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The eta invariant and the K-theory of odd dimensional spherical space forms

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0. Introduction

The eta invariant appears as a correction term in the Atiyah-Patodi-Singer index theorem for manifolds with boundary. It also gives rise to a C/Z valued twisted index [4-6]. If M is an odd-dimensional spin$_c$ manifold, we will use the twisted index to define a Q/Z valued bilinear form on the subring $\tilde{K}_{\mathrm{flat}}(M)$ of the reduced complex K-theory group $\tilde{K}(M)$ generated by bundles which admit a locally flat structure.

Let M be a spherical space form of odd dimension $2l-1$. We suppose $l>1$ henceforth. Let $G=\pi_1(M)$ be the fundamental group. G is finite and there exists a fixed point free representation τ of G into the unitary group $U(l)$ so that $M=S^{2l-1}/\tau(G)$. We refer to Wolf [30] for this and other facts concerning spherical space forms. Let $R(G)$ be the group representation ring and $R_0(G)$ the ideal of virtual representations of virtual dimension 0. Let $\alpha=\Sigma(-1)^p\cdot\Lambda^p(\tau)\in R_0(G)$. The natural map $R_0(G)\to\tilde{K}(M)$ is surjective [1, 10, 18, 25] so $\tilde{K}_{\mathrm{flat}}(M)=\tilde{K}(M)$.

This paper is divided into five sections. In the first section, we shall review the relevant analytic facts needed concerning the eta invariant. We will define the bilinear form and derive its basic properties. In the second section, we will relate this invariant for spherical space forms to certain Dedekind sums. If the fundamental group is Abelian, we will obtain combinatorial formulas in Q/Z which are polynomial in the defining data.

In the third section, we will show the eta invariant of the Dolbeault complex is non-degenerate on $\tilde{K}$ for spherical space forms. This will lead to the structure theorem $\tilde{K}(M)=R_0(G)/\alpha R(G)$. In the fourth section, we will use the bilinear form to obtain further information about $\tilde{K}$. Let $L(p;\vec{q})$ be a lens space with fundamental group Z_p. Let i: $L(p;q)\mapsto M$ be a finite covering. We will show that $V\in\tilde{K}(M)$ is zero iff $i^*(V)=0$ for all possible such finite coverings.

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This reduces the problem in a certain sense to the Abelian case. We will show $\tilde{K}(M)$ depends only on (G, l) and not on the particular defining representation τ.

In the final section, we will reduce the problem of computing $|\tilde{K}|$ to the case in which τ is irreducible. We will derive Kambe's theorem giving the structure of $\tilde{K}(L(p;\vec{q}))$ ([17]) from some results of Atiyah and Hirzebruch [3] regarding characteristic numbers. We will discuss the structure for metacyclic fundamental groups to illustrate using the eta invariant to simplify the calculations involved.

In the course of our investigations, we calculated a number of Dedekind sums; these are available upon request from the author. It is a pleasant task to thank both the University of Oregon and the University of New Mexico for making computer time available for these calculations. We would also like to thank Professors Karoubi, Seitz, and Wolf for sharing their knowledge of various fields with us and for being so patient in answering our questions.

1. The eta invariant

Let M be a compact Riemannian manifold without boundary of odd dimension $2l-1$. We shall rule out the case of the circle implicitly in what follows and assume $l>1$. Let P: $C^\infty(V)\to C^\infty(V)$ be a self-adjoint elliptic partial differential operator. Let $\{\lambda_\nu\}_{\nu=1}^\infty$ denote the spectrum of P where each eigenvalue is repeated according to multiplicity. Define:

$$\eta(z,P)=\mathrm{Tr}(P\cdot(P^2)^{-(z+1)/2})=\sum_{\lambda_\nu\neq 0}\mathrm{sign}(\lambda_\nu)|\lambda_\nu|^{-z} \quad \text{for } \mathrm{Re}(z)\gg 0.$$

The calculus of pseudo-differential operators depending upon a complex parameter developed by Seeley [29] can be used to show this series converges absolutely for $\mathrm{Re}(z)>>0$ to define a holomorphic function of z. η has a meromorphic extension to C with isolated simple poles on the real axis. The residue at such a pole is given by a local formula in the jets of the total symbol of P which is integrated over M.

A-priori, the value at $z=0$ is not regular and the corresponding local formula need not vanish identically [11]. Atiyah et al. [5] used K-theoretic methods to show the regularity if m is odd; there is a corresponding regularity theorem if m is even discussed in [12]. Let $N(P)$ be the dimension of the 0-eigenspace of P and define:

$$\eta(P)=\tfrac{1}{2}\{\eta(0,P)+N(P)\}\in R/Z.$$

If $P(t)$ is a smooth 1-parameter family of such operators, then $\eta(0,P(t))$ has integer jumps as spectral values cross the origin. The reduced invariant $\eta(P(t))$ is a smooth function of the parameter t and measures the spectral asymmetry.

We differentiate formally to calculate:

$$\frac{d}{dt}\eta(z,P(t))=-z\cdot\mathrm{Tr}\{P'(t)(P(t)^2)^{-(z+1)/2}\}.$$

The calculus developed by Seeley can be used to justify this step and to show $\mathrm{Tr}\{P'(t)\cdot(P(t)^2)^{-(z+1)/2}\}$ has a meromorphic extension to C with isolated simple poles having locally computable residue. Since the trace is multiplied by $-z$, we conclude $\frac{d}{dt}\eta(z,P(t))$ is regular at $z=0$. Although the value $\eta(P)$ is *not* locally computable, the value of the derivative is locally computable in terms of the jets of the total symbols of $P(t)$ and $P'(t)$.

It is not necessary to assume that P is self-adjoint to define η and in fact this invariant can be defined as long as P is elliptic with the leading symbol having no purely imaginary eigenvalues on the unit sphere bundle $S(T^*M)$. In this instance, $\eta(P)$ will take values in C/Z rather than R/Z.

We can use η to define a C/Z valued index. Let $\rho\colon \pi_1(M)\to GL(k,C)$ be a representation of the fundamental group and let V_ρ be the locally flat bundle defined by the representation. It is equipped with a natural connection ∇_ρ which has 0-curvature. The holonomy of ∇_ρ is ρ. This bundle has locally constant transition functions so we can define the operator P_ρ on $C^\infty(V\otimes V_\rho)$ to be locally isomorphic to k-copies of P on $C^\infty(V)$. Define:

$$\mathrm{ind}(\rho,P)=\eta(P_\rho)-k\eta(P)\in C/Z.$$

If $P(t)$ is a smooth 1-parameter family of such operators, then:

$$\frac{d}{dt}\mathrm{ind}(\rho,P(t))=\frac{d}{dt}\eta(P(t)_\rho)-k\frac{d}{dt}\eta(P(t)).$$

We noted above that the derivative is given by a local formula. Since the operator $P(t)_\rho$ is locally isomorphic to k-copies of the operator $P(t)$, the two local formulas cancel so the derivative is zero. This shows $\mathrm{ind}(\rho,P(t))$ does not depend on the parameter t and only depends on the homotopy class of the leading symbol of the operator P within this class of operators.

If the bundle defined by the representation is topological trivial, then the twisted index can be computed explicitly as an integral over $S(T^*M)$. Let $\mathrm{Todd}(M)$ be the real Todd class of M and let $\mathrm{Tch}(\rho)$ denote the secondary characteristic classes of the representation expressed in differential geometric terms [8]. Let $\Pi_+(P)$ be the sub-bundle of V over $S(T^*M)$ spanned by the eigenvectors of the symbol p which correspond to eigenvalues with positive real part. This measures the infinitesimal spectral asymmetry of P. Let $\mathrm{ch}(\Pi_+(P))$ be the Chern character of this bundle. Then the Atiyah-Patodi-Singer formula [3–5] in this case becomes:

$$\mathrm{ind}(\rho,P)=\int_{S(T^*M)}\mathrm{TODD}(M)\wedge\mathrm{Tch}(\rho)\wedge\mathrm{ch}(\Pi_+(P))\bmod Z.$$

There is an analogous result if $\dim(M)$ is even [12].

This formula shows $\mathrm{ind}(\rho,P)$ is not a topological invariant of the bundle V_ρ in general. It is easy to construct topological invariants as follows:

Lemma 1.1. *Let $P\colon C^\infty(V)\to C^\infty(V)$ be a self-adjoint elliptic partial differential operator and suppose $\pi_1(M)$ is finite. Let ρ, $\tilde\rho$ be two representations so that V_ρ*

$\simeq V_{\tilde\rho}$ as complex vector bundles. Then

$$|\pi_1(M)|\,\mathrm{ind}(\rho,P)=|\pi_1(M)|\,\mathrm{ind}(\tilde\rho,P) \quad \text{in } C/Z.$$

Proof. Let $\overline{M}$ be the universal cover of M then this is a compact manifold. Fix an isomorphism $V_\rho \simeq V_{\tilde\rho}$ and regard ρ, $\tilde\rho$ as giving two different locally flat structures to the same bundle. The operators P_ρ, $P_{\tilde\rho}$ have the same leading symbol and hence are homotopic. Thus $\mathrm{ind}(\rho,P)-\mathrm{ind}(\tilde\rho,P)$ is given by a local formula if we integrate the derivative with respect to the homotopy. When this is lifted to the universal cover, the local formulas are multiplied by $|\pi_1(M)|$. If P', ρ', $\tilde\rho'$ are the corresponding objects on the universal cover then

$$|\pi_1(M)|\{\mathrm{ind}(\rho,P)-\mathrm{ind}(\tilde\rho,P)\}=\mathrm{ind}(\rho',P')-\mathrm{ind}(\tilde\rho',P') \quad \text{in } C/Z$$

However on the universal cover, $\rho'=\tilde\rho'$ so this difference is zero in C/Z.

This invariant measures the extent to which $\mathrm{ind}(*,P)$ fails to be multiplicative under finite coverings. We shall give an example in Sect. 2 showing this is non-trivial. This will show $\mathrm{ind}(\rho,P)$ is *not* in general given in terms of primary and secondary characteristic classes since primary and secondary classes are multiplicative under finite coverings [7]. We refer to Atiyah et al. [5] for a further discussion of this invariant.

This invariant suffers from two difficulties. First, it requires one to assume $\pi_1(M)$ is finite. Second and more importantly from our point of view, it does fail to detect all of the torsion in certain K-theory groups. We will discuss this further in the second section. A different invariant can be constructed as follows. Let ρ_1, ρ_2 be two representations of the fundamental group. Define:

$$\mathrm{ind}(\rho_1,\rho_2,P)=\mathrm{ind}(\rho_1\otimes\rho_2,P)-\dim(\rho_1)\,\mathrm{ind}(\rho_2,P)-\dim(\rho_2)\,\mathrm{ind}(\rho_1,P).$$

Lemma 1.2. *Let P: $C^\infty(V)\to C^\infty(V)$ be a self-adjoint elliptic partial differential operator. Let ρ_1, $\tilde\rho_1$, ρ_2 be representations of the fundamental group. Suppose $V_{\rho_1} \simeq V_{\tilde\rho_1}$ as complex vector bundles. Then $\mathrm{ind}(\rho_1,\rho_2,P)=\mathrm{ind}(\tilde\rho_1,\rho_2,P)$ in C/Z.*

Proof. We compute from the definition:

$$\begin{aligned}\mathrm{ind}(\rho_1,\rho_2,P)&=\eta(P_{\rho_1\otimes\rho_2})+\dim(\rho_1)\dim(\rho_2)\,\eta(P)-\dim(\rho_1)\,\eta(P_{\rho_2})-\dim(\rho_2)\,\eta(P_{\rho_1})\\&=\eta\{(P_{\rho_1})_{\rho_2}\}-\dim(\rho_2)\,\eta(P_{\rho_1})-\dim(\rho_1)\{\eta(P_{\rho_2})-\dim(\rho_2)\,\eta(P)\}\\&=\mathrm{ind}(\rho_2,P_{\rho_1})-\dim(\rho_1)\,\mathrm{ind}(\rho_2,P).\end{aligned}$$

By hypothesis, the bundles V_{ρ_1} and $V_{\tilde\rho_1}$ are topologically isomorphic. Under such an isomorphism, then the operators P_{ρ_1} and $P_{\tilde\rho_1}$ become homotopic since they have the same leading symbol. This shows:

$$\mathrm{ind}(\rho_2,P_{\rho_1})=\mathrm{ind}(\rho_2,P_{\tilde\rho_1})$$
$$\dim(\rho_1)\,\mathrm{ind}(\rho_2,P)=\dim(\tilde\rho_1)\,\mathrm{ind}(\rho_2,P)$$

which completes the proof.

$\mathrm{Ind}(\rho,P)$ and $\mathrm{ind}(\rho_1,\rho_2,P)$ depend only on the homotopy class of the leading symbol of P. The first invariant is linear and the second bilinear. They change sign if we replace P by $-P$ and vanish if P is either positive or negative definite. Let $G=\pi_1(M)$ and let $R(G)$ be the group representation ring. Let $R_0(G)$ be the ideal of virtual representations of virtual dimension 0. Since $\eta(P_\rho)$ is additive with respect to the representation involved, we can extend $\eta(\rho, P)=\eta(P_\rho)$ to a map $R(G)\to C/Z$. It is immediate that:

$$\operatorname{ind}(\rho, P)=\eta(P_\rho) \quad \text{and} \quad \operatorname{ind}(\rho_1, \rho_2, P)=\eta(P_{\rho_1\otimes\rho_2}) \qquad \text{if } \rho_i\in R_0(G).$$

Let $K(M)$ denote the complex K-theory group of M and $\tilde{K}(M)$ the ideal generated by the virtual bundles of virtual dimension 0. The map $\rho\to V_\rho$ defines a map from $R_0(G)\to\tilde{K}(M)$. We denote the image by $\tilde{K}_{\text{flat}}(M)$: this is a sub-ring. We can reinterpret Lemmas 1.1 and 1.2 as follows:

Lemma 1.3. *Let $\tilde{K}_{\text{flat}}(M)$ be the sub-ring of $\tilde{K}(M)$ generated by the locally flat bundles of virtual dimension 0. Let P be an elliptic-self adjoint differential operator on M. Then $\operatorname{ind}(\rho_1, \rho_2, P)$: $R_0(G)\otimes R_0(G)\to C/Z$ extends naturally to a bilinear form $\operatorname{ind}(*, *, P)$: $\tilde{K}_{\text{flat}}(M)\otimes\tilde{K}_{\text{flat}}(M)\to C/Z$. If G is finite, then $|G|\operatorname{ind}(*, P)$ extends to a map $|G|\operatorname{ind}(*, P)$: $\tilde{K}_{\text{flat}}(M)\mapsto C/Z$. The values are actually contained in Q/Z in this situation.*

Proof. By Lemmas 1.1 and 1.2, it is clear the relevant invariants are defined in K-theory. We need therefore only show the values are in Q/Z. Let V be a vector bundle with a locally flat connection. The rational Chern classes can be computed in terms of the curvature of the connection and therefore all vanish, except in dimension 0. The Chern character gives an isomorphism

$$ch\colon \tilde{K}(M)\otimes Q\to\bigoplus_{q>0} H^{2q}(M;Q)$$

and consequently any element of $\tilde{K}_{\text{flat}}(M)$ is a torsion class. Since $\operatorname{ind}(*, *, P)$ is defined on $\tilde{K}_{\text{flat}}(M)$ it must lie in the torsion subgroup of C/Z which is Q/Z and that completes the proof. We remark that $\operatorname{ind}(\rho, P)$ is not necessarily rational if G isn't finite and refer to [12] for suitable examples.

In fact, it is not necessary to assume that the operator P is partial differential and it is possible to work with pseudo-differential operators. This permits us to regard $\operatorname{ind}(*, *, P)$ as a map:

$$\operatorname{ind}\colon \tilde{K}_{\text{flat}}(M)\otimes\tilde{K}_{\text{flat}}(M)\otimes\{K(S(T^*M))/K(M)\}\to Q/Z$$

using the arguments discussed in [12]. Since we will always be interested in the bilinear form arising from a specific operator, we shall not need this formulation.

There are several natural operators P which arise from the classical elliptic complexes. Let N be a smooth Riemannian manifold of dimension $2l$ such that $dN=M$. We do not assume N is compact. Near the boundary, N has a collared neighborhood $[0,1)\times M$ with $dN=O\times M$. Let n be the geodesic normal parameter and let Q: $C^\infty(V_1)\to C^\infty(V_2)$ be an elliptic complex over N. We use the geodesic normal parameter and the symbol of Q to identify $V_1=V_2=V$ near M. Suppose that Q is first order and express:

$$Q=\partial/\partial n+P_T$$

where P_T: $C^\infty(V) \to C^\infty(V)$ is a tangential first order operator over M. The ellipticity of Q implies P_T is elliptic and has no purely imaginary eigenvalues on $S(T^*M)$.

