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Riemann–Roch Theorems for Higher Algebraic K-Theory

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Bibliography

INTRODUCTION

The purpose of this paper is to define Chern classes for higher algebraic K-theory in the greatest possible generality and to generalize the Riemann–Roch theorems of Grothendieck [34] and Baum *et al.* [2, 3] to higher algebraic K-theory. The classical Grothendieck Riemann–Roch theorem asserts that if $f: X \to Y$ is a proper morphism between quasi-projective varieties over an algebraically closed field and $\mathscr{F}$ is a locally free $\mathscr{O}_X$ module then:

$$f_!(ch(\mathscr{F}) \cdot Td(X)) = ch(f_*[\mathscr{F}])\, Td(Y),$$

where the Chern character of $\mathscr{F}$ is an element of the cohomology ring $A^*(X)$ (A^* could be the Chow ring, integral cohomology, or one of several other theories), $f_!$ is the direct image, "Gysin homomorphism" $A^*(X) \to A^*(Y)$, $Td(X)$ and $Td(Y)$ are certain universal power series in the Chern classes of the tangent bundles of X and Y and $f_*[\mathscr{F}]$ is $\sum (-1)^i [R^i f_* \mathscr{F}]$ considered as an element of the Grothendieck group $K_0(Y)$. The generalization we prove (see Section 4 for the full statement and proof) asserts that given a suitable category of schemes $\mathscr{V}$ and a suitable cohomology theory on $\mathscr{V}: X \to H^*(X, \Gamma(*))$, there is a theory of Chern

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classes for the higher algebraic K theory of $\mathscr{V}$ with coefficients in $H^*(X, \Gamma(*))$, i.e., for all p, $i \geqslant 0$ there is a natural transformation of contravariant functors (d is a constant $= 1$ or 2):

$$C_{i,p}: K_p(\ ) \to H^{di-p}(\ , \Gamma(i)).$$

Using these classes one may construct a natural transformation of covariant functors on the category of projective morphisms in $\mathscr{V}$:

$$\tau_* = \bigoplus_{q \geqslant 0} \tau_q : \bigoplus_{q \geqslant 0} K'_q(\ ) \to \bigoplus_{i \geqslant 0} \bigoplus_{q \geqslant 0} H_{di+q}(\ , \Gamma(i)),$$

where $K'(X)$ for X in $\mathscr{V}$ is the "homology" algebraic K-theory of coherent sheaves on X and $H_*(\ , \Gamma(*))$ is the homology theory corresponding to the cohomology theory $\Gamma(*)$. In the case of a projective morphism between non-singular algebraic varieties over field $f: X \to Y$, the theorem reduces to the more classical looking formula:

$$f_!\,(ch(\alpha) \cdot Td(X)) = ch(f_*(\alpha))\; Td(Y)$$

for any $\alpha \in K_p(X)$.

In Section 1 we write down the axioms that we need for a graded cohomology–homology theory $\Gamma(*)$ on a category of schemes $\mathscr{V}$. The cohomology theory $X \to H^*(X, \Gamma(*))$ is the hypercohomology of a graded complex of sheaves of abelian groups on the big Zariski site $\mathscr{V}_{\mathrm{ZAR}}$ of $\mathscr{V}$. The axioms are enough to ensure, using the methods of Grothendieck ([23]), that there exist a theory of Chern classes for representations of sheaves of groups on a scheme X in $\mathscr{V}$ coming from universal classes, for $n \geqslant i$ $C_i \in H^{di}(B.\mathscr{GL}_n, \Gamma(i))$ where $B.\mathscr{GL}_n$ is the simplicial sheaf on $\mathscr{V}_{\mathrm{ZAR}}$ which restricts to $B.\mathscr{GL}_n(\mathscr{O}_X)$ on each X in $\mathscr{V}$. In the case where $\mathscr{V}$ has a final object S the C_i lie in the cohomology of the simplicial scheme $B.\mathbf{GL}_n/S$. Using the homotopy theory of simplicial sheaves developed by K. Brown ([9], [10]) together with a generalization of an idea of Quillen one sees that elements of the cohomology of $B.\mathbf{GL}_n/S$ correspond to elements in the cohomology groups of the simplicial sheaf on $\mathscr{V}_{\mathrm{ZAR}}$ which takes the value $\Omega B\mathscr{Q}\mathscr{P}_X$ (the infinite loop space whose homotopy groups are the K-theory of X) on X in $\mathscr{V}$, or equivalently characteristic classes for the K-theory functors on $\mathscr{V}$. This point of view is central to the proof of the Riemann–Roch theorem, and also shows that Chern classes exist for higher K-theory with values in cohomology theories not considered in the text, such as crystalline cohomology. This approach also allows the a priori construction of local Chern classes for higher K-theory as conjectured by Grothendieck for K_0 (in Section 6 we show, via an equivalent definition of K-theory due to Waldhausen, that our definition agrees with that of Iversen for K_0 [27]). The construction of Chern classes, the Chern character and the proofs of their basic properties are in Section 2. In Section 3 we prove the first of the three Riemann–Roch theorems in the paper, the Riemann–Roch theorem without denominators for a closed immersion of schemes smooth over a base scheme S. This theorem (Theorem 3.1), apart from its independent interest, is a key step in the proof of the main Riemann–Roch theorem of Section 4.

Our Riemann–Roch without denominators is a theorem "with supports" in the style of Proposition 6.1 of [27], and as such can be used to show that the Chern classes for higher K-theory define maps of coniveau spectral sequences, and to compute these maps at the E_1 level by studying the relationship between Chern classes and localization in K-theory. In Section 4 we state and prove the full Riemann–Roch theorem for (possibly singular) schemes quasi-projective over a fixed base. In Section 5 we prove an analogue of this theorem, which asserts the existence of a natural transformation relating K-theory to topological K-theory.

Finally Sections 6 and 7 form an Appendix in which we collect various results on K-theory not previously in the literature that we use in the text, and prove that the Chow ring satisfies the axioms of Section 1. These results are taken from the author's 1978 Harvard thesis.

1. Generalized Cohomology Theories on Categories of Schemes

DEFINITION 1.1. (i) A graded cohomology theory $\Gamma(*)$ on a category $\mathscr{V}$ of schemes consists of a graded complex of shaves of abelian groups $\underline{\Gamma}^*(*) = \bigoplus_{i \in \mathbf{Z}} \underline{\Gamma}^*(i)$ on the big Zariski site $\mathscr{V}_{\mathrm{ZAR}}$ of $\mathscr{V}$, together with a pairing in the derived category of graded complexes of abelian sheaves on $\mathscr{V}_{\mathrm{ZAR}}$:

$$\underline{\Gamma}^*(*) \overset{L}{\underset{\mathbf{Z}}{\otimes}} \underline{\Gamma}^*(*) \to \underline{\Gamma}^*(*),$$

which is associative with unit and (graded-) commutative.

(ii) Given such a cohomology theory, for each pair (Y, X) of schemes in $\mathscr{V}$, with Y a closed subscheme of X one may define the cohomology of X with coefficients in Γ and supports in Y by

$$H^i_Y(X, \Gamma(j)) = \mathbf{H}^i_Y(X, \underline{\Gamma}^*(j)).$$

Note that by construction the groups $H^i_Y(X, \Gamma(j))$ are contravariant functors in (X, Y); given $f: Z \to X$ there is a natural map for all i, j:

$$f^!: H^i_Y(X, \Gamma(j)) \to H^i_{f^{-1}(Y)}(Z, \Gamma(j)).$$

For every variety X over $\mathbf{C}$ there is a natural map of sheaves of topological spaces

$$\Omega B.\mathscr{Q}\mathscr{P} \simeq \mathbf{Z} \times \mathbf{Z}_{\infty} B.\mathscr{G}\mathscr{L}(\mathscr{O}_X) \to (\mathbf{Z} \times BU^{(\ )}) = \mathscr{K}\mathscr{U},$$

where

$$\mathscr{K}\mathscr{U}(V) = (\mathbf{Z} \times BU)^V$$

The sheaf $\mathscr{K}\mathscr{U}$ is pseudoflasque in the sense of [10] because every pair of Zariski open sets V, $W \subset X$ is an excisive couple. Hence for every open set $V \subset X$ there is a map

$$\Omega B.\mathscr{Q}\mathscr{P}(V) \to (\mathbf{Z} \times BU)^V,$$

which is compatible with inclusions of open sets $V \subset W \subset X$. The induced map $\tau^p: K_p \to KU^{-p}$ is the higher K-theory analogue of τ we want. Given such an inclusion $V \subset W$ we then have a map induced on the homotopy fibres of the restriction maps:

$$R\Gamma_{W-V}(W, \Omega B.\mathscr{Q}\mathscr{P}) \to (\mathbf{Z} \times BU)^{(W,V)}.$$

In the case of $W = M$ a smooth variety and $V = M - X$, where $X \subset M$ is a closed subvariety, the domain of this map is $\Omega B.\mathscr{Q}\mathscr{M}(X)$ and the induced map on homotopy groups is

$$\tau^M: K'_q(X) \to KU_X^{-q}(M) \simeq KU_q^{\mathrm{LC}}(X).$$

The rest of the proof is entirely parallel to that of Theorem 4.1. Note that in the proof of Theorem 4.1 the independence of the map τ_* from the smooth embedding used to define it was a consequence of Theorem 3.1; the analogue of this theorem for topological K-theory instead of Γ cohomology is again proved using the same methods as those of Section 3, the purity theorems used in Sections 3 and 4 being replaced by duality or the Thom isomorphism and the Dold–Thom computation of the cohomology of projective bundles being replaced by the analogous result for KU^*.

6. Waldhausen K-Theory of the Derived Category and Local Chern Classes

Let $\mathscr{E}$ be an exact category in the sense of [31], which we may view as a full exact subcategory of an abelian category $\mathscr{A}$. Then the category $C_b(\mathscr{E})$ of bounded homological complexes of objects in $\mathscr{E}$ has a natural exact category structure; $X. \to Y. \to Z.$ is exact if for each k ($\geqslant 0$ by hypothesis) $X_k \to Y_k \to Z_k$ is an exact sequence in $\mathscr{E}$. $\mathscr{C}_b(\mathscr{E})$ is a category with cofibrations and weak equivalences in the sense of [39]. A cofibration is an admissable monomorphism in the exact category $C_b(\mathscr{E})$, and a weak equivalence is a homology equivalence (the homology objects lie, a priori, in $\mathscr{A}$). If we denote the category of weak equivalences by h, then in the notation of Waldhausen [39], there is a homotopy Cartesian square:

$$\begin{array}{ccc} S.\mathscr{C}_b(\mathscr{E})^h & \longrightarrow & h.S.\mathscr{C}_b(\mathscr{E})^h \simeq * \\ \downarrow{\scriptstyle S.J} & & \downarrow{\scriptstyle h.S.J} \\ S.\mathscr{C}_b(\mathscr{E}) & \longrightarrow & h.S.\mathscr{C}_b(\mathscr{E}) \end{array} \tag{6.1}$$

where $\mathscr{C}_b(\mathscr{E})^h$ is the category of acyclic complexes and $J: \mathscr{C}_b(\mathscr{E})^h \to C_b(\mathscr{E})$ is the natural inclusion. Our object is to prove:

THEOREM 6.2. *There is a weak equivalence* $h.S.\mathscr{C}_b(\mathscr{E}) \simeq B.\mathscr{Q}\mathscr{E}$.

First we need a lemma:

LEMMA 6.3. *Let* $\mathscr{E}$ *be an exact category. Then* $S.\mathscr{E} \simeq B.\mathscr{Q}\mathscr{E}$.

