, d_\varepsilon)$ is locally computable, it is continuous in the parameter ε. Thus the index is continuous in ε. Since the index is $\mathbb{Z}$ valued, it is constant; this shows the index is a homotopy invariant. A simple parity argument shows the local invariants $a_m(x, d)$ vanish if m is odd. This shows the index is zero if m is odd so we shall restrict to even-dimensional manifolds for the most part. Suppose that $\pi : \widetilde{M} \to M$ is a covering projection with finite fiber F. Let $(\pi^*\mathcal{V}, \pi^* d)$ denote the pull-back to $\widetilde{M}$. Since a_n is locally computable, $a_m(\widetilde{x}, \widetilde{A}) = a_m(\pi x, A)$. Since integration is multiplicative under finite coverings, the index is multiplied by the cardinality of the fiber:

$$\mathrm{index}(\widetilde{\mathcal{V}}, \widetilde{d}) = |F| \cdot \mathrm{index}(\mathcal{V}, d).$$

Suppose that the elliptic complex is *natural* with respect to connected sums #; this will be the case for the de Rham, signature, and spin complexes; it will not be the case for the Dolbeault complex. Since integration is additive, we have

$$\begin{aligned}&\operatorname{index}(\mathcal{V}, d; M_1) + \operatorname{index}(\mathcal{V}, d; M_2)\\&\quad = \operatorname{index}(\mathcal{V}, d; M_1 \# M_2) + \operatorname{index}\big(\mathcal{V}, d; S^m\big).\end{aligned}$$

We shall only consider elliptic complexes of partial differential operators. If we were to consider elliptic complexes of pseudo-differential operators, there would exist exist nontrivial index problems in odd dimensions. For example, the shift operator defined on $C^\infty(S^1)$ by

$$P\big(e^{\sqrt{-1}n\theta}\big) = \begin{cases} ne^{\sqrt{-1}(n-1)\theta} & \text{if } n > 0,\\ ne^{\sqrt{-1}n\theta} & \text{if } n \leqslant 0\end{cases}$$

has index 1 on the circle; this is a pseudo-differential operator.

There are examples where axiom (b) for an elliptic complex is satisfied but axiom (a) is not satisfied. Thus

$$\ker \sigma_L(d_p)(x,\xi) = \operatorname{range} \sigma_L(d_{p-1})(x,\xi)$$

for $\xi \neq 0$ but we do not have $d^2 = 0$; this means that the complex is exact at the symbol level. For example, if M is an almost complex manifold, then M is holomorphic if and only if the Nirenberg–Neulander integrability condition $(d^{0,1})^2 = 0$ is satisfied. However, we can always "roll up" the elliptic complex to create a 2 term complex and define an index. Let

$$A : C^\infty(V^e) \to C^\infty(V^o),$$

where

$$A := d + d^*, \quad V^e := \bigoplus V^{2k}, \quad V^o := \bigoplus V^{2k+1};$$

this is an elliptic complex if axiom b) is satisfied. If axiom a) is satisfied, this new $\mathbb{Z}_2$ graded complex has the same index as the original $\mathbb{Z}$ graded complex. Thus this construction extends the index to this more general setting. We define the associated operators of Laplace type and heat invariants

$$\Delta_e := A^*A, \quad \Delta_o := AA^* \quad \text{and} \quad a_m(x,d) = a_m(x,\Delta_e) - a_m(x,\Delta_o).$$

The complex is elliptic if and only if the associated Laplacians are elliptic. Of the classical elliptic complexes, the de Rham and Dolbeault are $\mathbb{Z}$ graded; the remaining complexes are $\mathbb{Z}_2$ graded.

3. The classical elliptic complexes

We define the *de Rham complex* by taking exterior differentiation

$$d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M).$$

We have $d^2 = 0$. If $\xi \neq 0$, then $\mathrm{ext}(\xi)\omega = 0$ if and only if there exists ϕ so $\omega = \mathrm{ext}(\xi)\phi$. Thus this is an elliptic complex. The associated Laplacian is Δ_p and the associated cohomology groups are the de Rham cohomology groups. Thus the index is $\sum_p (-1)^p \dim H^p(M;\mathbb{C})$; this is the Euler–Poincaré characteristic $\chi(M)$ and is a combinatorial invariant. If M is given a simplicial structure or a cell structure with n_p simplices or cells of dimension p, then $\chi(M) = \sum_p (-1)^p n_p$. If M is odd dimensional, then $\chi(M) = 0$. Let $\mathbb{T}^n$ be the n torus. We have

$$\chi(S^{2n}) = 2, \quad \chi(\mathbb{RP}^{2n}) = 1, \quad \chi(\mathbb{T}^{2n}) = 0 \quad \text{and} \quad \chi(\mathbb{CP}^m) = m + 1.$$

Let M be oriented and even-dimensional. The *signature complex*

$$d + \delta : C^\infty(\Lambda^- M) \to C^\infty(\Lambda^+ M)$$

is defined by decomposing the exterior algebra into the ± 1 eigenvalues of the normalized orientation $c(\mathrm{orn})$; since M is even-dimensional, $c(\mathrm{orn})$ anti-commutes with $d + \delta$. The symbol of the signature complex is Clifford multiplication; the signature complex is an elliptic complex. We let $\mathrm{sign}(M)$ be the index; it is independent of the metric on M and changes sign if the orientation of M is reversed. Furthermore, $\mathrm{sign}(M) = 0$ if $m \equiv 2 \bmod 4$. The associated Laplacians are the restriction of the Bochner Laplacian. As with the de Rham complex, it is possible to give a topological interpretation. Let $m = 4k$. Only the middle dimension plays a role here. The *index form* $I(\omega_1, \omega_2) := \int_M \omega_1 \wedge \omega_2$ extends to a nonsingular bilinear form on the de Rham cohomology groups $H^{2k}(M;\mathbb{C})$ in the middle dimension. We have $\star = c(\mathrm{orn})$ on Λ^{2k}. We decompose the Laplacian $\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-$; I is positive definite on $\ker \Delta_{2k}^+$, I is negative definite on $\ker \Delta_{2k}^-$, and $\ker \Delta_{2k}^+$ is orthogonal to $\ker \Delta_{2k}^-$ with respect to the pairing defined by I. Thus $\mathrm{sign}(M)$ is the index of the form I. We have

$$\mathrm{sign}(S^{4k}) = 0, \quad \mathrm{sign}(\mathbb{T}^{4k}) = 0 \quad \text{and} \quad \mathrm{sign}(\mathbb{CP}^{2k}) = 1.$$

The *Yang–Mills* complex in dimension 4 arises from yet another decomposition of the exterior algebra. Let $\pi : \Lambda^2 M \to \Lambda^{2,-}(M)$ be orthogonal projection. The Yang–Mills complex is a 3 term elliptic complex given by:

$$d : C^\infty(\Lambda^0 M) \to C^\infty(\Lambda^1 M) \quad \text{and} \quad \pi d : C^\infty(\Lambda^1 M) \to C^\infty(\Lambda^{2,-} M).$$

Let $\mathcal{Y}(M)$ be the index of this elliptic complex; when twisted by a suitable coefficient bundle, it plays a crucial role in the study of the moduli space of anti-self dual connections in Donaldson theory. We have $\mathcal{Y}(M) = \frac{1}{2}(\chi(M) - \mathrm{sign}(M))$ so

$$\mathcal{Y}(S^4) = 1, \quad \mathcal{Y}(\mathbb{T}^4) = 0 \quad \text{and} \quad \mathcal{Y}(\mathbb{CP}^2) = 1.$$

The *Dolbeault* complex is the holomorphic analogue of the de Rham complex. Let $z := (z^1, \dots, z^m)$ be a system of local holomorphic coordinates on a holomorphic manifold M of complex dimension m where $z^j := x^j + \sqrt{-1}y^j$. In the complexifications $TM \otimes_{\mathbb{R}} \mathbb{C}$ and $\Lambda(M) \otimes_{\mathbb{R}} \mathbb{C}$ we define:

$$dz^j := dx^j + \sqrt{-1}\,dy^j, \qquad d\bar{z}^j := dx^j - \sqrt{-1}\,dy^j,$$

$$dz^I := dz^{i_1} \wedge \cdots \wedge dz^{i_p}, \qquad d\bar{z}^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

$$\Lambda^{p,q} M := \operatorname{span}_{|I|=p,|J|=q}\{dz^I \wedge d\bar{z}^J\},$$

$$\partial_j^z := \frac{1}{2}\left(\partial_j^x - \sqrt{-1}\partial_j^y\right), \qquad \partial_j^{\bar{z}} := \frac{1}{2}\left(\partial_j^x + \sqrt{-1}\partial_j^y\right),$$

$$d^{1,0} \sum_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^J := \sum_{j,I,J} \partial_j^z(f_{I,J})\, dz^j \wedge dz^I \wedge d\bar{z}^J,$$

$$d^{0,1} \sum_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^J := \sum_{j,I,J} \partial_j^{\bar{z}}(f_{I,J})\, d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J.$$