We may suppose $N=[0,1)\times M$. If M is oriented, we can construct the signature operator on N and let P_{sign} be the tangential part of the signature complex. The bundle V in this instance is $\Lambda(T^*M)$ and the resulting operator has the form

$$P_{\text{sign}} = \pm(*d - d*)$$

where we refer to [3] for precise details regarding the signs involved. This operator naturally decomposes into two operators $P_{\text{sign}}^{\text{even}} \oplus P_{\text{sign}}^{\text{odd}}$ on the space of even and odd forms and these two operators have the same eta invariant.

If N is a holomorphic manifold, then M inherits a Cauchy-Riemann structure and we let P_{DOL} be the tangential operator of the Dolbeault complex. More generally, if we assume M has a spin$_c$ structure, then we can define P_{DOL} using the twisted spin complex. If M (or N) has a spin structure, we let P_{spin} be the tangential operator of the spin complex. If W is an auxiliary coefficient bundle over M (or N) we can take these operators with coefficients in W. We assume henceforth that M admits a spin$_c$ structure, then the operators P_{DOL}^W corresponding to the Dolbeault complex with coefficients in W generate $K(S(T^*M))/K(M)$. If we assume only that M is oriented, then the operators P_{sign}^W do not detect 2-torsion well but detect torsion at the remaing primes; the P_{sign}^W generate $K(S(T^*M))/K(M)\otimes Z[2^{-1}, 2^{-2}, \ldots]$ localized at the prime 2.

2. Dedekind sums

The eta invariant can be computed explicitly for spherical space forms in terms of Dedekind sums. Let G be a finite group and let τ: $G \to U(l)$ be a unitary representation. We assume that τ is fixed point free-i.e. $\det(I-\tau(g)) \neq 0$ for $g \neq I$. The existence of such a representation places substantial restrictions on the group G. For example if G is Abelian, then it is necessarily cyclic. We refer to Wolf [30] for a discussion of the admissible groups.

τ is faithful and G acts on S^{2l-1} by deck transformations. Let $M=M_\tau = S^{2l-1}/\tau(G)$ be the resulting quotient manifold. M inherits a natural Riemannian metric of constant sectional curvature 1 and every compact odd dimensional manifold admitting such a metric arises in this way. Such M_τ are called spherical space forms and we restrict to $M=M_\tau$ henceforth. The only even dimensional compact manifolds admitting a metric of constant sectional curvature 1 are S^{2l} and RP^{2l}. We assume $l>1$ so $\pi_1(M) \simeq G$.

We must first obtain an explicit combinatorial formula for $\eta(P_\rho)$ where $P = P_{\text{sign}}$, P_{DOL}, or P_{spin} is the tangential operator of one of the three classic elliptic complexes. We review the Lefschetz fixed point formulas as follows: let $g \in U(l)$ with $\det(I-g) \neq 0$. Let g_r denote the corresponding element of $SO(2l)$. Let the complex eigenvalues of g be $\{\lambda_\nu\}_{\nu=1}^l$ and let the rotation angles of g_r be $\{\theta_\nu\}$. We define the following defects relative to the three complexes:

$$\operatorname{def}(g,\operatorname{sign})=\det(I-g_r)^{-1}\cdot\prod_\nu\{\bar\lambda_\nu-\lambda_\nu\}=\prod_\nu\{(\lambda_\nu+1)/(\lambda_\nu-1)\}$$
$$=(-i)^l\prod_\nu\cot(\theta_\nu/2)$$
$$\operatorname{def}(g,\operatorname{Dol})=\det(I-g_r)^{-1}\prod_\nu\{1-\lambda_\nu\}=\prod_\nu\{\lambda_\nu/(\lambda_\nu-1)\}$$
$$\operatorname{def}(g,\operatorname{spin})=\det(I-g_r)^{-1}\prod_\nu\{\sqrt{\bar\lambda_\nu}-\sqrt{\lambda_\nu}\}=\prod_\nu\{\sqrt{\lambda_\nu}/(\lambda_\nu-1)\}$$
$$=(2i)^{-l}\prod_\nu\operatorname{cosec}(\theta_\nu/2).$$

$\operatorname{Def}(g,\operatorname{sign})$ depends only on the rotation angles and is defined on the fixed point free elements of $SO(2l)$; cot is periodic with period π. Since cosec is *not* periodic modulo π, $\operatorname{def}(g,\operatorname{spin})$ depends upon the lift to $\operatorname{SPIN}(2l)$. Finally, $\operatorname{def}(g,\operatorname{Dol})$ is a $U(l)$ invariant. We add one note of caution. The usual formula for the Lefschetz number relative to the Dolbeault complex would define $\operatorname{def}(g,\operatorname{Dol})=\det(I-g)^{-1}$. We are thinking of all the representations as defined on $T(M)$ rather than $T^*(M)$ so it is better to replace $\bar\lambda_\nu$ by λ_ν. At worst we shall have replaced the Dolbeault complex by the complex conjugate. However, as we shall see shortly, this is the correct formula and makes the signs work out properly.

The Atiyah-Patodi-Singer index theorem has been generalized by Donnelly [9] to include Lefschetz fixed point formulas for manifolds with boundary. When these formulas are used and when the twisting representation is taken into account, the following Lemma can be proved. It also follows, of course, from the results of Millson [26, 27] and from Atiyah et al. [3–5].

Lemma 2.1. *Let τ: $G\to U(l)$ be a fixed point free representation and let $M=S^{2l-1}/\tau(G)$.*

(a) *M inherits a natural orientation from the orientation of S^{2l-l}. Let P_{sign} be the tangential operator of the signature complex on $M\times[0,1)$ and let ρ be a representation of G. Then:*

$$\eta(\rho,P_{\text{sign}})=\frac{1}{|G|}\sum_{g\in G,g\neq I}\operatorname{Tr}(\rho(g))\operatorname{def}(\tau(g),\operatorname{sign})$$

(b) *M inherits a natural Cauchy-Riemann (or* spin$_c$*) structure since the representation τ is unitary. Let P_{DOL} be the tangential operator of the Dolbeault complex on $M\times[0,1)$ and let ρ be a representation of G. Then:*

$$\eta(\rho,P_{\text{DOL}})=\frac{1}{|G|}\sum_{g\in G,g\neq I}\operatorname{Tr}(\rho(g))\operatorname{def}(\tau(g),\operatorname{Dol}).$$

(c) *Suppose there is a representation $\tilde\tau$ of G to* $\operatorname{SPIN}(2l)$ *which induces the given representation τ under the double cover* $\operatorname{SPIN}(2l)\to SO(2l)$. *Then M inherits a* spin *structure. Let P_{spin} be the tangential operator of the* Spin *complex on $M\times[0,1)$ and let ρ be a representation of G. Then:*

$$\eta(\rho,P_{\text{spin}})=\frac{1}{|G|}\sum_{g\in G,g\neq I}\operatorname{Tr}(\rho(g))\operatorname{def}(\tau(g),\operatorname{spin}).$$

Of course, not every space form admits a spin structure. If G is Abelian, for example, then M always admits a spin structure if $|G|$ is odd and the spin structure is unique. If $|G|$ is even, then M does *not* admit a spin structure if l is odd and admits exactly two inequivalent spin structures if l is even. If M is spin, then we can define $\tilde{\rho}=\sqrt{\det(\tau)}$ and then calculate easily

$$\eta(\rho, P_{\text{spin}})=\eta(\rho\otimes\tilde{\rho}^*, P_{\text{Dol}}).$$

Since $\tilde{\rho}$ is a unit in the group representation ring, we can always express $\eta(*, P_{\text{spin}})$ in terms of $\eta(*, P_{\text{DOL}})$ so we shall concentrate for the most part upon P_{Dol} and P_{sign}.

The remainder of this section is devoted to a discussion of some of the elementary number theoretic properties of these Dedekind sums. The reader who is only interested in the topological calculation of $\tilde{K}(M)$ can skip to the third section; we will use these results in the fourth section to calculate $\tilde{K}(M)$ explicitly in some examples. We refer to Zagier [31] and Hirzebruch and Zagier [14] for further details concerning Dedekind sums.

We suppose for the moment that G is Abelian and hence cyclic. Let $p=|G|$ and identify $G=Z_p$ with the p^{th} roots of unity. Let $\vec{q}=(q_1, \ldots, q_l)$ be a collection of integers coprime to p and for $\lambda^p=1$ define $\tau(\vec{q})(\lambda)=\operatorname{diag}(\lambda^{q_1}, \ldots, \lambda^{q_l})\in U(l)$. This defines a fixed point free representation of G; any fixed point free representation of G is conjugate to one of these. The quotient M_τ is the lensspace $L(p;\vec{q})$. If we permute the q's, we obtain isomorphic lensspaces. If we replace $\vec{q}$ by $k\vec{q}$ where k is coprime to p, we are just changing the generator of G and again obtain isomorphic lensspaces. It is clear that only the congruence class of $\vec{q}$ modulo p is important. p need not be prime in this notation.

If $\lambda^p=1$, let $\rho_s(\lambda)=\lambda^s$ for $0\leqq s<p$. The $\{\rho_s\}$ parametrize the irreducible representations of G and only the congruence class of s modulo p is important. Define:

$$\operatorname{def}(s, p, \vec{q}; \text{sign})=p\cdot\eta\{(P_{\text{sign}})_{\rho_s}\} = \sum_{\lambda^p=1, \lambda\neq 1} \lambda^s \prod_\nu \{(\lambda^{q_\nu}+1)/(\lambda^{q_\nu}-1)\}$$

$$\operatorname{def}(s, p, \vec{q}; \text{Dol})=p\cdot\eta\{(P_{\text{Dol}})_{\rho_s}\} = \sum_{\lambda^p=1, \lambda\neq 1} \lambda^s \prod_\nu \{\lambda^{q_\nu}/(\lambda^{q_\nu}-1)\}$$

as generalized defects.

Lemma 2.2. $p^l \operatorname{def}(s, p, \vec{q}, \text{sign})\in Z$ *and* $p^l \operatorname{def}(s, p, \vec{q}, \text{Dol})\in Z$.

Proof. We sum over the p^{th} roots of unity distinct from 1. This sum is invariant under the Galois group of the cyclotonic number field and is rational. If we can show $p/(\lambda-1)$ is an algebraic integer for $\lambda^p=1$, then $p^l\operatorname{def}(\ldots)$ will be a rational algebraic integer and hence an integer. Set $x=1/(\lambda-1)$. Then $(x+1)=\lambda/(\lambda-1)$ so $x^p=(x+1)^p$ and x satisfies the equation $px^{p-1}+\ldots+1=0$. Since all the coefficients are integers and the leading coefficient is p, px is an algebraic integer which completes the proof.

The basic tool used to study these Dedekind sums is Rademacher reciprocity. Let $L_k(x_0, \ldots)$ be the Hirzebruch polynomial, $Td_k(x_0, \ldots)$ be the Todd polynomial, and $\hat{A}_k$ the A-roof polynomial. For example:

$$L_1(x)=\frac{1}{3}\sum_i x_i^2 \qquad L_2(x)=\frac{1}{45}\{7\sum_{i<j} x_i^2 x_j^2-(\sum_i x_i^2)^2\}$$

$$Td_1(x)=\frac{1}{2}\sum_i x_i \qquad Td_2(x)=\frac{1}{12}\{\sum_{i<j} x_i x_j+(\sum_i x_i)^2\}$$

$$\hat{A}_1(x)=\frac{-1}{24}\sum_i x_i^2 \qquad \hat{A}_2(x)=\frac{1}{5{,}760}\{-4\sum_{i<j} x_i^2 x_j^2+7(\sum_i x_i^2)^2\}.$$

We refer to Hirzebruch [13] for further details. The generating function for L is $x/\tanh(x)$, the generating function for Td is $x/(1-\exp(-x))$, and the generating function for $\hat{A}$ is $\frac{x}{2}\Big/\sinh\left(\frac{x}{2}\right)$. We introduce an auxilary parameter s and define:

$$L_k(s;x)=\sum_{a+2b=k} 2^a s^a L_b(x)/a!$$

$$Td_k(s;x)=\sum_{a+b=k} s^a Td_b(x)/a!$$

$$\hat{A}_k(s;x)=\sum_{a+2b=k} s^a \hat{A}_b(x)/a!$$

We let $\mu(*)$ denote the least common denominator of such a polynomial.

Let $\vec{r}=(r_0,\dots,r_l)$ be a collection of positive integers which are mutually coprime and let $\vec{r}_j=(r_0,\dots,r_{j-1},r_{j+1},\dots,r_l)$ be the collection with r_j deleted.

Lemma 2.3. *Let $\vec{r}$ be as above and let $s\in Z$. There exist integers $a=a(s,\vec{r})$ and $b=b(s,\vec{r})$ such that:*

(a) $\sum_{\nu=0}^{l}\frac{1}{r_\nu}\operatorname{def}(s,r_\nu,\vec{r}_\nu,\mathrm{sign})=a-L_l(s;\vec{r})/(r_0\dots r_l)$.

(b) $\sum_{\nu=0}^{l}\frac{1}{r_\nu}\operatorname{def}(s,r_\nu,\vec{r}_\nu,\mathrm{Dol})=b-Td_l(s;\vec{r})/(r_0\dots r_l)$.

Proof. We apply the residue theorem to a suitable meromorphic function. Let

$$f(z)=\tfrac{1}{2}z^{s-1}\prod_{\nu=0}^{l}\{(z^{r_\nu}+1)/(z^{r_\nu}-1)\}.$$

f has poles at $z=0,1,\infty$. The remaining poles are at the r^{th} roots of unity. They are simple since the r's are assumed to be coprime. Let $\lambda^r=1$ for $\lambda\neq 1$. We use L'Hospitals rule to compute:

$$\lim_{z\to\lambda}(z-\lambda)/(z^r-1)=1/(r\lambda^{r-1})$$

$$\operatorname{Res}_{z=\lambda} f(z)\,dz=\frac{1}{2r}\lambda^{s-1}\lambda^{1-r}(\lambda^r+1)\prod_{r_\nu\neq r}\{(\lambda^{r_\nu}+1)/(\lambda^{r_\nu}-1)\}$$

$$=\frac{1}{r}\lambda^s\prod_{r_\nu\neq r}\{(\lambda^{r_\nu}+1)/(\lambda^{r_\nu}-1)\}.$$

Summing over the roots of unity yields the left hand side of the equation of (a).

Let $a(s,r) = -\mathrm{Res}_{z=0} f(z)\,dz - \mathrm{Res}_{z=\infty} f(z)\,dz$. First suppose $s=0$. The product becomes $(-1)^{l+1}$ at $z=0$ so the residue there is $(-1)^{l+1}\cdot\frac{1}{2}$. At infinity we replace z by $w=1/z$ and compute $\mathrm{Res}_{z=\infty} f(z)\,dz = -\mathrm{Res}_{w=0} f(1/w)\,dw/w^2 = -1/2$. Thus $a=0$ if l is odd and 1 if l is even. Next suppose $s<0$ so the residue at ∞ vanishes. Expand

$$(z^r+1)/(z^r-1) = -(z^r+1)(1+z^r+z^{2r}+\ldots) = -(1+2z^r+2z^{2r}+\ldots).$$

We multiply the power series together to get a power series with leading term 1 and all other coefficients even integers. Thus the residue is integral. The case $s>0$ is similar

Finally we compute the residue at 1. Substitute $z=\exp(2t)$ and evaluate at $t=0$:

$$\begin{aligned}\mathrm{Res}_{z=1} f(z)\,dz &= \mathrm{Res}_{t=0}\, 2\exp(2t) f(\exp(2t))\,dt\\ &= \mathrm{Res}_{t=0} \exp(2ts)\prod_\nu \{(\exp(2tr_\nu)+1)/(\exp(2tr_\nu)-1)\}\,dt.\end{aligned}$$

We use the identity:

$$\begin{aligned}(\exp(2x)+1)/(\exp(2x)-1) &= (\exp(x)+\exp(-x))/(\exp(x)-\exp(-x))\\ &= 1/\tanh(x)\end{aligned}$$

to rewrite the residue in the form:

$$\begin{aligned}\mathrm{Res}_{z=1} f(z)\,dz &= \mathrm{Res}_{t=0} \exp(2ts)\prod_\nu \{1/\tanh(r_\nu t)\}\,dt\\ &= (r_0\ldots r_l)^{-1}\,\mathrm{Res}_{t=0} \exp(2ts)\prod_\nu \{r_\nu t/\tanh(r_\nu t)\}\,dt/t^{l+1}.\end{aligned}$$

The Hirzebruch L-genus is defined by the power series:

$$\prod_\nu \{r_\nu t/\tanh(r_\nu t)\} = \sum_j t^{2j} L_j(\vec{r}).$$

We expand $\exp(2ts)$ in a power series and substitute this expansion. We compute the coefficient of t^l to show

$$\mathrm{Res}_{z=1} f(z)\,dz = L_l(s;\vec{r}) = \sum_{2j+k=l} (2s)^j L_k(\vec{r})/k!$$

Since the sum of the residues of a meromorphic function is zero, this proves (a).