Proof. A p-simplex in $B.\mathscr{Q}\mathscr{E}$ is a diagram

$$P_p \to P_{p-1} \to \cdots \to P_0$$

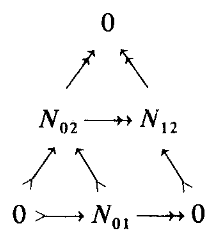
in $Q\mathscr{E}$. For each i, j, $0 \leqslant i < j \leqslant n$ we have a map $P_j \to P_i$ which is represented by a diagram in $\mathscr{E}$:

$$P_j \twoheadleftarrow Q_{ij} \rightarrowtail P_i.$$

We call the kernel N_{ij} of the map $Q_{ij} \twoheadrightarrow P_j$ the kernel of the map $P_j \to P_i$. If $k \geqslant j \geqslant i$ then there is an exact sequence e_{ijk}

$$N_{ij} \rightarrowtail N_{ik} \twoheadrightarrow N_{jk}.$$

The data $(\{N_{ij}\}_{i<j}, \{e_{ijk}\}_{i<j<k})$ represent a p-simplex in $S.\mathscr{E}$. It is easily checked that this defines a map $B.\mathscr{Q}\mathscr{E} \to S.\mathscr{E}$. There is a map $S.\mathscr{E} \to B.\mathscr{Q}\mathscr{E}$, defined as follows. Let $(\{N_{ij}\}, \{e_{ijk}\})$ be a p-simplex in $S.\mathscr{E}$. Then for each e_{ijk} there is a diagram E_{ijk} in $B.\mathscr{Q}\mathscr{E}$:



which represents a "singular" 2-simplex of $B.\mathscr{Q}\mathscr{E}$; i.e., a 2-simplex of $Ex^2(B.\mathscr{Q}\mathscr{E})$ [45]. The E_{ijk} all fit together to form a singular p-simplex of $B.\mathscr{Q}\mathscr{E}$ with vertices 0, and edges $0 \rightarrowtail N_{ij} \twoheadrightarrow 0$ for each $i<j$. Evidently this map is injective. For each vertex P in $B.\mathscr{Q}\mathscr{E}$ there is a unique path from the single vertex 0 of $S.\mathscr{E}$ to P

$$0 \rightarrowtail P$$

and for each edge in $B.\mathscr{Q}\mathscr{E}$

$$P_0 \twoheadleftarrow Q \rightarrowtail P_1$$

there is a homotopy ($N = \mathrm{Ker}(Q \twoheadrightarrow P_0)$):

$$\begin{array}{ccccc} P_0 & \twoheadleftarrow & Q & \rightarrowtail & P_1 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \twoheadleftarrow & N & \leftarrowtail & 0 \end{array}$$

This defines a canonical retraction of the 1-skeleton of $B.\mathscr{Q}\mathscr{E}$ onto the 1-skeleton of $S.\mathscr{E}$, which extends naturally to a retraction of $B.\mathscr{Q}\mathscr{E}$ onto $S.\mathscr{E}$.

Proof of theorem. By construction the map $S.J$ is an infinite loop map so it is enough to prove that there is a co-fibration sequence in the stable homotopy category $\mathscr{S}Ho$ [1]:

$$S.\mathscr{C}_b(\mathscr{E})^h \xrightarrow{S.J} S.\mathscr{C}_b(\mathscr{E}) \to S.\mathscr{E}.$$

By Lemma 6.3 this is equivalent to proving that the cofiber of the map $B\mathscr{Q}J: B.\mathscr{Q}C_b(\mathscr{E})^h \to B.\mathscr{Q}\mathscr{C}_b(\mathscr{E})$ is $B\mathscr{Q}\mathscr{E}$. First we identify the domain and codomain of BQJ as objects of $\mathscr{S}Ho$. The identity functor I on $\mathscr{C}_b(\mathscr{E})$ has a "cofiltration" by exact quotient functors:

$$I = F^0 \twoheadrightarrow F^1 \twoheadrightarrow F^2 \twoheadrightarrow \cdots \twoheadrightarrow F^n \twoheadrightarrow \cdots.$$

defined by $F^j(X.)_k = X_k$ if $k \geqslant j$ or 0 if $k<j$. For each $j \geqslant 0$ there is an exact sequence of functors:

$$P^j \rightarrowtail F^j \twoheadrightarrow F^{j+1},$$

where $P^j(X.)_k = 0$ if $j \neq k$ and X_k if $j=k$.

Hence by [30, Sect. 3] we have

$$\sum_{j=0}^{\infty} K_*(P^j) = Id: K_*(\mathscr{C}_b(\mathscr{E})) \to K_*(\mathscr{C}_b(\mathscr{E})). \tag{6.4}$$

(Note that any diagram in $\mathscr{D}\mathscr{C}_b(\mathscr{E})$ is mapped to zero by all but a finite number of the F^j, so the non-finiteness of the cofiltration $F^{\cdot}$ is not a problem.)

The functor $\sum_{j=0}^{\infty} P^j$ is the exact functor $\mathscr{C}_b(\mathscr{E}) \to \mathscr{C}_b(\mathscr{E})$ which replaces all the differentials of a complex by zero, so it may be regarded as the composition:

$$\begin{array}{ccc} \mathscr{C}_b(\mathscr{E}) & \xrightarrow{\ \sum P^j\ } & \mathscr{C}_b(\mathscr{E}) \\ & {\scriptstyle P}\searrow \quad \nearrow{\scriptstyle G} & \\ & \mathscr{G}_b(\mathscr{E}) & \end{array} \tag{6.5}$$

where $\mathscr{G}_b(\mathscr{E})$ is the exact category of graded objects $\oplus_{j \geqslant 0} X_j$ of $\mathscr{E}$, with $X_j \simeq 0$ for $j \gg 0$, P is the obvious forgetful functor and G the natural inclusion. Obviously $P.G = I_{\mathscr{G}_b}(\mathscr{E})$, so by (6.4) P and G are inverse weak equivalences. Since $\mathscr{G}_b(\mathscr{E})$ is a direct sum of exact categories, there is an isomorphism in $\mathscr{S}Ho$:

$$B.\mathscr{D}\mathscr{G}_b(\mathscr{E}) \simeq \bigvee_{j \geqslant 0} B.\mathscr{D}\mathscr{E}$$

(The join $\bigvee$ is the direct sum in the additive category $\mathscr{S}Ho$). Now we define a map (the "Euler characteristic")

$$E\colon \bigvee_{j \geqslant 0} B.\mathscr{D}\mathscr{E} \to B.\mathscr{D}\mathscr{E}$$

by $E = \sum_{i \geqslant 0} (-1)^j E_j$, where E_j is projection onto the jth factor of the domain of E.

Turning to $\mathscr{C}_b(\mathscr{E})^h$ we see that the identity functor $I^h\colon \mathscr{C}_b(\mathscr{E})^h \to \mathscr{C}_b(\mathscr{E})^h$ has a filtration by exact subfunctors L_k:

$$\rightarrowtail L_k \rightarrowtail \cdots \rightarrowtail L_1 \rightarrowtail L_0 = J$$

$$\begin{aligned} L_k(X.)_l &= X_1 && l > k, \\ &= Z_k && l = k, \\ &= 0 && l < k, \end{aligned}$$

where $Z_k = \ker(X_k \to X_{k-1})$. Note that a priori Z_k does not lie in $\mathscr{E}$. But we may suppose without changing its K-theory that $\mathscr{E}$ contains all objects in $\mathscr{A}$ with finite resolutions by objects in $\mathscr{E}$ [31, Sect. 4], and then Z_k does lie in $\mathscr{E}$. The quotients of the filtration $\{L.\}$ are the functors:

$$Z_k\colon \mathscr{C}_b(\mathscr{E})^h \to \mathscr{C}_b(\mathscr{E})^h$$

$$\begin{aligned} Z_k(X.)_l &= 0 && k \neq l,\ l+1, \\ &= Z_k && k = l,\ l+1 \end{aligned}$$

with the differential $Z_k \to Z_k$, the identity. There is a commutative triangle of functors

$$\begin{array}{ccc} \mathcal{Q}\mathcal{C}_b(\mathcal{E})^h & \xrightarrow{\oplus Z_j} & \mathcal{Q}\mathcal{C}_b(\mathcal{E})^h \\ & {\scriptstyle H}\searrow \quad \nearrow{\scriptstyle S} & \\ & \mathcal{G}_b(\mathcal{E}) & \end{array}$$

where

$$H(X.) = \bigoplus_{j \geq 0} Z_j(X.)_j,$$

$$S(X.)_j = X_j \oplus X_{j-1}.$$

As with (6.4) we see that S and H are weak equivalences. Hence we have a commutative diagram:

$$\begin{array}{ccccc} B.\mathcal{Q}\mathcal{C}_b(\mathcal{E})^h & \xrightarrow{B.\mathcal{Q}\mathcal{T}} & B.\mathcal{Q}\mathcal{C}_b(\mathcal{E}) & & \\ \downarrow{\wr H} & & \downarrow{\wr P} & & \\ B.\mathcal{Q}\mathcal{G}_b(\mathcal{E}) & \xrightarrow{BQ(P.J.S)} & B.\mathcal{Q}\mathcal{G}_b(\mathcal{E}) & & \\ \downarrow{\wr} & & \downarrow{\wr} & & \\ \bigvee_{j \geq 0} B.\mathcal{Q}\mathcal{E} & \xrightarrow{D} & \bigvee_{j \geq 0} B.\mathcal{Q}\mathcal{E} & \xrightarrow{E} & B.\mathcal{Q}\mathcal{E} \end{array}$$

where D is the map

$$(x_0, x_1, \ldots, x_n, \ldots) \to (x_0, x_0 + x_1, x_1 + x_2, \ldots, x_n + x_{n-1}, \ldots)$$

clearly $E.D$ is the zero map in $\mathcal{S}Ho$. Furthermore E is split by the map C: $(x) \to (x, 0, 0, \ldots, 0, \ldots)$ and since the matrix $(\delta_{i,j} + \delta_{i,j+1})$ is invertible over $\mathbf{Z}$ we see that

$$C \vee D: B.\mathcal{Q}\mathcal{E} \vee \bigvee_{j \geq 0} B.\mathcal{Q}\mathcal{E} \to \bigvee_{j \geq 0} B.\mathcal{Q}\mathcal{E} \simeq B.\mathcal{Q}\mathcal{C}_b(\mathcal{E})$$

is a weak equivalence, and E is projection onto the first factor of a direct sum decomposition the other factor of which is $B.\mathcal{Q}\mathcal{C}_b(\mathcal{E})^h$.

We now use the preceding proposition to relate the definition of local Chern classes in Section 2 with those of Iversen [27]. First observe that if X is a (noetheran, separated) scheme and $Y \subset X$ is a closed subscheme, then we have a homotopy Cartesian square [39]

$$\begin{array}{ccc} h_X S.\mathcal{C}_b(\mathcal{P}_X)^{h_U} & \longrightarrow & h_U S.\mathcal{C}_b(\mathcal{P}_X)^{h_U} \simeq * \\ \downarrow & & \downarrow \\ h_X S.\mathcal{C}_b(\mathcal{P}_X) & \longrightarrow & h_U S.\mathcal{C}_b(\mathcal{P}_X) \end{array}$$

APPENDIXES

7. K-THEORY OF SCHEMES

In this section we collect together the basic facts about K-theory that we needed in the main text.

First we need to introduce the language of "n-categories."

DEFINITION 7.1. (i) Let $\mathscr{D}$ be a category with finite inverse limits. By a *category object* C in $\mathscr{D}$ we mean a pair of objects $Ob(C)$, $Mor(C)$ in $\mathscr{D}$ together with a diagram in $\mathscr{D}$:

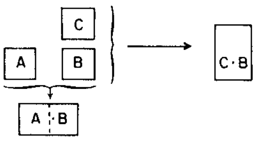
$$Ob(C) \underset{d_1}{\overset{\xrightarrow{s_0}}{\underset{\longleftarrow}{\xleftarrow{d_0}}}} Mor(C)$$

where $d_0 \cdot s_0 = d_1 \cdot s_0 = Id_{Ob(C)}.d_0$ is the "source" map, d_1 the "target" map and s_0 is the "identity morphism" map. We assume a "composition law" from $Mor(C) \times_{Ob(C)} Mor(C) \to^{\gamma} Mor(C)$ and we suppose that all these structure maps satisfy the usual compatibilities.

(ii) A simplicial object in $\mathscr{D}$ is a functor from $\Delta^{op} \to \mathscr{D}$, where Δ is the usual category of finite totally ordered sets and non-decreasing functions.

There is a natural fully faithful embedding of the category Cat ($\mathscr{D}$) of category objects in $\mathscr{D}$ into the category $[\Delta^{op}, \mathscr{D}]$ of simplificial objects in $\mathscr{D}$, called the "classifying space" functor. We shall make no notational distinction between a category object and its classifying space object, and by $\pi_i(C)$ for C in $\mathscr{C}at$ ($\mathscr{S}et$) we shall mean π_i of the associated simplicial set.

A bicategory has the same relationship to a category as a bisimplicial set (which is a functor $\Delta^{op} \times \Delta^{op} \to$ Sets) has to an ordinary category. A bicategory consists of the following data: Objects, Horizontal Morphisms, Vertical Morphisms and Bimorphisms. The bimorphisms can be thought of diagramatically as squares, and have two composition laws, vertical and horizontal:



The vertical and horizontal sides of the squares represent vertical and horizontal morphisms, and their corners represent the objects. A bicategory object in the category $\mathscr{D}$ will therefore consist of four objects (O, V, H, B)

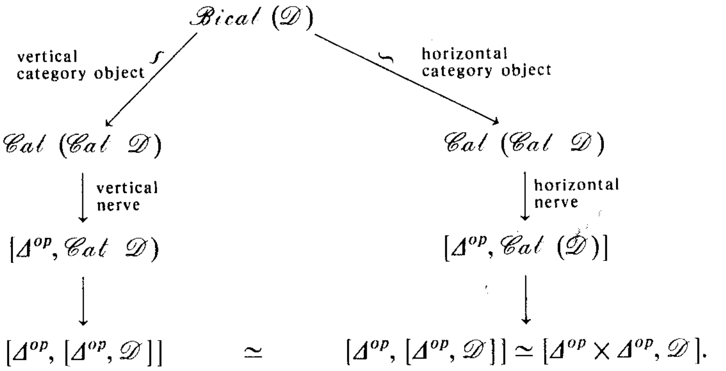
and certain maps between them. Associated to a bicategory object C in $\mathscr{D}$ we have two category objects in $\mathscr{C}at\,(\mathscr{D})$: the vertical and horizontal category objects; we only define the first as the second has a similar definition.