A complex function f is holomorphic if and only if $d^{0,1}f = 0$. Since $d = d^{1,0} + d^{0,1}$, we see $d^{0,1}d^{0,1} = 0$. Note that $d^{0,1}$ is often denoted by $\bar{\partial}$. The operators given above are invariantly defined. If ξ is a cotangent vector, decompose $\xi = \xi^{1,0} + \xi^{0,1}$. The leading symbol of $d^{0,1}$ is exterior multiplication by $\xi^{0,1}$ so the Dolbeault complex

$$d^{0,1} : C^\infty\left(\Lambda^{0,q}\right) \to C^\infty\left(\Lambda^{0,q+1}\right)$$

is an elliptic complex. The index of the Dolbeault complex is called the *arithmetic genus* of M and will be denoted by $\operatorname{Ag}(M)$. We have

$$\operatorname{Ag}\left(\mathbb{T}^{2n}\right) = 0 \quad \text{and} \quad \operatorname{Ag}\left(\mathbb{CP}^n\right) = 1.$$

If J is an almost complex structure on M, we can mimic this construction; J arises from a complex structure on M if and only if $(d^{0,1})^2 = 0$. To define an index problem in this setting, we "roll up" the complex. If $\delta^{0,1}$ is the adjoint of $d^{0,1}$, we take

$$\left(d^{0,1} + \delta^{0,1}\right) : C^\infty\left(\Lambda^{0,e}\right) \to C^\infty\left(\Lambda^{0,o}\right).$$

The sphere S^2 admits a complex structure. The sphere S^6 admits an almost complex structure; it is not known if S^6 admits a complex structure. No other sphere admits an almost complex structure.

The *spin complex* is defined for even-dimensional spin manifolds. Let $\mathcal{S}(M)$ be the spin bundle. Clifford multiplication defines a natural action of the cotangent bundle on the spin bundle $\mathcal{S}(M)$. We decompose $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$ into the *chiral spin bundles* where $c(\mathrm{orn}) = \pm 1$ on $\mathcal{S}^{\pm}(M)$. The Levi-Civita connection ∇ induces a natural connection called the spinor connection on $\mathcal{S}(M)$;

$$c \circ \nabla : C^{\infty}(\mathcal{S}^{+}(M)) \to C^{\infty}(\mathcal{S}^{-}(M))$$

defines the *spin complex*. The index of this elliptic complex is called the $\widehat{A}$-genus $\widehat{A}(M)$. The index of this elliptic complex vanishes if $m \equiv 2 \bmod 4$. Let K^4 be the Kummer surface; this is the set of points in $\mathbb{CP}^3$ satisfying the homogeneous equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$. We have

$$\widehat{A}(S^{4k}) = 0, \quad \widehat{A}(\mathbb{T}^{4k}) = 0 \quad \text{and} \quad \widehat{A}((K^4)^k) = 2^k.$$

There is a close relationship between the $\widehat{A}$ genus and the scalar curvature. The $\widehat{A}$ genus is a $\mathbb{Z}$ valued invariant which is defined if $m \equiv 0 \bmod 4$. It is possible to define a $\mathbb{Z}_2$ valued index if $m \equiv 1, 2 \bmod 8$. The formula of Lichnerowicz [18] shows that if M admits a metric of positive scalar curvature, then there are no harmonic spinors; consequently $\widehat{A}(M) = 0$. Stolz [20] has proven a partial converse: if M is a simply connected spin manifold of dimension $m \geqslant 5$ with $\widehat{A}(M) = 0$, then M admits a metric of positive scalar curvature. There are extensions of this result in the nonsimply connected setting, see [6] for details. Here the eta invariant plays a crucial role in giving the relevant characteristic numbers.

These elliptic complexes are multiplicative. Let $M := M_1 \times M_2$. When dealing with the signature complex, we assume the M_i are oriented; when dealing with the spin complex, we assume the M_i are spin; when dealing with the Dolbeault complex, we assume the M_i are holomorphic. We then have

$$\chi(M) = \chi(M_1)\chi(M_2), \quad \mathrm{sign}(M) = \mathrm{sign}(M_1)\mathrm{sign}(M_2),$$

$$\mathrm{Ag}(M) = \mathrm{Ag}(M_1)\mathrm{Ag}(M_2) \quad \text{and} \quad \widehat{A}(M) = \widehat{A}(M_1)\widehat{A}(M_2).$$

These elliptic complexes behave well with respect to finite coverings. Suppose that $\pi : \widetilde{M} \to M$ is a finite covering with finite fiber F. If M has an appropriate structure, there is a similar structure induced on $\widetilde{M}$. We have

$$\chi(\widetilde{M}) = \chi(M)|F|, \quad \mathrm{sign}(\widetilde{M}) = \mathrm{sign}(M)|F|,$$

$$\mathrm{Ag}(\widetilde{M}) = \mathrm{Ag}(M)|F|, \quad \text{and} \quad \widehat{A}(\widetilde{M}) = \widehat{A}(M)|F|.$$

Let # denote connected sum. The connected sum of two oriented manifolds is oriented and the connected sum of two spin manifolds is spin. However, the connected sum of two complex manifolds need not be complex. We have

$$\chi(M\#N) = \chi(M) + \chi(N) - 2,$$

$$\mathrm{sign}(M\#N) = \mathrm{sign}(M) + \mathrm{sign}(N) \quad \text{and} \quad \widehat{A}(M\#N) = \widehat{A}(M) + \widehat{A}(N).$$

The de Rham and Dolbeault complexes have nontrivial indexes in any even dimensions; the signature and spin complexes have nontrivial indexes only if $m \equiv 0 \bmod 4$. To get a nontrivial index if $m \equiv 2 \bmod 4$, we can twist these complexes by taking coefficients in an auxiliary bundle V. We assume V is equipped with a positive definite fiber metric and an auxiliary Riemannian connection ∇. We use ∇ and the Levi-Civita connection to covariantly differentiate tensors of all types. We define the following elliptic complexes with coefficients in V:

de Rham: $(c \otimes 1_V) \circ \nabla : C^\infty(\Lambda^e M \otimes V) \to C^\infty(\Lambda^o M \otimes V)$;
signature: $(c \otimes 1_V) \circ \nabla : C^\infty(\Lambda^+ M \otimes V) \to C^\infty(\Lambda^- M \otimes V)$;
Yang–Mills: $(c \otimes 1_V) \circ \nabla : C^\infty((\Lambda^0 M \oplus \Lambda^{2,-} M) \otimes V) \to C^\infty(\Lambda^1 M \otimes V)$;
spin: $(c \otimes 1_V) \circ \nabla : C^\infty(\mathcal{S}^+ M \otimes V) \to C^\infty(\mathcal{S}^- M \otimes V)$.

Let $\vec{s} = (s_1, \dots, s_r)$ be a local holomorphic frame for a holomorphic vector bundle V. The *twisted Dolbeault complex* with coefficients in V is defined by

$$d_V^{0,1} : C^\infty(\Lambda^{0,q} \otimes V) \to C^\infty(\Lambda^{0,q+1} \otimes V),$$

where

$$d_V^{0,1} \sum_{J,\nu} f_{J,\nu}\, d\bar{z}^J \otimes s_\nu := \sum_{j,J,\nu} \partial_j^{\bar{z}} f_{J,\nu}\, d\bar{z}^j \wedge d\bar{z}^J \otimes s_\nu.$$

Let $\chi(M, V)$, $\operatorname{sign}(M, V)$, $\mathcal{Y}(M, V)$, $\widehat{A}(M, V)$, and $\operatorname{Ag}(M, V)$ be the index of these elliptic complexes;

$$\chi(M, V) = \dim(V)\chi(M) \quad \text{and} \quad \mathcal{Y}(M, V) = \frac{1}{2}\bigl(\chi(M, V) - \operatorname{sign}(M, V)\bigr).$$

We note that it is necessary to "roll up" the Yang–Mills complex when twisting with a coefficient bundle; the following sequence is a complex if and only if the connection ∇ on the coefficient bundle V is anti-self dual:

$$d_\nabla : C^\infty(V) \to C^\infty(\Lambda^1 M \otimes V),$$

$$\pi \circ d_\nabla : C^\infty(\Lambda^1 \otimes V) \to C^\infty(\Lambda^{2,-} M \otimes V).$$

If M is spin, then we can write the de Rham and signature complexes in terms of the twisted spin complex. If M is holomorphic, then M is spin if and only if we can take a square root L of the canonical bundle $\Lambda^{0,m}$. Let $m = 2n$. We have

$$\chi(M, V) = (-1)^n \dim(V)\bigl\{\widehat{A}(M, \mathcal{S}^+) - \widehat{A}(M, \mathcal{S}^-)\bigr\},$$

$$\operatorname{sign}(M, V) = \widehat{A}(M, \mathcal{S} \otimes V), \qquad \operatorname{Ag}(M, V) = \widehat{A}(M, L \otimes V).$$

Let s_c be a spin^c structure on an even-dimensional manifold M. Let $\mathcal{S}_c$ be the associated spinor bundle and let $L = L(s_c)$ be the associated complex line bundle. Then M admits a spin structure if and only there is a square root of the line bundle L; if this is possible, then $\mathcal{S}_c = \mathcal{S} \otimes \sqrt{L}$. We define the *twisted* spin^c *complex* with coefficients in V using the diagram

$$(c \otimes 1) \circ \nabla : C^\infty(\mathcal{S}^+ \otimes V) \to C^\infty(\mathcal{S}^- \otimes V).$$

Let $\widehat{A}_c(M, V)$ be the index of this elliptic complex. The spin^c complex plays a crucial role in Seiberg–Witten theory if $m = 4$.