We consider the meromorphic function

$$g(z) = z^{s-1}\prod_\nu \{z^{r_\nu}/(z^{r_\nu}-1)\}$$

to prove (b). Again the poles of g at the roots of unity distinct from 1 are simple and:

$$\mathrm{Res}_{z=\lambda}\, g(z)\,dz = \frac{\lambda^s}{r}\prod_{r_\nu \neq r} \{\lambda^{r_\nu}/(\lambda^{r_\nu}-1)\} \qquad \text{for } \lambda^r=1,\ \lambda\neq 1.$$

Summing over such λ yields the left hand side of equation (b). At $z=0$, we expand $(z^r-1)^{-1}=-(1+z^r+\dots)$ so $g(z)$ has integral Laurant series and integral residue. The residue at infinity is similar. Thus $b=-\mathrm{Res}_{z=0}f(z)\,dz-\mathrm{Res}_{z=\infty}f(z)\,dz$ is an integer. At $z=1$, we substitute $z=\exp(t)$ and evaluate at $t=0$:

$$\begin{aligned}\mathrm{Res}_{z=1}\,g(z)\,dz &= \mathrm{Res}_{t=0}\exp(t)\,g(\exp(t))\,dt\\ &=\mathrm{Res}_{t=0}\exp(t\,s)\prod_\nu\{\exp(r_\nu t)/(\exp(r_\nu t)-1)\}\,dt\\ &=\mathrm{Res}_{t=0}\exp(t\,s)\prod_\nu\{1/(1-\exp(-r_\nu t))\}\,dt\\ &=\mathrm{Res}_{t=0}(r_0\dots r_l)^{-1}\exp(t\,s)\prod_\nu\{r_\nu t/(1-\exp(-r_\nu t))\}\,dt/t^{l+1}.\end{aligned}$$

The generating function for the Todd genus is $t/(1-\exp(-t))$ so we expand

$$\prod_\nu\{r_\nu t/(1-\exp(-r_\nu t))\}=\sum_j Td_j(\vec r)\,t^j.$$

Computing the coefficient of t^l in the product of this with $\exp(t\,s)$ shows that the residue is

$$\mathrm{Res}_{z=1}\,g(z)\,dz=\sum_{j+k=l}s^j\,Td_k(\vec r)/k!=Td_l(s;\vec r).$$

Since the sum of the residues is zero, this completes the proof of (b).

We can now improve Lemma 2.2:

Lemma 2.4. *Let $\mu(L_l(*;*))$ and $\mu(Td_l(*;*))$ denote the denominators of these polynomials in the variables $(s;\vec r)$. Then:*

(a) $\mu(L_l(*;*))\operatorname{def}(s,p,\vec q,\mathrm{sign})\in Z$. *If p is coprime to $\mu(L_l(*;*))$, then* $\operatorname{def}(s,p,\vec q,\mathrm{sign})\in Z$.

(b) $\mu(Td_l(*;*))\operatorname{def}(s,p,\vec q,\mathrm{Dol})\in Z$. *If p is coprime to $\mu(Td_l(*;*))$, then* $\operatorname{def}(s,p,\vec q,\mathrm{Dol})\in Z$.

Proof. We shall only prove (a) as the proof of (b) is similar. Use Dedekind's theorem to choose representatives $\operatorname{mod} p$ of $\vec q$ so the q's are all mutually coprime. We may also assume the q's are coprime to the denominator $\mu(L_l(*,*))$. We set $c=q_1\dots q_l$ and let $\vec r=(p,q_1,\dots,q_l)$ to apply Lemma 2.3. If $\vec q_\nu=(p,q_1,\dots,q_{\nu-1},q_{\nu+1},\dots)$ then Lemma 2.3 implies:

$$\frac{1}{p}\operatorname{def}(s,p,\vec q,\mathrm{sign})+\sum_\nu\frac{1}{q_\nu}\operatorname{def}(s,q_\nu,\vec q_\nu,\mathrm{sign})=a-(c\cdot p)^{-1}L_l(s;p,\vec q).$$

Multiply both sides of this identity by $c^{l+1}p$ to conclude:

$$c^{l+1}\operatorname{def}(s,p,\vec q,\mathrm{sign})+p\left\{\sum_\nu\frac{c^{l+1}}{q_\nu}\operatorname{def}(s,q_\nu,\vec q_\nu,\mathrm{sign})\right\}=p(c^{l+1}a)-c^l L_l(s;p,q).$$

By Lemma 2.2, $c^{l+1}\operatorname{def}(s,q_\nu,\vec q_\nu,\mathrm{sign})/q_\nu\in Z$ as q_ν divides c. Therefore:

$$c^{l+1}\operatorname{def}(s,p,\vec q,\mathrm{sign})-c^l L_l(s;p,\vec q)\in Z.$$

If $\mu_l=\mu(L_l(*;*))$ then $\mu_l c^l L_l(s;p,\vec q)\in Z$ so $c^{l+1}\mu_l\operatorname{def}(s,p,\vec q,\mathrm{sign})\in Z$.

By Lemma 2.2, the primes which divide the denominator of $\operatorname{def}(s, p, \vec{q}, \text{sign})$ must divide p. Since p is coprime to c, we conclude $\mu_l \operatorname{def}(s, p, \vec{q}, \text{sign}) \in Z$. If μ_l is coprime to p the same argument shows $\operatorname{def}(s, p, \vec{q}, \text{sign}) \in Z$.

We can now give a combinatorial formula for this defect.

Theorem 2.5. *Let $M = L(p, q_1, \ldots, q_l)$ be a lens space, where p need not be prime.*

(a) *Let P_s^{sign} denote the tangential operator of the signature complex with coefficients in the representation ρ_s. Choose the q's coprime to p and to the denominators occuring in $L_l(*, *)$. Choose the integer d so that $d q_1 \ldots q_l \equiv 1 \bmod p \cdot \mu(L_l(*, *))$. Then:*

$$\eta(P_s^{\text{sign}}) \equiv \frac{-d}{p} L_l(s; p, \vec{q}) \bmod Z.$$

(b) *Let P_s^{Dol} denote the tangential operator of the Dolbeault complex with coefficients in the representation ρ_s. Choose the q's coprime to p and to the denominators occuring in $Td_l(*, *)$. Choose the integer d so that $d q_1 \ldots q_l \equiv 1 \bmod p \cdot \mu(Td_l(*, *))$. Then:*

$$\eta(P_s^{\text{Dol}}) \equiv \frac{-d}{p} Td_l(s; p, \vec{q}) \bmod Z.$$

(c) *M admits a unique* spin *structure if p is odd; if p is even, M admits two* spin *structures if l is even and none if l is odd. For p, l even the two* spin *structures are defined by the residue class of the q's modulo $2p$.*

(i) *Let l be even and choose the q's to be coprime to $p \cdot 2 \cdot \mu(\hat{A}_l(*, *))$. Let P_s^{spin} be the tangential operator of the* spin *complex with coefficients in the representation ρ_s. Choose the integer d so that $d q_1 \ldots q_l \equiv 1 \bmod p \cdot 2 \cdot \mu(A_l(*, *))$. Then*

$$\eta(P_s^{\text{spin}}) \equiv \frac{-d}{p} \hat{A}_l \left(s + \frac{p}{2}, \vec{q}\right) \bmod Z.$$

(ii) *Let l be odd and choose the q's coprime to $p \cdot 2 \cdot \mu(\hat{A}_l(*, *))$. Let P_s^{spin} be the tangential operator of the spin complex with coefficients in the representation ρ_s. Choose the integer d so that $d q_1 \ldots q_l \equiv 1 \bmod p \cdot 2 \cdot \mu(A_l(*, *))$. Then:*

$$\eta(P_s^{\text{spin}}) \equiv \frac{-d}{p} \hat{A}_l(s, \vec{q}) \bmod Z.$$

Proof. In the proof of (a) it is clear only the residue class of the q's modulo $p \cdot \mu(L_l(*, *))$ matters and thus we may without loss of generality choose them to be mutually coprime. Let $c = q_1 \ldots q_l$ then Rademacher reciprocity implies:

$$c \cdot \operatorname{def}(s, p, \vec{q}, \text{sign}) + p \cdot \left\{ \sum_{\nu} \frac{c}{q_\nu} \operatorname{def}(s, q_\nu, \vec{q}_\nu, \text{sign}) \right\} = p(c\, a) - L_l(s, p, \vec{q}).$$

Since the q_ν are coprime to the denominators of L_l, Lemma 2.4 implies these defects are integers. We may therefore express $c \cdot \operatorname{def}(s, p, \vec{q}, \text{sign}) \equiv -L_l(s; p, \vec{q})$ modulo pZ. Since c is coprime to all the relevant denominators, we may multiply by d/p to conclude $\eta(P_s^{\mathrm{sign}}) = \frac{1}{p}\operatorname{def}(s,p,\vec{q},\mathrm{sign}) \equiv \frac{-d}{p} L(s;p,\vec{q}) \bmod Z$ which proves (a). The proof of (b) is similar and is therefore omitted.

We turn finally to the spin complex. Not every lensspace admits a spin structure. Let g_0 generate Z_p. If we fix the q's then the rotation angles of $\tau(g_0)$ are fixed and we can lift to $\mathrm{SPIN}(2l)$ to define $\tilde{\tau}(g_0)$. To have a representation we require $\{\tilde{\tau}(g_0)\}^p = 1$ or equivalently that $\sum q_\nu$ is even. If p is odd, this is always possible. If p is even, the q_ν are necessarily odd and this is possible only if l is even. Inequivalent spin structures are parametrized by $\mathrm{HOM}(Z_p, Z_2)$. This group has 1 element if p is odd and 2 elements if p is even.

Suppose for the moment that l is even and let the q's determine the spin structure. Define

$$\begin{aligned}\operatorname{def}(s,p,\vec{q},\mathrm{spin}) &= \tfrac{1}{2} \sum_{\lambda^{2p}=1,\lambda^2 \neq 1} \lambda^{2s} \prod_\nu \{\lambda^{q_\nu}/(\lambda^{2q_\nu}-1)\} \\ &= \operatorname{def}((s - \tfrac{1}{2}\textstyle\sum q_\nu), p, \vec{q}, \mathrm{Dol}).\end{aligned}$$

This identity just expresses the spin complex in terms of the Dolbeault complex. Set $\tilde{s} = s - \frac{1}{2}\sum q_\nu$ so $\eta(P_s^{\mathrm{spin}}) \equiv \frac{-d}{p} Td_l(\tilde{s};p,\vec{q}) \bmod Z$.

The generating function for $Td_*(\tilde{s};p,\vec{q})$ is:

$$\begin{aligned}&\exp(t\,s)\exp(-t\textstyle\sum q_\nu/2)\exp(t\,p)/(\exp(t\,p)-1)\prod_\nu\{\exp(t\,q_\nu)/(\exp(t\,q_\nu)-1)\} \\ &\quad = \exp(t\,s)\exp(t\,p/2)\cdot 1/2\sinh(t\,p/2)\cdot\prod_\nu\{1/2\sinh(t\,q_\nu/2)\}.\end{aligned}$$

Since this is the generating function for $\hat{A}_*\left(s+\frac{p}{2};p,\vec{q}\right)$ part $(c-i)$ follows.

If we assume that l is odd and choose the q's odd, then $\sum q_\nu$ is odd. Since p must be odd, we let $(p+q_1, q_2, \ldots)$ define the spin structure. Thus in this case we set $\tilde{s} = s - \frac{1}{2}(p + \sum q_\nu)$ so $\eta(P_s^{\mathrm{spin}}) \equiv \frac{-d}{p} Td_l(\tilde{s};p,\vec{q}) \bmod Z$. Since the factor of $\exp(-t\,p/2)$ is already present, it is not necessary to add it when computing in terms of sinh and therefore the generating function is $\hat{A}_*(s;p,\vec{q})$ which proves (c-ii).

We remark briefly that the operator P_{sign} decomposes as the direct sum of two operators with equal eta invariants. It is possible to improve the congruence (a) to be modulo $2Z$ instead of Z using this fact. This would show:

$$\eta(P_s^{\mathrm{sign}}) \equiv \frac{-d}{p} L_l(s;p,\vec{q}) + \sigma(s,l) \bmod 2Z$$

under suitable hypothesis where

$$\sigma(0,l) = \begin{Bmatrix} 0 & \text{if } l \text{ is odd} \\ 1 & \text{if } l \text{ is even}\end{Bmatrix}, \qquad \sigma(s,l) = \begin{Bmatrix} 0 & \text{if } l \text{ is even} \\ 1 & \text{if } l \text{ is odd}\end{Bmatrix} \qquad \text{for } 0<|s|<p.$$

Since we shall not need this improvement, we omit the details.

Let $\mu_l = \mu(L_l(*,*))$ be the least common denominator. Then $\mu_l L_l$ is an integral polynomial and $\mu_l L_l(s; p, \vec{q}) \equiv \mu_l L_l(s; \vec{q}) \bmod pZ$. Therefore:

$$\mu_l \eta(P_s^{\text{sign}}) \equiv \frac{-d}{p} \mu_l L_l(s; \vec{q}) \bmod Z.$$

If we fix $\vec{q}$ and take k coprime to $\vec{q}$, there is a k-fold covering $L(k p; \vec{q}) \to L(p; \vec{q})$ induced by the inclusion of the corresponding groups. This implies:

$$\mu_l \eta(P_s^{\text{sign}} \text{ on } L(p; \vec{q})) \equiv k \mu_l \eta(P_s^{\text{sign}} \text{ on } L(k p; q)) \bmod Z$$

so $\mu_l \eta$ is multiplicative under finite coverings.

Form the integral characteristic class $\mu_l L_l(V, T(M)) = \mu_l \sum_{i+2j=l} \mathrm{ch}_i(V) L_j(M)$. $T(M) \oplus 1$ inherits a natural locally flat structure. Since $\dim(M) = 2l-1$, this characteristic class vanishes on M. Let T denote the transgression and form the secondary characteristic class $T(\mu_l L_l(V_\rho, T(M)))$. This is an R/Z valued $l-1$ cohomology class and we refer to Cheeger-Simons [7] for more details concerning secondary characteristic classes. The Atiyah-Patodi-Singer index theorem for amnifolds with boundary leads immediately to the formula

$$\mu_l \eta(P_s^{\text{sign}}) = T(\mu_l L_l)[M] \in R/Z$$

when this R/Z class is evaluated on the fundamental cycle. The results of Millson [26, 27] can be used to calculate $T(\mu_l L_l)$ combinatorially and to give another derivation of the formula $\mu_l \eta(P_s^{\text{sign}}) \equiv \frac{-d}{p} L_l(s; \vec{q}) \mu_l \bmod Z$. Similar results hold, of course, for the other two elliptic complexes.

A lot of information is lost by clearing denominators. We discussed earlier the invariant $|G| \operatorname{ind}(\rho, P)$. We study the example: $P = P^{\text{sign}}$, $p = 3$, $\vec{q} = (1,1,1,1,1,1)$, and $l = 6$. Let $M = L(3; \vec{q})$ then we shall see $K_0(M) = Z_{27} \oplus Z_9$ with generators $V_{(\rho_1 - 1)}$ and $V_{(\rho_2 - 2\rho_1 + 1)}$ in Sect. 5. Theorem 2.5 permits us to compute:

$$\begin{aligned} 3 \cdot \operatorname{ind}(\rho_s, \text{sign}) &\equiv -(4 s^6 + 10 s^4 p_1(3, \vec{q}) + 2 s^2 (7 p_2 - p_1^2)(3, \vec{q}))/45 \\ &\equiv -(4 s^6 + 150 s^4 + 516 s^2)/45 \bmod Z. \end{aligned}$$

The numerator is always divisible by 5. We multiply by 5 to conclude

$$5 \cdot (3 \operatorname{ind}(\rho_s, \text{sign})) \equiv -(4 s^6 + 150 s^4 + 516 s^2)/9 \bmod Z.$$

Since $5 \cdot 2 = 10 = 1 \bmod 9$, we finally derive:

$$3 \cdot \operatorname{ind}(\rho_s, \text{sign}) \equiv (s^6 + 6 s^4 + 3 s^2)/9 \bmod Z.$$

Let $\rho = 3\rho_1 + 3\rho_2$ so that

$$3 \cdot \operatorname{ind}(\rho, \text{sign}) \equiv 6/9 = 2/3$$

and the bundle V_ρ is not topologically trivial. The cohomology of $L(3; \vec{q})$ is 3-torsion in positive even dimensions and vanishes in dimension 12. Let x

$=c_1(V_{\rho_1})=-c_1(V_{\rho_2})$. Then $c(V_\rho)=(1+x)^3(1-x)^3=1-x^6=1$ so this bundle has trivial Chern class.