DEFINITION 7.2. The *vertical category object* (O_V, M_V) in $\mathscr{C}at\,(\mathscr{C}at\,(\mathscr{D}))$ associated to a bicategory C has:

$$Ob(O_V) = Ob(C), \qquad Mor(O_V) = Hor\,Mor\,C$$

$$Ob(M_V) = Vert\,Mor(C), \qquad Mor(M_V) = Bimor\,C.$$

From the embedding $\mathscr{C}at\,(\mathscr{C}at\,\mathscr{D}) \to [\Delta^{op}, \mathscr{C}at\,(\mathscr{D})]$ we get a simplicial category object in $\mathscr{D}$, the *vertical nerve* of C. Similarly we have the horizontal nerve of C. We therefore have a diagram:



We therefore see the category of bicategory objects in $\mathscr{D}$ has a fully faithful embedding into the category of bisimplicial objects in $\mathscr{D}$, which is independent of whether one first takes horizontal or vertical nerves.

All of this has an even more messy generalization to n-categories for $n \geqslant 2$. These correspond to n-simplicial sets, and have n-morphisms corresponding to n-cubes. If $1 \leqslant k \leqslant n$ and C is an n-category object in $\mathscr{D}$ then the k-nerve of C is the simplicial $n-1$ category obtained by "forming nerves in the kth coordinate."

For more details and many examples of bicategories see [38].

We should also state a well-known lemma which we shall need later.

LEMMA 7.3. *Let $X. \to Y. \to Z.$ be a sequence of simplicial spaces such that $X. \to Z.$ is constant. Suppose that $X_n \to Y_n \to Z_n$ is a fibration up to homotopy for all n and that Z_n is connected for every n. Then $X. \to Y. \to Z.$ is a fibration up to homotopy.*

Proof. See Waldhausen [38].

Let $\mathscr{M}$ be a small exact category [31, p. 15]. Then Quillen constructed the groups $K_i(\mathscr{M})$ as the homotopy groups of a certain category $Q\mathscr{M}$ associated to $\mathscr{M}$.

DEFINITION 7.4. $\mathscr{Q}\mathscr{M}$ is the category:

$$Ob\,\mathscr{Q}\mathscr{M} = Ob\,\mathscr{M}$$

$\mathscr{Q}\mathscr{M}[M, N]$ = The set of isomorphism classes of diagrams, for fixed M and N:

$$M \overset{e}{\twoheadleftarrow} P \overset{i}{\rightarrowtail} N.$$

The morphism (e, i) may also be described by the length 2 filtration $\mathrm{Ker}(e) \rightarrowtail P \rightarrowtail N$ of N, together with an isomorphism $\bar{e}$ of the layer $P/\mathrm{Ker}(e)$ with M.

DEFINITION 7.5. By an *admissible filtration F.* of the object N of an exact category $\mathscr{M}$, we mean an isomorphism class of diagrams

$$F_0 \rightarrowtail F_1 \rightarrowtail F_2 \rightarrowtail \cdots \rightarrowtail F_{k-1} \rightarrowtail N = F_k,$$

where k is the length of the filtration. The quotient F_{i+1}/F_i is called the ith layer of F.

Clearly $\mathscr{M} \mapsto \mathscr{Q}\mathscr{M}$ is a functor from the category of small exact categories ($\mathscr{E}xact$) and exact functors to $\mathscr{C}at$ ($\mathscr{S}et$). Hence $\mathscr{M} \to \pi_*(\mathscr{Q}\mathscr{M})$ is a functor $\mathscr{E}xact \to \mathscr{G}raded\ \mathscr{A}belian\ groups$.

DEFINITION 7.6. $K_i(\mathscr{M}) = \pi_{i+1}(\mathscr{Q}\mathscr{M})$, $i \geqslant 0$.

Waldhausen [38] has described an iterated Q construction in which

$$\mathscr{Q}^k\mathscr{M} = \overbrace{\mathscr{Q} \cdots \mathscr{Q}}^{k\text{-times}}\mathscr{M}$$

is a k-category and $\mathscr{Q}^k\mathscr{M}$ is a delooping of $\mathscr{Q}^{k-1}\mathscr{M}$. Since this construction is of central importance in this chapter, we shall describe it in detail.

DEFINITION 7.7. $\mathscr{Q}^k\mathscr{M}$ is the following k-category:

$$Ob(\mathscr{Q}^k\mathscr{M}) = Ob(\mathscr{M})$$

k-$Mor(\mathscr{Q}^k\mathscr{M})$ = the set of objects of $\mathscr{M}$ with k-fold admissible filtrations of length 2.

Note. A k-fold admissible filtration $F_{i_1 \cdots i_k}$ (of length two) of an object M is a set of admissible subobjects $F_{i_1 \cdots i_k} \rightarrowtail M$ $(i_j = 0, 1, 2)$ such that if $i_j < r_j$ for all $j = 1 \cdots n$, $F_{i_1 \cdots i_k} \rightarrowtail F_{r_1 \cdots r_k}$ and if $r_j = \mathrm{Min}(s_j, t_j)$ for $j = 1 \cdots k$ we have a fibre product:

$$\begin{array}{ccc} F_{r_1 \cdots r_k} & \rightarrowtail & F_{s_1 \cdots s_k} \\ \downarrow & & \downarrow \\ F_{t_1 \cdots t_k} & \rightarrowtail & M = F_{2, \ldots, 2}, \end{array}$$

together with choices of quotients.

For example, $k = 2$

$$\begin{array}{ccccc} F_{00} & \rightarrowtail & F_{01} & \rightarrowtail & F_{02} \\ \downarrow & & \downarrow & & \downarrow \\ F_{10} & \rightarrowtail & F_{11} & \rightarrowtail & F_{12} \\ \downarrow & & \downarrow & & \downarrow \\ F_{20} & \rightarrowtail & F_{21} & \rightarrowtail & F_{22} = M \end{array}$$

We can also write this

$$\begin{array}{ccccc} F_{11}/F_{01} + F_{10} & \twoheadleftarrow & F_{11}/F_{01} & \rightarrowtail & F_{12}/F_{02} \\ \uparrow & & \uparrow & & \uparrow \\ F_{11}/F_{10} & \twoheadleftarrow & F_{11} & \rightarrowtail & F_{12} \\ \downarrow & & \downarrow & & \downarrow \\ F_{21}/F_{20} & \twoheadleftarrow & F_{21} & \rightarrowtail & F_{22}. \end{array}$$

The horizontal and vertical "faces" of this bimorphism correspond to the four edges of the second square.

THEOREM 7.8. *$\mathscr{Q}^{k+1}\mathscr{M}$ is a delooping of $\mathscr{Q}^k\mathscr{M}$, $k \geqslant 1$.*

Proof. We generalize Waldhausen's proof for the case $k = 2$ [38].

Let $\mathscr{L}\mathscr{M}$ be the category whose objects are the admissible monomorphisms in $\mathscr{M}$, and where a morphism from x to y is a commutative diagram

$$\begin{array}{ccc} X & \overset{x}{\rightarrowtail} & X' \\ \uparrow & & \searrow \\ Y & \overset{y}{\rightarrowtail} & Y' \end{array}$$

There is a natural functor $\mathcal{L}\mathcal{M} \to \mathcal{Q}\mathcal{M}$, which acts on objects and morphisms as follows:

$$\text{objects: } X \rightarrowtail X' \mapsto X'/X$$

$$\text{morphisms: } Y \rightarrowtail X \rightarrowtail X' \rightarrowtail Y' \mapsto X'/X \twoheadleftarrow X'/Y \rightarrowtail Y'/Y.$$

We can apply the Q-construction in a natural way to $\mathcal{L}\mathcal{M}$, forming $\mathcal{Q}^k\mathcal{L}\mathcal{M}$ for all $k \geqslant 1$. A $(k+1)$-morphism will be a filtered object $F_{i_1 \cdots i_{k+1}}$ of $\mathcal{M}$ where $i_1 \cdots i_k = 0, 1, 2$ and $i_{k+1} = 0, 1, 2, 3$. The F_i are required to satisfy compatibility conditions paralleling those for $\mathcal{Q}^{k+1}\mathcal{M}$. We now have the machinery for the proof of the theorem.

Consider the sequence

$$\mathcal{Q}^k\mathcal{M} \to \mathcal{Q}^k\mathcal{L} \to \mathcal{Q}^{k+1}\mathcal{M}$$

of $(k+1)$-categories, where $\mathcal{Q}^k\mathcal{L}\mathcal{M} \to \mathcal{Q}^{k+1}\mathcal{M}$ is the functor induced by the functor $\mathcal{L}\mathcal{M} \to \mathcal{Q}\mathcal{M}$, and $\mathcal{Q}^k\mathcal{M}$ is regarded as a degenerate $(k+1)$-category whose $(k+1)$-morphisms all have $(k+1)$st coordinate the identity. $\mathcal{Q}^k\mathcal{M}$ is therefore the full sub$(k+1)$-category of $\mathcal{Q}^k\mathcal{L}\mathcal{M}$ which is sent to the zero object of $\mathcal{Q}^{k+1}\mathcal{M}$.

We will be done if we can now show that this sequence is a fibration up to homotopy for all $k \geqslant 1$ and that $\mathcal{Q}^k\mathcal{L}\mathcal{M}$ is contractible for all $k \geqslant 1$. First we do the fibration part. Taking $(k+1)$-nerves of the sequence, we obtain a sequence of simplicial k-categories which in degree n is equivalent to:

$$\mathcal{Q}^k\mathcal{M} \to \mathcal{Q}^k\mathcal{F}_{2n+2}\mathcal{M} \to \mathcal{Q}_k\mathcal{F}_{2n+1}\mathcal{M},$$

where $\mathcal{F}_m\mathcal{M}$ is the exact category with objects admissible filtrations of length m; $F_0 \rightarrowtail \cdots \rightarrowtail F_m M = M$. By the Exactness Theorem

$$\mathcal{Q}\mathcal{F}_m\mathcal{M} \simeq (\mathcal{Q}\mathcal{M})^m$$

and so, assuming the theorem true for $j = 1, \ldots, k-1$,

$$\mathcal{Q}^k\mathcal{F}_m\mathcal{M} \simeq (\mathcal{Q}^k\mathcal{M})^m.$$

Hence, up to homotopy, our sequence becomes in degree n:

$$\mathcal{Q}^k\mathcal{M} \to (\mathcal{Q}^k\mathcal{M})^{2n+2} \to (\mathcal{Q}^k\mathcal{M})^{2n+1}.$$

This is clearly a fibration with connected base, and so by Lemma 7.3 we know that $\mathcal{Q}^k\mathcal{M} \to \mathcal{Q}^k\mathcal{L}\mathcal{M} \to \mathcal{Q}^{k+1}\mathcal{M}$ is a fibration up to homotopy (so long as $\mathcal{Q}^{j+1}\mathcal{M}$ is a delooping of $\mathcal{Q}^j\mathcal{M}$ for $j = 1 \cdots k-1$). Finally, both to start the induction off and to provide the inductive step we need:

LEMMA 7.9. *$\mathcal{Q}^k\mathcal{L}\mathcal{M}$ is contractible for all k.*

Proof of lemma. We have an explicit null homotopy of $\mathscr{L}\mathscr{M}$ via the pair of natural transformations:

$$(X \rightarrowtail X') \rightarrow (0 \rightarrowtail X') \leftarrow (0 \rightarrowtail 0).$$

These then induce the required nullhomotopy of $\mathscr{Q}^k\mathscr{L}\mathscr{M}$ by the functorality of the Q-construction. This also exhibits an explicit homotopy equivalence $\mathscr{Q}^k\mathscr{M} \simeq \Omega\mathscr{Q}^{k+1}\mathscr{M}$.

DEFINITION 7.10. $\mathscr{K}_{\mathscr{M}}$ is the $C - W\ \Omega$-spectrum which in degree n is:

$$\begin{aligned}(\mathscr{K}_{\mathscr{M}})_n &= \Omega^{-(n-1)}\mathscr{Q}\mathscr{M} \qquad & n \leqslant 0,\\ &= \mathscr{Q}^n\mathscr{M} \qquad & n \geqslant 1\end{aligned}$$

and has maps

$$S\mathscr{K}_{\mathscr{M},n} \rightarrow \mathscr{K}_{\mathscr{M},n+1}$$

adjoint to the maps $\mathscr{Q}^n\mathscr{M} \rightarrow \Omega\mathscr{Q}^{n+1}$ of the theorem.