If M is a complex manifold, there is a canonical spin^c structure on M and we may identify $\Lambda^{0,e} = \mathcal{S}^+$ and $\Lambda^{0,o} = \mathcal{S}^-$. Under these isomorphisms, the operators of the Dolbeault complex and of the spin^c complex agree if the metric is Kähler. Although they do not agree in general, they have the same leading symbol and hence the same index;

$$\mathrm{Ag}(M, V) = \widehat{A}_c(M, V).$$

We put all these elliptic complexes in a common framework as follows. Let M be an even-dimensional oriented manifold. Let c be a linear map from the cotangent bundle T^*M to the bundle of endomorphisms of a complex vector bundle E so that $c(\xi)^2 = -|\xi|^2$. We extend c to the Clifford algebra bundle generated by T^*M to define the endomorphism $c(\mathrm{orn})$ of E. We choose a unitary connection ∇ on E so that $\nabla c = 0$; such connections always exist. We decompose $E = E^+ \oplus E^-$ into the ± 1 eigenbundles of $c(\mathrm{orn})$ and define

$$d^\pm := c \circ \nabla : C^\infty(E^\pm) \to C^\infty(E^\mp).$$

The $d^\pm$ are elliptic first order operators with $(d^\pm)^* = d^\mp$. The associated second order operators $\Delta^\pm := d^\mp d^\pm$ have scalar leading symbol given by the metric tensor and are said to be of *Laplace type*. We consider the elliptic complex

$$d^+ : C^\infty(E^+) \to C^\infty(E^-);$$

this is an elliptic complex which is said to be of *Dirac type*. It is immediate that the signature, spin, and spin^c complexes are of Dirac type.

Let M be a complex manifold. Let $\xi^{1,0}$ and $\xi^{0,1}$ be the projections of a real cotangent vector ξ to $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$. We define

$$c(\xi) := \sqrt{2}\{\mathrm{ext}(\xi^{0,1}) - \mathrm{int}(\xi^{1,0})\}; \qquad c(\xi)^2 = -|\xi|^2 \cdot I.$$

Modulo a suitable normalizing constant, c is leading symbol of the Dolbeault operator $d^{0,1} + \delta^{0,1}$. The $\mathbb{Z}_2$ grading of $\Lambda^{0,*}$ given by $c(\mathrm{orn})$ is the standard decomposition $\Lambda^{0,e} \oplus \Lambda^{0,o}$. If the metric on M is Kähler, then we have $d^{0,1} + \delta^{0,1} = c \circ \nabla$ where ∇ is the Levi-Civita connection. For general metrics, this operator differs from the operator of the Dolbeault complex by a 0-th order term.

There is a 4 fold decomposition of $\Lambda(M)$ into forms of even and odd degrees as well as into ± 1 chirality. We define two elliptic complexes of Dirac type:

$$\mathcal{E}_1 := (d + \delta) : C^\infty(\Lambda^{e,+}M) \to C^\infty(\Lambda^{o,-}M),$$

$$\mathcal{E}_2 := (d+\delta) : C^\infty(\Lambda^{o,+}M) \to C^\infty(\Lambda^{e,-}M).$$

We can twist with a coefficient bundle. The signature complex is given by the formal sum $\mathcal{E}_1 + \mathcal{E}_2$ while the de Rham complex is given by the formal difference $\mathcal{E}_1 - \mathcal{E}_2$. The Yang–Mills complex is $-\mathcal{E}_2$. Thus we have

$$\chi(M, V) = \text{index}(\mathcal{E}_1) - \text{index}(\mathcal{E}_2),$$

$$\mathcal{Y}(M, V) = -\text{index}(\mathcal{E}_2) \quad \text{and} \quad \text{sign}(M, V) = \text{index}(\mathcal{E}_1) + \text{index}(\mathcal{E}_2).$$

This shows that the de Rham and Yang–Mills complexes are also of Dirac type. Suppose that M is spin. If $m \equiv 0 \bmod 4$, then $\mathcal{E}_1$ is the spin complex with coefficients in $\mathcal{S}^+$ and $\mathcal{E}_2$ is the spin complex with coefficients in $\mathcal{S}^-$; if $m \equiv 2 \bmod 4$, then $\mathcal{E}_1$ is the spin complex with coefficients in $\mathcal{S}^-$ and $\mathcal{E}_2$ is the spin complex with coefficients in $\mathcal{S}^+$.

4. Characteristic classes of vector bundles

The Stiefel–Whitney classes take values in $H^*(M; \mathbb{Z}_2)$; the Chern and Pontrjagin classes take values in $H^*(M; \mathbb{Z})$. We can complexify to define Chern and Pontrjagin classes taking values in $H^*(M; \mathbb{C})$ and to regard them as elements of de Rham cohomology. These classes can be computed in terms of curvature. We refer to Eguchi et al. [10], Hirzebruch [13], and Husemoller [16] for further details concerning the material of this section.

A connection ∇ on a real or complex vector bundle V is a generalization of the notion of a directional derivative. It is a first order partial differential operator

$$\nabla : C^\infty(V) \to C^\infty(T^*M \otimes V)$$

which satisfies the Leibnitz rule $\nabla(fs) = df \otimes s + f\nabla s$. There is a natural extension

$$\nabla : C^\infty(\Lambda^p M \otimes V) \to C^\infty(\Lambda^{p+1} M \otimes V)$$

defined by setting

$$\nabla(\omega_p \otimes s) = d\omega_p \otimes s + (-1)^p \omega_p \wedge \nabla s.$$

In contrast to ordinary exterior differentiation, ∇^2 need not vanish. However,

$$\nabla^2(fs) = ddf \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s$$

so ∇^2 is a 0-th order partial differential operator called the curvature Ω. Let (s_i) be a local frame. We sum over repeated indices to expand $\nabla s_i = \omega_i^j \otimes s_j$. Then

$$\nabla^2 s_i = \left(d\omega_i^j - \omega_i^k \wedge \omega_k^j\right) \otimes s_k \quad \text{and} \quad \Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j.$$

If $\tilde{s} = g_i^j s_j$ is another local frame, we compute $\tilde{\omega} = dgg^{-1} + g\omega g^{-1}$. We say that ∇ is a *Riemannian connection* if we have

$$(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2).$$

We restrict to such a connection henceforth. Relative to a local orthonormal frame, the curvature is skew-symmetric. We can always embed V in a trivial bundle of dimension ν; let π_V be the orthogonal projection on V. We project the flat connection to V to define a natural connection on V. For example, if M is embedded isometrically in Euclidean space $\mathbb{R}^\nu$, this construction gives the Levi-Civita connection on the tangent bundle TM. We summarize:

$$\Omega = d\omega - \omega^2 = \pi_V \, d\pi_V \, d\pi_V, \quad \Omega + \Omega^* = 0 \quad \text{and} \quad \tilde{\Omega} = g\Omega g^{-1}.$$

Let $P(A)$ be a homogeneous polynomial of order n defined on the set of $r \times r$ complex matrices $M_r(\mathbb{C})$ which is invariant, i.e. $P(gAg^{-1}) = P(A)$ for all g in $\mathrm{Gl}(r, \mathbb{C})$ and for all A in $M_r(\mathbb{C})$. We define $P(\Omega) \in C^\infty(\Lambda^{2n} M)$ by substitution; this is invariantly defined and independent of the particular local frame field chosen. We polarize P to define a multilinear invariant symmetric function $P(A_1, \ldots, A_n)$ so that $P(A, \ldots, A) = P(A)$. Then $dP(\Omega) = nP(d\Omega, \Omega, \ldots, \Omega)$ is invariantly defined. Fix $x_0 \in M$. We can always choose a local frame field so $\omega(x_0) = 0$ and thus $d\Omega(x_0) = 0$. This shows that $dP(\Omega)(x_0) = 0$ and hence $P(\Omega)$ is a closed differential form. Let $[P(\Omega)]$ denote the corresponding representative in de Rham cohomology. Let $\nabla(\varepsilon) := \varepsilon\nabla_1 + (1 - \varepsilon)\nabla_0$ be an affine homotopy between two connections on V. Let $\theta := \omega_1 - \omega_0$. Then $\tilde{\theta} = g\theta g^{-1}$ so θ transforms like a tensor. Since θ is a 1 form valued endomorphism of V and Ω is a 2 form valued endomorphism, $P(\theta, \Omega(\varepsilon), \ldots, \Omega(\varepsilon))$ is an invariantly defined $2j - 1$ form. One computes that

$$P(\Omega_1) - P(\Omega_0) = nd\left\{ \int_0^1 P\big(\theta, \Omega(\varepsilon), \ldots, \Omega(\varepsilon)\big)\, d\varepsilon \right\}$$

so $[P(\Omega_0)] = [P(\Omega_1)]$ in $H^{2n}(M; \mathbb{C})$; we denote this common value by $P(V)$.