The invariant $|G|\,\mathrm{ind}(*,P)$ detects vector bundles not detected by the Chern classes. On S^{11} this invariant is necessarily trivial. $3\cdot\mathrm{ind}(*,\mathrm{sign})$ is surjective to Z_9 so this invariant is not multiplicative under finite coverings and consequently cannot be expressed in terms of secondary characteristic classes. If we clear the denominators in the L-polynomial, this amounts to studying $9\cdot\mathrm{ind}(*,P)$ and this is a secondary characteristic class.

$|\pi_1|$ detects elements of order 9 in K_0 in this example. We shall see in the fifth section that $K_0\simeq Z_{27}\oplus Z_9$ for $L(3;1,1,1,1,1,1)$. The bilinear form $\mathrm{ind}(*,*,P)$ is a more subtle invariant which completely detects this group for $P=P^{\mathrm{sign}}$, P^{spin}, or P^{Dol} as we discuss in the next section.

3. K-theory of odd dimensional spherical space forms

Let M be a manifold and let $K_{\mathrm{tor}}(M)$ denote the torsion subgroup of $K(M)$. There exist simply connected manifolds so $K_{\mathrm{tor}}(M)\neq 0$. Thus $\tilde K_{\mathrm{flat}}(M)\neq K_{\mathrm{tor}}(M)$ in general [19]. Fortunately, the situation is much simpler for spherical space forms:

Theorem 3.1 (Karoubi). *Let τ: $G\to U(l)$ be a fixed point free representation of a fnite group. Let $M=S^{2l-1}/\tau(G)$, then the natural map $R_0(G)\to\tilde K(M)$ is surjective.*

Proof. This is well known. See for example [1, 10, 18, 25].

Let τ: $G\to U(l)$ be a fixed point representation of a finite group and let $M=S^{2l-1}/\tau(G)$ henceforth. Using this result, we will use the index form discussed in the first two sections to compute $\tilde K(M)$. For $\rho_i\in R(G)$, define:

$$\mathrm{ind}_\tau(\rho_1,\rho_2)=\frac{1}{|G|}\sum_{g\in G,\,g\neq I}\mathrm{Tr}(\rho_1\otimes\rho_2)(g)\det(I-\tau)^{-1}(g).$$

Let $J=J(\tau)$ be the ideal:

$$J=\{\rho_1\in R_0(G)\text{ so that }\mathrm{ind}_\tau(\rho_1,\rho_2)\in Z\text{ for all }\rho_2\in R_0(G)\}.$$

It is clear from the definition that $\mathrm{ind}_\tau(\rho_1\otimes\rho_3,\rho_2)=\mathrm{ind}_\tau(\rho_1,\rho_2\otimes\rho_3)$ so J is an ideal of $R(G)$. We first show J is a principal ideal:

Lemma 3.2. *Let $\alpha=\sum(-1)^p\Lambda^p(\tau)\in R_0(G)$, then $J=\alpha R(G)$.*

Before proving Lemma 3.2, we first review some facts from the theory of finite groups:

Lemma 3.3. *Let G be a finite group*

(a) *$R(G)$ is the free Z-group on equivalence classes of irreducible unitary representations.*

(b) *If $\rho\in R(G)$, then $\mathrm{Tr}(\rho(g))=0$ for all $g\in G$ implies $\rho=0$. Thus we may identify $R(G)$ with the ring of virtual characters. This is a subring of the ring of class functions.*

(c) *Let f_i be class functions on G. Define* $\operatorname{ind}_0(f_1,f_2)=\frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g)$. *A class function f is a virtual character if and only if* $\operatorname{ind}_0(f,\operatorname{Tr}(\rho))\in Z$ *for all* $\rho\in R(G)$.

Proof. See for example [16]. We have defined the inner product without the usual complex conjugate to make it symmetric; this makes no difference and will be more convenient for our purposes.

We now prove Lemma 3.2. From the definition we have the identity:

$$\operatorname{Tr}\alpha(g)=\det(I-\tau(g)).$$

Let $\tilde{\rho}\in R_0(G)$. Then $\tilde{\rho}(I)=0$ so that

$$\begin{aligned}\operatorname{ind}_\tau(\alpha,\tilde{\rho})&=\frac{1}{|G|}\sum_{g\in G,\, g\neq I}\operatorname{Tr}(\tilde{\rho}(g))\operatorname{Tr}(\alpha(g))\det(I-\tau(g))^{-1}\\ &=\frac{1}{|G|}\sum_{g\in G,\, g\neq I}\operatorname{Tr}(\tilde{\rho}(g))=\frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}(\tilde{\rho}(g))\\ &=\operatorname{ind}_0(\tilde{\rho},1)\in Z\end{aligned}$$

so that $\alpha\in J$. Conversely, let $\rho\in J$ and define the class function:

$$f(g)=\begin{cases}\operatorname{Tr}(\rho(g))\cdot\operatorname{Tr}(\alpha(g))^{-1} & \text{if } g\neq I\\ -\sum_{g\in G,\, g\neq I}\operatorname{Tr}(\rho(g))\cdot\operatorname{Tr}(\alpha(g))^{-1} & \text{if } g=I\end{cases}.$$

If we can show f is a generalized character with $f(g)=\operatorname{Tr}(\beta(g))$ then clearly $\operatorname{Tr}(\alpha\otimes\beta)(g)=\operatorname{Tr}(\rho(g))$ for $g\neq I$. However $\operatorname{Tr}(\alpha\otimes\beta)(I)=\operatorname{Tr}(\rho(I))=0$ so that $\alpha\otimes\beta=\rho$.

We defined f so $\operatorname{ind}_0(f,1)=0$. Thus it suffices to check $\operatorname{ind}_0(f,\operatorname{Tr}(\tilde{\rho}))\in Z$ for all $\tilde{\rho}\in R_0(G)$. However we defined f so $\operatorname{ind}_0(f,\operatorname{Tr}(\tilde{\rho}))=\operatorname{ind}_\tau(\rho,\tilde{\rho})$ is an integer by hypothesis and this completes the proof.

We now relate ind_τ to the invariants of the first and second section:

Lemma 3.4. *Let $\tau\colon G\to U(l)$ be a fixed point free representation of a finite group G and let $M=S^{2l-1}/\tau(G)$ be an odd dimensional spherical space form. Let* $\operatorname{ind}(*,*,\mathrm{Dol})$ *and* $\operatorname{ind}(*,*,\mathrm{sign})$ *be the Q/Z valued bilinear forms on $R_0(G)$ defined before. Let $\alpha=\sum(-1)^p\Lambda^p(\tau)\in R_0(G)$, let $J=\alpha R(G)$ and let $\rho\in R_0(G)$.*

(a) $\operatorname{ind}(\rho,\tilde{\rho},\mathrm{Dol})\equiv 0$ *in Q/Z for all $\tilde{\rho}\in R_0(G)$* iff $\rho\in J$

(b) *$R_0(G)/J$ is a finite group and $|R_0(G)/J|$ divides $|G|^{(l+1)(|G|-1)}$.*

(c) *Let $|G|$ be odd.* $\operatorname{ind}(\rho,\tilde{\rho},\mathrm{sign})\equiv 0$ *in Q/Z for all $\rho\in R_0(G)$* iff $\rho\in J$.

Proof. Set $\beta=\Lambda^l(\tau)$. This is a one-dimensional representation and hence a unit in $R(G)$. The results of sections one and two show $\operatorname{ind}(\rho,\tilde{\rho},\mathrm{Dol})=\pm\operatorname{ind}_\tau(\rho,\beta\otimes\tilde{\rho})$. Thus the first assertion follows from Lemma 3.2. Let $r=|G|$. If λ is an eigenvalue of $\tau(g)$ then $\lambda^r=1$. In section two we noted $r/(\lambda-1)$ is an algebraic integer so $r^l\det(I-\tau(g))$ is an algebraic integer. This implies $r^{l+1}\operatorname{ind}_\tau(\rho,\tilde{\rho})$ is an algebraic integer. Since it is rational, it is an integer so $r^{l+1}R_0(G)$ is contained in J; the order of any element of $R_0(G)/J$ must divide r^{l+1}. Since $R_0(G)$ is generated by at most $|G|-1$ elements, (b) follows.

Define $\gamma=\sum \Lambda^p(\tau)$ so $\operatorname{ind}(\rho,\tilde\rho,\text{sign})=\pm\operatorname{ind}_\tau(\rho,\gamma\otimes\tilde\rho)$. This shows $\operatorname{ind}(\rho,\tilde\rho,\text{sign})\equiv 0 \bmod Z$ if $\rho\in J$. Unfortunately, γ is not a unit in $R(G)$ so (c) does not follow directly (and in fact (c) is false if $G=Z_2$ as we shall shortly see). Let $r=|G|$ be odd and expand

$$(1+x)(1-x+x^2-\ldots+x^{r-1})=1+x^r.$$

Let $g\in G$ and let $\{\lambda_\nu\}$ be the complex eigenvalues of $\tau(g)$. Define the class function:

$$f(g)=(-1)^l\prod_\nu\{1-\lambda_\nu+\lambda_\nu^2-\ldots+\lambda_\nu^{r-1}\}$$

so that:

$$f(g)\operatorname{def}(\tau(g),\text{sign})=\prod_\nu\{(1+\lambda_\nu^r)/(1-\lambda_\nu)\}=2^l\det(I-\tau(g))^{-1}.$$

Let $s_i(\tau(g))=\operatorname{Tr}(\Lambda^i(g)))$ be the i^{th} elementary symmetric function of the eigenvalues. $f(g)$ is a symmetric polynomial in the eigenvalues with integral coefficients so we can write $f(g)=F(s_1(\tau(g)),\ldots,s_l(\tau(g)))$ in terms of the elementary symmetric functions. This proves f is a generalized character so $f(g)=\operatorname{Tr}(\delta(g))$. This has been chosen so that:

$$\operatorname{ind}(\rho,\delta\otimes\tilde\rho,\text{sign})=2^l\operatorname{ind}_\tau(\rho,\tilde\rho).$$

If $\operatorname{ind}(\rho,\tilde\rho,\text{sign})\equiv 0$ in Q/Z for all $\tilde\rho\in R_0(G)$ then $2^l\rho$ is 0 in $R_0(G)/J$. This group has odd order if $|G|$ is odd and thus $\rho\in J$ which completes the proof.

The element $\alpha=\sum(-1)^p\Lambda^p(\tau)$ plays a distinguished role in this discussion. Let $s(x)=x\in C^l$ for $x\in S^{2l-1}$. This gives a τ invariant section to the bundle $S^{2l-1}\times C^l$ and defines a global section to V_τ over M. We write $V_\tau=1\oplus V_1$ topologically. This is just the geometric fact that $T(M)\oplus 1=V_\tau^{\text{real}}$ which defines the Cauchy-Riemann structure on M. Thus $\Lambda^p(V_\tau)=\Lambda^p(V_1)\oplus\Lambda^{p-1}(V_1)$ and $V_\alpha=0$ in $\tilde K(M)$. This proves:

Lemma 3.5. *Let $\tau\colon G\to U(l)$ be a fixed point free representation of a finite group G and let $M=S^{2l-1}/\tau(G)$. Let $\alpha=\sum(-1)^p\Lambda^p(\tau)$, then $V_\alpha=0$ in $\tilde K(M)$.*

We can now determine the K-theory of an odd dimensional space form

Theorem 3.6. *Let $\tau\colon G\to U(l)$ be a fixed point free representation of a finite group G and let $M=S^{2l-1}/\tau(G)$. Let $\alpha=\sum(-1)^p\Lambda^p(\tau)$. The natural map $R_0(G)\to\tilde K(M)$ is surjective and induces an isomorphism $R_0(G)/\alpha R(G)\simeq\tilde K(M)$. The Q/Z valued bilinear form* $\operatorname{ind}(\rho_1,\rho_2,\text{Dol})$ *is non-singular on $\tilde K(M)$. If M admits a spin structure, the form* $\operatorname{ind}(\rho_1,\rho_2,\text{spin})$ *is non-singular as well. If $|G|$ is odd, the form* $\operatorname{ind}(\rho_1,\rho_2,\text{sign})$ *is non-singular. $\tilde K(M)$ is a finite group and $|\tilde K|$ divides $|G|^{(l+1)(|G|-1)}$.*

Proof. By Lemma 3.1 and 3.5, the map $R_0(G)/\alpha R(G)\to\tilde K(M)$ is well defined and surjective. Fix $\rho\in R_0(G)$. If $V_\rho=0$ in $\tilde K(M)$ then $\operatorname{ind}(\rho,\tilde\rho,\text{Dol})\equiv 0$ for all $\tilde\rho\in R_0(G)$ by Theorem 1.3. This implies $\rho\in J=\alpha R(G)$ so the map is injective and $\operatorname{ind}(\rho,\tilde\rho,\text{DOL})$ is non-singular. If M is spin, then $\operatorname{ind}(\rho,\tilde\rho,\text{spin})=\operatorname{ind}(\rho,\beta\otimes\tilde\rho,\text{Dol})$ for some unit β of $R(G)$ so this is non-singular as well. The remainder of the theorem follows directly from Lemma 3.4.

We specialize briefly to the case $G=Z_2$. $R_0(G)$ is generated by a single element $x=\rho_1-1$ and it is immediate that for the lensspace $RP_{2l-1}=S^{2l-1}/\pm I$ that:

$$\mathrm{ind}(kx,x,\mathrm{Dol})=\tfrac{1}{2}(k(-1-1)^2(-1)^l/(-1-1)^l)=k\cdot 2^{-l+1}$$

so $\tilde{K}(RP_{2l-1})$ is a cyclic group of order 2^{l-1}; the ring structure is given by $x^2=-2x$. On the other hand, it is also immediate that:

$$\mathrm{ind}(kx,jx,\mathrm{sign})=0$$

so the signature complex does not detect 2-torsion well as noted previously. We will discuss in more detail $\tilde{K}(L(p;\vec{q}))$ where p is prime in section 5.

The structure of $\tilde{K}(M)$ is well known. For G Abelian it is in [2]; for non-Abelian G it follows from the equivariant case discussed in [1]. We also refer to [10, 17, 18, 20–25] for other results in this area. What is new is the use of the eta invariant to get at rather easily the structure of $\tilde{K}_{\mathrm{flat}}(M)$ using the group representation theory. One then needs to use the somewhat deeper topological fact that $\tilde{K}(M)=\tilde{K}_{\mathrm{flat}}(M)$ to complete the proof. We hope the non-degeneracy will hold true in greater generality and will facilitate computations in other instances. We are presently studying the case of flat manifolds using these techniques; i.e. manifolds with 0 sectional curvature metrics.

4. The reduction to the Abelian case

Theorem 3.6 shows $\tilde{K}(M)=R_0(G)/\alpha R(G)$ and reduces the calculation to an algebraic problem. In this section, we will use the index form to reduce the problem in a certain sense to the Abelian case. Let H be a subgroup of G and let i: $H\to G$ be the inclusion. Restriction defines a map i^*: $R(G)\to R(H)$. Frobenius reciprocity defines a map i_*: $R(H)\to R(G)$; these two maps are dual on R_0 with respect to the forms ind_τ and ind_0. We will use these two additive morphisms to prove:

Theorem 4.1. *Let τ: $G\to U(l)$ be a fixed point free representation of a finite group G. Let $\alpha=\Sigma(-1)^k\Lambda^k(\tau)$ and $M=S^{2l-1}/\tau(G)$. $\tilde{K}(M)\simeq R_0(G)/\alpha R(G)$.*

(a) *Let $\rho\in R_0(G)$. Then $\rho\in\alpha R(G)$ if and only if $i_H^*(\rho)\in i_H^*(\alpha)R(H)$ for every Abelian subgroup of G with $|H|$ a prime power.*

(b) *Let $\tilde{\tau}$: $G\to U(l)$ be another representation. Let $\tilde{\alpha}=\Sigma(-1)^k\Lambda^k(\tilde{\tau})$. Then $\tilde{\alpha}\in\alpha R(G)$. If $\tilde{\tau}$ is fixed point free, then $\alpha\in\tilde{\alpha}R(G)$. Thus $\tilde{K}(M)$ depends only on (G,l) and not on the particular defining τ chosen.*

(c) *Let $\{\rho_v\}_{1\leq v\leq l}\in R_0(G)$. Then $\Pi\rho_v\in\alpha R(G)$ so the index of nilpotency for $\tilde{K}$ is at most l.*

Theorem 4.1(a) can be rephrased topologically. Let $V\in\tilde{K}(M)$, then $V=0$ iff $i^*(V)=0$ in $\tilde{K}(\tilde{M})$ for every lens space $\tilde{M}$ which covers M. Thus the calculation of $\tilde{K}$ in general reduces in a sense to that of lensspaces $L(p;\vec{q})$ with p a prime power.

