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The eta Invariant and $\tilde{K}O$ of Lens Spaces

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0. Introduction

Let $Z_n = \{\lambda \in C: \lambda^n = 1\}$ be the cyclic group of order n and let $\tau: Z_n \to O(v)$ be a fixed point free representation of Z_n in the orthogonal group. τ lets Z_n act without fixed points on the unit sphere S^{v-1} and the quotient manifold M of dimension $m = v-1$ is called a lens space. If m is even, then either $n=1$ and $M = S^m$ is the sphere or $n=2$ and $M = RP^m$ is real projective space. As S^m and RP^m are well understood, we restrict henceforth to the case m odd and v even. Since v is even, τ is conjugate to a representation $\tau: Z_n \to U(k) \subseteq O(2k=v)$. (See Wolf [13].) Let $K(M)$ and $KO(M)$ denote the complex and real K-theory rings, and let $R(Z_n)$ and $RO(Z_n)$ denote the corresponding representation rings. Let $\rho_s(\lambda) = \lambda^s$, then $\{\rho_s\}_{0 \leqq s < n}$ parametrize the irreducible complex representations of Z_n so $R(Z_n) = Z[\rho_1]/(\rho_1^n - 1)$. The structure of $RO(Z_n)$ is a bit more complicated and will be discussed in Sect. 1. Let $c: RO(Z_n) \to R(Z_n)$ and $c: KO(M) \to K(M)$ denote complexification; this is a ring morphism. Let $r: R(Z_n) \to RO(Z_n)$ and $r: K(M) \to KO(M)$ denote the operation of forgetting the complex structure; this is an additive morphism but not a ring morphism.

If ρ is a representation of Z_n, let V_ρ denote the bundle over M corresponding to ρ. The map $\rho \to V_\rho$ defines maps $\theta_c: R(Z_n) \to K(M)$ and $\theta_r: RO(Z_n) \to KO(M)$ which are ring morphisms. Atiyah [2] proved the map $R(Z_n) \to K(M)$ is surjective. We refer to Gilkey-Karoubi [9] for

Theorem 0.1. *Let $M = S^{2k-1}/\tau(Z_n)$ be a lens space of dimension $m = 2k-1$.*

(a) *If n is even, $\theta_c: R(Z_n) \to K(M) \to 0$ and $\theta_r: RO(Z_n) \to KO(M) \to 0$.*

(b) *If n is odd, there are split short exact sequences*

$$R(Z_n) \xrightarrow{\theta_c} K(M) \to \tilde{K}(S^m) \to 0 \quad \text{and} \quad RO(Z_n) \xrightarrow{\theta_r} KO(M) \to \tilde{K}O(S^m) \to 0$$

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Remark. Since $\tilde{K}(S^m)=0$, we recover Atiyah's theorem. Furthermore:

$$\tilde{K}O(S^m)=\begin{Bmatrix} Z_2 \text{ if } m\equiv 1(8) \\ 0 \text{ if } m \text{ is odd otherwise}\end{Bmatrix}$$

so that θ_r has cokernel Z_2 precisely when $m\equiv 1(8)$ and n is odd. Let $K_{\text{flat}}(M)$ and $KO_{\text{flat}}(M)$ denote the image of θ_c and θ_r; these are the subrings of $K(M)$ and $KO(M)$ generated by bundles which stably admit flat structures. We let "tilde" denote the corresponding objects of virtual dimension 0 and we let $R_0(Z_n)$ and $RO_0(Z_n)$ denote the augmentation ideals of representations of virtual dimension 0. Theorem 0.1 implies:

$$K(M)=Z\oplus\tilde{K}_{\text{flat}}(M)$$
$$KO(M)=Z\oplus\tilde{K}O_{\text{flat}}(M)\oplus\begin{Bmatrix} Z_2 \text{ if } n \text{ is odd and } m\equiv 1(8) \\ 0 \text{ otherwise}\end{Bmatrix}$$

Consequently to compute $K(M)$ and $KO(M)$ it suffices to calculate $\tilde{K}_{\text{flat}}(M)$ and $\tilde{K}O_{\text{flat}}(M)$.

The eta invariant of Atiyah et al. [3] is an R/Z valued spectral invariant measuring the spectral asymmetry of a self-adjoint elliptic operator. We refer to the first section for further details. Let $M=S^{2k-1}/\tau(Z_n)$ be a lens space, then M inherits a natural unitary and spin^c structure. Let P denote the tangential operator of the spin^c complex. P is a first order self-adjoint elliptic differential operator. If ρ is a representation of Z_n, the transition functions of V_ρ are locally constant so the operator P_ρ with coefficients in V_ρ is canonically defined and locally isomorphic to the direct sum of $\dim(\rho)$ copies of P. The eta invariant is additive with respect to direct sums so $\eta(P_\rho)\in R/Z$ is well defined for $\rho\in R_0(Z_n)$. We refer to Gilkey [6] for the following result.

Theorem 0.2. *Let $M=S^{2k-1}/\tau(Z_n)$ be a lens space of dimension $m=2k-1$. Then $\tilde{K}(M)=R_0(Z_n)/R_0(Z_n)^k$ and $|\tilde{K}(M)|=n^{k-1}$. $\tilde{K}(M)$ depends only on (n,m) and not the particular τ. Let P be the tangential operator of the* spin^c *complex. Define a bilinear R/Z valued form on $R_0(Z_n)\otimes R_0(Z_n)$ by $\eta(\rho_1,\rho_2)=\eta(P_{\rho_1\otimes\rho_2})$. This extends to a bilinear form:*

$$\eta\colon \tilde{K}_{\text{flat}}(M)\otimes\tilde{K}_{\text{flat}}(M)\to R/Z$$

which is non-degenerate. In other words, if $\rho\in R_0(Z_n)$ and $V=V_\rho$, then $V=0$ in $\tilde{K}(M)$ if and only if $\eta(\rho,\rho_1)=0\ \forall\ \rho_1\in R_0(Z_n)$.

Remark. Since $\tilde{K}_{\text{flat}}(M)$ is pure torsion, the eta invariant takes values in Q/Z in Theorem 0.2.