Complexification gives a natural map $H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{Z})$. We can complexify the integral Chern classes to define the complex Chern classes and compute them in terms of de Rham cohomology using curvature. The total *Chern form* of a Riemannian connection ∇ on a complex vector bundle of complex dimension r is given by

$$c(\Omega) := \det\left(I + \frac{\sqrt{-1}}{2\pi} \Omega \right) = 1 + c_1(\Omega) + c_2(\Omega) + \cdots + c_r(\Omega).$$

The complex Chern classes are $\mathbb{C}$ characteristic classes of a complex vector bundle V. They are characterized by the properties:

(a) If $\dim(V) = r$, then $c(V) = 1 + c_1(V) + \cdots + c_r(V)$ for $c_i \in H^i(M; \mathbb{C})$.

(b) If $f : M_1 \to M_2$, then $f^*(c(V)) = c(f^*V)$.

(c) We have $c(V \oplus W) = c(V)c(W)$.

(d) If L is the classifying line bundle over S^2, then $\int_{S^2} c_1(L) = -1$.

The first three properties are immediate from the definition; we check the final property as an example. Let $\mathbb{CP}^1 = S^3/S^1$ be the set of complex lines through the origin in $\mathbb{C}^2$. Let $\langle\xi\rangle \in \mathbb{CP}^1$ be the line determined by the point $0 \neq \xi \in \mathbb{C}^2$. The classifying line bundle L over $\mathbb{CP}^1$ is given by

$$L = \{(\langle\xi\rangle, \lambda) \in \mathbb{CP}^1 \times \mathbb{C}^2\colon \lambda \in \langle\xi\rangle\};$$

L is a sub-bundle of the trivial 2 plane bundle. Let $s(z) := (\langle z, 1\rangle, (z, 1))$ be the canonical section to L over $\mathbb{C} \subset \mathbb{CP}^1$; s is a meromorphic section to L with a simple pole at ∞. We compute:

$$\nabla(s) = dz \otimes \pi_L\{(1,0)\} = (1+|z|^2)^{-1}\bar{z}\,dz \otimes s,$$

$$\nabla^2(s) = (1+|z|^2)^{-2}\,d\bar{z} \wedge dz \otimes s,$$

$$c_1(\Omega) = -\tfrac{1}{\pi}(1+x^2+y^2)^{-2}\,dx \wedge dy,$$

$$\int_{\mathbb{CP}^1} c_1(L) = \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} -r(1+r^2)^{-2}\,d\theta\,dr = -1.$$

Let L be a holomorphic line bundle over a Riemann surface M. Choose a meromorphic section s to L. Let n_s and p_s be the number of zeros and poles of s. The calculation performed for the classifying line bundle over $\mathbb{CP}^1$ can be used to show that:

$$\int_M c_1(L) = n_s - p_s.$$

The total *Chern character* is defined by the formal sum

$$\mathrm{ch}(\Omega) := \mathrm{tr}(e^{\sqrt{-1}\Omega/2\pi}) = \sum_\nu \frac{(\sqrt{-1})^\nu}{(2\pi)^\nu \nu!}\mathrm{tr}(\Omega^\nu) = \mathrm{ch}_0 + \mathrm{ch}_1 + \cdots;$$

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W) \quad \text{and} \quad \mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W).$$

The Chern class lifts from $H^*(M;\mathbb{C})$ to $H^*(M;\mathbb{Z})$; it is an integral class. The Chern character lifts from $H^*(M;\mathbb{C})$ to $H^*(M;\mathbb{Q})$; it is a rational class but not an integral class. Let $KU(M)$ be the K theory group of M; we refer to Karoubi [17] for further details concerning K theory. The Chern character extends to a ring isomorphism from $KU(M)\otimes\mathbb{Q}$ to $H^e(M;\mathbb{Q})$ which is a natural equivalence of functors; modulo torsion, K theory and cohomology are the same functors.

Let q be a linear map from $\mathbb{R}^{2j+1}$ to the set of $\nu \times \nu$ complex self-adjoint matrices so that $q(x)^2 = |x|^2$; for example, if $j = 1$, we could let

$$q_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad q_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

define $q(x) = x^0 q_0 + x^1 q_1 + x^2 q_2$. Let V be the eigenbundles of q over the sphere S^{2j};

$$V_\pm := \{(x, \xi) \in S^{2j} \times \mathbb{C}^\nu \colon q(x)\xi = \pm\xi\}.$$

We wish to compute $\int_{S^{2j}} \operatorname{ch}_j(V_\pm)$. Let

$$\pi_\pm := \frac{1}{2}\big(1 \pm q(x)\big)$$

be orthogonal projection on the bundles $V_\pm$. We project the flat connection on $S^{2j} \times \mathbb{C}^\nu$ to define connections $\nabla_\pm$ on the bundles $V_\pm$; this is analogous to the construction of the Levi-Civita connection on a hypersurface by projection of the Euclidean connection. Fix a point $P \in S^{2j}$ and let $\vec{s}_\pm(P)$ be a basis for $V_\pm$. Extend this basis to a local frame by defining $\vec{s}_\pm(x) := \pi_\pm(x)\vec{s}_\pm(P)$. We compute the curvatures:

$$\nabla_\pm(x)\vec{s}_\pm(x) = \pi_\pm(x)\, d\pi_\pm(x)\vec{s}_\pm(P),$$

$$\Omega_\pm(x)\vec{s}_\pm(x) = \pi_\pm(x)\, d\pi_\pm(x)\, d\pi_\pm(x)\vec{s}_\pm(P),$$

so

$$\Omega_\pm = \pi_\pm d\pi_\pm d\pi_\pm.$$

Choose oriented orthonormal coordinates for $\mathbb{R}^{2j+1}$ so that the point P in question is the north pole. Expand

$$q(x) = x^0 q_0 + \cdots + x^{2j} q_{2j}; \quad q_i q_j + q_j q_i = 2\delta_{ij}.$$

Note that $\widehat{q} := \sqrt{-1}q$ extends to the Clifford algebra; modulo a suitable normalizing the evaluation of $\widehat{q}$ on the normalized orientation form is given by $q_0 \cdots q_{2j}$; thus this product is invariantly defined and does not depend upon the choice of P nor upon the orthonormal coordinate system chosen for $\mathbb{R}^{2j+1}$; it does depend, of course on the orientation. Thus in particular

$$\mathcal{T} := \operatorname{tr}(q_0 \cdots q_{2j})$$

is invariantly defined. We compute at P that:

$$dvol = dx^1 \wedge \cdots \wedge dx^{2j},$$

$$\pi_+(P) = \frac{1}{2}(1 + q_0),$$

$$d\pi_+(P) = \frac{1}{2} dx^i q_i,$$

$$\Omega_+(P) = \frac{1}{8}(dx^i \wedge dx^j)(1+q_0)q_i q_j,$$

$$\Omega_+(P)^{2j} = 2^{-2j-1}(2j)!(dx^1 \wedge \cdots \wedge dx^{2j})(1+q_0)q_1 \cdots q_{2j},$$

$$\operatorname{tr}\{\Omega_+(P)^{2j}\} = 2^{-2j-1}(2j)!\,dvol\operatorname{tr}\{(1+q_0)q_1 \cdots q_{2j}\},$$

$$\operatorname{ch}_j(\Omega_+)(P) = \frac{(\sqrt{-1})^j(2j)!}{(2\pi)^j j! 2^{2j+1}} \mathcal{T}\, dvol.$$

Since P was arbitrary, this identity holds in general. Since the volume of S^{2j} is $j!\pi^j 2^{2j+1}/(2j)!$, we conclude

$$\int_{S^{2j}} \operatorname{ch}_j(V_+) = 2^{-j}(\sqrt{-1})^j \operatorname{tr}(q_0 \cdots q_{2j}).$$

If $q = \sqrt{-1}c$ is defined by the spin representation, $\nu = 2^j$,

$$2^{-j}(\sqrt{-1})^j \operatorname{tr}(q_0 \cdots q_{2j}) = \pm 1 \quad \text{and} \quad \int_{S^{2j}} \operatorname{ch}_j(V_+) = \pm 1.$$

The corresponding element $[V_+]$ in K theory is called the Bott element; it and the trivial line bundle generate the K theory of S^{2j}, i.e.