The remainder of this section is devoted to the proof of Theorem 4.1. We will be using Frobenius reciprocity extensively in what follows so review briefly the facts we shall need. Let G be a finite group and H a subgroup. Let f: $H \to C$; f is a class function if $f(xhx^{-1})=f(x)$ for all $x, h \in H$. f is said to be a generalized character of H if $f(h)=\mathrm{Tr}(\rho(h))$ for some $\rho \in R(H)$ and all h. The orthogonality relations imply ρ is unique and the map $\rho \to \mathrm{Tr}(\rho)$ embeds $R(H)$ as a subring of the ring of class functions on H.

Let $f^0(g)=f(g)$ for $g \in H$ and 0 otherwise. This need not be a class function. Let

$$i_* f(g)=f^G(g)=\frac{1}{|H|}\sum_{x\in G} f^0(xgx^{-1})=\sum_{x\in G/H} f^0(xgx^{-1}).$$

This is a class function on G and it is immediate (see [16])

$$\mathrm{ind}_0^G(i_* f, e)=\frac{1}{|G|}\sum_{x\in G} f^G(x)\,e(x)=\frac{1}{|H|}\sum_{h\in H} f(h)\,e(h)=\mathrm{ind}_0^H(f, i^* e)$$

for any class function e on G. Consequently, if f is a generalized character of H then f^G is a generalized character of G. If $\rho \in R(H)$, we let $\rho^G=i_*(\rho)$ be the induced representation defined by $\mathrm{Tr}(\rho^G)(g)=(\mathrm{Tr}(\rho)^G)(g)$. We will often use "1" as both the constant class function and as the trivial representation. We note $(1_H)^G \neq 1_G$ in general.

Let τ: $G \to U(l)$ be a fixed point free representation. Let $\alpha=\Sigma(-1)^p \Lambda^p(\tau)$, $\rho \in R_0(G)$, and $\tilde\rho \in R_0(H)$. Let $f(h)=\mathrm{Tr}(\tilde\rho(h))$ and $e(I)=0$, $e(g)=\mathrm{Tr}(\rho(g))/\mathrm{Tr}(\alpha(g))$ for $g \neq I$. Then:

$$\mathrm{ind}_\tau(\rho, i_*\tilde\rho)=\mathrm{ind}_0^G(e, i_* f)=\mathrm{ind}_0^H(i^* e, f)=\mathrm{ind}_{i^*(\tau)}(i^*(\rho), \tilde\rho)$$

so i^* and i_* are also adjoint with respect to ind_τ.

For any group, the right regular representation is defined by:

$$tr(r_G(g))=\begin{Bmatrix} |G| & \text{if } g=I \\ 0 & \text{if } g\neq I \end{Bmatrix}.$$

Suppose $G=H\oplus H'$ is a direct sum. We extend representations on one factor to be trivial on the other so $R(G)=R(H)\otimes R(H')$. Let r_H, $r_{H'}$ be the right regular representations of these two subgroups. If $\rho \in R(H)$ then $\rho^G=\rho\otimes r_{H'}$. $i^* i_*=|H'|$ on $R(H)$.

We begin our study of $\tilde K$ with the Abelian case. Let $|G|=p$ be Abelian where p need not be prime. Identify G with the p^{th} roots of unity in C and let $\rho_s(\lambda)=\lambda^s$ where $\lambda \in C$, $\lambda^p=1$. The $\{\rho_s\}_{0\leq s<p}$ parametrize the irreducible representations of G which are all one dimensional. Let τ: $G\to U(l)$; we choose a basis for C^l which diagonalizes τ to decompose $\tau=\tau_{q_1}\oplus\ldots\oplus\tau_{q_l}$. τ is fixed point free if and only if the q's are coprime to p. We set $\vec q=(q_1,\ldots,q_l)$; the resulting quotient is the lens space $L(p;\vec q)$. In this situation:

$$\alpha=\Sigma(-1)^k \Lambda^k(\tau)=\Pi(1-\rho_{q_\nu}).$$

The identities:

$$(1-\rho_q)=(1-\rho_1)(1+\rho_1+\ldots+\rho_{q-1}) \quad \text{and}$$
$$(1-\rho_1)=(1-\rho_1^{qk})=(1-\rho_q)(1+\rho_q+\rho_{2q}+\ldots+\rho_{q(k-1)}) \quad \text{where } kq\equiv 1\,(p)$$

implies $\alpha R(G)=(1-\rho_1)^l R(G)$ so $\tilde{K}$ only depends on (p,l) in this situation. We may therefore set all the q's$=1$. We define $x=\rho_1-1$. The ρ_s-1 generate $R_0(G)$ as an Abelian group. The identity $\rho_s-1=\sum_{1\leqq k\leqq s}\binom{s}{k}(\rho_1-1)^k$ implies $xR(G)=R_0(G)$. If we take $s=p$ then:

$$p\cdot x+\sum_{2\leqq k\leqq p}\binom{p}{k}x^k=0 \qquad \text{so } px\in x^2R(G).$$

Therefore $|R_0(G)/x^2R(G)|=|xR(G)/x^2R(G)|\leqq p$. Let $\tau_l=\rho_1\oplus\ldots\oplus\rho_1$ (l-times) and $\mathrm{ind}_l=\mathrm{ind}_{\tau_l}$. It is immediate that:

$$\mathrm{ind}_l(x\rho,\tilde{\rho})=-\mathrm{ind}_{l-1}(\rho,\tilde{\rho}).$$

We compute:

$$\mathrm{ind}_2(jx,x)=\frac{1}{p}\sum_{\lambda^p=1,\lambda\neq 1}j(\lambda-1)^2/(1-\lambda)^2\equiv\frac{-j}{p}\bmod Z$$

so the elements $\{jx\}_{0\leqq j<p}$ are distinct in $R_0(G)/x^2R(G)$. This shows $R_0(G)/x^2R(G)\simeq Z_p$.

The natural map x: $x^{l-1}R(G)/x^lR(G)\to x^lR(G)/x^{l+1}R(G)$ is surjective. Let $\rho\in R_0(G)$ and suppose $x\rho\in x^{l+1}R(G)$. Then $\mathrm{ind}_l(\rho,\tilde{\rho})=-\mathrm{ind}_{l+1}(x\rho,\tilde{\rho})\equiv 0\bmod Z$ for all $\tilde{\rho}\in R_0(G)$ so $\rho\in x^lR(G)$ and the map is bijective. We use induction starting with the case $l=2$ to conclude that $|x^{l-1}R(G)/x^lR(G)|=p$ for $l\geqq 2$. The short exact sequence:

$$0\to Z_p\simeq x^{l-1}R(G)/x^lR(G)\to R_0(G)/x^lR(G)\to R_0(G)/x^{l-1}R(G)\to 0$$

shows $|R_0(G)/x^lR(G)|=p^{l-1}$. We summarize these calculations in:

Lemma 4.2. *Let $G\simeq Z_p$ be a cyclic group of order p (where p need not be prime). Let $\tau=\rho_1\otimes 1^l=\rho_1\oplus\ldots\oplus\rho_1$ (l-times) and $\alpha=\Sigma(-1)^k\Lambda^k(\tau)=(1-\rho_1)^l=(-1)^lx^l$.*

(a) *$|R_0(G)/\alpha R(G)|=p^{l-1}$. If $l=2$ then $R_0(G)/\alpha R(G)\simeq Z_p$.*

(b) *Let $\tilde{\tau}$: $G\to U(l)$ be another representation not necessarily fixed point free. Let $\tilde{\alpha}=\Sigma(-1)^k\Lambda^k(\tilde{\tau})$. Then $\tilde{\alpha}\in\alpha R(G)$. If $\tilde{\alpha}$ is fixed point free then $\alpha\in\tilde{\alpha}R(G)$ so $\tilde{K}(S^{2l-1}/\tau(G))\simeq\tilde{K}(S^{2l-1}/\tilde{\tau}(G))$.*

(c) *$\{R_0(G)\}^l=\alpha R(G)$.*

For Abelian fundamental groups, we can study each prime separately:

Lemma 4.3. *Let $G=Z_m\oplus Z_n$ where m and n are coprime. Let τ: $G\to U(l)$ be a fixed point free representation. Let $\alpha=\Sigma(-1)^k\Lambda^k(\tau)$. We have natural inclusions i_m: $Z_m\to G$ and i_n: $Z_n\to G$ which induce dual maps i_m^*: $R(G)\to R(Z_m)$ and i_n^*: $R(G)\to R(Z_n)$. Then $\rho\in R_0(G)$ satisfies $\rho\in\alpha R(G)$ if and only if $i_m^*(\rho)\in i_m^*(\alpha)R(Z_m)$ and $i_n^*(\rho)\in i_n^*(\alpha)R(Z_n)$. Thus the map*

$$i_m^*\oplus i_n^*:\ R_0(G)/\alpha R(G)\to R_0(Z_m)/i_m^*(\alpha)R(Z_m)\oplus R_0(Z_n)/i_n^*(\alpha)R(Z_n)$$

is a ring isomorphism. The index form decomposes as a direct sum

$$\mathrm{ind}_\tau=a\cdot\mathrm{ind}_{i_m^*(\tau)}\oplus b\cdot\mathrm{ind}_{i_n^*(\tau)}$$

where a and b are integers so $a\cdot n\equiv 1(m^l)$ and $b\cdot m\equiv 1(n^l)$.

Proof. Since G decomposes as a direct sum, i_m^*: $R(G)\to R(Z_m)$ and i_n^*: $R(G)\to R(Z_n)$ are surjective. Thus i_m^*: $R_0(G)/\alpha R(G)\to R_0(Z_m)/i_m^*(\alpha)R(Z_m)$ and i_n^*: $R_0(G)/\alpha R(G)\to R_0(Z_m)/i_m^*(\alpha)R(Z_m)$ are surjective separately. The orders of the two image groups are m^{l-1} and n^{l-1} which are coprime. Therefore $i_m^*\oplus i_n^*$ is surjective. Both the image and range are finite groups of order $(mn)^{l-1}$ so $i_m^*\oplus i_n^*$ is bijective.

The inverse map is given by Frobenius reciprocity. We regard $R(Z_m)$ and $R(Z_n)$ as subrings which generate $R(G)$. Let r_m and r_n be the right regular representations of Z_m and Z_n. Let $a\cdot n\equiv 1(m^l)$ and $b\cdot m\equiv 1(n^l)$. Set $\sigma_m=a\cdot r_n$ and $\sigma_n=b\cdot r_m$. Then $\sigma_m\sigma_n R_0(G)=0$ so the ideals $\sigma_m R_0(G)$ and $\sigma_n R_0(G)$ are orthogonal with respect to ind_τ. Furthermore:

$$\operatorname{ind}_\tau(\sigma_m\rho,\sigma_m\tilde\rho)=a^2 n\cdot\operatorname{ind}_{i_m^*(\tau)}(i_m^*(\rho),i_m^*(\tilde\rho))$$
$$\operatorname{ind}_\tau(\sigma_n\rho,\sigma_n\tilde\rho)=b^2 m\cdot\operatorname{ind}_{i_n^*(\tau)}(i_n^*(\rho),i_n^*(\tilde\rho)).$$

If $\rho\in R_0(Z_m)$ then $i_m^*\sigma_m\rho=a\cdot n\rho$. Since $a\cdot n\equiv 1\bmod |R_0(Z_m)/i_m^*(\alpha)R(Z_m)|$ we conclude $i_m^*\sigma_m=1$ and $\operatorname{ind}_\tau=a\operatorname{ind}_{i_m^*(\tau)}$ on this subring. The index n is the same which completes the proof.

Before studying the general non-Abelian case, we must first review some general facts concerning groups which admit fixed point free representations. We refer to [28, 30] for details.

Lemma 4.4. *Let G be a finite group and suppose there exists τ: $G\to U(l)$ which is a fixed point free representation.*

(a) *If $|G|>2$ and if τ is irreducible, then τ is not real.*

(b) *If G is Abelian, then G is cyclic.*

(c) *Let G be a p-group for an odd prime p. Then G is cyclic.*

(d) *Let G be a 2-group which is not cyclic. Then $G=Q(a)$ is a generalized quaternionic group for some $a\geqq 3$. $Q(a)$ is generated by 2 elements A and B with: $A^m=B^4=I$, $BAB^{-1}=A^{-1}$, $B^2=A^{m/2}$ where $m=2^{a-1}\cdot|Q(a)|=2^a$.*

(e) *A subgroup H of G is said to be elementary if $H=H_1\oplus H_2$ where H_1 is a p-group and H_2 is cyclic. The elementary subgroups of such a G are either cyclic or of the form $H=Q(a)\oplus Z_m$ for m-odd.*

Brauer's theorem (see [16]) can be used to induction over elementary subgroups:

Lemma 4.5. *Let f: $G\to C$ be a class function. f is a generalized character if and only if $i_H^*(f)$ is a generalized character of H for every elementary subgroup H of G.*

The groups $Q(a)$ play a distinguished role in the study of groups admitting fixed point representations since they are the only non-cyclic Sylow subgroups possible. We now establish some technical facts concerning the $Q(a)$:

Lemma 4.6. *Let $G=Q(a)$ for $a\geqq 3$. Let $m=2^{a-1}$ and $\lambda=\exp(2\pi i/m)$. Define:*

$$\pi_k(A)=\begin{pmatrix}\lambda^k & 0\\ 0 & \lambda^{-k}\end{pmatrix},\ \rho_0(A)=1,\ \rho_1(A)=-1,\ \rho_2(A)=1,\ \rho_3(A)=-1$$

$$\pi_k(B)=\begin{pmatrix}0 & 1\\ (-1)^k & 0\end{pmatrix},\ \rho_0(B)=1,\ \rho_1(B)=1,\ \rho_2(B)=-1,\ \rho_3(B)=-1.$$

(a) *The conjugacy classes of G are represented by the $\frac{m}{2}+3$ elements $\{A^j\}_{0\leqq j\leqq m/2}$, B, AB.*

(b) *The π_k, ρ_j extend to representations of G. $\pi_0=\rho_0\oplus\rho_2$, $\pi_{m/2}=\rho_1\oplus\rho_3$. The $\{\pi_k\}_{0<k<m/2}$, $\{\rho_j\}_{0\leqq j\leqq 3}$ are irreducible, inequivalent, and parametrize all the irreducible representations of G. π_k is fixed point free if and only if k is odd.*

(c) *Let $\rho\in R(G)$. Then $\rho\in\operatorname{span}\{\pi_k\}$ if and only if $\operatorname{Tr}(\rho(B))=\operatorname{Tr}(\rho(AB))=0$.*

(d) *Let H_1, H_2, H_3 be the cyclic subgroups of G generated by A, B, AB respectively. Let i_j: $H_j\to G$ be the inclusions and $(i_j)_*$: $R(H_j)\to R(G)$ given by Frobenius reciprocity. Then $R_0(G)$ is generated additively by $\{(i_j)_* R_0(H_j)\}_{1\leqq j\leqq 3}$.*

(e) *Let τ: $G\to U(l)$ be a fixed point free representation of G. Let $\alpha=\Sigma(-1)^k\Lambda^k(\tau)$. Let $\rho\in R_0(G)$, then $\rho\in\alpha R(G)$ iff $i_j^*(\rho)\in i_j^*(\alpha)R(H_j)$ for $1\leqq j\leqq 3$.*

Proof. (a) and (b) are elementary computations. They are well known [28, 30] and we omit the proofs. In (c), we permit the values $k=0$ and $k=m/2$. Since $\pi_0=\rho_0+\rho_2$ and $\pi_{m/2}=\rho_1+\rho_3$, we can express any $\rho\in R(G)$ in the form:

$$\rho=\sum_{0\leqq k\leqq m/2} n_k\pi_k+c_0\rho_0+c_1\rho_1.$$

$\operatorname{Tr}(\pi_k)$ vanishes on both (B) and (AB) so

$$\operatorname{Tr}(\rho)(B)=c_0+c_1,\qquad \operatorname{Tr}(\rho)(AB)=c_0-c_1.$$

These both vanish if and only if $c_0=c_1=0$ which proves (c).

Let $\mathscr{R}=\operatorname{span}(i_j)_*(R_0(H_j))_{1\leqq j\leqq 3}$. Since H_1 is a normal subgroup of G of index 2, we compute easily that if $\beta_k(A)=\lambda^k$ then $\beta_k^G=\pi_k$ so $\pi_j-\pi_k\in\mathscr{R}$ for any (j,k).