Define $\eta\colon \tilde{K}O_{\text{flat}}(M)\otimes\tilde{K}(M)\to R/Z$ by $\eta(V,W)=\eta(c(V),W)$. This extends these invariants to $\tilde{K}O_{\text{flat}}(M)$. The extent to which c fails to be injective is a matter of 2-torsion. One can refine the eta invariant just enough using the reality condition to detect $\operatorname{Ker}(c)$ and consequently to completely detect $\tilde{K}O_{\text{flat}}(M)$ using the eta invariant. The major result of this paper is the following:

Theorem 0.3. *Let $M=S^{2k-1}/\tau(Z_n)$ be a lens space.*

(a) *Let n be odd. Then $c\colon \tilde{K}O_{\text{flat}}(M)\to\tilde{K}(M)$ is injective and the invariants of Theorem 0.2 completely detect $\tilde{K}O_{\text{flat}}(M)$.*

(b) *Let n be even and let* $m=7(8)$. *Then* $c\colon KO(M)\to K(M)$ *is injective so the invariants of Theorem* 0.2 *completely detect* $\tilde{K}O_{\text{flat}}(M)$.

(c) *If n is even and if m is not congruent to* 7 mod 8, *then* $\ker(c)=Z_2$. *Let* $y=r(\rho_0-\rho_1)\in RO_0(Z_n)$. *If* $m\equiv 1{,}5(8)$, *let* $m=2(2j+1)-1$ *and let* $W(m)$ *correspond to the representation* $y^j\cdot(\rho_{n/2}-\rho_0)$. *If* $m\equiv 3(8)$, *let* $m=2(2j)-1$ *and let* $W(m)$ *correspond to the representation* y^j. *Then* $W(m)$ *is an element of order* 2 *which generates* Ker(c).

(d) *Let n be even and let P be the tangential operator of the* spin^c *complex over M.*

(i) *If* $m\equiv 1{,}5(8)$ *the map* $\rho\to\eta(P_\rho)$ *extends to a map* $\tilde{K}O(M)\to R/Z$.

(ii) *If* $m\equiv 3(8)$ *the map* $\rho\to\frac{n}{2}\cdot\eta(P_\rho)$ *extends to a map* $\tilde{K}O(M)\to R/Z$.

(e) *Let n be even. If* $m\equiv 1{,}5(8)$, *then* $\eta(P_{W(m)})=0.5$. *If* $m\equiv 3(8)$, *then* $\frac{n}{2}\cdot\eta(P_{W(m)})=0.5$. *The invariants of* (d) *together with the invariants of Theorem* 0.2 *completely detect* $\tilde{K}O(M)$.

(f) *If* τ *and* $\bar{\tau}$ *are two fixed point free representations, then* $\tilde{K}O(S^{2k-1}/\tau(Z_n))=\tilde{K}O(S^{2k-1}/\bar{\tau}(Z_n))$.

Remark. The proof of (b) will rely on a result of Yasuo [14].

The orders of these groups are given by

Theorem 0.4. *Let* $M=S^{2k-1}/\tau(Z_n)$ *be a lens space. Let* $\varepsilon=1$ *if n is odd and* $\varepsilon=2$ *if n is even. Let* $m=2k-1$. *Then:*

(a) *If* $m\equiv 1(4)$ *so* $k=2j+1$ *is odd, then* $|c(\tilde{K}O(M))|=\varepsilon^j\cdot n^j$,

(b) *If* $m=3(4)$ *so* $k=2j$ *is even, then* $|c(\tilde{K}O(M))|=\varepsilon^j\cdot n^{j-1}$,

(c) *If* $m\equiv 1{,}5(8)$ *so* $k=2j+1$ *then* $|\tilde{K}O_{\text{flat}}(M)|=\varepsilon^{j+1}\cdot n^j$,

(d) *If* $m\equiv 3(8)$ *so* $k=2j$ *then* $|\tilde{K}O_{\text{flat}}(M)|=\varepsilon^{j+1}\cdot n^{j-1}$,

(e) *If* $m\equiv 7(8)$ *so* $k=2j$ *then* $|\tilde{K}O_{\text{flat}}(M)|=\varepsilon^j\cdot n^{j-1}$.

Remark. Since $\tilde{K}(RP^{2k-1})$ and $\tilde{K}O(RP^{2k-1})$ are groups with only *one* generator, this gives the additive structure if $M=RP^{2k-1}$.

This paper is divided into three sections and an appendix. In the first section, we review the analytic facts concerning the eta invariant and prove Theorem 0.3(d). In the second section, we will complete the proof of Theorem 0.3. In the third section, we will prove Theorem 0.4. In the appendix, we give a list of some of these groups calculated on a computer using the eta invariant; further details are available from the author upon request.

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1. The eta Invariant

Let M be a compact Riemannian manifold without boundary of odd dimension $m=2k-1$ and let $P\colon C^\infty(V)\to C^\infty(V)$ be a self-adjoint first order partial differential operator. Let $\{\lambda_\nu\}_{\nu=1}^{\infty}$ denote the spectrum of P where each eigenvalue is repeated according to its multiplicity. Define:

$$\eta(z, P) = \mathrm{Tr}(P \cdot (P^2)^{-(z+1)/2}) = \sum_{\lambda_\nu \neq 0} \mathrm{sign}(\lambda_\nu) |\lambda_\nu|^{-z}$$

for $\mathrm{Re}(z) \gg 0$. $\eta(z, P)$ has a meromorphic extension to C with isolated simple poles on the real axis and $z=0$ is a regular value. Let

$$\eta(P) = \tfrac{1}{2}\{\eta(0, P) + \dim\ker(P)\} \in R/Z$$

be a measure of the spectral asymmetry of P. If $P(t)$ is a smooth 1-parameter family of such operators, then $\eta(O, P(t))$ has integer jumps as spectral values cross the origin. The reduction mod Z yields a smooth function of the parameter t. We refer to Atiyah et al. [3] or Gilkey [7] for details.

If M is a lens space and if P is the tangential operator of the spin^c complex, the eta invariant can be calculated in terms of generalized Dedekind sums. We refer to Donnelly [5] for a proof of the following lemma.

Lemma 1.1. *Let $M = S^{2k-1}/\tau(Z_n)$ be a lens space and let P be the tangential operator of the* spin^c *complex over M. Let $\rho \in R(Z_n)$.* Then

$$\eta(P_\rho) = \frac{1}{n} \cdot \sum_{g \in Z_n,\, g \neq 1} \mathrm{Tr}(\rho(g)) \cdot \det(I - \tau(g))^{-1} \bmod Z.$$

We prove Theorem 0.3(d–i) as follows. Let $M = S^{2k-1}/\tau(Z_n)$ be a lens space of dimension $m \equiv 1(4)$, let P be the tangential operator of the spin^c complex on M, and let V be a real vector bundle over M. Fix a Riemannian connection ∇ on V and let P_V denote the operator P with coefficients in V defined by the connection. The leading symbol of P is unique, but the 0^{th} order terms depend upon the connection chosen. We define $\eta(V) = \eta(P_V)$. We must show this is independent of the choice of connection; since the eta invariant is additive with respect to direct sums it extends to $KO(M)$.