$$KU(S^{2j}) = [1] \cdot \mathbb{Z} \oplus [V_+] \cdot \mathbb{Z}.$$

The Chern character is defined by the exponential function. There are other characteristic classes which appear in the index theorem which are defined using other generating functions. Let $\vec{x} := (x_1, \ldots)$ be a collection of indeterminates. Let $s_\nu(\vec{x})$ be the ν-th elementary symmetric function;

$$\prod_\nu (1 + x_\nu) = 1 + s_1(\vec{x}) + s_2(\vec{x}) + \cdots.$$

Let $f(\vec{x})$ be a symmetric polynomial or more generally a formal power series which is symmetric. We can express $f(\vec{x}) = F(s_1(\vec{x}), \ldots)$ in terms of the elementary symmetric functions. For a diagonal matrix $A := \operatorname{diag}(\lambda_1, \ldots)$, let $x_j := \sqrt{-1}\lambda_j/2\pi$ be the normalized eigenvalues. Then

$$c(A) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}A\right) = 1 + s_1(\vec{x}) + \cdots.$$

We define $f(\Omega) = F(c_1(\Omega), \dots)$ by substitution. For example, if $f(\vec{x}) := \sum_\nu e^{x_\nu}$, then $f(\Omega) = \mathrm{ch}(\Omega)$ is the Chern character. The *Todd class* is defined using a different generating function:

$$\mathrm{td}(\vec{x}) := \prod_\nu x_\nu \left(1 - e^{-x_\nu}\right)^{-1} = 1 + \mathrm{td}_1(\vec{x}) + \cdots .$$

If V is a real vector bundle with a Riemannian connection ∇, the total *Pontrjagin form* is defined by

$$p(\Omega) := \det\left(1 + \frac{1}{2}\pi/\Omega\right) = 1 + p_1(\Omega) + p_2(\Omega) + \cdots,$$

where the $p_i(\Omega)$ are closed differential forms of degree $4i$; since $\Omega + \Omega^t = 0$, the forms of degree $4i + 2$ vanish. Let $p_i(V) = [p_i(\Omega)]$ denote the corresponding elements of de Rham cohomology; these are independent of the particular Riemannian connection which is chosen. Let $[\cdot]$ be the greatest integer function. The Pontrjagin classes are characterized by the properties:

(a) If $\dim(V) = r$, then $p(V) = 1 + p_1(V) + \cdots + p_{[r/2]}(V)$ for $p_i \in H^{4i}(M; \mathbb{C})$;
(b) If $f : M_1 \to M_2$, then $f^*(p(V)) = p(f^*V)$;
(c) We have $p(V \oplus W) = p(V)p(W)$;
(d) We have $\int_{\mathbb{CP}^2} p_1(T\mathbb{CP}^2) = 3$.

The Pontrjagin classes can be lifted to $\mathbb{Z}$ integral classes by defining

$$p_i(V) := (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C});$$

the formula in (c) only holds modulo elements of order 2 over $\mathbb{Z}$. Let x_2 generate $H^2(\mathbb{CP}^n; \mathbb{Z}) \subset H^2(\mathbb{CP}^n; \mathbb{C})$. The formula in (d) follows from the observations

$$p\left(T\mathbb{CP}^n\right) = \left(1 + x_2^2\right)^{n+1} \quad \text{and} \quad \int_{\mathbb{CP}^n} x_2^n = 1.$$

We can define some additional characteristic classes using formal power series. Let $\{\pm\sqrt{-1}\lambda_1, \dots\}$ be the nonzero eigenvalues of a skew-symmetric matrix A. We set $x_j = -\lambda_j/2\pi$ and define the *Hirzebruch polynomial* L and the $\widehat{A}$ *genus* by:

$$L(\vec{x}) := \prod_\nu \frac{x_\nu}{\tanh(x_\nu)} = 1 + L_1(\vec{x}) + L_2(\vec{x}) + \cdots,$$

$$\widehat{A}(\vec{x}) := \prod_\nu \frac{x_\nu}{2\sinh\left(\frac{1}{2}x_\nu\right)} = 1 + \widehat{A}_1(\vec{x}) + \widehat{A}_2(\vec{x}) + \cdots .$$

The generating functions $x/\tanh(x)$ and $\frac{1}{2}x/\sinh(\frac{1}{2}x)$ are even functions of x so the ambiguity in the choice of sign plays no role. This defines characteristic classes

$$L_i(V) \in H^{4i}(M; \mathbb{C}) \quad \text{and} \quad \widehat{A}_i(V) \in H^{4i}(M; \mathbb{C}).$$

We summarize below some useful properties of these classes:

$$p_j(V) = (-1)^j c_{2j}(V \otimes_{\mathbb{R}} \mathbb{C}),$$

$$c_1(\Omega) = \frac{\sqrt{-1}}{2\pi}\operatorname{tr}(\Omega), \qquad c_2(\Omega) = \frac{1}{8\pi^2}\{\operatorname{tr}(\Omega^2) - \operatorname{tr}(\Omega)^2\},$$

$$p_1(\Omega) = -\frac{1}{8\pi^2}\operatorname{tr}(\Omega^2),$$

$$\operatorname{ch}(V) = \dim(V) + c_1(V) + \frac{1}{2}(c_1^2 - 2c_2)(V) + \cdots,$$

$$\operatorname{td}(V) = 1 + \frac{1}{2}c_1(V) + \frac{1}{12}(c_1^2 + c_2)(V) + \frac{1}{24}(c_1c_2)(V) + \cdots,$$

$$\int_{\mathbb{CP}^k} \operatorname{td}(\Lambda^{0,1}\mathbb{CP}^k) = 1,$$

$$\widehat{A}(V) = 1 - \frac{1}{24}p_1(V) + \frac{1}{5760}(7p_1^2 - 4p_2)(V) + \cdots, \qquad \int_K \widehat{A}_4(TK^4) = 2,$$

$$L(V) = 1 + \frac{1}{3}p_1(V) + \frac{1}{45}(7p_2 - p_1^2)(V) + \cdots, \qquad \int_{\mathbb{CP}^{2k}} L_k(T\mathbb{CP}^{2k}) = 1,$$

$$\operatorname{td}(V \oplus W) = \operatorname{td}(V)\operatorname{td}(W), \quad \widehat{A}(V \oplus W) = \widehat{A}(V)\widehat{A}(W) \quad \text{and}$$

$$L(V \oplus W) = L(V)L(W).$$

There is one final characteristic class which will play an important role in our analysis. While a real anti-symmetric matrix A of shape $2n \times 2n$ cannot be diagonalized, it can be put in block diagonal form with 2×2 off diagonal elements

$$\begin{pmatrix} 0 & \lambda_\nu \\ -\lambda_\nu & 0 \end{pmatrix}.$$

The top Pontrjagin class $p_n(A) = x_1^2 \cdots x_n^2$ is a perfect square. The *Euler class* $e_{2n}(A) := x_1 \cdots x_n$ is the square root of p_n. If V is an oriented vector bundle of dimension $2n$, then $e_{2n}(V) \in H^{2n}(M; \mathbb{C})$ is a well defined characteristic class satisfying $e_{2n}(V)^2 = p_n(V)$. If V is the underlying real oriented vector bundle of a complex vector bundle W, then $e_{2n}(V) = c_n(W)$. If M is an even-dimensional manifold, let $e_m(M) := e_m(TM)$. If we reverse the local orientation of M, then $e_m(M)$ changes sign. Consequently $e_m(M)$ is a measure rather than an m form; we use the Riemannian measure on M to regard $e_m(M)$ as a scalar. Let R_{ijkl} be the components of the curvature of the Levi-Civita connection with respect to some local orthonormal frame field; we adopt the convention that $R_{1221} = 1$ on the standard sphere S^2 in $\mathbb{R}^3$. If $\varepsilon^{I,J} := (e^I, e^J)$ is the totally antisymmetric tensor, then

$$E_{2n} := \sum_{I,J} \varepsilon^{I,J} \frac{1}{(8\pi)^n n!} R_{i_1 i_2 j_2 j_1} \cdots R_{i_{m-1} i_m j_m j_{m-1}}.$$

Let $\mathcal{R} := R_{ijji}$ and $\rho_{ij} := R_{ikkj}$ be the *scalar curvature* and the *Ricci tensor*. Then

$$E_2 = \frac{1}{4\pi}\mathcal{R} \quad \text{and} \quad E_4 = \frac{1}{32\pi^2}\big(\mathcal{R}^2 - 4|\rho|^2 + |R|^2\big);$$

these are the integrands of the Chern–Gauss–Bonnet theorem discussed in Section 2.