H_2 is generated by B so $H_2\simeq Z_4$. We let $\gamma_j(B)=(\sqrt{-1})^j$ parametrize the irreducible representations of B. If f a class function on B, then:

$$\begin{aligned} &f^G(\pm I)=\frac{m}{2}f(\pm I) && f^G(A^j)=0 \quad \text{for } 1\leqq j<m/2\\ &f^G(AB)=0 && f^G(B)=f(B)+f(B^3).\end{aligned}$$

Consequently, we can compute a character table:

	I	$-I$	B	AB	$A^j(1\leqq j<m/2)$
$\operatorname{Tr}(\gamma_0^G)$	$\frac{m}{2}$	$\frac{m}{2}$	2	0	0
$\operatorname{Tr}(\gamma_1^G)$	$\frac{m}{2}$	$-\frac{m}{2}$	0	0	0
$\operatorname{Tr}(\rho_0)$	1	1	1	1	1
$\operatorname{Tr}(\rho_1)$	1	1	1	-1	$(-1)^j$
$\operatorname{Tr}(\rho_2)$	1	1	-1	-1	1
$\operatorname{Tr}(\rho_3)$	1	1	-1	1	$(-1)^j$

Consequently, $\operatorname{Tr}(\gamma_0^G-\rho_0-\rho_1)$ and $\operatorname{Tr}(\gamma_1^G)$ vanish on B and on AB so are in $\operatorname{span}\{\pi_j\}$. From this it follows $(\gamma_0^G-\gamma_1^G-\rho_0-\rho_1+\pi_{m/2})\in\operatorname{span}\{\pi_j-\pi_k\}\in\mathscr{R}$. Since

$\gamma_0^G-\gamma_1^G\in\mathscr{R}$ we conclude $\rho_0+\rho_1-\pi_{m/2}=\rho_0+\rho_1-\rho_1-\rho_3=\rho_0-\rho_3\in\mathscr{R}$. If we work with the subgroup generated by AB instead of B, we must reverse the roles of ρ_1 and ρ_3 so we conclude $\rho_0-\rho_1\in\mathscr{R}$. Since $\rho_2-\rho_0=(\rho_1-\rho_0)+(\rho_3-\rho_0)+(\rho_0+\rho_2)-(\rho_1+\rho_3)$ we conclude $\rho_2-\rho_0\in\mathscr{R}$. Then $\pi_k-2\rho_0=\pi_k-\pi_0+(\rho_2-\rho_0)\in\mathscr{R}$ so $\mathscr{R}=R_0(G)$ which proves (d).

Finally, one direction of (e) is clear. Conversely, let $\rho\in R_0(G)$ satisfy $i_j^*(\rho)\in i_j^*(\alpha)R(H_j)$. Then $\mathrm{ind}_\tau(\rho,\tilde\rho^G)=\mathrm{ind}_{i_j^*(\tau)}(i_j^*(\rho),\tilde\rho)=0$ for all $\tilde\rho\in R_0(H_j)$. By (d) this implies $\mathrm{ind}_\tau(\rho,\tilde\rho)=0$ for all $\tilde\rho\in R_0(G)$ which completes the proof of (e).

We now consider a group of the form $G=Q(a)\oplus Z_q$ where q is odd. This is the most general possible non-cyclic elementary subgroup of a group admitting a fixed point free representation. The following lemma is somewhat technical and will be used to study the general case using Brauer's theorem.

Lemma 4.7. *Let $a\geqq 3$ and let q be odd. Let $G=Q(a)\oplus Z_q$ and let C generate Z_q. Let (A,B,π_k,ρ_j) be as in Lemma 4.6. Let $\rho_s^q(C)=\exp(2\pi i s/q)$. The $\{\pi_k,\rho_j,\rho_s^q\}$ generate $R(G)$.*

(a) *Let $\tau_{k,s}^1=\pi_k\rho_s$ and $\alpha_{k,s}^1=\Sigma(-1)^\nu\Lambda^\nu(\tau_{k,s}^1)$. $\tau_{k,s}^1$ is an irreducible fixed point free 2-dimensional representation if k is odd and s coprime to q. For any (k,s) $\alpha_{k,s}^1\in\alpha_{1,1}^1R(G)$. If $\tau_{k,s}^1$ is fixed point free $\alpha_{1,1}^1\in\alpha_{k,s}^1R(G)$.*

(b) *$(1-\rho_1^q)^2\in\alpha_{1,1}^1R(G)$ and $2(1-\rho_j)\in\alpha_{1,1}^1R(G)$.*

(c) *Let $\tau,\tilde\tau\colon G\to U(l)$ be arbitrary fixed point free representations. Let $\alpha=\Sigma(-1)^\nu\Lambda^\nu(\tau)$ and $\tilde\alpha=\Sigma(-1)^\nu\Lambda^\nu(\tilde\tau)$ then $\tilde\alpha\in\alpha R(G)$ so $R_0(G)/\alpha R(G)$ depends only on (G,l) and not on the particular τ chosen.*

(d) *Let $i_1\colon Q(a)\to G$ and $i_2\colon Z_q\to G$. Let (τ,α) be as in (c). Let $\rho\in R_0(G)$. Then $\rho\in\alpha R(G)$ if and only if $i_1^*(\rho)\in i_1^*(\alpha)R(Q(a))$ and $i_2^*(\rho)\in i_2^*(\alpha)R(Z_q)$.*

Proof. Let $M=S^3/\tau_{1,1}^1(G)$. Let $\tau=\tau_{1,1}^1$ and $\tilde\tau=\tau_{k,s}^1$. Let $V_{\tilde\tau}$ be the bundle over M. If we can construct a global non-vanishing section over M, the same argument given to prove Lemma 3.5 will show $V=0$ in $\tilde K(M)$ and prove (a) by Theorem 3.6. Choose $u\in Z$ so $u\equiv k \bmod m$ and $u\equiv s\bmod q$. We use u to define:

$$\tilde s(z_1,z_2)=(z_1,z_2,z_1^u,z_2^u)\colon\ S^3\to S^3\times C^2.$$

It is clear $\tilde s$ is non-vanishing. If we can show that $\tilde s$ is equivariant with respect to the action of $\tau\oplus\tilde\tau=\tau_{1,1}^1\oplus\tau_{k,s}$ then it will descend to define the desired section on M to the bundle V. We let $\mu=\exp(2\pi i/q)$ and $\lambda=\exp(2\pi i/m)$ then:

$$\begin{aligned}(\tau\oplus\tilde\tau)(A)\cdot(\tilde s(z_1,z_2))&=(\lambda z_1,\bar\lambda z_2,\lambda^k z_1^u,\bar\lambda^k z_2^u)=(\lambda z_1,\bar\lambda z_2,(\lambda z_1)^u,(\bar\lambda z_2)^u)\\&=\tilde s(\tau(A)\cdot(z_1,z_2)),\\(\tau\oplus\tilde\tau)(B)\cdot(\tilde s(z_1,z_2))&=(z_2,-z_1,z_2^u,(-1)^k z_1^u)=(z_2,-z_1,z_2^u,(-z_1)^u)\\&=\tilde s(\tau(B)\cdot(z_1,z_2)),\\(\tau\oplus\tilde\tau)(C)\cdot(\tilde s(z_1,z_2))&=(\mu z_1,\mu z_2,\mu^s z_1^u,\mu^s z_2^u)=(\mu z_1,\mu z_2,(\mu z_1)^u,(\mu z_2)^u)\\&=\tilde s(\mu(C)\cdot(z_1,z_2))\end{aligned}$$

which proves the first assertion of (a). If k is odd and s coprime to q, we can interchange the roles of τ, $\tilde\tau$ by choosing new generators for G. This proves (a).

Let $\alpha=\alpha^1_{1,1}$. Since $\Lambda^2(\pi_1)=1$ we conclude $\alpha^1_{1,s}=(\pi_1\rho^q_s-1-\rho^q_{2s})\in\alpha R(G)$. Let $s=-1$ and compute:

$$\begin{aligned}(\rho^q_{-1}\pi_1-1-\rho^q_{-2})&=\rho^q_{-1}(\pi_1-2)+\rho^q_{-1}(2-\rho^q_1-\rho^q_{-1})\\&=\rho^q_{-1}(\pi_1-2)+\rho^q_{-2}(2\rho^q_1-1-\rho^q_2)\\&=\rho^q_{-1}(\pi_1-2)-\rho^q_{-2}(1-\rho^q_1)^2\end{aligned}$$

where "1" denotes the trivial representation of G. Since ρ^q_{-2} is a unit, this identity shows $(1-\rho^q_1)^2\in\alpha R(G)$. This proves the first part of (b). If k is odd then $\Lambda^2(\pi_k)=1$ so $\pi_k-2\in\alpha R(G)$. We compute:

$$\rho_2(\pi_1-2)=\pi_1-2\rho_2=(\pi_1-2)+2(1-\rho_2)$$

$$\rho_1(\pi_1-2)=\pi_{\left(1+\frac{m}{2}\right)}-2\rho_1=\{\pi_{\left(1+\frac{m}{2}\right)}-2\}+2(1-\rho_1)$$

$$\rho_3(\pi_1-2)=\pi_{\left(1+\frac{m}{2}\right)}-2\rho_3=\{\pi_{\left(1+\frac{m}{2}\right)}-2\}+2(1-\rho_3)$$

which completes the proof of (b).

To prove (c) we let $\tau,\tilde\tau$: $G\to U(l)$ be arbitrary fixed point free representations. Then l is even and we can decompose:

$$\begin{aligned}\tau&=\tau^1_{k_1,s_1}\oplus\ldots\oplus\tau^1_{k_v,s_v} &\quad \alpha&=\alpha^1_{k_1,s_1}\ldots\alpha^1_{k_v,s_v}\\ \tilde\tau&=\tau^1_{j_1,t_1}\oplus\ldots\oplus\tau^1_{j_v,t_v} &\quad \tilde\alpha&=\alpha^1_{j_1,t_1}\ldots\alpha^1_{j_v,t_v}.\end{aligned}$$

By (a) we can express

$$\alpha^1_{j,t}=\beta\alpha^1_{k,s}\qquad\text{for } \beta=\beta(j,t,k,s)$$

so $\tilde\alpha\in\alpha R(G)$ and similarly $\alpha\in\tilde\alpha R(G)$. This proves (c).

Let $\tau^1=\tau^1_{1,1}$ and $\alpha^1=\alpha^1_{1,1}$. Let τ: $G\to U(l)$ be a fixed point free representation. By (c) we may suppose $\tau=\tau^1\otimes 1^v$ and $\alpha=(\alpha^1)^v$ where $2v=l$. Let $m=2^{a-1}$. By (a) and (b) we know $(\pi_k-2)\in\alpha^1 R(G)$ if k is odd and $2(\rho_j-1)\in\alpha^1 R(G)$. If k is even, $\Lambda^2(\pi_k)=\rho_2$. The identity $2(\pi_k-1-\rho_2)=2(\pi_k-2)-2(\rho_2-1)$ shows $2(\pi_k-2)\in\alpha^1 R(G)$ so $2R_0(Q(a))$ is contained in $\alpha^1 R(G)$. Let r_1 be the right regular representation of $Q(a)$. Then $2(r_1-2m)\in\alpha^1 R(G)$. It is immediate $r_1^2=2mr_1$ so $(r_1-2m)^2=-2m(r_1-2m)$ and inductively we can find j so $2^j(r_1-2m)\in\alpha R(G)$.

Similarly $q(1-\rho^q_1)\in(1-\rho^q_1)^2R(Z_q)$ so $q(1-\rho^q_1)\in\alpha^1R(G)$. Since $(1-\rho^q_1)$ generates $R_0(Z_q)$, $qR_0(Z_q)$ is contained in $\alpha^1R(G)$. Let r_2 be the right regular representation of Z_q. The same argument given above permits us to find j so $q^j(r_2-q)\in\alpha R(G)$. Let $\gamma_1=q^jr_2$ and $\gamma_2=2^jr_1$. Let $\rho\in R_0(G)$ satisfy $i_1^*(\rho)\in i_1^*(\alpha)R(Q(a))$ and $i_2^*(\rho)\in i_2^*(\alpha)R(Z_q)$. Let $\tilde\rho\in R_0(G)$ and use Frobenius reciprocity:

$$\begin{aligned}\operatorname{ind}_\tau(\gamma_1\rho,\tilde\rho)&=q^j\operatorname{ind}_{i_2^*(\tau)}(i_1^*(\rho),i_1^*(\tilde\rho))=0\\ \operatorname{ind}_\tau(\gamma_2\rho,\tilde\rho)&=2^j\operatorname{ind}_{i_1^*(\tau)}(i_2^*(\rho),i_2^*(\tilde\rho))=0\end{aligned}$$

since the support of γ_2 is on $Q(a)$ and the support of γ_1 is on Z_q. By Lemma 3.2 we conclude $\gamma_1\rho\in\alpha R(G)$ and $\gamma_2\rho\in\alpha R(G)$. Let $b=2^j|Q(a)|+Q^{j+1}$ then $\gamma_1+\gamma_2-b\in\alpha R(G)$ so we conclude $b\rho\in\alpha R(G)$. However b is coprime to $|G|$

and therefore to $|R_0(G)/\alpha R(G)|$ by Lemma 3.4. This implies $\rho \in \alpha R(G)$ and completes the proof.

We have now established the technical lemmas we shall need to prove Theorem 4.1. Let G be a finite group and τ: $G \to U(l)$ be a fixed point free representation. Let $\alpha = \Sigma(-1)^k \Lambda^k(\tau)$ and let $\rho \in R_0(G)$. One direction in (a) is clear so we suppose $i_H^*(\rho) \in i_H^*(\alpha) R(H)$ for every cyclic subgroup which has a prime power order. We must show $\rho \in \alpha R(G)$.

Let H be an arbitrary subgroup of G and r_H the right regular representation. Let $\alpha_H = i_H^*(\alpha)$ and $\rho_H = i_H^*(\rho)$. The support of $\mathrm{Tr}(r_H)$ is cocentrated at I;

$$\mathrm{Tr}(r_H(h)) = \begin{cases} |H| & \text{if } h = I \\ 0 & \text{if } h \neq I \end{cases}.$$

Since $\mathrm{Tr}(\alpha_H(I)) = 0$ we conclude $\mathrm{Tr}(r_H \alpha_H) \equiv 0$ so $r_H \alpha_H = 0$. We note $\mathrm{Tr}(\alpha_H(h)) \neq 0$ for $h \neq I$. Let $\beta \in R(H)$ be such that $\beta \alpha_H = 0$. Then the support of $\mathrm{Tr}(\beta)$ is concentrated at I so $\beta = c \cdot r_H$. The orthogonality relations imply c is an integer so $\beta = k r_H$ for $k \in Z$. If $\gamma \alpha_H = \tilde{\gamma} \alpha_H$ then $\gamma = \tilde{\gamma} + k r_H$. By adding a suitable multiple of r_H we can adjust $\mathrm{Tr}(\gamma(I))$ arbitrarily within a given congruence class modulo $|H|$.

Let $G(p)$ be a Sylow p-subgroup of G; any two such for the same prime p are conjugate. If p is odd, $G(p)$ is cyclic so $i^*(\rho) \in i^*(\alpha) R(G(p))$ by hypothesis. If $p = 2$, then either $G(2)$ is cyclic or $G(2)$ is quaternionic. Since $i^*(\rho) \in i^*(\alpha) R(H)$ for all the cyclic subgroups of $G(2)$, Lemma 4.6 lets us conclude $i^*(\rho) \in i^*(\alpha) R(G(2))$ as well. We solve the equation $i^*(\rho) = \beta(p) i^*(\alpha)$ for $\beta(p) \in R(G(p))$. Let $k(p) = \mathrm{Tr}(\beta(p)(I)) \in Z$. Using the Chineese remainder theorem, we can find $k \in Z$ so $k \equiv k(p) \bmod |G(p)|$ for all primes p dividing $|G|$. Define the class function:

$$f(g) = \begin{cases} k & \text{if } g = I \\ \mathrm{Tr}(\rho(g))/\mathrm{Tr}(\alpha(g)) & \text{if } g \neq I \end{cases}.$$

If we can find $\beta \in R(G)$ so $\mathrm{Tr}(\beta) = f$ then $\rho = \alpha \beta$ which will prove (a).

By Brauer's theorem it suffices to show f is a generalized character on H for all elementary subgroups. Suppose first H is cyclic and decompose

$$H = H(p_1) \oplus \ldots \oplus H(p_v)$$

into cyclic prime power subgroups. The p's are necessarily distinct and each $H(p_j)$ is contained in a subgroup conjugate to $G(p_j)$. Thus the restriction of ρ to each $H(p_j)$ lies in the appropriate ideal. Inducation and Lemma 4.3 then implies $i_H^*(\rho) \in i_H^*(\alpha) R(H)$. Set $i_H^*(\rho) = \beta_H i_H^*(\alpha)$. If we restrict to $H(p_j)$ the same equation holds true and thus $\mathrm{Tr}(\beta_H(I)) \equiv k \bmod |H(p_j)|$. Therefore $\mathrm{Tr}(\beta_H(I)) \equiv k \bmod |H|$. We adjust the choice of β_H so $\mathrm{Tr}(\beta_H(I)) = k$ and observe therefore that $\mathrm{Tr}(\beta_H) = f$ and f is a generalized character.