We therefore can regard the K-theory of the category $\mathscr{M}$ as the (stable) homotopy of the spectrum $\mathscr{K}_{\mathscr{M}}$, i.e., $K_i(\mathscr{M}) = \pi_i(\mathscr{K}_{\mathscr{M}})$.

We now describe Waldhausen's construction of products, and show that it is compatible with the infinite loop space structure of Definition 7.10. Throughout we shall be considering a biexact functor $\mathscr{M} \times \mathscr{N} \rightarrow^{\varphi} \mathscr{P}$.

Consider the pair of categories $\mathscr{Q}\mathscr{M}$, $\mathscr{Q}\mathscr{N}$. Then we have a natural bicategory $\mathscr{Q}\mathscr{M} \otimes \mathscr{Q}\mathscr{N}$, homotopy equivalent to $\mathscr{Q}\mathscr{M} \times \mathscr{Q}\mathscr{N}$, in which a bimorphism is pair (μ, ν) of morphisms from $\mathscr{Q}\mathscr{M}$, $\mathscr{Q}\mathscr{N}$, respectively. We then have a natural functor $\mathscr{Q}\mathscr{M} \otimes \mathscr{Q}\mathscr{N} \rightarrow \mathscr{Q}\mathscr{Q}\mathscr{P}$ which takes the bimorphism (F_i, G_j) in $\mathscr{Q}\mathscr{M} \otimes \mathscr{Q}\mathscr{N}$ (remember F_i and G_j are length 2 admissible filtrations of objects in $\mathscr{M}$, $\mathscr{N}$, respectively) to the bimorphism $\varphi(F_i, G_j)$ of $\mathscr{Q}\mathscr{Q}\mathscr{P}$; here $\varphi(F_i, G_j)$ is a bifiltered object of $\mathscr{P}$. We can extend this to a whole family of products $\varphi_{m,n}$ for $m, n \geqslant 1$:

$$\mathscr{Q}^m\mathscr{M} \otimes \mathscr{Q}^n\mathscr{N} \xrightarrow[m,n]{\varphi} \mathscr{Q}^{m+n}\mathscr{P}.$$

These take the pair (F_i, G_j), where now F_i and G_j are objects of $\mathscr{M}$ and $\mathscr{N}$ with m and n-fold filtration respectively, to the $(m+n)$-fold filtered object $\varphi_{m,n}(F_i, G_j)$.

Since $\varphi_{m,n}(0, N) = \varphi_{m,n}(M, 0) = 0$ for all M, N in $Ob(\mathscr{Q}^m\mathscr{M})$, $Ob(\mathscr{Q}^n\mathscr{N})$ these products induce maps on spaces (i.e., simplicial sets)

$$\mathscr{Q}^m\mathscr{M} \wedge \mathscr{Q}^n\mathscr{N} \rightarrow \mathscr{Q}^{m+n}\mathscr{P}.$$

We want to patch these together to obtain a pairing of spectra, so we must check that all these products are compatible with the deloopings.

LEMMA 7.11. *We have commutative diagrams*

$$\begin{array}{ccc} \mathcal{Q}^m\mathcal{M}\otimes\mathcal{Q}^n\mathcal{N} & \longrightarrow & \mathcal{Q}^{m+n}\mathcal{P} \\ | & & | \\ \mathcal{Q}^m\mathcal{L}\mathcal{M}\otimes\mathcal{Q}^n\mathcal{N} & \longrightarrow & \mathcal{Q}^m\mathcal{L}\mathcal{Q}^n\mathcal{P} \\ | & & | \\ \mathcal{Q}^{m+1}\mathcal{M}\otimes\mathcal{Q}^n\mathcal{N} & \longrightarrow & \mathcal{Q}^{m+n+1}\mathcal{P} \end{array} \qquad \begin{array}{ccc} \mathcal{Q}^m\mathcal{M}\otimes\mathcal{Q}^n\mathcal{N} & \longrightarrow & \mathcal{Q}^{m+n}\mathcal{P} \\ | & & | \\ \mathcal{Q}^m\mathcal{M}\otimes\mathcal{Q}^n\mathcal{L}\mathcal{N} & \longrightarrow & \mathcal{Q}^{m+n}\mathcal{L}\mathcal{P} \\ | & & | \\ \mathcal{Q}^m\mathcal{M}\otimes\mathcal{Q}^{n+1}\mathcal{N} & \longrightarrow & \mathcal{Q}^{m+n+1}\mathcal{P}. \end{array}$$

Proof. Obvious.

Checking the difference between the isomorphisms

$$\pi_k(\mathcal{Q}^{m+n}\mathcal{P}) \simeq \pi_{k+1}(\mathcal{Q}^{m+n+1}\mathcal{P})$$

induced by the two deloopings of $\mathcal{Q}^{m+n}\mathcal{P}$ in the lemma, we find that if $a \in \pi_k(\mathcal{Q}^{m+1}\mathcal{M})$, $\beta \in \pi_j\mathcal{Q}^n\mathcal{N}$ and $\Omega\alpha \in \pi_{k-1}\mathcal{Q}^m\mathcal{M}$, then $\varphi(\Omega\alpha, \beta) = (-1)^n\,\Omega\varphi(\alpha, \beta)$, and $\varphi(\alpha, \Omega b) = \Omega\varphi(\alpha, \beta)$.

We can summarize the results of this section in:

THEOREM 7.12. *Let $\mathcal{M}\times\mathcal{N}\to\mathcal{P}$ be a biexact functor. Then there is a natural pairing of spectra*

$$\mathcal{K}_{\mathcal{M}}\wedge\mathcal{K}_{\mathcal{N}}\to\mathcal{K}_{\mathcal{P}}$$

described explicitly in degrees $m, n \geqslant 0$ by the products $\varphi_{m,n}$.

Recall Quillen's localization theorem:

THEOREM 7.13. *Let $\mathcal{A}$ be an abelian category, $\mathcal{S}\subset\mathcal{A}$ a Serre subcategory, $\mathcal{A}/\mathcal{S}$ the associated quotient category. Then*

$$\mathcal{Q}\mathcal{S}\to\mathcal{Q}\mathcal{A}\to\mathcal{Q}\mathcal{A}/\mathcal{S}$$

is a fibration up to homotopy.

Proof. See [31].

Obviously $\mathcal{Q}^k\mathcal{S}\to\mathcal{Q}^k\mathcal{A}\to\mathcal{Q}^k\mathcal{A}/\mathcal{S}$ is a delooping of the fibration for all $k \geqslant 1$, and so we can rephrase Quillen's theorem:

THEOREM 7.13′. *Let $\mathcal{A}$, $\mathcal{S}$, $\mathcal{A}/\mathcal{S}$ be as above. Then*

$$\mathcal{K}_{\mathcal{S}}\to\mathcal{K}_{\mathcal{A}}\to\mathcal{K}_{\mathcal{A}/\mathcal{S}}$$

is a cofibre sequence of spectra.

This has the obvious but important:

COROLLARY 7.14. *Let $\mathscr{S}$, $\mathscr{A}$, $\mathscr{A}/\mathscr{S}$; $\mathscr{S}'$, $\mathscr{A}'$, $\mathscr{A}'/\mathscr{S}'$ be localizations of abelian categories, as in Theorem 7.13, and $\mathscr{M}$ an exact category such that there are compatible products (i.e., the horizontal maps are biexact):*

$$\begin{array}{ccc} \mathscr{S}\times\mathscr{M} & \longrightarrow & \mathscr{S}' \\ \downarrow & & \downarrow \\ \mathscr{S}\times\mathscr{M} & \longrightarrow & \mathscr{A}' \\ \downarrow & & \downarrow \\ \mathscr{A}/\mathscr{S}\times\mathscr{M} & \longrightarrow & \mathscr{A}'/\mathscr{S}'. \end{array}$$

Then

(i) $\mathscr{K}_{\mathscr{S}}\wedge\mathscr{K}_{\mathscr{M}}\to\mathscr{K}_{\mathscr{A}}\wedge\mathscr{K}_{\mathscr{A}/\mathscr{S}}\wedge\mathscr{K}_{\mathscr{M}}$ *is a cofibre sequence (also true for $\mathscr{A}'$, $\mathscr{S}'$, $\mathscr{A}'/\mathscr{S}'$).*

(ii) *The following diagram commutes:*

$$\begin{array}{ccc} \mathscr{K}_{\mathscr{S}}\wedge\mathscr{K}_{\mathscr{M}} & \longrightarrow & \mathscr{K}_{\mathscr{S}'} \\ \downarrow & & \downarrow \\ \mathscr{K}_{\mathscr{A}}\wedge\mathscr{K}_{\mathscr{M}} & \longrightarrow & \mathscr{K}_{\mathscr{A}'} \\ \downarrow & & \downarrow \\ \mathscr{K}_{\mathscr{A}/\mathscr{S}}\wedge\mathscr{K}_{\mathscr{M}} & \longrightarrow & \mathscr{K}_{\mathscr{A}'/\mathscr{S}'} \\ \downarrow & & \downarrow \\ S\mathscr{K}_{\mathscr{S}}\wedge\mathscr{K}_{m} & \longrightarrow & S\mathscr{K}_{\mathscr{S}'} \end{array}$$

If we look at the bottom square of the diagram of the lemma, we see:

$$\begin{array}{ccc} K_p(a/s)\times K_q(m) & \longrightarrow & K_{p+q}(a/s') \\ \Big\downarrow{\scriptstyle \partial+1} & \circlearrowright & \Big\downarrow{\scriptstyle \partial} \\ K_{p-1}(s)\times K_q(m) & \longrightarrow & K_{p+q-1}(s') \end{array}$$

Note. If we multiplied on the left by $K_q\mathscr{M}$ this would commute up to $(-1)^q$ [40, p. 274].

Suppose that $\mathscr{A}^{\cdot}$ is an abelian category filtered by Serre subcategories $\mathscr{A} \supset \cdots \mathscr{A}^{i} \supset \mathscr{A}^{i+1} \supset \cdots$. Then $\mathscr{K}_{\mathscr{A}}$ has an induced filtration $\mathscr{K}_{\mathscr{A}} \supset \cdots \supset \mathscr{K}_{\mathscr{A}^i} \supset \cdots$ such that for all $i \geqslant j$ we have cofiberings:

$$\mathscr{K}_{\mathscr{A}^i} \to \mathscr{K}_{\mathscr{A}^j} \to \mathscr{K}_{\mathscr{A}^j/\mathscr{A}^i}.$$

Associated to this filtration of $\mathscr{K}_{\mathscr{A}}$ there is a spectral sequence

$$E_1^{pq} = K_{-p-q}(\mathscr{A}^p/\mathscr{A}^{p+1}) \Rightarrow K_{-p-q}(\mathscr{A}).$$

If $\mathscr{A} = \mathscr{A}^0$ then E_1^{pq} is zero unless $p \geqslant 0$ and $p+q \leqslant 0$.

Note. If we have an increasing filtration $\cdots \mathscr{A}_i \subset \mathscr{A}_{i+1} \cdots$ we have a spectral sequence

$$E^1_{pq} = K_{p+q}(\mathscr{A}_p/\mathscr{A}_{p-1}) \Rightarrow K_{p+q}(\mathscr{A}),$$

which if $i \geqslant 0$ is concentrated in $p \geqslant 0$, $p+q \geqslant 0$.

For details see [1].

THEOREM 7.15. (a) *Let* $\mathscr{M}^{(i)} \times \mathscr{N}^{(j)} \to^{\mu} \mathscr{P}^{(k)}$ *be a biexact functor between abelian categories filtered by Serre subcategories, such that* $\mu(\mathscr{M}^{(i)} \times \mathscr{N}^{(j)}) \subset \mathscr{P}^{(i+j)}$. *Then there is a naturally induced pairing of spectral sequences*

$$E_r^{p,q}(\mathscr{M}^{\cdot}) \otimes E_r^{p',q'}(\mathscr{N}^{\cdot}) \to E_r^{p+p',q+q'}(\mathscr{P}^{\cdot}).$$

(b) *Let* $\mathscr{M}^{(i)} \times \mathscr{N} \to \mathscr{P}^{(i)}$ *be a pairing with* $\mathscr{M}^{\cdot}$ *and* $\mathscr{P}^{\cdot}$ *as in* (a), *and* $\mathscr{N}$ *exact. Then there is a product*

$$E_r^{p,q}(\mathscr{M}^{\cdot}) \otimes K_n(\mathscr{N}) \to E_r^{p,q-n}(\mathscr{P}^{\cdot}).$$

Proof. (a) For all n, q, $r \geqslant 0$ we have products

$$\mathscr{K}_{\mathscr{M}^n/\mathscr{M}^{n+r}} \wedge \mathscr{K}_{\mathscr{N}^q/\mathscr{N}^{q+r}} \xrightarrow{\mu} \mathscr{K}_{\mathscr{P}^{n+q}/\mathscr{P}^{n+q+r}}.$$

Following [41] these products induce the required product of spectral sequences so long as they satisfy the following two conditions.

