Invent. math. 66, 191–197 (1982)

Inventiones mathematicae

On the Slice Genus of Knots

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Given a knot K in the 3-sphere, the genus of K, denoted $g(K)$, is defined to be the minimal genus for a Seifert surface for K. The slice genus $g_s(K)$ is defined to be the minimal genus of an oriented surface G admitting a smooth proper embedding in the 4-ball which maps ∂G to K. If we insist the embedding of G have no local maximum with respect to the radial function, we obtain the ribbon genus $g_r(K)$ instead. Thus a knot is slice (ribbon) if and only if $g_s(K)=0$ ($g_r(K)=0$). It is clear that $g_s(K)\leqq g_r(K)\leqq g(K)$.

There are well known lower bounds on $g_s(K)$ given by invariants of a Seifert matrix for K. These are all included in the invariant $m(K)$ [8] defined by Taylor. It gives the best possible bound based on a Seifert matrix. $m(K)$ vanishes if and only if the Seifert pairing is metabolic. If this is the case, K is called algebraically slice. The work of Casson and Gordon [1, 2, 5] showed that certain algebraically slice knots are not in fact slice.

We generalize the main theorem of [1]. As an application, we give a sequence of algebraically slice knots Q_n such that $g_s(Q_n)=g(Q_n)=n$. We also study the slice genus of $K_t \# K_t$ where K_t denotes the t twisted double of the unknot. We show for example that $g_s(K_{12}\# K_{12})=2$. K_{12} is algebraically slice but not slice by [1]. Section 1 has some preliminaries on the linking form. In Sect. 2, we state and prove our main theorem. In Sect. 3 we give our examples. In this paper, all manifolds are oriented. We use e to denote the Euler characteristic.

We wish to thank Andrew Casson, Pierre Conner and Larry Taylor for helpful discussions.

§1. Linking Forms

A linking form on a finite abelian group A is a bilinear symmetric map $\alpha: A \times A \to \mathbb{Q}/\mathbb{Z}$ which is nonsingular. Here nonsingular means that the correlation map $c: A \to A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. It will be convenient for our

* Partially supported by the N.S.F.

0020-9910/82/0066/0191/$01.40

purposes to define a dual form α^* on A^* by the formula $\alpha^*(cx, cy) = \alpha(x, y)$. If H is subgroup of A, define

$$H^{\perp} = \{x \in A \mid \alpha(x, h) = 0 \, \forall h \in H\}.$$

If there is a subgroup H such that $H = H^{\perp}$, α is called metabolic and H is called a metabolizer.

We now discuss the notion of a presentation for linking forms. Let L be a free $\mathbb{Z}$ module of finite rank and $\langle\ ,\ \rangle$ a nondegenerate bilinear symmetric form $L \times L \to \mathbb{Z}$. Nondegenerate means the correlation $L \to L^* = \mathrm{Hom}(L, \mathbb{Z})$ is injective. We can extend $\langle\ ,\ \rangle$ to a form $V \otimes \mathbb{Q} \times V \otimes \mathbb{Q} \to \mathbb{Q}$ and let $L^{\#} = \{x \in V \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z},\ \forall y \in L\}$. We have $L \subset L^{\#}$ and $L^{\#}/L$ is a finite abelian group. One can define a linking form α on $L^{\#}/L$ by

$$\alpha(xL, yL) = \langle x, y \rangle \bmod \mathbb{Z}$$

$\langle\ ,\ \rangle$ is said to be a presentation of α. Every linking form has such an even presentation [9] (Theorem 6).

Suppose M is a rational homology 3-sphere and consider the geometric linking form l defined on $H_1(M)$. Let β denote $-l^*$ defined on $(H_1(M))^*$, the set of characters $\chi\colon H_1(M) \to \mathbb{Q}/\mathbb{Z}$.

Lemma 1. *If M is a boundary of a 4-manifold V then $\beta = \beta_1 \oplus \beta_2$ where β_2 is metabolic and β_1 has a presentation with rank* $\dim H_2(V, \mathbb{Q})$ *and signature* Sign V. *Moreover, the set of characters which extend to $H_1(V)$ forms a metabolizer for β_2. If V is spin, the presentation of β_1 can be taken to be even.*

Proof. We will consider the long exact sequence for the pair (V, M)

$$0 \to H_2(V) \xrightarrow{i} H_2(V, M) \xrightarrow{\partial} H_1(M) \to H_1(V) \xrightarrow{j} H_1(V, M) \to 0.$$