If H is an elementary subgroup, either H is cyclic or $H = Q(a) \oplus Z_q$ for q-odd. We know the restriction of ρ to $Q(a)$ is in the appropriate ideal and have just shown the restriction of ρ to H is in the appropriate ideal so Lemma 4.7 shows we can solve the equation $i_H^*(\rho) = \beta_H i_H^*(\alpha)$ for $\beta_H \in R(Q(a) \oplus Z_q)$. The same argument given above shows $\mathrm{Tr}(\beta_H(I)) \equiv k \bmod |H|$ so we can adjust β_H so

$\mathrm{Tr}(\beta_H(I))=k$ and thereby show f is a generalized character on H. This completes the proof of (a). We remark that to solve $\rho=\alpha\beta$, $\mathrm{Tr}(\beta(g))$ is determined for $g\neq I$ so it is only the trace at the identity which must be adjusted suitably in each case.

We now prove (b). Let $\tilde{\tau}$: $G\to U(l)$ be an arbitrary representation and set $\tilde{\alpha}=\Sigma(-1)^k\Lambda^k(\tau)$. To show $\tilde{\alpha}\in\alpha R(G)$, it suffices to establish this in the Abelian case by (a). This follows from Lemma 4.2, and proves (b). Again, (c) follows directly from the corresponding assertion for the Abelian case and this completes the proof of Theorem 4.1.

5. Further results on $\tilde{K}$

In this final section, we will use the index form to obtain some results concerning the structure of $\tilde{K}$ for Abelian, metacyclic, and quaternionic groups. There is a vast literature concerning the specific structure of $\tilde{K}$ for spherical space forms. We refer to [10, 15, 17, 18, 20–25] for other results of this type.

In Lemma 4.2, we computed $|\tilde{K}|$ if the fundamental group is Abelian. In this lemma, we reduce the computation of $|\tilde{K}|$ in general to the case in which the defining representation is irreducible.

Lemma 5.1. *Let G be a finite group and let τ_0: $G\to U(l)$ be an irreducible fixed point free representation. Any two such have the same degree. Let τ: $G\to U(v)$ be an arbitrary fixed point free representation. Then $v=kl$ for $k\in Z$. Let $\alpha_0=\Sigma(-1)^j\Lambda^j(\tau_0)$ and $\alpha=\Sigma(-1)^j\Lambda^j(\tau)$. Then $\alpha R(G)=\alpha_0^k R(G)$ and*

$$|R_0(G)/\alpha R(G)|=|R_0(G)/\alpha_0 R(G)|^k\cdot|G|^{k-1}.$$

Proof. We can decompose $\tau=\tau_1\oplus\ldots\oplus\tau_k$ as a direct sum of irreducibles. Each τ_μ is necessarily fixed point free. By Wolf (Theorem 7.2.18) [30] the τ_μ all act on representation spaces of the same dimension which must be l and therefore $v=kl$. $\alpha R(G)=\alpha_0^k R(G)$ by Theorem 4.1. Let $n=|R_0(G)/\alpha_0 R(G)|$ and let $J=\alpha_0 R(G)$. There is a short exact sequence for $k>1$:

$$0\to J^{k-1}/J^k\to R_0(G)/J^k\to R_0(G)/J^{k-1}\to 0.$$

If we can show $|J^{k-1}/J^k|=n\cdot|G|$ then the lemma will follow by induction.

The natural map α_0^{k-2}: $J/J^2\to J^{k-1}/J^k\to 0$. Let $\rho\in R_0(G)$ and suppose $\alpha_0^{k-2}\rho\in J^k$. Let $\tau_\mu=\tau_0\otimes 1^\mu$ and $\mathrm{ind}_\mu=\mathrm{ind}_{\tau_\mu}$. Then $\mathrm{ind}_k(\alpha_0^{k-2}\rho,\tilde{\rho})=\mathrm{ind}_2(\rho,\tilde{\rho})=0$ for $\tilde{\rho}\in R_0(G)$. Lemma 3.2 shows $\rho\in J^2$ so this map is $1-1$. Consequently it suffices to show $|J/J^2|=n\cdot|G|$ to complete the proof of the lemma.

Multiplication by α induces a map f: $R(G)\to J/J^2\to 0$. If $f(\rho)=0$ then $\alpha\rho=\alpha^2\beta$. Thus $\mathrm{Tr}(\alpha(\rho-\alpha\beta))(g)=0$ implies $\mathrm{Tr}(\rho-\alpha\beta)(g)=0$ for $g\neq I$. If r_G is the right regular representation of G then $\rho=\alpha\beta+jr_G$ for some $j\in Z$. Conversely since $\alpha r_G=0$ such an element is necessarily in the kernel of f so $\ker(f)=r_G R(G)+\alpha R(G)$. The short exact sequence:

$$0\to R_0(G)\to R(G)\to Z\to 0$$

together with the observation $\mathrm{Tr}(\alpha(I))=0$ and $\mathrm{Tr}(r_G(I))=|G|$ induces a short exact sequence:

$$0\to R_0(G)/\alpha R(G)\to R(G)(r_G R(G)\oplus\alpha R(G))\to Z_{|G|}\to 0$$

so finally:

$$|J/J^2|=|R(G)(r_G R(G)\oplus\alpha R(G))=|R_0(G)/\alpha R(G)|\cdot|G|$$

which completes the proof.

Lemma 4.3 reduces the calculation of $\tilde{K}$ for an Abelian group to the case in which $|G|$ is a prime power. The structure in general is somewhat complicated. We can, however, obtain fairly easily the following result which generalizes a theorem of Kambe [17].

Lemma 5.2. *Let p be prime and let $m=p^a$ for $a\geqq 1$ be a prime power. Let $G=Z_m=\{\lambda\in C\,|\,\lambda^m=1\}$ and let $\rho_s(\lambda)=\lambda^s$. The $\{\rho_s\}_{0\leqq s<m}$ parametrize the irreducible representations of G. Let $x=\rho_1-1$ so $R_0(G)=xR(G)$. Let $\tau\colon G\to U(l)$ be a fixed point free representation. Let $\alpha=\Sigma(-1)^j\Lambda^j(\tau)$ so $\alpha R(G)=x^l R(G)$. If $\rho\in R_0(G)$, let $\mathrm{ord}_l(\rho)$ be the order of ρ in $R_0(G)/\alpha R(G)$. This is a power of p. Let $\mathrm{int}(*)$ denote the greatest integer function. Decompose $l-1=u(p-1)+v$ for $1\leqq v\leqq p-1$. $u=u(l)=\mathrm{int}\{(l-2)/(p-1)\}$. $v=v(l)=l-1-u(l)(p-1)$. Then:*

(a) $\mathrm{ord}_l(\rho)\leqq m p^u=p^{a+u}$.

(b) *Let $l\leqq p$. Then $R_0(G)/\alpha R(G)\simeq Z_m\oplus\ldots\oplus Z_m(l-1$ times) with generators $\{x^j\}_{1\leqq j<l}$.*

(c) *Let $l>p$ and let $\mathscr{R}$ be the subgroup of $R_0(G)/\alpha R(G)$ generated by $\{x^j\}_{1\leqq j<p}$. Then $\mathrm{ord}_l(x^j)=m p^u$ for $1\leqq j\leqq v$ and $\mathrm{ord}_l(x^j)=m p^{u-1}$ for $v<j<p$. As an Abelian group, $\mathscr{R}\simeq Z_{mp^u}\oplus\ldots\oplus Z_{mp^u}\oplus Z_{mp^{u-1}}\oplus\ldots\oplus Z_{mp^{u-1}}$ where there are v of the first factors and $p-1-v$ of the second factors. If $v=p-1$ the second factors don't appear.*

(d) *If $a=1$ so m is prime, then $\mathscr{R}=R_0(G)/\alpha R(G)$ and* (b) and (c) *give the complete structure of $\tilde{K}$ in this case.*

Proof. The proof is based on several fairly simple binomial identities. It is, however, a bit technical and may be skipped by the reader without loosing the general flow of the paper. If $l=2$ then $R_0(G)/\alpha R(G)\simeq Z_m$ and the lemma is immediate. We will use induction on l. Let $\tau_l=\rho_1\otimes 1^l$. By Theorem 4.1 we may assume without loss of generality that $\tau=\tau_l$ so $\alpha=\pm x^l$. We let $\mathrm{ind}_l=\mathrm{ind}_{\tau_l}$.

We first prove (a). The defining algebraic equation for x is:

$$\sum_{1\leqq j\leqq m}\binom{m}{j}x^j=0.$$

We decompose:

$$\binom{m}{j}=\frac{m}{m-j}\cdot\frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\ldots\cdot\frac{m-j}{j}.$$

Since m is a power of p, the exponent of p appearing in j is the same exponent which appears in $m-j$. Decompose $j=sp^t$ for $s,t\in Z$ and s coprime to p. Then $\binom{m}{j}=mp^{-t}c_j$ where c_j is coprime to p. Assume (a) true for all smaller values of l. Let $\rho, \tilde\rho \in R_0(G)$. Decompose $\rho = \beta x$ for some $\beta \in R(G)$. We compute:

$$\operatorname{ind}_l(m p^u \rho, \tilde\rho) = \operatorname{ind}_l(m p^u \beta x, \tilde\rho) = -\sum_{2 \leqq j \leqq m} \operatorname{ind}_l(m p^{u-t(j)} c_j \beta x^j, \tilde\rho)$$
$$= \sum_{2 \leqq j \leqq m} \pm c_j \operatorname{ind}_{l-(j-1)}(x, \beta \tilde\rho).$$

If we can show $u(l) - t(j) \geqq u(l-(j-1))$, then all the terms on the right hand side will vanish by induction. This will show $\operatorname{ind}_l(m p^u \rho, \tilde\rho) = 0$ so $m p^u \rho \in \alpha R(G)$ which will prove (a).

The greatest integer function int(*) is non-decreasing. Thus if $t(j) = 0$ $u(l) \geqq u(l-(j-1))$. We therefore assume $t \geqq 1$. It is easy to see that $s p^{t-1} \geqq t$. We estimate:

$$u(l-(j-1)) = \operatorname{int}\{(l-2-(j-1))/(p-1)\} = \operatorname{int}\{(l-2-(s p^t - 1))/(p-1)\}$$
$$\leqq \operatorname{int}\{(l-2-(s p^t - s p^{t-1}))/(p-1)\}$$
$$= \operatorname{int}\{(l-2)/(p-1)\} - s p^{t-1} \leqq u(l) - t$$

which completes the proof of (a). We note the order must be a power of p by Lemma 3.4.

Next suppose $l \leqq p$ so that $u = 0$. The $\{x^j\}_{1 \leqq j < l}$ generate $R_0(G)/\alpha R(G)$. The order of this group is m^{l-1} by Lemma 4.2. Each of these $(l-1)$ elements has order at most m by (a). Consequently each element has order exactly m and they generate a free Z_m module of rank $(l-1)$. This proves (b).

We use induction to prove (c). We proceed by a series of reductions. Assume it is true for all smaller values; there may not be any such if $l = p+1$ of course. We must first improve the upper bound of (a). We know $\operatorname{ord}_l(x^j) \leqq m p^u$ by (a). Suppose $1 \leqq v < j < p$ and let $\rho \in R_0(G)$. We compute:

$$\operatorname{ind}_l(m p^{u-1} x^j, \rho) = -\operatorname{ind}_{l-1}(m p^{u-1} x^{j-1}, \rho).$$

If we can show the right hand side vanishes, then $m p^{u-1} x^j \in \alpha R(G)$ and we will have shown $\operatorname{ord}_l(x^j) \leqq m p^{u-1}$ in this case. We distinguish two cases. If $v = 1$ then $u(l-1) = u(l) - 1$ so $m p^{u-1} x^{j-1} \in x^{l-1} R(G)$ by (a). If $v > 1$ then $l > p+1$ and we can use the induction hypothesis. $u(l-1) = u(l)$ and $v(l-1) = v(l) - 1$ so $j - 1 > v(l-1)$ and $m p^{u-1} x^{j-1} \in x^{l-1} R(G)$ by (c) applied to $l-1$. Thus indeed the right hand side vanishes and we have an upper bound $\operatorname{ind}_l(x^j) \leqq m p^{u-1}$ for $v < j < p$.

Let $y = \sum_{1 \leqq j < p} n_j x^j \in \mathscr{R}$. The results of the previous paragraph show $y = 0 \bmod \alpha R(G)$ provided $n_j \equiv O(m p^u)$ for $1 \leqq j \leqq v$ and $n_j \equiv O(m p^{u-1})$ for $v < j < p$. If we can show the converse assertion is true, we will have proved (c), as this will give the desired structure for $\mathscr{R}$ and give a lower bound on the orders involved. Suppose $y \in \alpha R(G)$ then trivially $y \in x^{l-1} R(G)$. We show first this implies all the congruences except $j = v$. We distinguish 3 cases. If $l = p+1$ then $u = v = 1$. $y \in x^{l-1} R(G) = x^p R(G)$ implies $n_j \equiv O(m)$ by (b). Thus $n_j \equiv O(m p^{u-1})$ for $1 < j < p$. Next suppose $u > 1$ but $v = 1$. Then $u(l-1) = u - 1$ and $v(l-1) = p - 1$. Then the induction hypothesis implies $n_j \equiv O(m p^{u-1})$ for $1 \leqq j < p$. Finally suppose $v > 1$. Then $u(l-1) = u$ and $v(l-1) = v(l) - 1$. The induction hypothesis implies $n_j \equiv O(m p^u)$ for $1 \leqq j \leqq v - 1$ and $n_j \equiv O(m p^{u-1})$ for $v \leqq j < p$.

This analysis shows $n_j \equiv O(mp^u)$ for $1 \leqq j < v$ and $n_j \equiv O(mp^{u-1})$ for $v < j < p$. Consequently $y - \sum_{j \neq v} n_j x^j \in \alpha R(G)$ using the upper bounds on the order already established. This shows that to complete the proof of (c) it suffices to establish that $nx^v \in \alpha R(G)$ implies $n \equiv O(mp^u)$. Since the order of x^v is divisible by p and is at most mp^u, it suffices to show $mp^{u-1}x^v \notin \alpha R(G)$. Suppose $v > 1$, then $\operatorname{ind}_l(mp^{u-1}x^v, \rho) = -\operatorname{ind}_{l-1}(mp^{u-1}x^{v-1}, \rho)$. Since $u(l-1) = u(l)$ and $v(l-1) = v(l) - 1$ we can apply the induction hypothesis to choose $\rho \in R_0(G)$ so the right hand side is non-zero. This completes the proof of (c) in this instance.

We will establish a recursion relationship of the form:

$$\operatorname{ind}_l(mp^{u-1}x, \rho) = c \cdot \operatorname{ind}_{l-(p-1)}(mp^{u-2}x, \rho)$$

when $v = 1$ where c is coprime to p. If $u = 1$ then $l - (p-1) = 2$. Since by Lemma 4.2 the order$_2$ of x is m, we can find ρ so $\operatorname{ind}_2(mp^{-1}x, \rho) \neq 0$ in Q/Z. If $u > 1$ we can find ρ so $\operatorname{ind}_2(mp^{u-2}x, \rho) \neq 0$ since $\operatorname{ord}_{l-(p-1)}(x) = mp^{u-1}$ by induction. Since the relevant denominators are powers of p and c is coprime to p, this implies the left hand side is non-zero which will complete the proof of (c).

We expand:

$$\operatorname{ind}_l(mp^{u-1}x, \rho) = \sum_{2 \leqq j \leqq m} c_j \operatorname{ind}_{l-(j-1)}(mp^{u-1-t(j)}x, \rho)$$

as was done in the proof of (a). If $j = p$, then this is the desired term in the recursion relationship. We must therefore show all the terms $j \neq p$ vanish in Q/Z. If $t(j) = 0$ then $u(l-(j-1)) < u$ as $v = 1$ and thus $u(l-(j-1)) \leqq u-1$ and this vanishes by (a). We may therefore assume $t(j) \geqq 1$. As before, we estimate:

$$\begin{aligned} u(l-(j-1)) &= \operatorname{int}\{(l-2-(j-1))/(p-1)\} = \operatorname{int}\{(u(p-1)-(sp^t-1))/(p-1)\} \\ &= u + \operatorname{int}\{-(sp^t - sp^{t-1} + sp^{t-1} - \ldots - s + s - 1)/(p-1)\} \\ &= u - sp^{t-1} - sp^{t-2} - \ldots - s + \operatorname{int}\{-(s-1)/(p-1)\}. \end{aligned}$$

Suppose first $t(j) \geqq 2$. Then $sp^{t-1} + sp^{t-2} \geqq t+1$ so $u(l-(j-1)) \leqq u-t-1$ as claimed. Next suppose $s \geqq 2$. Then $sp^{t-1} \geqq t$ and $\operatorname{ind}\{-(s-1)/(p-1)\} \leqq -1$ again implies the desired conclusion. We are therefore only left with the term $s = t = 1$ so $j = p$ as desired.