Let ∇_0 and ∇_1 be two different connections and let ch denote the Chern character. Let Tch denote the transgression of the Chern character so that

$$\mathrm{ch}(\nabla_1) - \mathrm{ch}(\nabla_0) = d\,T\mathrm{ch}(\nabla_1, \nabla_0)$$

(see Chern et al. [4] or Gilkey [8]). Let V_0 and V_1 denote V with the two connections. Let L be the bundle corresponding to the representation $\det(\tau)$; this is the complex line bundle of the spin^c structure. If $\hat{A}$ is the A-proof genus, we proved Gilkey [8] that we can lift $\eta(P_{V_1}) - \eta(P_{V_2})$ from R/Z to R so that

$$\eta(P_{V_1}) - \eta(P_{V_0}) = \int_M \hat{A} \cdot \mathrm{ch}(L) \cdot T\mathrm{ch}(\nabla_1, \nabla_0).$$

Since L has a flat structure and since M is conformally flat, $\mathrm{ch}(L) \cdot \hat{A} = 1$. Since ∇_0 and ∇_1 are *real* Riemannian connections, $T\mathrm{ch}(\nabla_1, \nabla_0)$ has components only in degrees $j \equiv 3(4)$. Since $m \equiv 1(4)$, we conclude that this integrand is 0 so $\eta(P_{V_1}) = \eta(P_{V_0})$ which completes the proof of Theorem 0.3(d–i).

Suppose next that $m\equiv 3(8)$ and that $n=2v$ is even. Let $\gamma_i\in RO(Z_n)$ define isomorphic *real* vector bundles. We must show that the normalized eta invariants coincide in R/Z. We compute as above:

$$v\cdot\{\eta(P_{\gamma_1})-\eta(P_{\gamma_0})\}=v\cdot\int_M \hat{A}\cdot\mathrm{ch}(L)\cdot T\mathrm{ch}(V_{\gamma_1},V_{\gamma_0})=v\cdot\int_M T\mathrm{ch}(V_{\gamma_1},V_{\gamma_0}).$$

This is a local formula. If we lift it to the universal cover, we multiply by the order of the covering so that

$$v\cdot\{\eta(P_{\gamma_1})-\eta(P_{\gamma_2})\}=\tfrac{1}{2}\cdot\int_{S^m} T\mathrm{ch}(V_{\gamma_1},V_{\gamma_0}).$$

Since we are working in R/Z we complete the proof by showing

$$\int_{S^m} T\mathrm{ch}(V_{\gamma_1},V_{\gamma_0})\in 2\cdot Z.$$

Since the sphere is simply connected, all locally flat bundles have natural trivializations. An isomorphism between V_{γ_1} and V_{γ_0} over S^m is equivalent to a map $g\colon S^m\to GL(\dim(V),R)$; $T\mathrm{ch}$ is the pull back of a suitable expression in the Maurer-Cartan form. Let W_g be the real vector bundle over $S^{2k=m+1}$ defined by the clutching function g. Then

$$\int_{S^m} T\mathrm{ch}(\gamma_1,\gamma_0)=\int_{S^{m+1}}\mathrm{ch}(W_g\otimes C).$$

(We refer to [7] for details.) The Atiyah-Singer index theorem implies

$$\int_{S^{m+1}}\mathrm{ch}(W)\in Z\ \forall\ W\in K(S^{m+1}).$$

Since $m\equiv 3(8)$, the map $c\colon \tilde{K}O(S^{m+1})=Z\to\tilde{K}(S^{m+1})=Z$ is multiplication by two (see [10] for example). This proves

$$\int_{S^m} T\mathrm{ch}(\gamma_1,\gamma_0)\in 2Z$$

which completes the proof of Theorem 0.3(d).

2. The Real K-Theory of Spherical Space Forms

If ρ is a representation of Z_n, let ρ^* denote the dual representation; $\rho_s^*=\rho_{-s}$. We have $c(r(\rho))=\rho+\rho^*$, $r(\rho)=r(\rho^*)$, and $r(c(\rho))=2\rho$. We say that ρ is self-dual if $\rho=\rho^*$. If n is even, there are two self-dual irreducible representations $\rho_{n/2}$ and ρ_0; if n is odd, ρ_0 is the only irreducible self-dual representation. We note that $c\colon RO(Z_n)\to R(Z_n)$ is injective. We begin the proof of Theorem 0.3 with

Lemma 2.1. *Let $A=\{\rho\in R(Z_n)\colon \rho=\rho^*\}$ and let $A_0=A\cap R_0(Z_n)$. A is a subalgebra of $R(Z_n)$ and A_0 is an ideal of A.*

(a) *Let $n=2v+1$. Then additive generators for $RO(Z_n)$ and A are given by $\{\rho_0,r(\rho_1),\ldots,r(\rho_v)\}$ and $\{\rho_0,\rho_s+\rho_{-s}\}_{0<s\leq v}$* respectively.

(b) *Let* $n=2v$. *Then additive generators for* $RO(Z_n)$ *and* A *are given by* $\{\rho_0, r(\rho_1), \ldots, r(\rho_{v-1}), \rho_v\}$ *and* $\{\rho_0, \rho_v, \rho_s+\rho_{-s}\}_{0<s<v}$ *respectively.*

(c) *Let* $\delta=\sum_s \rho_s$ *be the regular representation then* $\delta\cdot R_0(Z_n)=0$. *Let* τ *be fixed point free and let* $\alpha=\sum_k(-1)^k\Lambda^k(\tau)=\det(I-\tau)\in R_0(Z_n)$. *If* $\beta\cdot\alpha=0$, *then* β *is an integer multiple of* δ. $\alpha\cdot R(Z_n)=R_0(Z_n)^k$.