(i) If $n \geqslant n'$, $q \geqslant q'$, $n+r \geqslant n'+r'$, $q+r \geqslant q'+r'$, then

$$\begin{array}{ccc}
\mathscr{K}_{\mathscr{M}^n/\mathscr{M}^{n+r}} \wedge \mathscr{K}_{\mathscr{N}^q/\mathscr{N}^{q+1}} & \xrightarrow{\mu} & \mathscr{K}_{\mathscr{P}^{n+q}/\mathscr{P}^{n+q+r}} \\
\Big| & & \Big| \\
\mathscr{K}_{\mathscr{M}^{n'}/\mathscr{M}^{n'+r'}} \wedge \mathscr{K}_{\mathscr{N}^{q'}/\mathscr{N}^{q'+r'}} & \longrightarrow & \mathscr{K}_{\mathscr{P}^{n'+q'}/\mathscr{P}^{n'+q'+r'}}.
\end{array}$$

(ii) For all $n, q \geqslant 0$, $r \geqslant 1$ in the diagram:

$$
\begin{array}{ccccc}
\mathcal{K}_{\mathcal{M}^n/\mathcal{M}^{n+r}} \wedge \mathcal{K}_{\mathcal{N}^q/\mathcal{N}^{q+r}} & & \xrightarrow{\ \mu_r\ } & & \mathcal{K}_{\mathcal{P}^{n+q}/\mathcal{P}^{n+q+r}} \\
 & \searrow^{\eta\wedge\partial} & & & \\
\downarrow{\scriptstyle \partial\wedge\eta} & & \mathcal{K}_{\mathcal{M}^n/\mathcal{M}^{n+1}} \wedge \mathcal{K}_{\mathcal{N}^{q+r}/\mathcal{N}^{q+r+1}} & & \downarrow{\scriptstyle \partial} \\
 & & & \searrow^{\mu_1} & \\
S\mathcal{K}_{\mathcal{M}^{n+r}/\mathcal{M}^{n+r+1}} \wedge \mathcal{K}_{\mathcal{N}^q/\mathcal{N}^{q+r}} & & \xrightarrow{\ \mu_1\ } & & S\mathcal{K}_{\mathcal{P}^{n+q+r}/\mathcal{P}^{n+q+r+1}}
\end{array}
$$

One has $\partial \cdot \mu_r = \mu_1(\eta \wedge \partial) + \mu_1(\partial \wedge \eta)$.

Note that at the level of homotopy groups, for the product

$$\mu_1 : \mathcal{K}_{\mathcal{M}^n/\mathcal{M}^{n+1}} \wedge S\mathcal{K}_{\mathcal{N}^q/\mathcal{N}^{q+r+1}} \to S\mathcal{K}_{\mathcal{P}^{n+q+r}/\mathcal{P}^{n+q+r+1}}$$

we must add a factor of $(-1)^q$:

$$\mu_1 : K_i(\mathcal{M}^n/\mathcal{M}^{n+1}) \otimes K_{j-1}(\mathcal{N}^q/\mathcal{N}^{q+r+1}) \xrightarrow{(-1)^q} K_{i+j-1}(\mathcal{P}^{n+q+r}/\mathcal{P}^{n+q+r+1}).$$

The truth of (i) is obvious. As for (ii), this involves tedious checking and use of [1, Sect. 9] (properties of products of spectra).

Associated to a commutative noetherian ring with unit, R, there are two categories.

$\mathcal{M}_R$ = Category of all f. g, R-modules. This is an abelian category, and we give it the naturally induced exact category structure.

$\mathcal{P}_R$ = Full subcategory of $\mathcal{M}_R$ of projective R-modules. We make it an exact category by selecting those sequences $P' \rightarrowtail P \twoheadrightarrow P''$ which are exact in $\mathcal{M}_R$. Thus all epis are admissible and monomorphisms are if they have projective cokernels.

If X is a noetherian separated scheme, we again have associated exact categories.

$\mathcal{M}_X$ = Abelian category of coherent $\mathcal{O}_X$-modules

$\mathcal{P}_X$ = Exact category of locally free modules.

Clearly if $X = \operatorname{Spec}(R)$, then $\mathcal{M}_R = \mathcal{M}_X$ and $\mathcal{P}_R = \mathcal{P}_X$.

Finally we set $K_i(X) = K_i(\mathcal{P}_X)$, $K_i(R) = K_i(\mathcal{P}_R)$, $K'_i(X) = K_i(\mathcal{M}_X)$ and $K'_i(R) = K_i(\mathcal{M}_R)$.

The groups K_i and K'_i have the following natural properties, which are only stated for schemes X leaving the affine case implicit. Proofs are either obvious or may be found in Quillen [31].

(1) The $K_i(X)$ are contravariant functors on the category of all noetherian separated schemes.

(2) The $K_i'(X)$ are covariant for those morphisms $X \to^f Y$ for which every coherent sheaf on X has a finite resolution by $R^{\cdot}f_*$-acyclic coherent sheaves, and contravariant with respect to flat morphisms.

(3) Both K_i and K_i' are presheaves in the Zariski, étale and flat topologies. We denote the associated sheaves by $\mathscr{K}_i$ and $\mathscr{K}'$.

(4) The natural exact functor $\mathscr{P}_X \to \mathscr{M}_X$ induces homomorphisms $K_i(X) \to K_i'(X)$ which are isomorphisms if X is regular.

(5) $K_i'(X_{\text{red}}) = K_i'(X)$.

(6) The biexact functors $\mathscr{P}_X \times \mathscr{M}_X \to^{\otimes \mathcal{O}_X} \mathscr{M}_X$ and $\mathscr{P}_X \times \mathscr{P}_X \to^{\otimes \mathcal{O}_X} \mathscr{P}_X$ induce pairings on the associated graded presheaves, and their sheafifications.

(7) If X and Y are schemes over a field k, there is a biexact functor:

$$\mathscr{M}_X \times \mathscr{M}_Y \xrightarrow{\otimes_k} \mathscr{M}(X \times_k Y)$$

which induces an external pairing

$$K'_*(X) \otimes K'_*(Y) \to K'_*(X \times Y).$$

On a (noetherian, separated) scheme X we have a filtration of $\mathscr{M}_X$ by Serre subcategories $\mathscr{M}^i_X$:

$\mathscr{M}^i_X$ = Category of sheaves with support of codimension at least i.

Let $E_r^{pq}(X)$ be the associated spectral sequence, then we have the following theorem of Quillen [31]:

THEOREM 7.16. *Let $X^{(p)}$ be the set of points of codimension p on X. Then*

(i) $E_1^{pq}(X) = \oplus_{x \in X^{(p)}} K_{-p-q}(k(x))$.

(ii) $E_r^{pq}(X) \Rightarrow K'_{-p-q}(X)$.

(iii) E_r^{pq} *is contravariant for flat morphisms.*

(iv) *If $X = \varprojlim X_i$ where $i \mapsto X_i$ is a filtered projective system with affine flat transition morphisms, then*

$$E_r^{pq}(X) = \varinjlim E_r^{pq}(X_i).$$

Note that we could have filtered $\mathscr{M}_X$ by dimension of support, and obtained a spectral sequence $E^r_{pq}(X)$ having essentially the same properties (though in (iii) and (iv) above one must take into account the relative dimension of the morphisms concerned). If X is of finite type over a field and of pure dimension d, then:

$$E^r_{p,q}(X) = E_r^{d-p,-d-q}(X).$$

The E_1 term of the spectral sequence of Theorem 2.1 breaks up into a family of complexes R_q^* $(q \geqslant 0)$:

$$R_q^p = \bigoplus_{x \in X^{(p)}} K_{q-p}(k(x))$$

each of which comes with an augmentation $K_q(X) \to R_q^*(X)$. By (iii) above this situation may be sheafified in the Zariski topology, to obtain for each $q \geqslant 0$ an augmented complex of sheaves,

$$\mathscr{K}_q \to \mathscr{R}_q^*$$

whose stalk at a point $x \in X$ is by (iv) of the theorem:

$$K_q(\mathscr{O}_{X,x}) \to R_q^*(\mathscr{O}_{X,x}).$$

The utility of these complexes comes from:

GERSTEN'S CONJECTURE. *If $X = \mathrm{Spec}(A)$, A regular local, then R_q^* is a resolution of K_q, for all q.*

THEOREM 7.17 (Quillen [31]) *The above conjecture is true if A is a semi-local ring on a scheme of finite type over a field.*

COROLLARY 7.18. *If X is a regular scheme of finite type over a field*

$$E_2^{p,-q}(X) = H^p_{\mathrm{ZAR}}(X, \mathscr{K}_q).$$

Proof. By (i) of Theorem 7.16, $\mathscr{R}_q^*$ is a flasque sheaf. By Gersten's Conjecture and the comment preceding it, it is a resolution of $\mathscr{K}_q$. Hence

$$H^p_{\mathrm{ZAR}}(X, \mathscr{K}_q) = \mathbf{H}^p(X, \mathscr{R}_q^*) = H^q(E_1^{p,*}(X)) = E_2^{p,-q}(X).$$

From (i) of Theorem 7.16 we can see that if $Y \subset X$ is a closed codim d subset of X:

$$\Gamma_Y(\mathscr{R}_p^*) = R_{p-d}^{*-d}(Y).$$

Therefore if X/k is regular, of finite type over a field:

$$H_Y^p(X, \mathscr{K}_q) = H^p(\Gamma_Y(\mathscr{R}_q^*)) = \mathbf{H}^{p-d}(Y, \mathscr{R}_{q-d}^*).$$

The geometric content of this computation lies in the following theorem:

THEOREM 7.19 (Quillen, Grayson). *Let X be a scheme. Then the differential in the spectral sequence*:

$$\bigoplus_{x \in X^{(p-1)}} K_1 k(x) \xrightarrow{d_1} \underbrace{\bigoplus_{x \in X^{(p)}} K_0(k(x))}_{\simeq Z^p(X) \text{ the group of codimension } p \text{ cycles on } X}$$

$$\wr\Big|$$

$$\bigoplus_{x \in X^{(p-1)}} k(x)^*$$

associates to every rational function $f \in k(x)^*$ its principal Cartier divisor on $\overline{\{x\}}$, which then determines a codim p cycle on X.

Proof. [31, 42].

COROLLARY 7.20. $\mathbf{H}^p(X, \mathscr{K}_p^*) \simeq CH^p(X)$ *the Chow group of cycles of codimension p on X, modulo rational equivalence* (*cf.* *Fulton* [17]).

Let R be a commutative ring, S a multiplicative set in R. Then the S-torsion modules form a Serre subcategory $\mathscr{E}_S$ of $\mathscr{M}_A$, with quotient category $\mathscr{M}_{S^{-1}R}$. Hence we have a localization sequence:

$$\xrightarrow{\partial} K_i(\mathscr{E}_S) \to K_i(\mathscr{M}_R) \to K_i(\mathscr{M}_{S^{-1}\mathscr{A}}) \xrightarrow{\partial} K_{i-1}(\mathscr{E}_S) \to .$$

In the special case of a one-dimensional local domain A with quotient field F and residue field k, we get (setting $S = R - \{0\}$)

$$\xrightarrow{\partial} K_i(k) \to K_i'(R) \to K_i(F) \xrightarrow{\partial} K_{i-1}(k) \to .$$

As is well known, $K_1(R)$ splits as a direct sum: $K_1(R) = R^{\cdot} \oplus SK_1 R$ for a general commutative R. For $f \in R^{\cdot}$ we denote the corresponding element of $K_1(R)$ by $\{f\}$. If $*$ denotes the K-theory product, we write products of the form $\{f_1\} * \cdots * \{f_p\} \in K_p(R)$ as $\{f_1, \ldots, f_p\}$ and refer to them as "p-symbols." The question to be answered in this section is: "How does ∂ act on symbols?" If $p = 1$ we find

$$\partial : K_1 F = F^* \to K_0 k = \mathbf{Z}$$

is the unique homomorphism having the property that if $f \in R$, $\partial\{f\} = l(A/fR)$. This is just a rephrasing of Theorem 7.19. For $p \geqslant 2$ we treat only the case of a discrete valuation ring. So let R be as above. Set $R/\pi R = k$, $\pi^{-1}R = F$ the residue and quotient fields, respectively. Since every $f \in R$ may be written

$$f = \pi^{v(f)} g \qquad g \in R^*$$

we have

$$\{f_1,\ldots,f_p\} = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_r\leqslant p} \{g_1,\ldots,\pi^{v(f_{i_1})},\ldots,g_k,\ldots,\pi^{v(f_{i_r})},\ldots,g_p\}.$$