We have given a fairly combinatorial proof which is entirely self-contained. In fact, this lemma is intimately connected with a result of Atiyah and Hirzebruch [3] regarding the denominators appear in the Todd polynomial. Let Td_j be the Todd polynomial and let $\mu(Td_j)$ be the relevant denominator. Then [3] shows that the power of p dividing $\mu(Td_j)$ is precisely $\operatorname{int}(j/(p-1))$. We compute on the lens space $L(m; 1, \ldots, 1)$ without loss of generality. Using the formulas of the second section:

$$\operatorname{ind}((\rho_s - 1) \otimes (\rho_t - 1), P_{\mathrm{DOL}}) \equiv \frac{-1}{m} \sum_{i+j+k=l, i>0, j>0} s^i t^j \, Td_k(m, 1, \ldots, 1)/i!j!$$

$\operatorname{mod} Z$. An easy computation with factorials together with the Atiyah and Hirzebruch result shows the power of p appearing in the denominator is at most $a+u$ and thus the order of any element in $R_0(G)/x^l R(G)$ is at most mp^u. This provides another proof of (a). On the other hand, the results of (c) lead easily to show the power of p appearing in Td_{k-2} must be at least u so these two results are quite closely related.

We now discuss the structure of $\tilde{K}$ for a non-Abelian group. Let T: $G \to G$ be an automorphism of a finite group. We let $T^*(\rho)(g)=\rho(Tg)$ to define a ring isomorphism T^*: $R(G) \to R(G)$ preserving the ideal $R_0(G)$. If τ: $G \to U(l)$ is a fixed point free representation then $T^*(\tau)$ is also fixed point free since T is assumed to be an isomorphism. It is immediate that $T^*(\alpha(\tau))=\alpha(T^*(\tau))$ so T^* preserves the ideal $\alpha R(G)$ by Theorem 4.1. We therefore get a natural map T^*: $R(G)/\alpha R(G) \to R(G)/\alpha R(G)$. If H is a group of automorphisms of G, we let $\{R(G)/\alpha R(G)\}^H$ denote the subring which is invariant under this action.

Lemma 5.3. *Let G be a finite group and τ: $G \to U(l)$ be a fixed point free representation. Suppose all the Sylow subgroups of G are cyclic. Then there exist coprime integers (m, n) and a short exact sequence $0 \to Z_m \to G \to Z_n \to 0$. Choose a non-cannonical splitting to regard Z_n as a subgroup of G. Z_m is normal and Z_n acts on Z_m by conjugation. Let: $\alpha = \Sigma(-1)^j \Lambda^j(\tau)$ then:*

$$i_m^* \oplus i_n^*: R_0(G)/\alpha R(G) \to \{R_0(Z_m)/i_m^*(\alpha) R(Z_m)\}^{Z_n} \oplus R_0(Z_n)/i_n^*(\alpha) R(Z_n).$$

is a ring isomorphism.

Proof. We remark that although ind_τ splits as a direct sum under this decomposition, the restriction to each factor is not simply a rescale of $\mathrm{ind}_{i_m^*(\tau)}$ and $\mathrm{ind}_{i_n^*(\tau)}$ in general as was the case for Abelian groups. We also note that each factor is independent of the particular embedding of Z_n into G by Theorem 4.1.

The structure of such a group G follows from the classification given by Wolf (Theorem 5.4.1 [30]). Such G are the only groups which can arise if the complex dimension l is odd. Every Sylow subgroup is conjugate to a subgroup of either Z_m or Z_n so the map:

$$i_m^* \oplus i_n^*: R_0(G)/\alpha R(G) \to R_0(Z_m)/i_m^*(\alpha) R(Z_m) \oplus R_0(Z_n)/i_n^*(\alpha) R(Z_n)$$

is injective. Since $R_0(G)$ is invariant under conjugation by any element of G, we conclude the image of i_m^* lies in $\{R_0(Z_m)/i_m^*(\alpha) R(Z_m)\}^{Z_n}$. Thus the map in question is well defined and injective.

Let π_n: $G \to Z_n$ be the projection. Since $\pi_n i_n = 1_{Z_n}$ we conclude $i_n^* \pi_n^* = 1_{Z_n}$ and thus i_n^* is surjective. Since the two groups in question have coprime orders, $i_m^* \oplus i_n^*$ is surjective to the second factor.

Let A generate Z_n and let $\rho \in R_0(Z_m)$. Suppose $A^*(\rho) \equiv \rho$ in $R_0(Z_m)/i_m^*(\alpha) R(Z_m)$. Then $A^*(\rho)=\rho+i_m^*(\alpha)\beta$. We let $\tilde{\rho} = \sum_{0 \leqq j < n} (A^*)^j(\rho) = n \cdot \rho + i_m^*(\alpha)\tilde{\beta}$. Then $A^*(\tilde{\rho})=\tilde{\rho}$. Since this is invariant under conjugation by Z_n and Z_m is normal, it is invariant under conjugation by all the elements of G. We conclude therefore that the induced character $\tilde{\rho}^G = n\tilde{\rho}^0 \in R_0(G)$ as class functions. Since Z_m and Z_n only intersect in the identity, $i_n^*(\tilde{\rho}^G)=0$. Therefore $(i_m^* \oplus i_n^*)(\tilde{\rho}^G) = n\tilde{\rho} = n^2 \rho + i_m^*(\alpha)\hat{\beta}$. This shows $i_m^* \oplus i_n^*$ is surjective to n^2 times the first factor. Since n is coprime to the order of the first factor, we conclude $i_m^* \oplus i_n^*$ is an isomorphism as claimed.

Let τ: $G \to U(l)$ be a fixed point free representation and let $\alpha = \Sigma(-1)^j \Lambda^j(\tau)$. In Theorem 4.1, we showed the index of nilpotency to be at most l for the ring $R_0(G)/\alpha R(G) = \tilde{K}(S^{2l-1}/\tau(G))$. Lemma 5.3 shows the index of nilpotency is exactly l if every Sylow subgroup of G is cyclic since G contains a split Abelian factor. This is always the case if $|G|$ is odd.

We conclude with a brief discussion of the structure of $\tilde{K}$ when $G = Q(a)$. These are the only non-cyclic Sylow subgroups which can occur. For a complete calculation when $a = 3, 4$ we refer to [25].

Lemma 5.4. *Let $G = Q(a)$ be the generalized quaternionic group. We adopt the notation of Lemma 4.6; $a \geqq 3$. Let τ: $G \to U(l)$ be a fixed point free representation. Let $\alpha = \Sigma(-1)^j \Lambda^j(\tau)$, and let $l = 2k$.*

(a) *If $l = 2$ then $R_0(G)/\alpha R(G) \simeq Z_2 \oplus Z_2$ with trivial ring structure. It is generated by $\{\rho_1 - 1, \rho_2 - 1\}$.*

(b) $|R_0(G)/\alpha R(G)| = 2^{2+(a+2)(k-1)}$

(c) *Let H be a cyclic subgroup of G generated by some element C. Let $n = |H| \geqq 4$, and let $\delta_j(C) = \exp(2\pi ij/n)$ parametrize the irreducible representations of H. Let i: $H \to G$ be the natural inclusion. Let the ideal $J = \frac{n}{2} R_0(H) + R_0(H)^2$. Then $i^* R_0(G)$ is contained in J. Furthermore, J^{k+1} is contained in $J \cdot R_0(H)^{2k}$.*

(d) *Let $\{\gamma_j \in R_0(Q(a))\}_{0 \leqq j \leqq k}$. Then $\gamma = \Pi_j \gamma_j \in \alpha R(G)$. Furthermore, $(\pi_1 - 2)^{v-1}(\rho_2 - 1) \notin \alpha R(G)$. Thus the index of nilpotency of $R_0(Q(a))/\alpha R(Q(a))$ is $k+1$.*

Proof. Suppose first $l = 2$. In Lemma 4.7 we computed:

$$2(1 - \rho_j) \in \alpha R(G) \quad \text{and} \quad 1 + \Lambda^2(\pi_j) - \pi_j \in \alpha R(G).$$

We computed $\Lambda^2(\pi_j) = 1$ if j is odd and ρ_2 if j is even. Taking $j = m/2$ yields $1 + \rho_2 - \rho_1 - \rho_3 \in \alpha R(G)$ so $R_0(G)/\alpha R(G) = \tilde{K}$ is generated additively by $\{\rho_1 - 1, \rho_2 - 1\}$ and the group has order at most 4. Since

$$(\rho_1 - 1)^2 = -2(\rho_1 - 1), \quad (\rho_2 - 1)^2 = -2(\rho_2 - 1), \quad (\rho_1 - 1)(\rho_2 - 1) = \rho_3 + 1 - \rho_1 - \rho_2$$

$\tilde{K}$ has trivial ring structure. Let H be the subgroup generated by B and let i: $H \to G$ be the natural inclusion. Then:

$$\begin{aligned} &i^*(\rho_2 - 1) = \delta_2 - 1 = 2(\delta_1 - 1) + (\delta_1 - 1)^2 \notin i^*(\alpha) R_0(H) \\ &i^*(\rho_1 - 1) = 0. \end{aligned}$$

Thus if $a(\rho_1 - 1) + b(\rho_2 - 1) \in \alpha R(G)$ we conclude $b \equiv O(2)$ so $a(\rho_1 - 1) \in \alpha R(G)$. If we let $\bar{H}$ be the subgroup generated by AB then $\bar{i}^*(\rho_1 - 1) = \delta_2 - 1$ so we also conclude $a \equiv O(2)$. This shows $\tilde{K} \simeq Z_2 \oplus Z_2$ and completes the proof of (a).

To prove (b), we decompose $l = 2k$ and apply Lemma 5.1:

$$|R_0(G)/\alpha R(G)| = 4^k \cdot 2^{a(k-1)} = 2^{2k + ak - a} = 2^{2+(k-1)(a+2)}$$

which proves (b).

(c) is a technical result to be used in the proof of (d). If $\rho \in i^* R_0(G)$, then $\mathrm{Tr}(\rho(A)) = \mathrm{Tr}(\rho(A^{-1}))$. Thus image i^* is generated by $\{\delta_k + \delta_{-k} - 2, \delta_{n/2} - 1\}$. We can express $(\delta_k+\delta_{-k}-2)=-(\delta_k-1)(\delta_{-k}-1)\in R_0(H)^2=(\delta_1-1)^2 R(H)$, which is contained in the ideal J. Similarly, we compute:

$$(\delta_{n/2}-1)=\sum_{1\leqq j\leqq n/2}\binom{n/2}{j}(\delta_1-1)^j=\frac{n}{2}(\delta_1-1)+\sum_{2\leqq j\leqq n/2}\binom{n/2}{j}(\delta_1-1)^j\in J.$$

This shows $i^* R_0(G)$ is contained in J. The defining relation:

$$n(\delta_1-1)+n(n-1)/2(\delta_1-2)^2+\sum_{3\leqq j\leqq n}\binom{n}{j}(\delta_1-1)^j=0$$

implies

$$n(\delta_1-1)+n(n-1)/2(\delta_1-1)^2\in(\delta_1-1)^3 R(H).$$

We compute therefore:

$$J^2=n\cdot\frac{n}{4}(\delta_1-1)^2 R(G)+\frac{n}{2}(\delta_1-1)^3 R(G)+(\delta_1-1)^4 R(G)$$

$$\subseteq\frac{n}{2}(\delta_1-1)^3 R(G)+(\delta_1-1)^4 R(G)=J(\delta_1-1)^2 R(G).$$

by the previous identity. This proves (c) if $k=1$ and the rest of (c) follows easily from this case by induction.

Let $\gamma=\prod_{0\leqq j\leqq k}\gamma_j$ for $\gamma_j\in \mathrm{R}_0(\mathrm{G})$ and let H be the subgroup of G generated by either A, B, or AB. Then $i^*(\gamma)\in J\cdot(\delta_1-1)^{2k} R(H)$ by (c). This is contained in $i^*(\alpha) R(H)=(\delta_1-1)^{2k} R(H)$. Therefore $\gamma\in R(G)$ by Theorem 4.1. Let $\pi_k=\pi_1\otimes 1^k$ and $\mathrm{ind}_k=\mathrm{ind}_{\tau_k}$. $\mathrm{ind}_k((\pi_k-2)^{k-1}(\rho_2-1),\tilde\rho)=\pm\mathrm{ind}_1(\rho_2-1,\tilde\rho)$ does not vanish identically for all $\tilde\rho\in R_0(G)$ by (a) and therefore $(\pi_k-2)^{k-1}(\rho_2-1)\notin\alpha_k R(G)$ $=\alpha R(G)$. This completes the proof of (d).

References

1. Atiyah, M.F.: Bott periodicity and the index of elliptic operators. Q.J. Math. **74**, 113–140 (1968)
2. Atiyah, M.F.: *K*-theory. New York: Benjamin 1967
3. Atiyah, M.F., Hirzebruch, F.: Cohomologie Operationen und characterische Klassen. Math. Z. **77**, 139–187 (1961)
4. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry I. Math. Proc. Camb. Phil. Soc. **77**, 43–69 (1975)
5. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry II. Math. Proc. Camb. Phil. Soc. **78**, 405–432 (1975)
6. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry III. Math. Proc. Camb. Phil. Soc. **79**, 71–99 (1976)
7. Cheeger, J., Simons, J.: Differential characters and geometric invariants. Preprint
8. Chern, S.S., Simons, J.: Characteristic forms and Geometric Invariants. Ann. of Math. **99**, 48–69 (1974)
9. Donnelly, H.: Eta invariants for *G*-spaces. Indiana Univ. Math. J. **27**, 889–918 (1978)
10. Folkman, J.: Equivariant maps of spheres into the classical groups. Memoirs of the AMS, vol. 95, (1970)
11. Gilkey, P.: The residue of the local eta function at the origin. Math. Ann. **240**, 183–189 (1979)

12. Gilkey, P.: The residue of the global eta function at the origin. Adv. in Math. **40**, 290–307 (1981)
13. Hirzebruch, F.: Topological methods in algebraic geometry (3rd ed). Berlin-Heidelberg-New York: Springer 1966
14. Hirzebruch, F., Zagier, D.: The Atiyah-Singer theorem and elementary number theory. Publish or Perish Press vol. 3, 1974
15. Hoffman, P., Zabrodsky, A.: Thin Lens spaces. Canad. Math. Bull. **21**, 31–35 (1978)
16. Isaacs, M.: Character theory of finite groups. New York: Academic Press 1976
17. Kambe, T.: The structure of K-rings of the lens space and their applications. J. Math. Soc. Japan **18** (No. 2), 135–146 (1966)
18. Karoubi, M.: K-theorie equivariante des fibres en spheres. Topology **12**, 275–281 (1973)
19. Karoubi, M.: Private communication
20. Kobyashi, T., Sugawara, M.: K-rings of Lens Spaces $L^n(4)$. Hiroshima Math. J. **1**, 253–271 (1971)
21. Kobyashi, T., Sugawara, M.: K and KO rings of the lens space $L^n(p^2)$ for odd prime p. Hiroshima Math. J. **1**, 273–286 (1971)
22. Mahammed, N.: A propos de la K-theorie des espaces lenticulares. C.R. Acad. Sc. Paris **271**, 639–642 (1970)
23. Mahammed, N.: K-theorie des espaces lenticulaires. C.R. Acad. Sci. Paris **272**, 1363–1365 (1971)
24. Mahammed, N.: K-theorie des formes spheriques tetredriques. C.R. Acad. Sci. Paris **281**, 141–144 (1975)
25. Mahammed, N.: K-theorie des formes spheriques. Deuxime These Universite des Sciences et Techniques de Lille, vol. 339, 1975
26. Millson, J.: Chern-Simons invariants of constant curvature manifolds. Ph.D. Thesis University of California. Berkeley 1973
27. Millson, J.: Examples of nonvanishing Chern-Simons invariants. J. Diff. Geo. **10**, 589–600 (1974)
28. Passman, D.S.: Permutation groups. New York: Benjamin 1968
29. Seeley, R.: Complex powers of an elliptic operator. Proc. Symp. Pure Math. **10**, 288–307 (1966)
30. Wolf, J.: Spaces of constant curvature. New York: McGraw-Hill (1967)
31. Zagier, D.: Equivariant Pontrjagin classes and applications to orbit spaces. Lecture Notes in Mathematics, vol. 290. Berlin-Heidelberg-New York: Springer 1972

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