(d) *Image* $(c)=A$. *Let* $y=r(\rho_0-\rho_1)$ *and let* $\alpha=c(y)=2\cdot\rho_0-\rho_1-\rho_{-1}$. *Then*

$$R_0(Z_n)^{2j}=\alpha^j\cdot R(Z_n) \quad \text{and} \quad R_0(Z_n)^{2j+1}=\alpha^j\cdot R_0(Z_n)$$
$$A\cap R_0(Z_n)^{2j}=\alpha^j\cdot A \quad \text{and} \quad A\cap R_0(Z_n)^{2j+1}=\alpha^j\cdot A_0.$$

Proof. Assertions (a) and (b) are immediate. Since $\mathrm{Tr}(\delta(\lambda))=0$ for $\lambda\neq 1$, $\mathrm{tr}(\delta\cdot\rho(\lambda))=0 \ \forall\lambda\in Z_n$, $\forall\rho\in R_0(Z_n)$. Therefore $\delta\cdot R_0(Z_n)=0$. Conversely, let $\beta\cdot\det(I-r)=0$. Since $\det(I-\tau)(\lambda)\neq 0$ for $\lambda\neq I$, we conclude $\mathrm{tr}(\beta(\lambda))=0$ for $\lambda\neq I$ and the orthogonality relations imply β is an integer multiple of δ. The remaining assertion of (c) are immediate. Image $(r)=A$ by (a) and (b). Let $\rho\in A\cap R_0(Z_n)^{2j}$ and let $\tau=j\cdot\rho_1+j\cdot\rho_{-1}$. Since τ is a fixed point free representation of Z_n, $R_0(Z_n)^{2j}=\det(I-\tau)\cdot R(Z_n)=\alpha^j\cdot R(Z_n)$. We decompose $\rho=\alpha^j\cdot\beta$. Since ρ and α are self-dual, we conclude $\alpha^j\cdot\beta^*=\rho=\rho^*=\alpha^j\cdot\beta^*$ so that $\alpha^j\cdot(\beta-\beta^*)=0$. This implies $\beta-\beta^*=c\cdot\delta$. As δ is self-dual, $\beta-\beta^*=\beta^*-\beta$ which implies β is self dual. Consequently $A\cap R_0(Z_n)^{2j}=\alpha\cdot A$. Similarly, let $\rho\in A\cap R_0(Z_n)^{2j+1}$ and decompose $\rho=\alpha^j\cdot\beta$ for $\beta\in R_0(Z_n)$. Then as β must be self-dual, $\rho\in\alpha^j\cdot A_0$ which completes the proof.

We can now prove Theorem 0.3(a). As $\tilde{K}(M)$ is a finite group of odd order, it is clear that $\tilde{K}O(S^{2k-1})\subseteq\mathrm{Ker}(c)$. We must therefore show that c is injective on $\tilde{K}O_{\mathrm{flat}}(M)$. Let $\beta\in RO_0(Z_n)$ and let $V=V_\beta\in\tilde{K}O_{\mathrm{flat}}(M)$. Suppose $c(V)=0$. We must show $V=0$. By Theorem 0.2, $c(\beta)\in R_0(Z_n)^k\cap A$. Let $y=r(\rho_0-\rho_1)$ so that $c(y)=2\cdot\rho_0-\rho_1-\rho_{-1}$. First let $m\equiv 1(4)$ so $k=2j+1$ is odd. Use Lemma 2.1 to decompose $c(\beta)=c(y^j)\cdot\gamma$ for some $\gamma\in A_0$. As γ is self dual, we may decompose

$$\gamma=\sum_{0<s\leqq v} c_s\cdot(\rho_s+\rho_{-s}-2\cdot\rho_0)=\gamma_1+\gamma_1^* \quad \text{for } \gamma_1=\sum_{0<s\leqq v}(\rho_s-\rho_0).$$

Since $c(y^j)\cdot\gamma_1\in R_0(Z_n)^k$, the bundle defined by this representation is zero. Since

$$c(\beta)-c(r(c(y^j)\cdot\gamma_1))=c(\beta)-c(y^j)\cdot\gamma_1-c(y^j)\cdot\gamma_1^*=0,$$

and since $c\colon RO(Z_n)\to R(Z_n)$ is injective, $\beta=r(c(y^j)\cdot\gamma_1)$ and $V_\beta=r(V_{c(y^j)\cdot\gamma_1})=0$. Next let $m\equiv 3(4)$ so $k=2j$ is even. Use Lemma 2.1 to decompose $c(\beta)=c(y^j)\cdot\gamma$ for some $\gamma\in A$. Decompose

$$\gamma=c_0\cdot\rho_0+\sum_{0<s<n} c_s\cdot\rho_s$$

where $c_s=c_{-s}$. As $\delta\cdot c(y)=0$, we may replace γ by $\gamma+\delta$ if necessary to assume without loss of generality that c_0 is *even*. Let $\gamma_1=(c_0/2)\cdot\rho_0+\sum_{0<s\leqq v} c_s\cdot\rho_s$ so that $\gamma=\gamma_1+\gamma_1^*$. Since $c(y^j)\cdot\gamma_1\in R_0(Z_n)^k$, the bundle defined by this representation is zero. Since

$$c(\beta)-c(r(c(y^j)\cdot\gamma_1))=c(y^j)\cdot\gamma-c(y^j)\cdot\gamma_1-c(y^j)\cdot\gamma_1^*=0$$

$\beta=r(c(y^j)\cdot\gamma_1)$ and $V_\beta=0$. This completes the proof of Theorem 0.3(a).

If $m \equiv 7(8)$, c is injective if n is a power of 2 by Yasuo [14]. More generally, decompose n into its prime power decomposition and consider the corresponding direct sum decomposition of $\tilde{K}_{\text{flat}}$. The 2-primary piece injects by Yasuo. The odd primary piece injects using the same argument as that given above to prove Theorem 0.3(a). This completes the proof of Theorem 0.3(b).

We now prove Theorem 0.3(c) and 0.3(e); we recall that 0.3(d) was proved in section one. Suppose first $m \equiv 1(4)$ so $k=2j+1$ is odd. Let $y=r(\rho_0-\rho_1)$ and let W correspond to the representation $y^j \cdot (\rho_v - \rho_0)$. Since $c(y^j \cdot (\rho_v - \rho_0)) \in R_0(Z_n)^k$, $c(W)=0$ so $W \in \text{Ker}(c)$. Let $\beta \in RO_0(Z_n)$ and let $V=V_\beta \in \tilde{K}O_{\text{flat}}(M)$. Suppose $c(V)=0$ so $c(\beta) \in R_0(Z_n)^{2j+1}$. By Lemma 2.1, decompose $c(\beta)=c(y^j)\cdot\gamma$ for $\gamma \in A_0$. We modify the argument given for n odd to take into account the one extra self-dual representation. Decompose $\gamma = \sum_{0<s\leqq v} c_s\{(\rho_s-\rho_0)+(\rho_{-s}-\rho_0)\}$ $+e\cdot(\rho_v-\rho_0)$ where $e=0,1$ reflects the parity with which $\rho_v-\rho_0$ appears in γ. Let $\gamma_1 = \sum_{0<s\leqq v} c_s(\rho_s-\rho_0)$. Since

$$c(\beta)=c(r(c(y^j)\cdot\gamma_1))+e\cdot c(y^j)\cdot(\rho_v-\rho_0)=c\{r(c(y^j)\cdot\gamma_1)+e\cdot y^j\cdot(\rho_v-\rho_0)\}$$

$\beta=r(c(y^j)\cdot\gamma_1)+e\cdot y^j\cdot(\rho_v-\rho_0)$. Since $c(y^j)\cdot\gamma_1 \in R_0(Z_n)^k$, the bundle defined by this representation is 0. We replace β by $\beta - r(c(y^j)\cdot\gamma_1)$ without changing the given bundle V. Thus without loss of generality, we may assume $\beta = e\cdot y^j\cdot(\rho_v-\rho_0)$ for $e=0,1$ so that Ker(c) is a subgroup of order at most 2 generated by W.