So by Corollary 7.14

$$\partial\{f_1,\ldots,f_p\} = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_r\leqslant P} \partial\{\pi^{v(f_{i_1})},\ldots,\pi^{v(f_{i_r})}\}^{(-1)^{\epsilon_{i_1\cdots i_r}}}$$
$$\times\{\bar{g}_1,\ldots,\hat{g}_{i_1},\ldots,\bar{g}_k,\ldots,\hat{g}_{i_r},\ldots,\bar{g}_P\},$$

where $\bar{g}_i$ is the image of g_i in k^* and $\varepsilon_{i_1\cdots i_r}$ is the sign of the permutation $(i_1,\ldots,i_r,\ 1,\ldots,\hat{\imath}_1,\ldots,\ \hat{\imath}_r,\ldots,p)$. Now to compute the $\partial\{\pi,\ldots,\pi\}$ portion of this formula, we first observe that $\{\pi,\pi\}=\{\pi,-1\}$, since $\{\pi,1-\pi\}=1$. Rather than justify this directly, we shall use the fact that Waldhausen's product agrees with Loday's [28, 38]; now it is known for Loday's product that the symbols in K_2 defined via the product $K_1\times K_1\to K_2$ are the same as those defined by Milnor. Hence $\{\pi,\ldots,\pi\}=\{\pi,-1,\ldots,-1\}\in K_r(F)$ so $\partial\{\pi,\ldots,\pi\}=\{-1,\ldots,-1\}\in K_{r-1}(F)$. Summarizing, we have:

THEOREM 7.21. *Let R, F, k, v, f_i, g_i, $\bar{g}_i$ $(i=1\cdots p)$, be as above, then*

$$\partial\{f_1\cdots f_P\} = \sum_{i\leqslant i_1\leqslant\cdots\leqslant i_r\leqslant p} \{-1,\ldots,-1\}^{(-1)^{\epsilon_{i_1\cdots i_r}}\sum_{j=1}^{1} v(f_{i_j})}$$
$$\times\{\bar{g}_1,\ldots,\hat{g}_{i_1},\ldots,\bar{g}_k,\ldots,\hat{g}_{i_r},\ldots,\bar{g}_p\}.$$

THEOREM 7.22. *Let $f: X\to Y$ be a proper morphism of noetherian schemes.*

(i) *If there is an ample line bundle L on X, then there is a natural homomorphism of spectral sequences*

$$f_*: E^r_{pq}(X)\to E^r_{pq}(Y)$$

compatible with the map on the abutments

$$f_*: K'_*(X)\to K'_*(Y).$$

(ii) *In general there may be no such L on X and therefore we do not know how to construct a trace map on E^r for all r. However there is always a trace map between the E^2 terms of the spectral sequences*:

$$f_*: E^2_{pq}(X)\to E^2_{pq}(Y).$$

This, however, is only a homomorphism of bigraded groups.

Proof. (i) Let $\mathcal{N}_X \subset \mathcal{M}_X$ be the full subcategory consisting of those sheaves F for which $R^i f_* F = 0$, $i > 0$. Let $\mathcal{N}_i \subset \mathcal{M}_i$ be the induced filtration by dimension of support. By the same argument as in Quillen [31, p. 42], the inclusion $\mathcal{N}_i \subset \mathcal{M}_i$ is a homotopy equivalence, and therefore induces an isomorphism

$$E^r_{pq}(\mathcal{N}X) \simeq E^r_{pq}(\mathcal{M}X).$$

But we have an exact functor, compatible with the topological filtrations: $\mathcal{N}X \to \mathcal{M}Y$. Hence we have the desired homomorphism:

$$E^r_{pq}(X) \simeq E^r_{pq}(\mathcal{N}X) \to E^r_{pq}(Y).$$

An immediate corollary of (i) is the following generalization of a result of Weil.

Reciprocity Law for Curves. *Let X/k be a complete curve over a field, and $d_v: K_i(k(X)) \to K_{i-1}(k(v))$ the differential in the localization sequence for the local ring at the point v of X. Then $\forall \alpha \in K_i(k(X))$:*

$$\sum_v Nm_{k(v)/k}(d_v(\alpha)) = 0,$$

where $Nm_{k(v)/k}: K_{i-1}(k(v)) \to K_{i-1}(k)$ is the trace map for the finite field extension $k(v)/k$.

Proof. Consider the projection $p: X \to \mathrm{Spec}(k)$. $\alpha \in K_i(k(X))$ defines an element in $E^1_{1,i-1}(X)$. Now

$$\sum_v Nm_{k(v)/k}(d_v(\alpha)) = p_*(d^1_{1,i-1}(\alpha))$$

which by existence of the trace map $= d^1_{1,i-1} p_*(\alpha)$. But $p_*(\alpha)$ lies in $E^1_{1,i-1}(\mathrm{Spec}(k)) = 0$.

Using the reciprocity law we now prove (ii) of the theorem.

First observe that for each p we have natural additive functors

$$f_!: \mathcal{M}_p(X)/\mathcal{M}_{p-1}(X) \to \mathcal{M}_p(Y)/\mathcal{M}_{p-1}(Y).$$

However, $f_!$ is exact, for suppose $\mathcal{F}$ is a coherent sheaf supported on closed subset $Z \subset X$ of dimension at most p. Then $f(Z)$ either has dimension $<p$ or $f|_Z$ is generically finite, and in both cases $R^j f_* \mathcal{F}$ has support of dimension at most $p-1$ for all $j \geqslant 1$. Hence there is a homomorphism of bigraded groups

$$f_!: E^1_{pq}(X) \to E^1_{pq}(Y).$$

It will be sufficient to check that this homomorphism is compatible with the differentials:

$$\begin{array}{ccc} \bigoplus_{x \in X_p} K_{p+q} k(x) & \xrightarrow{d^1} & \bigoplus_{u \in X_{p-1}} K_{p+q-1}(k(u)) \\ \downarrow{f_*} & & \downarrow{f_*} \\ \bigoplus_{y \in Y_p} K_q k(y) & \xrightarrow{d^1} & \bigoplus_{v \in Y_{p-1}} K_{p+q-1}(k(v)). \end{array}$$

So we take $\alpha \in K_q k(x)$, some $x \in X_p$, and chase it round the diagram. There are two cases:

(a) $k(x)/k(f(x))$ is finite; i.e., $\{\bar{x}\} \to \{\bar{y}\}$ $(y = f(x))$ is generically finite. The non-finite locus has codim $\geqslant 2$ on $\{y\}$, so all the $u \in \{\bar{x}\} \cap X_{p-1}$ are finite over their images. So to check that $f_* d(\alpha) = df_*(\alpha)$ we may restrict ourselves to the open subset U of x in $\{\bar{x}\}$ on which f is finite; since $f_*: \mathscr{M}_U \to \mathscr{M}_{\{\bar{y}\}}$ is exact we have a map $E^r_{**}(U) \to E^r_{**}(\{\bar{y}\})$ for all $r \geqslant 1$, and the desired identity follows.

(b) $k(x)/k(f(x))$ transcendental. Since this means $f_*(\alpha) = 0$, we have to check $f_* \cdot d^1(\alpha) = 0$. If t: dim $k(x)/k(f(x)) \geqslant 2$ then all divisors u of $\{\bar{x}\}$ have relative dimension at least 1 over their images and so $f_* d^1(\alpha) = 0$. If t: dim $k(x) \mid k(f(x))$ is 1, then all divisors u which are finite over their images lie in the generic fibre of f: $\{\bar{x}\} \to \{\bar{y}\}$ which is a curve. The result now follows from the reciprocity law for curves.

As a corollary of this we obtain the following well known algebraic fact: (cf. Fulton [17, p. 6]).

COROLLARY 7.23. *Let A be a one-dimension local noetherian domain with maximal ideal P and quotient field K. Let L be a finite extension of K, B a finite A-algebra whose quotient field is L. Let $P_1 \cdots P_r$ be the prime ideals of B lying over P, $B_i = Bp_i$. Suppose $t \in B$ and $N_{L/K}(t) \in A$. Then*

$$l_A(A/N(t)A) = \sum_i [B_i/P_iB_i : A/P] \, l_{B_i}(B_i/tB_i).$$

Proof. Just use the theorem for $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ together with Theorem 7.14.

For a general scheme X, the biexact functor:

$$\mathscr{M}_X \times \mathscr{P}_X \xrightarrow{\otimes \mathscr{O}_X} \mathscr{M}_X$$

is compatible with the topological filtration on $\mathscr{M}_X$, so that we have pairings:

$$E_r^{pq}(X) \otimes K_i(X) \to E_r^{p,q+i}(X),$$

$$E^r_{pq}(X) \otimes K_i(X) \to E^r_{p,q+i}(X)$$

and hence also a pairing of complexes of sheaves:

$$\mathscr{K}_i \otimes \mathscr{R}_p^* \to \mathscr{R}_{p+i}^*.$$

This last pairing is compatible with the augmentations $\mathscr{K}_p \to \mathscr{R}_p^*$ and $\mathscr{K}'_p \to \mathscr{R}_p^*$.

Similarly for the external product for schemes of finite type over a field we get:

$$E_r^{pq}(X) \otimes E_r^{p',q'}(Y) \to E_r^{p+p',q+q'}(X \times Y).$$

In particular the product on E_1 terms provides pairings of complexes, for each $q, q' \geqslant 0$:

$$\tau^{*,-q,*,-q'}: E_1^{*,-q}(X) \otimes E_1^{*,-q'}(Y) \to E_1^{*,-q-q'}(X \times Y)$$

compatible with the differentials in the sense that:

$$d_1^{p+p',-q-q'}(\tau(x \otimes y)) = \tau(d_1^{p-q}(x) \otimes y) + (-1)^{p-q}\tau(x \otimes d_1^{p'-q'}(y)).$$

Using Theorem 7.16 we may write this pairing down explicitly; the pairing

$$\left[\bigoplus_{x \in X^{(i)}} K_{q-i}(\mathbf{k}(x))\right] \otimes \left[\bigoplus_{y \in Y^{(j)}} K_{q'-j}(\mathbf{k}(y))\right] \to \bigoplus_{z \in (X \times Y)^{(i+j)}} K_{q+q'-(i+j)}(\mathbf{k}(z))$$

is composed of the maps

$$K_{q-i}(\mathbf{k}(x)) \otimes K_{q-j}(\mathbf{k}(y)) \to \bigoplus_{z \in (\{\bar{x}\} \times \{\bar{y}\})^{(0)}} K_{q+q'-(i+j)}(\mathbf{k}(z))$$

induced by the pairing of rings

$$\mathbf{k}(x) \otimes \mathbf{k}(y) \to \bigoplus_z \mathbf{k}(z).$$

In particular $\tau^{q,-q,q',-q'}$ is the external product on cycles.

When, for each $q \geqslant 0$, we view $E_1^{*,-q}(X)$ as a complex $R_q^*(X)$ and similarly write $E_1^{*,-q'}(Y)$ as $R_{q'}^*(Y)$ we find that the product τ induces a pairing not from $R_q^*(X) \otimes R_{q'}^*(Y)$ to $R_{q+q'}^*(X \times Y)$ but between the complexes $I_q^*(X)$ and $I_{qi}^*(Y)$ where $I_p^{i-p} = E_1^{i,-p}$ for $p = q, q'$. Using a trick of Grayson we can modify the pairing in order to circumvent the failure of τ to be a pairing between the complexes R^*. Define a pairing

$$\mu_{q,q'}^{*,*}: R_q^*(X) \otimes R_{q'}^*(Y) \to R_{q+q'}^*(X \times Y)$$

by $\mu_{q,q'}^{i,j'} = (-1)^{qj}\tau^{i,-q,j,-q'}$. It is easily checked that $d(\mu(x \otimes y)) = \mu(dx \otimes y) + (-1)^{|x|}\mu(x \otimes dy)$, so that μ is indeed a pairing between complexes. We may sheafify the situation in order to obtain a pairing of complexes of sheaves on $X \times Y$ (where p_X, p_Y are the natural projections):

$$\mu : (p_X^* \mathscr{R}_q^*) \otimes (p_Y^* \mathscr{R}_{q'}^*) \to \mathscr{R}_{q+q'}^*$$

compatible, via the augmentations $\mathscr{K}_p \to \mathscr{R}_p^*$ for $p = q, q'$, with the product $p_X^* \mathscr{K}_q \otimes p_Y^* \mathscr{K}_{q'} \to \mathscr{K}_{q+q'}$.