5. Characteristic classes of principal bundles

Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G. Let $\pi : P \to M$ be a principal G bundle over M. For $p \in P$, let

$$\mathcal{V}_p := \ker \pi_* : T_p P \to T_{\pi p} M \quad \text{and} \quad \mathcal{H}_p := \mathcal{V}_p^{\perp}$$

be the vertical and horizontal distributions of the projection π. We assume the metric on P is chosen to be G invariant and so that $\pi_* : \mathcal{H}_p \to T_p M$ is an isometry; thus π is a *Riemannian submersion*. If F is a tangent vector field on M, let $\mathcal{H}F$ be the corresponding vertical lift. Let $\rho_{\mathcal{V}}$ be orthogonal projection on the distribution $\mathcal{V}$. The curvature is defined by:

$$\Omega(F_1, F_2) = \rho_{\mathcal{V}}\big[\mathcal{H}(F_1), \mathcal{H}(F_2)\big];$$

the horizontal distribution $\mathcal{H}$ is integrable if and only if the curvature vanishes. Since the metric is G invariant, $\Omega(F_1, F_2)$ is invariant under the group action. We may use a local section s to P over a contractable coordinate chart $\mathcal{O}$ to split $\pi^{-1}\mathcal{O} = \mathcal{O} \times G$. This permits us to identify $\mathcal{V}$ with TG and to regard Ω as a $\mathfrak{g}$ valued 2 form. If we replace the section s by a section $\tilde{s}$, then $\tilde{\Omega} = g\Omega g^{-1}$ changes by the adjoint action of G on $\mathfrak{g}$. If V is a real or complex vector bundle over M, we can put a fiber metric on V to reduce the structure group to the orthogonal group $O(r)$ in the real setting or the unitary group $U(r)$ in the complex setting. Let P_V be the associated frame bundle. A Riemannian connection ∇ on V induces an invariant splitting of $TP_V = \mathcal{V} \oplus \mathcal{H}$ and defines a natural metric on P_V; the curvature Ω of the connection ∇ defined in Section 4 agrees with the definition given above in terms of principal bundles in this setting.

Let $\mathcal{Q}(G)$ be the algebra of all polynomials on $\mathfrak{g}$ which are invariant under the adjoint action. If $Q \in \mathcal{Q}(G)$, then $Q(\Omega)$ is well defined. It is not difficult to show that $dQ(\Omega) = 0$ and that the de Rham cohomology class $Q(P) := [Q(\Omega)]$ is independent of the particular connection chosen. Let BG be the associated classifying space. For example

$$BU(1) = \mathbb{CP}^\infty, \quad B\mathbb{T}^r = \big(\mathbb{CP}^\infty\big)^r, \quad BO(1) = \mathbb{RP}^\infty \quad \text{and} \quad BSU(2) = \mathbb{HP}^\infty.$$

Let C be a coefficient group. Let $f_P : M \to BG$ be the classifying map for a principal G bundle P over M. Let θ be a cohomology class in $H^\nu(BG; C)$. Since f_P is well defined up to homotopy, we use pullback to define

$$\theta(P) := f_P^*(\theta) \in H^\nu(M; C).$$

If $\nu = m$, we can evaluate θ on the fundamental class $[M]$ to define a *characteristic number* $\theta(P)[M] \in C$. The map $Q \to [Q(\Omega)]$ defines an isomorphism from $\mathcal{Q}$ to $H^*(BG; \mathbb{C})$ which is called the *Chern–Weil* isomorphism. For example, we have:

$$H^*(BU(r); \mathbb{C}) = \mathcal{Q}(U(r)) = \mathbb{C}[c_1, \ldots, c_r], \quad \text{where } \deg(c_i) = 2i,$$

$$H^*(BSU(r); \mathbb{C}) = \mathcal{Q}(SU(r)) = \mathbb{C}[c_2, \ldots, c_r],$$

$$H^*(B\mathbb{T}^r; \mathbb{C}) = \mathcal{Q}(\mathbb{T}^r) = \mathbb{C}[x_1, \ldots, x_r], \quad \text{where } \deg(x_i) = 1,$$

$$H^*(BO(r); \mathbb{C}) = \mathcal{Q}(O(r)) = \mathbb{C}[p_1, \ldots, p_{[r/2]}], \quad \text{where } \deg(p_i) = 4i,$$

$$H^*(BSO(2s); \mathbb{C}) = \mathcal{Q}(SO(2s)) = \mathbb{C}[p_1, \ldots, p_s] \oplus \mathbb{C}[p_1, \ldots, p_s]e_{2s},$$

$$H^*(BSO(2s+1); \mathbb{C}) = \mathcal{Q}(SO(2s+1)) = \mathbb{C}[p_1, \ldots, p_s].$$

The natural inclusion of the torus in the unitary group $U(r)$ induces a pull-back morphism from $\mathcal{Q}(U(r))$ to $\mathcal{Q}(\mathbb{T}^r)$; the pull back of the Chern class c_j is the j-th elementary symmetric function in the x_ν variables. Similarly, the pull-back of the Pontrjagin class p_j under the natural inclusion of the torus in the special orthogonal group is the j-th elementary symmetric function in the x_ν^2 variables; the Euler class pulls back to the polynomial $x_1 \cdots x_r$.

The natural inclusions $\mathbb{R}^m \to \mathbb{R}^{m+1}$ and $\mathbb{C}^m \to \mathbb{C}^{m+1}$ induce natural inclusions and dual homomorphisms

$$O(r) \to O(r+1), \qquad \mathcal{Q}(O(r+1)) \to \mathcal{Q}(O(r)),$$

$$SO(r) \to SO(r+1), \qquad \mathcal{Q}(SO(r+1)) \to \mathcal{Q}(SO(r)),$$

$$U(r) \to U(r+1), \qquad \mathcal{Q}(U(r+1)) \to \mathcal{Q}(U(r)).$$

The Chern and Pontrjagin classes are stable characteristic classes. This means that

$$p_j(V \oplus 1) = p_j(V) \quad \text{and} \quad c_j(V \oplus 1) = c_j(V);$$

they are preserved by the restrictions maps defined above. In contrast, the Euler class is an unstable characteristic class; the Euler class cannot be extended from $\mathcal{Q}(SO(2r))$ to $\mathcal{Q}(SO(2r+1))$.

There are other coefficient groups one can use. We have

$$H^*(BO(r); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_r] \quad \text{and} \quad H^*(BU(r); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_r].$$

The Chern classes lift from $\mathbb{C}$ to $\mathbb{Z}$ in a natural fashion. Let $L_c = L \otimes \mathbb{C}$ be the complexification of the classifying line bundle over $\mathbb{RP}^m$ for $m \geqslant 2$. Then $H^2(\mathbb{RP}^m; \mathbb{C}) = 0$ so c_1 in de Rham cohomology yields no information. However $c_1(L_c) \neq 0$ in $H^2(\mathbb{RP}^m; \mathbb{Z}_2) = \mathbb{Z}_2$; information concerning torsion is lost in passing from the integral Chern classes to de Rham cohomology in this instance. Since the index is a $\mathbb{Z}$ valued invariant, we can work with de Rham cohomology in computing the index of an elliptic complex.

Bordism and characteristic classes are intimately related. We give a brief introduction to the subject and refer to Stong [21] for further details. Let $MO(m)$ be the set of all m-dimensional compact manifolds modulo the bordism relationship that $[M_1] = [M_2]$ if there exists a compact manifold N with boundary the disjoint union of M_1 and M_2. The Stiefel–Whitney numbers are the characteristic numbers of $MO(m)$. This means that $[M_1] = [M_2]$ in $MO(m)$ if and only if $\omega(M_1) = \omega(M_2)$ for all $\omega \in H^m(BO(m); \mathbb{Z}_2)$. For example, $\mathbb{RP}^n \times \mathbb{RP}^n$ and $\mathbb{CP}^n$ have the same Stiefel–Whitney numbers so they are bordant; there exists a compact manifold W so that the boundary of W is the disjoint union of $\mathbb{RP}^n \times \mathbb{RP}^n$ and $\mathbb{CP}^n$. We refer to Conner and Floyd [8] for details; see also Stong [22].

Let $MSO(m)$ be the set of all m-dimensional compact oriented manifolds modulo the bordism relationship $[M_1] = [M_2]$ if there exists a compact oriented manifold N with oriented boundary the disjoint union of M_1 and $-M_2$. The Stiefel–Whitney numbers and Pontrjagin numbers are the characteristic numbers of $MSO(m)$. This means that $[M_1] = [M_2]$ in $MSO(m)$ if and only if $\omega(M_1) = \omega(M_2)$ in $\mathbb{Z}_2$ for all $\omega \in H^m(BO(m); \mathbb{Z}_2)$ and $\sigma(M_1) = \sigma(M_2)$ in $\mathbb{Z}$ for all $\sigma \in H^m(BO(m); \mathbb{Z})$; the Euler class is an unstable characteristic class and plays no role in this theory. If m is even, the stable tangent bundle is $TM \oplus 1^2$; if m is odd, the stable tangent bundle is $TM \oplus 1$. We say that M admits a stable almost complex structure if the stable tangent bundle of M admits an almost complex structure. Let $MU(m)$ be the set of all m-dimensional compact manifolds with stable almost complex structures modulo a suitable bordism relationship. The Chern numbers are the characteristic numbers of $MU(m)$; we have $[M_1] = [M_2]$ in $MU(m)$ if and only if $\omega(M_1) = \omega(M_2)$ in $\mathbb{Z}$ for all $\omega \in H^m(BU(m); \mathbb{Z})$. Thus in particular, $MU(m) = 0$ if m is odd. One can also define spin bordism; the characteristic numbers arise from real K theory as well as from cohomology.