We use Theorem 0.3(d) to show W is non-trivial in $\tilde{K}O_{\text{flat}}(M)$. Let $\tau \cdot Z_n \to U(k)$ be a fixed point free representation. Since $\det(I-\tau)\cdot R(Z_n)$ $=R_0(Z_n)^k = c(y^j)\cdot(\rho_1-\rho_0)$, we may choose $\gamma \in R(Z_n)$ so that $\gamma\cdot\det(I-\tau)$ $=c(y^j)\cdot(\rho_1-\rho_0)$. Let U be the complex bundle defined by γ. By Lemma 1.1:

$$\begin{aligned}
\eta(P_{W\otimes U}) &= \frac{1}{n}\sum_{\lambda^n=1,\lambda\neq 1} \text{Tr}(\gamma(\lambda))\cdot\text{Tr}(c(y^j)(\lambda))\cdot\text{Tr}(\rho_v-\rho_0)(\lambda)\cdot\det(I-\tau(\lambda))^{-1}\\
&= \frac{1}{n}\sum_{\lambda^n=1,\lambda\neq 1}(\lambda^v-1)\cdot(\lambda-1)^{-1} = \frac{1}{n}\sum_{\lambda^n=1,\lambda\neq 1}(\lambda^{v-1}+\ldots+\lambda^0)\\
&= \frac{1}{n}\sum_{\lambda^n=1}(\lambda^{v-1}+\ldots+\lambda^0) - \frac{v}{n} = \frac{v}{n} = \frac{1}{2} \bmod Z.
\end{aligned}$$

Let $\gamma = c\cdot\rho_0+\gamma_0$ for $\gamma_0 \in R_0(Z_n)$ and let U_0 correspond to the representation γ_0. Then $\eta(P_{W\otimes U}) = c\cdot\eta(P_W)+\eta(P_{W\otimes U_0})$. As the map $(W,U)\to\eta(P_{W\otimes U})$ extends to a map in K-theory by Theorem 0.2 and since $W=0$ in $\tilde{K}_{\text{flat}}(M)$, we conclude $\eta(P_{W\otimes U})=0$ so that $c\cdot\eta(P_W)=0.5$. Since Ker(c) has at most 2 elements, W is an element of order at most 2 in $\tilde{K}O_{\text{flat}}(M)$. Since $W\to\eta(P_W)$ is well defined as a map in KO-theory, $\eta(P_W)$ is an element of order 2 in R/Z. This implies c is odd and $\eta(P_W)=0.5$. This shows W is non-trivial and completes the proof of Theorem 0.3(c, e) in this case.

Next we consider $m\equiv 3(8)$. Let $m=4j-1$, let $y=r(\rho_0-\rho_1)$ and let W be the bundle corresponding to the representation y^j. Since $c(y^j)\in R_0(Z_n)^k$, $c(W)=0$ so $W\in\text{Ker}(c)$. Let $\beta\in RO_0(Z_n)$ satisfy $c(V_\beta)=0$. Decompose $c(\beta)=c(y^j)\cdot\gamma$ for $\gamma\in R(Z_n)$; γ is self-dual so $\gamma=\sum_{0\leqq s<n} c_s\cdot\rho_s$ for $c_s=-c_{-s}$. By adding an appropri-

ate multiple of δ, we may assume without loss of generality that c_v is even. We may therefore decompose $\gamma=\gamma_1+\gamma_1^*+e\cdot\rho_0$ where $e=0,1$ reflects the parity of c_0. By replacing β by $\beta-r(c(y^j)\cdot\gamma_1)$ if necessary, we may assume without loss of generality that $c(\beta)=e\cdot c(y^j)$ so that $V=e\cdot W$. This shows $\operatorname{Ker}(c)$ has at most 2 elements, that $\operatorname{Ker}(c)$ is generated by W, and that W has order at most 2. Choose γ so $\gamma\cdot c(y)^v=\det(I-\tau)$. Let U be the bundle corresponding to the representation γ, then

$$\begin{aligned}\eta(P_{W\otimes U})&=\frac{1}{2}\cdot\sum_{\lambda^n=1,\lambda\neq 1}\{\operatorname{Tr}(\gamma\cdot c(y^j))(\lambda)\}\cdot\det(I-\tau(\lambda))^{-1}\\&=\frac{1}{2}\sum_{\lambda^n=1,\lambda\neq 1}1=\frac{n-1}{2}=0.5.\end{aligned}$$

The same argument as that given in the case $m\equiv 1(4)$ then shows $\eta(P_W)=0.5$ which completes the proof of Theorem 0.3(c, e).

Finally, let τ_i define lens spaces $M_i=S^m/\tau_i(Z_n)$. Let $\beta\in R(Z_n)$ define bundles V_i over M_i. Suppose $V_2=0$ so that $c(V_2)=0$. This implies $c(\beta)\in R_0(Z_n)^k$ for $2k-1=m$ and consequently $c(V_1)=0$ as well. This completes the proof if n is odd or if $m\equiv 7(8)$. In the exceptional cases, we conclude $V_i=c\cdot W$ where $c=0,1$ does not depend on i and can be determined from the representation β. Since $V_2=0$, $c=0$ which shows $V_1=0$. This complete the proof of Theorem 0.3.

3. The Orders of the Real K-Theory Groups

We shall need the following lemma in the proof of Theorem 0.4.

Lemma 3.1. *Let* $M=S^{2k-1}/\tau(Z_n)$ *be a lens space. Let* $\alpha=(\rho_0-\rho_{-1})\cdot(\rho_0-\rho_1)$ $=c(y)$ and let A and A_0 be as defined in Lemma 2.1. Set $\varepsilon=1$ if n is odd and $\varepsilon=2$ if n is even. Then $|A_0/\alpha\cdot A|=\varepsilon$ and $|\alpha\cdot A/\alpha\cdot A_0|=n$.