THEOREM 7.24 (Projection Formula). *Let $f: X \to Y$ be a proper morphism, then we have a commutative diagram of graded complexes:*

$$\begin{array}{ccc} E^1_{*,q}(X) \otimes K_p(Y) & \longrightarrow & E^1_{*,q+p}(X) \\ \big\downarrow{\scriptstyle f_* \otimes 1} & & \big\downarrow{\scriptstyle f_*} \\ E^1_{*,q}(Y) \otimes K_p(Y) & \longrightarrow & E^1_{*,q+p}(Y) \end{array}$$

Proof. We have, for each $i \geqslant 0$, a commutative diagram of exact functors:

$$\begin{array}{ccc} [\mathscr{M}_i(X)/\mathscr{M}_{i-1}(X)] \times \mathscr{P}(Y) & \xrightarrow{\otimes \mathscr{O}_Y} & \mathscr{M}_i(X)/\mathscr{M}_{i-1}(X) \\ \big\downarrow{\scriptstyle f_* \times 1} & & \big\downarrow{\scriptstyle f_*} \\ [\mathscr{M}_i(Y)/\mathscr{M}_{i-1}(Y)] \times \mathscr{P}(Y) & \xrightarrow{\otimes \mathscr{O}_Y} & \mathscr{M}_i(X)/\mathscr{M}_{i-1}(X) \end{array}$$

The vertical arrows give the desired maps of complexes by Theorem 7.22 while the horizontal arrows do too, since the pairing "$\otimes_{\mathscr{O}_Y}$" respects the topological filtrations on $\mathscr{M}(X)$ and $\mathscr{M}(Y)$.

8. Chow Theories on the Categories of Schemes

Recall from Theorems 7.16 and 7.22 that the two Quillen spectral sequences (corresponding to the dimension and codimension filtrations) have the following properties:

(i) E^2_{pq} is covariant with respect to proper morphisms.

(ii) $E^2_{p,-q}(X) \simeq CH_p(X)$, the Chow group of dimension p cycles modulo rational equivalence.

(iii) If X is regular and of finite type over a field, then $E_1^{pq}(X)$ breaks up into complexes R_p^* which are resolutions of $\mathscr{K}_p$, so that $E_2^{p,q}(X) \simeq H^p(X, \mathscr{K}_{-q})$.

The significance of these three properties is that (i) and (ii) suggest that the functor $X \to E^2_{pq}(X)$ forms a homology theory on the category of schemes, which contains the "classical" Chow homology theory (cf. Fulton

[17]) as a direct summand. Turning to (iii), this we may regard as saying that we have a "duality" isomorphism between the groups $E^2_{pq}(X)$ and $H^{n-p}(X, \mathscr{K}_{q-n})$ for a pure n-dimensional regular variety. Because the functors K_q are contravariant on the category of all schemes we see that the $H^p(X, \mathscr{K}_q)$ form a bigraded contravariant functor with a product structure induced by the K-theory product.

The object of the rest of this section will be to show that these two functors form a homology-cohomology theory, with values in the category of abelian groups, possessing suitable properties. Such a theory can be referred to as a "Chow theory."

We now give a general description of a Chow theory, and list the main properties such a theory should have, which are more extensive than the axioms of Section 1.

Homology

Homology should be a covariant functor from the category of schemes and proper morphisms to the category of bigraded abelian groups:

$$X \to CH_{r,s}(X).$$

This functor should have the following properties.

(1) Given $Y \to^f X$ of pure relative dimension d, there should be Gysin maps $CH_{r,s}(X) \to CH_{r+d,s+d}(Y)$ if f is either flat or a regular immersion.

(2) The Gysin homomorphism $CH_{r,s}(X) \to CH_{r+1,s+1}(\mathbf{A}^1_X)$ should be an isomorphism (i.e., CH_{**} should satisfy homotopy).

(3) Given a pair (X, U), U open in X, there should be a long exact sequence

$$\xrightarrow{\partial} CH_{r,s}(X-U) \to CH_{r,s}(X) \to CH_{r+d,s+d}(U) \xrightarrow{\partial} CH_{r-1,s}(X-U) \to$$

(d is the relative dimension of U/X).

(4) Given a filtered projective system $\{X_\alpha, \tau_{\alpha\beta}\}$ with affine flat transition morphisms, the natural map

$$\varinjlim_\alpha CH_{**}(X_\alpha) \to CH_{**}(\varprojlim_\alpha X_\alpha)$$

should be an isomorphism.

(5) There should be a specialization map from the homology of the general fibre of a flat family to the homology of the special fibre.

(6) Homology should only depend on the reduced structure of a scheme, i.e.,

$$CH_{r,s}(X) = CH_{r,s}(X_{\text{red}}).$$

(7) If X is irreducible of pure dimension n, then $CH_{n,+n}(X)$ should be of rank 1 with a canonical generator $[X]$. Hence for each closed codimension p subscheme $Y \subset X$ we should get a class $[Y] \in CH_{n-p,n-p}(X)$; this "cycle class" map should define a natural transformation from the classical Chow group functor [17] to the theory CH_{**}.

Cohomology

This should be a contravariant functor from the category of all pairs of schemes (X, U), $U \subset X$ open, to the category of bigraded abelian groups:

$$(X, U) \to A^{p,q}_{X-U}(X) = A^{p,q}_Y(X),$$

where $Y = X - U$. Cohomology should have cup products and cap products, both compatible with supports and with each other:

$$A^{p,q}_Y(X) \otimes A^{p',q'}_Z(X) \xrightarrow{\cup} A^{p+p',q+q'}_{Y \cap Z}(X)$$

$$A^{p,q}_Y(X) \otimes CH_{r,s}(X) \xrightarrow{\cap} CH_{r-p,s-q}(Y).$$

Cohomology should also have the following properties:

(1) For X regular, of finite type over a field and of pure dimension n, cap product should induce an isomorphism:

$$A^{p,q}_Y(X) \xrightarrow{\cap [X]} CH_{d-p,d-q}(Y).$$

(2) If $p: X \to Y$ is proper and if $x \in CH_{**}(X)$, $y \in A^{**}(Y)$:

$$p_*(x \cap p^*(y)) = p_*(x) \cap y$$

(The projection formula).

(3) If X is of finite type over a field, smooth and quasi-projective, then the cup product on $A^{p,p}$ should coincide, via duality, with the classical intersection product.

(4) There should be a theory of Chern classes satisfying the usual axioms (as described in Grothendieck's article [21]).

Definition 8.1. From now on, by Chow homology and cohomology we shall mean the theory defined as follows:

$CH_{p,q} = E^2_{p,-q}(X)$; homology graded by dimension,

$A^{p,q}_Y(X) = H^p_Y(X, \mathscr{K}_q)$; cohomology with supports in Y.

(To keep track of the indices, recall that

$$E^1_{p,q}(X) = \bigoplus_{\dim(\bar{x}) = p} K_{p+q}(k(x)).)$$

The Cup product is the product induced by the pairings of sheaves $\mathscr{K}_p \otimes \mathscr{K}_q \to \mathscr{K}_{p+q}$. To define the cap product we use the pairing (Section 7)

$$E^1_{r,s} \otimes K_q \to E^1_{r,s+q},$$

which induces a pairing of sheaves, and hence of hypercohomology:

$$H^p_Y(X, \mathscr{K}_q) \otimes \mathbf{H}^{-r}(E^1_{\cdot,\cdot}) \to \mathbf{H}^{p-r}_Y(E^1_{\cdot,\cdot}),$$

i.e., $H^p_Y(X, \mathscr{K}_q) \otimes E^2_{r,-s}(X) \to E^2_{r-p,-s+q}$. This gives us our product

$$A^{p,q}_Y(X) \otimes CH_{r,s}(X) \xrightarrow{\cap} CH_{r-p,s-q}(X).$$

We can now check some of the properties directly from the definitions. The homology Gysin map for flat morphisms becomes the flat contravariance of the *B-G-Q* spectral sequence (Theorem 7.16). Similarly, compatibility with inverse flat limits is Theorem 7.17. For cohomology, we know that excision is a general feature of the local cohomology of sheaves of abelian groups.

The long exact sequence for the homology of a pair (X, U) can be viewed in two ways. Either as the long exact sequence for local hypercohomology of the graded complex $\mathscr{E}^1_{**}$ of flasque sheaves, together with the isomorphism $\Gamma_Y(\mathscr{E}^1_{pq}) \simeq E^1_{pq}(Y)$, or as coming from the fibration $\mathscr{M}(X-U) \to \mathscr{M}(X) \to \mathscr{M}(U)$ of filtered categories.

From Corollary 7.20 we have, for X regular irreducible and of finite type over a field, the duality isomorphism

$$A^{p,q}_Y(X) \simeq CH_{n-p,-n+q}(Y) \qquad n = \dim X.$$

In particular, if Y is irreducible of codimension d:

$$A^{d,d}_Y(X) \simeq CH_{n-d,-(n-d)}(Y) \simeq \mathbf{Z}[Y].$$

So we have a fundamental class $\eta_X(Y) \in A^{d,d}_Y(X)$ such that $\eta_X(Y) \cap [X] = [Y]$.

THEOREM 8.2. *Let $Y \subset Z$ be a pair of closed subschemes of the scheme X, then given elements $\alpha \in A^{p,q}_Y(X)$, $\beta \in A^{r,s}_Z(X)$, $\gamma \in CH_{i,j}(X)$ we have:*

$$(\alpha \cup \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma) \in CH_{i-p-r,j-q-s}(Y).$$

Note that if $\alpha \in A^{p,q}_Y(X)$ and $\beta \cap \gamma \in CH_{i-r,j-s}(Z)$, we define $\alpha \cap (\beta \cap \gamma)$ as $(i^*\alpha) \cap (\beta \cap \gamma)$, where $i\colon Z \to X$ is the natural inclusion.

Proof. Since the K-theory product is associative for all triexact functors

$\mathcal{M} \times \mathcal{M}' \times \mathcal{M}'' \to \mathcal{N}$, and $\mathcal{K}_p \otimes E^1_{*,*} \to E^1_{*,*}$ is a map of graded complexes, we have a commutative diagram:

$$\begin{array}{ccccc} & & \mathscr{E}^1_{*,*} \otimes \mathcal{K}_s \otimes \mathcal{K}_q & & \\ & \swarrow & & \searrow & \\ \mathscr{E}^1_{*,*+s} \otimes \mathcal{K}_q & & & & \mathscr{E}^1_{*,*} \otimes \mathcal{K}_{s+q} \\ & \searrow & & \swarrow & \\ & & \mathscr{E}^1_{*,*+s+q} & & \end{array}$$

The theorem now follows by applying general properties of local cohomology.

THEOREM 8.3. *Let $p: T \to X$ be a flat map whose fibres are all affine spaces of dimension d (note that p is therefore surjective). Then $p^*: CH_{r,s}(X) \to CH_{r+d,s+d}(T)$ is an isomorphism.*

Proof. (following Quillen's proof for K' [31, p. 44]). If Z is a closed subset of X with complement U, then because f is flat we have a map of exact sequences

$$\begin{array}{ccccccc} \longrightarrow & CH_{**}(Z) & \longrightarrow & CH_{**}(X) & \longrightarrow & CH_{**}(U) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & CH_{**}(T_Z) & \longrightarrow & CH_{**}(T) & \longrightarrow & CH_{**}(T_U) & \longrightarrow \end{array}$$

which comes from the commutative diagram of filtered categories:

$$\begin{array}{ccccc} \mathcal{M}(Z) & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \mathcal{M}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}(T_Z) & \longrightarrow & \mathcal{M}(T_X) & \longrightarrow & \mathcal{M}(T_U). \end{array}$$

By the five lemma the proposition is true for X if it is true for Z and U. Using noetherian induction we can assume the proposition holds for all closed subsets $Z \neq X$. Further, we can suppose X irreducible, for if $X = Z_1 \cup Z_2$ with $Z_i \neq X$, $i = 1, 2$, then the proposition holds for Z_1 and $X - Z_1 = Z_2 - (Z_1 \cap Z_2)$ by induction and the five lemma. Also we can assume X reduced since $CH_{**}(X) = CH_{**}(X_{red})$.

Taking the inductive limit over all closed proper subschemes Z of X we get

$$\varinjlim CH_{**}(U) = CH_{**}(\mathrm{Spec}(k(X)))$$

and

$$\varinjlim CH_{**}(T_U) = CH_{**}(T_{k(X)}).$$

Both of these equalities follow from Theorem 7.16.

We are therefore reduced to the case of $\mathbf{A}_k^d$ for k a field; further by induction on d, we can take $d=1$. This is the only point at which we depart from the proof for K'-theory.