6. The index theorem

The Atiyah–Singer index theorem [3,4] expresses the index of any elliptic complex in terms of characteristic classes. We first discuss this formula for the classical elliptic complexes. We then give the general formulation.

The index of the twisted de Rham complex is the Euler–Poincarè characteristic $\chi(M, V)$. Since $\chi(M, V) = \dim(V)\chi(M)$, no new information is added by twisting the de Rham complex with a coefficient bundle. If the Atiyah–Singer index theorem is applied to this setting, one gets the *Chern–Gauss–Bonnet theorem* [7]. Let E_m be the Euler class defined in Section 4. Then

$$\chi(M) = \int_M E_m |dvol|.$$

The index of the twisted signature complex is the L genus $L(M, V)$. Let L_k be the Hirzebruch polynomial. If the Atiyah–Singer index theorem is applied to this setting, one gets the *Hirzebruch signature formula*

$$\operatorname{sign}(M, V) = \int_M \sum_{2j+4k=m} 2^j \operatorname{ch}_j(V) \wedge L_k(TM).$$

Let M be an orientable manifold of even dimension $m = 2n$. The Chern character gives an isomorphism between $KU(M) \otimes \mathbb{C}$ and $H^e(M; \mathbb{C})$. Thus there exists V so that $\operatorname{ch}_n(V) \neq 0$. The Hirzebruch signature formula shows that $\operatorname{sign}(M, k \cdot V) \neq 0$ for k sufficiently large. If $m \equiv 2 \bmod 4$, then $\operatorname{sign}(M) = 0$; however, the twisted index will be nonzero for suitably chosen V; there always exists a nontrivial index problem over M.

The index of the twisted spin complex with coefficients in an auxiliary bundle V is the $\widehat{A}$ genus $\widehat{A}(M, V)$. If the Atiyah–Singer index theorem is applied to this setting, one gets the formula

$$\widehat{A}(M, V) = \int_M \sum_{2j+4k=m} \operatorname{ch}_j(V) \wedge \widehat{A}_k(TM).$$

The index of the twisted Yang–Mills complex in dimension $m = 4$ with coefficients in an auxiliary bundle V is $\mathcal{Y}(M, V)$. If the Atiyah–Singer index theorem is applied to this setting, one gets the formula

$$\mathcal{Y}(M, V) = \int_M \left\{ \frac{\dim(V)}{2}(E_4 - L_1) + \left(2c_2 - c_1^2\right)(V) \right\}.$$

A spin^c structure on a manifold M defines an auxiliary complex line bundle L. If the Atiyah–Singer index theorem is applied to this setting, one gets the formula

$$\widehat{A}_c(M, V) = \int_M \sum_{2j+4k+2\ell=m} 2^{-\ell} \operatorname{ch}_j(V) \wedge \widehat{A}_k(TM) \wedge \operatorname{ch}_\ell(L).$$

The index of the twisted Dolbeault complex is the arithmetic genus $\operatorname{Ag}(M, V)$. If the Atiyah–Singer index theorem is applied to this setting, one gets the *Riemann–Roch formula*

$$\operatorname{Ag}(M, V) = \int_M \sum_{2j+2k=m} \operatorname{ch}_j(V) \wedge \operatorname{td}_k\left(\Lambda^{0,1} M\right).$$

We give a single example to illustrate the use of the index formula to prove nonexistence results; there are many such examples. When the signature formula, the Chern–Gauss–Bonnet formula, and the Riemann–Roch formula are combined for an almost complex manifold of real dimension 4, one gets the formula

$$\mathrm{Ag}(M^4) = \frac{1}{4}\{\chi(M^4) + \mathrm{sign}(M^4)\}.$$

If we take $M = S^4$, then $\frac{1}{4}\{\chi(S^4) + \mathrm{sign}(S^4)\} = \frac{1}{4}(2+0)$ is not an integer; thus S^4 does not admit an almost complex structure. More generally, let M_i be complex surfaces. We show that $M_1\#M_2$ does not admit an almost complex structure by computing:

$$\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - 2,$$

$$\mathrm{sign}(N) = \mathrm{sign}(M_1) + \mathrm{sign}(M_2),$$

so

$$\mathrm{Ag}(N) = \mathrm{Ag}(M_1) + \mathrm{Ag}(M_2) - \frac{2}{4}.$$

All the formulas described so far can be put into a common framework. Let c be a map from the cotangent bundle of M to the bundle of endomorphisms of a complex vector bundle V so that $c(\xi)^2 = -|\xi|^2 I_V$. We choose a compatible connection ∇ on V; this means that ∇ is Riemannian and that $\nabla c = 0$, such connections always exist. Let $c \circ \nabla$ be the associated operator of Dirac type; the elliptic complex is then said to be a *compatible elliptic complex of Dirac type*. The chiral splitting of $V = V^+ \oplus V^-$ into the ± 1 eigenvalues of $c(\mathrm{orn})$ defines an elliptic complex of Dirac type. The Chern character of the spin bundle $\mathrm{ch}(\mathcal{S})$ is a well defined characteristic class even if M is not spin; the ambiguity in defining the spin bundles is a flat $\mathbb{Z}_2$ ambiguity which does not affect the characteristic polynomials in the curvature tensor. Since $\mathrm{ch}_0(\mathcal{S}) \neq 0$, this characteristic class is invertible. The index of this elliptic complex of Dirac type is given by

$$\sum_{4j+2k+2\ell=m} \int_M \widehat{A}_j(TM) \wedge \mathrm{ch}^{-1}(\mathcal{S})_k \wedge \mathrm{ch}_\ell(V).$$

Note that the particular Clifford module structure is not important in this formulation as only the Chern character of V enters. Thus when considering an elliptic complex of Dirac type, it is only necessary to identify the underlying vector bundle. When this formula is applied to the de Rham, twisted signature, twisted spin, and twisted Yang–Mills complexes, the formulas given above result. When considering the twisted Dolbeault complex, the resulting operator has the same leading symbol and thus the index is unchanged.

We now discuss the *index theorem* of Atiyah and Singer in complete generality. Let $A: C^\infty(V_0) \to C^\infty(V_1)$ be an elliptic complex. In this framework, we permit A to be a pseudo-differential operator. Let a be the leading symbol of A; for $0 \neq \xi \in T_x^*M$, $a(x, \xi)$

is an isomorphism from the fiber of V_0 over x to the fiber of V_1 over x. Let $S(M)$ be the unit sphere bundle of T^*M and let $D^{\pm}(M)$ be two copies of the unit disk bundle of T^*M. We use the symbol a as a clutching function to glue V_0 over $S^+(M)$ to V_1 over $S^-(M)$; this defines a vector bundle $\Sigma(V_0, V_1, a)$ over

$$\Sigma(M) := D^+(M) \cup_{S(M)} D^-(M)$$

which encodes all the relevant information. Give $\Sigma(M)$ a suitable orientation. The *Atiyah–Singer index formula* then becomes:

$$\operatorname{index}(V_0, V_1, A) = \int_{\Sigma(M)} \sum_{2k+2l=2m} \operatorname{td}_k(TM \otimes \mathbb{C}) \wedge \operatorname{ch}_l\big(\Sigma(V_0, V_1, a)\big).$$

If V_i are trivial bundles, the index can be expressed in terms of *secondary characteristic classes*. In this case a is matrix valued and we define the pull-back via a of the normalized Maurer–Cartan form

$$\Theta_{2\ell-1} := \frac{(\sqrt{-1})^{\ell}(\ell-1)}{(2\pi)^{\ell}(2\ell-1)!}\operatorname{tr}\big\{\big(a^{-1}da\big)^{2\ell-1}\big\}.$$

When a suitable orientation of the sphere bundle $S(M)$ is chosen, we have

$$\operatorname{index}(V_0, V_1, A) = \int_{S(M)} \sum_{4k+2\ell=2m} \operatorname{td}_k(TM \otimes \mathbb{C}) \wedge \Theta_{2\ell-1}.$$

The original proof of the Atiyah–Singer index theorem [4] was topological in nature and used bordism. Since then, a number of other proofs have been given. We are somewhat partial to the heat equation proof, see [12] for details. We sketch this proof as follows; it uses the local formula $\operatorname{index}(\mathcal{V}, d) = \int_M a_m(x, d)$ for the index described in Section 2. If $\{\mathcal{V}, d\}$ is a compatible elliptic complex of Dirac type, one can use invariance theory to show that $a_m(x, d)$ is given in terms of characteristic classes; the method of universal examples then shows that $a_m(x, d)$ is given by the characteristic form described above. This proves the Atiyah–Singer index theorem for elliptic complexes of Dirac type; a simple K theory argument then derives it in general.