Proof. If n is odd, then A_0 is generated additively by

$$\begin{aligned}\{2\cdot\rho_0-\rho_s-\rho_{-s}\}&=\{(\rho_0-\rho_s)\cdot(\rho_0-\rho_{-s})\}\\&=\{(\rho_0-\rho_1)\cdot(\rho_0-\rho_{-1})\cdot(\rho_0+\ldots+\rho_{s-1})\cdot(\rho_0+\ldots+\rho_{1-s})\}\\&=\alpha\cdot A.\end{aligned}$$

Therefore $|A_0/\alpha A|=1$ if n is odd. If $n=2v$ is even, there is an additional element $\rho_0-\rho_v$ not considered if n is odd so $A_0/\alpha\cdot A$ is generated by $\rho_0-\rho_v$. Let c be the order of $\rho_0-\rho_v$ in $A_0/\alpha\cdot A$. Since $\alpha\cdot A=R_0(Z_n)^2\cap A$ by Lemma 2.1, $c\cdot(\rho_0-\rho_v)\in R_0(Z_n)^2$. Let $\tau=2\cdot\rho_1$ and let

$$\operatorname{ind}_2(\beta,\gamma)=\frac{1}{n}\sum_{\lambda^n=1,\lambda\neq 1}\operatorname{Tr}(\beta(\lambda)\cdot\gamma(\lambda))\cdot\det(I-\tau(\lambda))^{-1}.$$

By Theorem 0.2 and Lemma 1.1, $c\cdot(\rho_0-\rho_v)\in R_0(Z_n)^2$ if and only if $\operatorname{ind}_2(c\cdot(\rho_0-\rho_v),\rho_0-\rho_s)=0\in R/Z$ for $0<s<n$. We compute

$$\begin{aligned}\operatorname{ind}_2(c\cdot(\rho_0-\rho_v),\rho_0-\rho_s)&=\frac{c}{n}\sum_{\lambda_n=1,\lambda\neq 1}(1-\lambda^v)\cdot(1-\lambda^s)\cdot(1-\lambda)^{-2}\\&=\frac{c}{n}\sum_{\lambda^n=1,\lambda\neq 1}(\lambda^{v-1}+\ldots+1)\cdot(\lambda^{s-1}+\ldots+1)\\&=\frac{-vsc}{n}+\frac{1}{n}\sum_{\lambda^n=1}(\lambda^{v-1}+\ldots+1)\cdot(\lambda^{s-1}+\ldots+1)=\frac{-sc}{2}\end{aligned}$$

using the orthogonality relations. Since $sc\equiv 0(2)$ for all values of s, $c=2$ and $\rho_0-\rho_v$ is an element of order 2 in $A_0/\alpha\cdot A$. This completes the proof of the first assertion. We prove the second assertion by decomposing $A=Z\cdot\rho_0+A_0$ so that $\alpha A=\alpha\cdot Z\oplus\alpha\cdot A_0$. This shows α generates $\alpha A/\alpha\cdot A_0$. Let c be the order of α in $\alpha\cdot A/\alpha\cdot A_0$. Then $c\cdot\alpha\in\alpha\cdot A_0=R_0(Z_n)^3\cap A$. Let $\tau_3=2\cdot\rho_1+\rho_{-1}$ so that $\det(I-\tau_3)=\alpha\cdot(\rho_0-\rho_1)$. We let $\operatorname{ind}_3(c\cdot\alpha,\ \rho_0-\rho_s)$ be the corresponding Dedekind sum, then $c\cdot\alpha\in R_0(Z_n)^3$ if and only if $\operatorname{ind}_3(c\cdot\alpha,\rho_0-\rho_s)=0$ for all s. We compute:

$$\begin{aligned}\operatorname{ind}_3(c\cdot\alpha,\rho_0-\rho_s)&=\frac{c}{n}\sum_{\lambda^n=1,\lambda\neq 1}\alpha(\lambda)(1-\lambda^s)\cdot(\alpha(\lambda)\cdot(1-\lambda))^{-1}\\&=\frac{c}{n}\sum_{\lambda^n=1,\lambda\neq 1}(\lambda^{s-1}+\ldots+1)=\frac{-sc}{n}\bmod Z\end{aligned}$$

so that $c=n$ which completes the proof.

We prove Theorem 0.4(a) by induction. Let $k=2j+1$ and $M=S^{2k-1}/\tau(Z_n)$. If $j=0$, then $M=S^1$ and $\tilde{K}(M)=0$. We may therefore assume $m>1$. Consider the short exact sequence:

$$0\to\alpha^{j-1}A_0/\alpha^jA_0\to A_0/\alpha^jA_0\to A_0/\alpha^{j-1}A_0\to 0$$

so $|A_0/\alpha^jA_0|=|A_0/\alpha^{j-1}A_0|\cdot|A_0/\alpha\cdot A_0|$. Multiplication by α^{j-1} induces an isomorphism between $A_0/\alpha A_0$ and $\alpha^{j-1}\mathrm{A}_0/\alpha^j\mathrm{A}_0$. The short exact sequence

$$0\to\alpha A/\alpha A_0\to A_0/\alpha A_0\to A_0/\alpha A\to 0$$

shows $|A_0/\alpha A_0|=\varepsilon\cdot n$ by Lemma 3.1. Consequently

$$\begin{aligned}|c\tilde{K}O_{\mathrm{flat}}(M^m)|&=|A_0/(A_0\cap R_0(Z_n)^k)|=|A_0/\alpha^jA_0|\\&=|A_0/\alpha^{j-1}A_0|\cdot\varepsilon n=\varepsilon n|c\tilde{K}O_{\mathrm{flat}}(M^{m-4})|.\end{aligned}$$

This recursion relation together with the initial value for $m=1$ completes the proof of Theorem 0.4(a).

To prove Theorem 0.4(b), we let $k=2j$ so $m\equiv 3(4)$. If $m=3$, then $|\tilde{K}O_{\mathrm{flat}}(M)|=|A_0/\alpha\cdot A|=\varepsilon$ by Lemma 3.1. This completes the proof if $j=1$. For $j>1$, we use the short exact sequence

$$0\to\alpha^{j-1}A/\alpha^jA\to\{c(\tilde{K}O_{\mathrm{flat}}(M))=A_0/\alpha^jA\}\to A_0/\alpha^{j-1}A_0\to 0.$$

This shows

$$|c(\tilde{K}O(M))|=\varepsilon\cdot|A_0/\alpha^{j-1}\cdot A_0|=\varepsilon\cdot(\varepsilon\cdot n)^{j-1}$$

by Theorem 0.4(a). This completes the proof of Theorem 0.4(b). The remaining parts of Theorem 0.4 now follow from the calculation of Ker(c) in Theorem 0.3.