LEMMA 8.4. *Let k be a field. Then*

$$CH_{r+1,s+1}(\mathbf{A}_k^1) \simeq CH_{r,s}(\mathrm{Spec}(k)).$$

Proof of lemma. By duality, we rewrite this as

$$\left\{\begin{matrix} p=-r \\ q=-s \end{matrix}\right\}: H^p(\mathbf{A}_k^1, \mathscr{K}_q) = H^p(\mathrm{Spec}(k), \mathscr{K}_q),$$

and start with the local cohomology sequence where $P = \mathbf{P}_k^1 - \mathbf{A}_k^1$)

$$\xrightarrow{\partial} H_P^p(\mathbf{P}_k^1, \mathscr{K}_q) \to H^p(\mathbf{P}_k^1, \mathscr{K}_q) \to H^p(\mathbf{A}_k^1, \mathscr{K}_q) \xrightarrow{\partial}. \qquad (*)$$

Again by duality

$$\begin{aligned} H_P^p(\mathbf{P}_k^1, \mathscr{K}_q) = H^{p-1}(\mathrm{Spec}(k), \mathscr{K}_{q-1}) &= 0, && p>1, \\ &= K_{q-1}(k), && p=1, \\ &= 0, && p=0. \end{aligned}$$

Clearly we must compute $H^p(\mathbf{P}_k^1, \mathscr{K}_q)$.

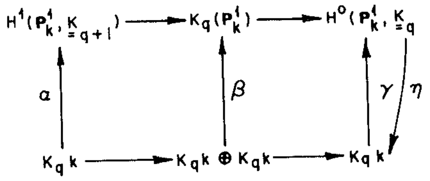
SUBLEMMA 8.5.

$$\begin{aligned} H^p(\mathbf{P}_k^1, \mathscr{K}_q) &= K_q(k), && p=0, \\ &= K_{q-1}(k), && p=1, \\ &= 0, && p>1. \end{aligned}$$

Proof of sublemma. The Quillen spectral sequence for a curve degenerates at E^2, and splits up into exact sequences:

$$0 \to H^1(\mathbf{P}_k^1, \mathscr{K}_{q+1}) \to K_q(\mathbf{P}_k^1) \to H^0(\mathbf{P}_k^1, \mathscr{K}_q) \to 0.$$

Since $K_q\mathbf{P}^1_k \simeq K_q k \oplus K_q k$ (Quillen [31]) we have a commutative diagram:



where

$\alpha = p^*(\ ) \cup \xi, \xi \in H^1(\mathbf{P}^1_k, \mathscr{K}_1)$ the tautological Cartier divisor,

$\beta = p^*(\ )([\mathscr{O}_{\mathbf{P}^1}] - [\mathscr{O}_{\mathbf{P}^1}(-1)]) \oplus p^*$,

$\gamma = p^*$,

$\eta =$ evaluation at any k-rational point.

From this diagram we immediately see that $H^1(\mathbf{P}^1_k, \mathscr{K}_{q+1}) \simeq H^0(\mathbf{P}^1_k, \mathscr{K}_q) \simeq K_q(k)$, and so we are done.

Returning to the lemma we see that

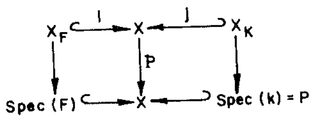
$$H^1_P(\mathbf{P}^1_k, \mathscr{K}_q) \simeq H^1(\mathbf{P}^1_k, \mathscr{K}_q)$$

and so the long exact sequence (*) becomes:

$$\begin{aligned} 0 \to H^0(\mathbf{P}^1_k, \mathscr{K}_q) \to H^0(\mathbf{A}^1_k, \mathscr{K}_q) \to H^1_P(\mathbf{P}_k, \mathscr{K}_q) \\ \simeq H^1_P(\mathbf{P}_k, \mathscr{K}_q) \simeq H^1(\mathbf{P}_k, \mathscr{K}_q) \to H^1(\mathbf{A}_k, \mathscr{K}_q) \to 0. \end{aligned}$$

(Note that $H^0_P(\mathbf{P}^1_k, \mathscr{K}_q) = 0$ since $\Gamma_P(\mathscr{R}^i_q) = 0$ for $i = 0$.) Thus we have completed the proof of both the lemma and the theorem.

Let $X \to^p \mathrm{Spec}(R)$ be a flat family, where R is a discrete valuation ring with quotient field F and residue field k. So:



We wish to define a homomorphism ("specialization") from the homology of the general fibre X_F of p to that of the special fibre.

The closed point P of $\mathrm{Spec}(R)$ is a Cartier divisor with local equation π where π is any generator of the maximal ideal of R. Then η_P, the fundamental class of $[P]$ is the image of the class $\{\pi\}$ under

$$F^* = K_1 F = H^0(\mathrm{Spec}(F), \mathscr{K}_1) \xrightarrow{\partial} H^1(\mathrm{Spec}(R), \mathscr{K}_1).$$

DEFINITION 8.6. The specialization map $CH_{r,s}(X_F) \to^{\sigma} CH_{r,s}(X_k)$ is given by

$$\alpha \xrightarrow{\sigma} \partial(p^*\{\pi\} \cap \alpha),$$

where

$$CH_{r,s} \xrightarrow{p^*\{\pi\}\cap} CH_{r,s-1}(X_F) \xrightarrow{\partial} CH_{r,s}(X_k).$$

Note that the long exact sequence for the pair (X, X_F) has a dimension shift:

$$CH_{r,s}(X) \to CH_{r-1,s-1}(X_F) \to CH_{r-1,s}(X_k).$$

The specialization map has the following property:

THEOREM 8.7. *Let $\alpha \in CH_{r,s}(X)$. Then*

$$\sigma(i^*(\alpha)) = p^*\eta_P \cap \alpha.$$

Proof. This equality is the same as the commutativity of the diagram:

$$\begin{array}{ccc} H^0(X_F, \mathscr{K}_1) & \xrightarrow{\partial} & H^1_{X_k}(X, \mathscr{K}_1) \\ \otimes & & \otimes \\ \mathbf{H}^{n-r}(X, \mathscr{R}^*_{s+n}) & \longrightarrow & \mathbf{H}^{n-r}(X, \mathscr{R}^*_{s+n}) \\ \downarrow & & \downarrow \\ \mathbf{H}^{n-r}(X_F, \mathscr{R}^*_{s+n+1}) & \xrightarrow{\partial} & \mathbf{H}^{n-r+1}_{X_k}(X, \mathscr{R}^*_{s+n+1}). \end{array}$$

This, however, is just a statement about local cohomology and the product $\mathscr{R}^*_{s+n} \otimes \mathscr{K}_1 \to \mathscr{R}^*_{s+n+1}$.

Note that a similar construction gives a specialization map for K'_*.

Let $Y \to^i X$ be a codimension p local complete intersection. By [2] we have a family $D_{X/Y}$ over $A^1_{\mathbf{Z}}$ which is constructed as follows:

If $\mathscr{I}_{X/Y} \subset \mathscr{O}_X$ is the sheaf of ideals defining Y, set $D_{X/Y} = \operatorname{Spec}_{\mathscr{O}_X[t]}((\oplus_{n\geq 0}(\mathscr{I}_{X/Y}, t)^n)_{(t)})_{\deg=0}$. While the family $D_{X/Y}$ need not be flat over $A^1_{\mathbf{Z}}$ we do know that t is not a zero divisor on $D_{X/Y}$, and that the special fibre over $t = 0$ is equal to $C_{X/Y}$ ($N_{X/Y}$ in our situation). The complement of the special fibre is $X \times_{\mathbf{Z}} \mathbf{G}_m$ which is flat over X. So if $f: X \times \mathbf{G}_m \to X$ and $p: N_{X/Y} \to Y$ are the projections our Gysin map is the composition:

$$CH_{r,s}(X) \xrightarrow{f^*} CH_{r+1,s+1}(X \times \mathbf{G}_m) \xrightarrow{p^*\{t\}\cap(\ )} CH_{r+1,s}(X \times \mathbf{G}_m)$$

$$\xrightarrow{\partial} CH_{r,s}(N_{X/Y}) \xrightarrow{p^{*-1}} CH_{r-p,s-p}(Y).$$

Note that p^{*-1} is defined because $N_{X/Y}$ is a vector bundle over Y.

This construction is a slight generalization of that of Verdier [14].

The Projection Formula

We now verify the obvious analogue for our situation of the classical projection formula.

THEOREM 8.8. *Let $X \to^p Y$ be a proper morphism and $x \in CH_{**}(X)$, $y \in A^{**}(Y)$. Then*

$$p_*(x \cap p^* y) = p_*(x) \cap y.$$

Proof. This follows immediately from Theorem 7.24 after "sheafifying" the commutative diagram there and taking hypercohomology.

Finally, we know that the cup production on the groups $A^p(X) \simeq H^p(X, \mathscr{K}_p)$ is compatible with the classical intersection product, when this is defined.

THEOREM 8.9. *Let X be a smooth, n-dimensional, irreducible variety of finite type over the field K. Suppose Y and Z and are cycles on X, of codimension p, q, respectively, intersecting properly. Then using Serre's intersection theory we may define $Y.Z$ as a cycle on $S = Y \cap Z$. If $\eta(Y)$ and $\eta(Z)$ are the fundamental classes of Y and Z in $H^p_Y(X, \mathscr{K}_q)$ and $H^q_Z(X, \mathscr{K}_q)$ (i.e., $\eta(Y) \cap [X] = [Y] \in CH_{n-p}(Y)$ and $\eta(Z) \cap [X] = [Z] \in CH_{n-q}(Z)$) their cup product $\eta(Y) \cup \eta(Z)$ lies in $H^{p+q}_{Y \cap Z}(X, \mathscr{K}_{p+q})$ and so defines a class $(\eta(Y) \cup \eta(Z)) \cap [X] \in CH_{n-p-q}(S)$. Then as cycles on S, we have:*

$$(\eta(Y) \cup \eta(Z)) \cap [X] = (-1)^{pq} [Y.Z].$$

Proof. [43, 44].

Finally in order to construct Chern classes and prove the Riemann–Roch theorem we need:

THEOREM 8.10. *Let E be a vector bundle over a regular variety X. Then*

$$A^{p,q}(\mathbf{P}(E)) \simeq \bigoplus_{i=0}^{n-1} A^{p-i,q-i}(X),$$

where $A^{p-i,q-i}$ is understood to be zero if $i > p, q$.

Proof. Consider the Leray Spectral Sequence for the projection $\pi: \mathbf{P}(E) \to X$. By the compatibility of E_2 of the B-G-Q spectral sequence with flat inverse limits,

$$(R^j \pi_* K_q)_x \simeq H^j(\mathbf{P}^{n-1}_{\mathscr{O}_{X,x}}, \mathscr{K}_q) \qquad \forall x \in X.$$

LEMMA 8.11. *Let R be a regular local ring. Then*

$$H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) \simeq \bigoplus_{i=0}^{n-1} H^{p-i}(\mathrm{Spec}(R), \mathscr{K}_{q-i}).$$

Proof of lemma. We have the localization sequence

$$H^{p-1}(\mathbf{A}_R^{n-1}, \mathscr{K}_q) \xrightarrow{\partial} H_H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) \to H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) \to H^p(\mathbf{A}_R^{n-1}, \mathscr{K}_q),$$

where H is the hyperplane at ∞ on $\mathbf{P}^{n-1}$.

By Theorem 8.3 $\pi: \mathbf{A}_R^{n-1} \to \mathrm{Spec}(R)$ induces an isomorphism on cohomology, hence the long exact sequence above splits into short exact sequences:

$$\begin{array}{ccccccc} 0 \to & H_H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) & \to & H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) & \to & H^p(A_R^{n-1}, \mathscr{K}_q) \to 0 \\ & \wr\!\!\mid & & \uparrow \pi^* & \nearrow \pi^* \wr & \\ & H^{p-1}(\mathbf{P}^{n-2}, \mathscr{K}_{q-1}) & & H^p(\mathrm{Spec}(R), \mathscr{K}_q) & & \end{array}$$

The lemma now follows by induction on n.

Observe that if $\xi_H \in H_H^1(\mathbf{P}_R^{n-1}, \mathscr{K}_1)$ is the tautological divisor, then we have a commutative diagram:

$$\begin{array}{ccccc} H^p(\mathrm{Spec}(R), \mathscr{K}_q) & \xrightarrow{\pi^*} & H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) & \xrightarrow{\cup \xi} & H_H^{p+1}(\mathbf{P}_R^{n-1}, \mathscr{K}_{q+1}) \\ & \searrow^{\pi^*} & & & \wr\!\!\mid \\ & & & & H^p(\mathbf{P}^{n-2}, \mathscr{K}_q) \end{array}$$

So we can rewrite the direct sum:

$$H^p(\mathbf{P}_R^{n-1}, \mathscr{K}_q) \simeq \bigoplus_{i=0}^{n-1} \xi^i \cup \pi^* H^{p-i}(\mathrm{Spec}(R), \mathscr{K}_q).$$

Returning to the theorem, the result now follows by the standard Leray Spectral Sequence argument.

Note. (i) The same theorem is true for CH^{**} without the regularity assumption.

(ii) For the case of $A^{p,p}$ this is only a minor modification of the classical argument (cf. Verdier [14]).

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