It is possible to state an *equivariant index theorem*. We shall restrict to the classical elliptic complexes in the interests of simplicity. Let $\Psi : M \to M$ be a smooth map. When considering the de Rham complex, we make no additional assumptions. When considering the signature complex, we assume Ψ is an orientation preserving isometry. When considering the spin complex, we assume Ψ is an orientation preserving isometry which also preserves the spin structure. When considering the Dolbeault complex, we assume Ψ is holomorphic. Then Ψ induces an action on the appropriate cohomology groups $H^*(M; \mathcal{V})$ and we define the Lefschetz number

$$\mathcal{L}_{\mathcal{V}}(\Psi) := \sum_p (-1)^p \operatorname{tr}\big(\Psi \text{ on } H^p(M; \mathcal{V})\big).$$

To simplify the discussion, we shall assume Ψ has isolated fixed points and that $\det(I - d\Psi(x_\nu)) \neq 0$ at any fixed point x_ν.

If Ψ is an orientation preserving isometry, let $0 < \theta_{j,\nu} < 2\pi$ be the rotation angles of $d\Psi(x_\nu)$. If Ψ is holomorphic, let $\lambda_{j,\nu}$ be the complex eigenvalues of the complex Jacobian $d_c\Psi$. Define:

$$\mathcal{L}_{\text{de Rham}}(T) = \sum_\nu \operatorname{sign}\det\big(I - dT(x_\nu)\big),$$

$$\mathcal{L}_{\text{sign}}(T) = \sum_\nu \prod_j \big\{-\sqrt{-1}\cot(\theta_{j,x_\nu}/2)\big\},$$

$$\mathcal{L}_{\text{spin}}(T) = \sum_\nu \prod_j \left\{-\frac{1}{2}\sqrt{-1}\csc(\theta_{j,\nu}/2)\right\},$$

$$\mathcal{L}_{\text{Dol}}(T) = \sum_q (-1)^q \operatorname{tr}\big(T^* \text{ on } H^{0,q}(M;\mathbb{C})\big) = \sum_\nu \prod_j \big(1 - \bar{\lambda}_j(x_\nu)\big)^{-1}.$$

The generalized Lefschetz fixed point formula then becomes

$$\mathcal{L}_{\mathcal{V}} = \sum_{\{x:\ \Psi(x)=x\}} \mathcal{L}\big(d\Psi(x)\big).$$

It is worthwhile considering a few examples; we work with the de Rham complex for simplicity.

(a) Let $T(z) = \sqrt{-1}z$ mapping $S^2 = \mathbb{CP}^1$ to itself. This has two isolated fixed points at 0 and ∞; the rotation angles are $\pi/2$ at 0 and $-\pi/2$ at ∞. Since T is an orientation preserving isometry, it acts trivially on $\ker\Delta_0 = 1 \cdot \mathbb{C}$ and $\ker\Delta_2 = \text{orn} \cdot \mathbb{C}$ and T^* is the identity on de Rham cohomology so the Lefschetz number is 2. We have $\operatorname{sign}\det(I - dT)(x) = 1$ at $x = 0$ and $x = \infty$ and the fixed point formula yields $2 = 1 + 1$.

(b) We can also consider T as a map from the square 2 torus $\mathbb{T}^2$ to itself. We have T^* acts as the identity on $H^0(\mathbb{T}^2;\mathbb{C}) = \mathbb{C}$ and $H^2(\mathbb{T}^2;\mathbb{C}) = \mathbb{C}$ and as a rotation through an angle of $\pi/2$ on $H^1(\mathbb{T}^2;\mathbb{C}) = \mathbb{C}\oplus\mathbb{C}$. Thus the Lefschetz number is $1 - 0 + 1 = 2$. The map T has two fixed points $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$ and the contribution at each fixed point is 1. This example shows that the equivariant index can be nonzero even if the Euler characteristic is zero.

(c) Let $T(x) = -x$ map $\mathbb{T}^3$ to itself. Then T acts as the identity on H^0 and H^2 and as minus the identity on H^1 and H^3. Thus the Lefschetz number is $1 - (-3) + 3 - (-1) = 8$. This map has 8 fixed points at $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ where $\varepsilon_i = 0, \frac{1}{2}$ and the contribution of $\operatorname{sign}\det(I - dT)$ is $+1$ at each fixed point. This example shows that the equivariant index can be nonzero even in odd dimensions.

(d) Let G be a compact connected nontrivial Lie group. Let g be an element of G distinct from the identity element. Let $T(x) = gx$ be left multiplication by g. Then T has no fixed points so the Lefschetz number is 0. On the other hand, the Lefschetz number relative to the de Rham complex is a homotopy invariant. Let $\gamma(t)$ be a path in G from the identity to g. Then the Lefschetz number of T is the Lefschetz number of the identity map. Thus $\chi(G)=0$.

Let M be a compact smooth Riemannian manifold with smooth boundary dM. We assume $\dim(M)=m$. We assume for the moment that dM is a totally geodesic submanifold of M. Then the Chern–Gauss–Bonnet formula continues to hold;

$$\chi(M)=\int_M E_m,$$

where E_m is the Euler form discussed previously. The Hirzebruch signature formula does not extend to this setting. There is an additional correction term:

$$\operatorname{sign}(M)=\int_M L_k+\eta(dM).$$

The Novikov additivity for the signature shows $\eta(dM)$ depends only on the boundary dM. It is a global invariant, it is not locally computable. Atiyah, Patodi, and Singer [2] showed it was in fact a spectral invariant. We describe this invariant as follows. Let P be a self-adjoint elliptic first order partial differential operator. Let $\{\phi_\nu,\lambda_\nu\}$ be the spectral resolution of P. We define

$$\eta(s,P):=\frac{1}{2}\Big\{\dim\ker(P)+\sum_{\lambda_\nu\neq 0}\operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s}\Big\}.$$

This series converges absolutely for the real part of s very positive; it has a meromorphic extension to $\mathbb{C}$ which is regular at $s=0$. We define

$$\eta(P):=\eta(s,P)|_{s=0}$$

as a measure of the spectral asymmetry of P. Let $A: C^\infty(V_0)\to C^\infty(V_1)$ be a compatible elliptic complex of Dirac type. Near dM, we can use the leading symbol of A applied to the normal covector to identify V_0 with V_1 and express $A=\partial_n+P$. The de Rham complex admits local boundary conditions (absolute or relative). However there is a topological obstruction to the existence of local boundary conditions in general; the signature, spin, Yang–Mills, and Dolbeault complexes do not admit local boundary conditions. However, for an arbitrary elliptic complex of Dirac type, there exist *spectral boundary conditions*; these are pseudo-differential boundary conditions defined by the vanishing of the projection in L^2 on space spanned by the eigensections of P with nonnegative eigenvalues for $f\in C^\infty(V_0)$ and positive eigenvalues for $f\in C^\infty(V_1)$; there is a slight bit of fuss dealing with the zero mod spectrum. With these boundary conditions, the index theorem for manifolds with boundary becomes:

$$\operatorname{index}(A)=\sum_{4j+2k+2\ell=m}\int_M \widehat{A}_j(M)\wedge \operatorname{ch}^{-1}(\mathcal{S})_k\wedge \operatorname{ch}_\ell(V)-\eta(P).$$

There is also an equivariant index theorem for manifolds with boundary; we refer to Donnelly [9] for details.

In Section 5, we noted that the characteristic numbers completely detected the bordism groups. Thus, for example, a compact orientable manifold M^m without boundary is the boundary of a compact orientable manifold N^{m+1} if and only if $\omega(M) = 0$ in $\mathbb{Z}_2$ for all $\omega \in H^m(BO(m); \mathbb{Z}_2)$ and $\omega(M) = 0$ in $\mathbb{Z}$ for all $\omega \in H^m(BO(m); \mathbb{Z})$; in the first instance, ω is a homogeneous polynomial in the Stiefel–Whitney classes and in the second instance, ω is a homogeneous polynomial in the Pontrjagin classes. There are, however, relations among the characteristic classes given by the index theorem. For example, the Hirzebruch signature theorem shows that $p_1(M)$ is divisible by 3 if $m = 4$. Other integrality results for MSO can be obtained by twisting the signature complex with coefficients in bundles determined by a representation of $SO(4)$. Similarly, if M is a complex surface, we apply the index theorem to see $\mathrm{Ag}(M) = (c_2 + c_1^2)[M]/12$ and thus $(c_2 + c_1^2)(M)$ is divisible by 12 if M is a complex surface. The *Hattori–Stong theorem* [14,23] shows that all such universal integrality relations in MSO or MU are the result of the index theorem; there is a similar result for spin bordism that is more difficult to state.

Let BG be the classifying space for a spherical space form group; for example, we could take G to be a finite cyclic group. The eta invariant defines $\mathbb{Q}/\mathbb{Z}$ valued characteristic numbers of the reduced equivariant bordism groups $\widetilde{MU}_m(BG)$ which completely detect these groups. Thus the eta invariant can be thought of as a secondary index; it is sometimes expressible as a secondary characteristic class. We refer to [11] for details.

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