Appendix

It is often useful to have lists of K-theory groups to check various conjectures. Using the results of this paper and a computer program, we have computed the additive structures of the K and KO groups listed below. If $n=p\cdot q$ for p and q coprime, then $\tilde{K}(S^m/Z_n)=\tilde{K}(S^m/Z_q)\oplus\tilde{K}(S^m/Z_p)$ and $\tilde{K}O_{\text{flat}}(S^m/Z_n)=\tilde{K}O_{\text{flat}}(S^m/Z_p)\oplus\tilde{K}O_{\text{flat}}(S^m/Z_q)$ so in computing these groups, we may assume n is a prime power without loss of generality.

k	n	$\tilde{K}(S^k/Z_n)$	$\tilde{K}O_{\text{flat}}(S^k/Z_n)$
1	2	–	Z_2
3	2	Z_2	Z_4
5	2	Z_4	Z_8
7	2	Z_8	Z_8
9	2	Z_{16}	Z_{32}
11	2	Z_{32}	Z_{64}
13	2	Z_{64}	Z_{128}
15	2	Z_{128}	Z_{128}
17	2	Z_{256}	Z_{512}
19	2	Z_{512}	Z_{1024}
1	4	–	Z_2
3	4	Z_4	$Z_2\oplus Z_2$
5	4	$Z_8\oplus Z_2$	$Z_8\oplus Z_2$
7	4	$Z_{16}\oplus Z_2\oplus Z_2$	$Z_8\oplus Z_2$
9	4	$Z_{32}\oplus Z_4\oplus Z_2$	$Z_{32}\oplus Z_4$
11	4	$Z_{64}\oplus Z_4\oplus Z_4$	$Z_{32}\oplus Z_8$
13	4	$Z_{128}\oplus Z_8\oplus Z_4$	$Z_{128}\oplus Z_8$
15	4	$Z_{256}\oplus Z_8\oplus Z_8$	$Z_{128}\oplus Z_8$
17	4	$Z_{512}\oplus Z_{16}\oplus Z_8$	$Z_{512}\oplus Z_{16}$
19	4	$Z_{1024}\oplus Z_{16}\oplus Z_{16}$	$Z_{512}\oplus Z_{32}$
1	8	–	Z_2
3	8	Z_8	$Z_2\oplus Z_2$
5	8	$Z_{16}\oplus Z_4$	$Z_{16}\oplus Z_2$
7	8	$Z_{32}\oplus Z_4\oplus Z_4$	$Z_{16}\oplus Z_2$
9	8	$Z_{64}\oplus Z_8\oplus Z_4\oplus Z_2$	$Z_{64}\oplus Z_4\oplus Z_2$
11	8	$Z_{128}\oplus Z_8\oplus Z_8\oplus Z_2\oplus Z_2$	$Z_{64}\oplus Z_4\oplus Z_2\oplus Z_2$
13	8	$Z_{256}\oplus Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2$	$Z_{256}\oplus Z_8\oplus Z_2\oplus Z_2$
15	8	$Z_{512}\oplus Z_{16}\oplus Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$	$Z_{256}\oplus Z_8\oplus Z_2\oplus Z_2$
17	8	$Z_{1024}\oplus Z_{32}\oplus Z_{16}\oplus Z_4\oplus Z_2\oplus Z_2\oplus Z_2$	$Z_{1024}\oplus Z_{16}\oplus Z_4\oplus Z_2$
19	8	$Z_{2048}\oplus Z_{32}\oplus Z_{32}\oplus Z_4\oplus Z_4\oplus Z_2\oplus Z_2$	$Z_{1024}\oplus Z_{16}\oplus Z_4\oplus Z_4$
1	16	–	Z_2
3	16	Z_{16}	$Z_2\oplus Z_2$
5	16	$Z_{32}\oplus Z_8$	$Z_{32}\oplus Z_2$
7	16	$Z_{64}\oplus Z_8\oplus Z_8$	$Z_{32}\oplus Z_2$
9	16	$Z_{128}\oplus Z_{16}\oplus Z_8\oplus Z_4$	$Z_{128}\oplus Z_8\oplus Z_2$
11	16	$Z_{256}\oplus Z_{16}\oplus Z_{16}\oplus Z_4\oplus Z_4$	$Z_{128}\oplus Z_8\oplus Z_2\oplus Z_2$
13	16	$Z_{512}\oplus Z_{32}\oplus Z_{16}\oplus Z_4\oplus Z_4\oplus Z_4$	$Z_{512}\oplus Z_{16}\oplus Z_4\oplus Z_2$
15	16	$Z_{1024}\oplus Z_{32}\oplus Z_{32}\oplus Z_4\oplus Z_4\oplus Z_4\oplus Z_4$	$Z_{512}\oplus Z_{16}\oplus Z_4\oplus Z_2$
17	16	$Z_{2048}\oplus Z_{64}\oplus Z_{32}\oplus Z_8\oplus Z_4\oplus Z_4\oplus Z_4\oplus Z_2$	$Z_{2048}\oplus Z_{32}\oplus Z_4\oplus Z_4\oplus Z_2$
19	16	$Z_{4096}\oplus Z_{64}\oplus Z_{64}\oplus Z_8\oplus Z_8\oplus Z_4\oplus Z_4\oplus Z_2\oplus Z_2$	$Z_{2048}\oplus Z_{32}\oplus Z_4\oplus Z_4\oplus Z_2\oplus Z_2$

1	32	–	Z_2
3	32	Z_{32}	$Z_2 \oplus Z_2$
5	32	$Z_{64} \oplus Z_{16}$	$Z_{64} \oplus Z_2$
7	32	$Z_{128} \oplus Z_{16} \oplus Z_{16}$	$Z_{64} \oplus Z_2$
9	32	$Z_{256} \oplus Z_{32} \oplus Z_{16} \oplus Z_8$	$Z_{256} \oplus Z_{16} \oplus Z_2$
11	32	$Z_{512} \oplus Z_{32} \oplus Z_{32} \oplus Z_8 \oplus Z_8$	$Z_{256} \oplus Z_{16} \oplus Z_2 \oplus Z_2$
13	32	$Z_{1024} \oplus Z_{64} \oplus Z_{32} \oplus Z_8 \oplus Z_8 \oplus Z_8$	$Z_{1024} \oplus Z_{32} \oplus Z_8 \oplus Z_2$
15	32	$Z_{2048} \oplus Z_{64} \oplus Z_{64} \oplus Z_8 \oplus Z_8 \oplus Z_8 \oplus Z_8$	$Z_{1024} \oplus Z_{32} \oplus Z_8 \oplus Z_2$
17	32	$Z_{4096} \oplus Z_{128} \oplus Z_{64} \oplus Z_{16} \oplus Z_8 \oplus Z_8 \oplus Z_8 \oplus Z_4$	$Z_{4096} \oplus Z_{64} \oplus Z_8 \oplus Z_8 \oplus Z_2$
19	32	$Z_{8192} \oplus Z_{128} \oplus Z_{128} \oplus Z_{16} \oplus Z_{16} \oplus Z_8 \oplus Z_8 \oplus Z_4 \oplus Z_4$	$Z_{4096} \oplus Z_{64} \oplus Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2$
1	3	–	–
3	3	Z_3	–
5	3	$Z_3 \oplus Z_3$	Z_3
7	3	$Z_9 \oplus Z_3$	Z_3
9	3	$Z_9 \oplus Z_9$	Z_9
11	3	$Z_{27} \oplus Z_9$	Z_9
13	3	$Z_{27} \oplus Z_{27}$	Z_{27}
15	3	$Z_{81} \oplus Z_{27}$	Z_{27}
17	3	$Z_{81} \oplus Z_{81}$	Z_{81}
19	3	$Z_{243} \oplus Z_{81}$	Z_{81}
1	9	–	–
3	9	Z_9	–
5	9	$Z_9 \oplus Z_9$	Z_9
7	9	$Z_{27} \oplus Z_9 \oplus Z_3$	Z_9
9	9	$Z_{27} \oplus Z_{27} \oplus Z_3 \oplus Z_3$	$Z_{27} \oplus Z_3$
11	9	$Z_{81} \oplus Z_{27} \oplus Z_3 \oplus Z_3 \oplus Z_3$	$Z_{27} \oplus Z_3$
13	9	$Z_{81} \oplus Z_{81} \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$	$Z_{81} \oplus Z_3 \oplus Z_3$
15	9	$Z_{243} \oplus Z_{81} \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$	$Z_{81} \oplus Z_3 \oplus Z_3$
17	9	$Z_{243} \oplus Z_{243} \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$	$Z_{243} \oplus Z_3 \oplus Z_3 \oplus Z_3$
19	9	$Z_{729} \oplus Z_{243} \oplus Z_9 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$	$Z_{243} \oplus Z_3 \oplus Z_3 \oplus Z_3$
1	27	–	–
3	27	Z_{27}	–
5	27	$Z_{27} \oplus Z_{27}$	Z_{27}
7	27	$Z_{81} \oplus Z_{27} \oplus Z_9$	Z_{27}
9	27	$Z_{81} \oplus Z_{81} \oplus Z_9 \oplus Z_9$	$Z_{81} \oplus Z_9$
11	27	$Z_{243} \oplus Z_{81} \oplus Z_9 \oplus Z_9 \oplus Z_9$	$Z_{81} \oplus Z_9$
13	27	$Z_{243} \oplus Z_{243} \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9$	$Z_{243} \oplus Z_9 \oplus Z_9$
15	27	$Z_{729} \oplus Z_{243} \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9$	$Z_{243} \oplus Z_9 \oplus Z_9$
17	27	$Z_{729} \oplus Z_{729} \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9$	$Z_{729} \oplus Z_9 \oplus Z_9 \oplus Z_9$
19	27	$Z_{2187} \oplus Z_{729} \oplus_{27} \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_9 \oplus Z_3$	$Z_{729} \oplus Z_9 \oplus Z_9 \oplus Z_9$
1	5	–	–
3	5	Z_5	–
5	5	$Z_5 \oplus Z_5$	Z_5
7	5	$Z_5 \oplus Z_5 \oplus Z_5$	Z_5
9	5	$Z_5 \oplus Z_5 \oplus Z_5 \oplus Z_5$	$Z_5 \oplus Z_5$
11	5	$Z_{25} \oplus Z_5 \oplus Z_5 \oplus Z_5$	$Z_5 \oplus Z_5$
13	5	$Z_{25} \oplus Z_{25} \oplus Z_5 \oplus Z_5$	$Z_{25} \oplus Z_5$
15	5	$Z_{25} \oplus Z_{25} \oplus Z_{25} \oplus Z_5$	$Z_{25} \oplus Z_5$
17	5	$Z_{25} \oplus Z_{25} \oplus Z_{25} \oplus Z_{25}$	$Z_{25} \oplus Z_{25}$
19	5	$Z_{125} \oplus Z_{25} \oplus Z_{25} \oplus Z_{25}$	$Z_{25} \oplus Z_{25}$

1	25	–	–
3	25	Z_{25}	–
5	25	$Z_{25}\oplus Z_{25}$	Z_{25}
7	25	$Z_{25}\oplus Z_{25}\oplus Z_{25}$	Z_{25}
9	25	$Z_{25}\oplus Z_{25}\oplus Z_{25}\oplus Z_{25}$	$Z_{25}\oplus Z_{25}$
11	25	$Z_{125}\oplus Z_{25}\oplus Z_{25}\oplus Z_{25}\oplus Z_{5}$	$Z_{25}\oplus Z_{25}$
13	25	$Z_{125}\oplus Z_{125}\oplus Z_{25}\oplus Z_{25}\oplus Z_{5}\oplus Z_{5}$	$Z_{125}\oplus Z_{25}\oplus Z_{5}$
15	25	$Z_{125}\oplus Z_{125}\oplus Z_{125}\oplus Z_{25}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}$	$Z_{125}\oplus Z_{25}\oplus Z_{5}$
17	27	$Z_{125}\oplus Z_{125}\oplus Z_{125}\oplus Z_{125}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}$	$Z_{125}\oplus Z_{125}\oplus Z_{5}\oplus Z_{5}$
19	25	$Z_{625}\oplus Z_{125}\oplus Z_{125}\oplus Z_{125}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}\oplus Z_{5}$	$Z_{125}\oplus Z_{125}\oplus Z_{5}\oplus Z_{5}$
1	7	–	–
3	7	Z_7	–
5	7	$Z_7\oplus Z_7$	Z_7
7	7	$Z_7\oplus Z_7\oplus Z_7$	Z_7
9	7	$Z_7\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_7\oplus Z_7$
11	7	$Z_7\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_7\oplus Z_7$
13	7	$Z_7\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_7\oplus Z_7\oplus Z_7$
15	7	$Z_{49}\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_7\oplus Z_7\oplus Z_7$
17	7	$Z_{49}\oplus Z_{49}\oplus Z_7\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_{49}\oplus Z_7\oplus Z_7$
19	7	$Z_{49}\oplus Z_{49}\oplus Z_{49}\oplus Z_7\oplus Z_7\oplus Z_7$	$Z_{49}\oplus Z_7\oplus Z_7$

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