We can describe l on the image of ∂ as follows. Let $x, y \in H_2(V, M)$, then pick $\tilde{x}, \tilde{y} \in H_2(V)$ such that $i\tilde{x} = rx$ and $i\tilde{y} = sy$ where r and s are integers. An easy geometric argument shows

$$(*) \qquad l(\partial x, \partial y) = -\frac{1}{rs}\langle x, y \rangle.$$

Here $\langle\ ,\ \rangle$ denotes the intersection pairing on $H_2(V)$. Pick a free subgroup F of $H_2(V, M)$ so that $H_2(V, M) = F \oplus \mathrm{Tor}\, H_2(V, M)$. Let $B = \partial F$ and $D = \partial\, \mathrm{Tor}\, H_2(V, M)$. Let $l_1 = -l/B$ then by $(*)$ (B, l_1) is nonsingular and has presentation with rank $=$ rank F and signature Sign V. This presentation is even if V is spin. By [9] Lemma 1,

$$(H_1(M), -l) = (B, l_1) \oplus (B^{\perp}, l_2).$$

The formula $(*)$ shows that $D \subset B^{\perp}$ and $D \subset D^{\perp}$.

Since $H_1(M)$ is torsion, the kernel of j is a subgroup of $\mathrm{Tor}\, H_1(V)$. Thus, sitting in the exact sequence for (V, M) above, we can find

$$0 \to \mathrm{Tor}\, H_2(V) \to \mathrm{Tor}\, H_2(V, M) \to B^{\perp} \to \mathrm{Tor}\, H_1(V) \to \mathrm{Tor}\, H_1(V, M) \to 0.$$

By Poincaré duality and the universal coefficient theorem, we have isomorphisms

$$\operatorname{Tor} H_2(V) \approx \operatorname{Tor} H_1(V, M)$$

and

$$\operatorname{Tor} H_2(V, M) \approx \operatorname{Tor} H_1(V).$$

Therefore, $|D|^2 = |B^\perp|$. Since $D \subset D^\perp$, D is a metabolizer for l_2. Now $\beta = -l^* = l_1^* \oplus l_2^*$. Define $\beta_1 = l_1^*$ and $\beta_2 = l_2^*$. β_1 is noncanonically isomorphic to l_1 so it has a presentation of the required type. A character $\chi \in [H_1(M)]^*$ extends to $H_1(V)$ if and only if it vanishes on $B \oplus D$. This means $\chi \in B^{\perp *}$ and χ vanishes on D. It is not hard to see that the set of such characters forms a metabolizer for β_2. □

§2

Given $\chi \in H_1(M)^*$, Casson and Gordon [1, 5] have defined some invariants $\sigma_\lambda(\tau(K, \chi)) \in \mathbb{Q}$ where $\lambda \in S^1$. $M - K$ has a finite cyclic cover defined by χ and an infinite cyclic cover (the infinite cyclic cover of $S^3 - K$). Putting these together, one has a $C_m \times C_\infty$ cover of $M - K$. Define $\mu(K, \chi) = \dim H_1^t(M - K, \mathbb{C}(t))$. Here we use the notation of [5] for homology with twisted coefficients. By [1] Lemma 4 (or Lemma 2 below), if χ has prime power order $\mu(K, \chi) = 0$. Let $\sigma(K)$ denote the ordinary signature of a knot as defined by Murasugi and Trotter. We now state and prove our main result. For $g = 0$, this is a theorem of Casson and Gordon.

Theorem 1. *If $g_s(K) = g$, then $(H_1(M)^*, \beta)$ can be written as a direct sum $\beta_1 \oplus \beta_2$ such that* 1) *β_1 has an even presentation with rank $2g$ and signature $\sigma(K)$ and* 2) *β_2 has a metabolizer H such that if $\chi \in H$ has prime power order, then $|\sigma_\lambda \tau(K, \chi) + \sigma(K)| \leqq 4g$ for all $\lambda \in S^1$. If $g_r(K) = g$, then* 2) *can be changed to* 2)′ *β_2 has a metabolizer H such that*

$$|\sigma_\lambda \tau(K, \chi) + \sigma(K)| \leqq 4g + \mu(K, \chi)$$

for all $\chi \in H$ and $\lambda \in S^1$.

Proof. Let Y denote D^4 minus an open tubular neighborhood of our surface G of genus g. The Thom isomorphism and excision show that Y has the homology of S^1 wedge $2g$ 2-spheres. Let W be the double cover of Y. Then $e(W) = 4g$. To compute the rational homology of W one should consider the $+1$ and -1 eigenspaces for the action of the covering transformation and recall that the transfer maps the homology of the base isomorphically to the $+1$ eigenspace. Propositions (1.4) and (1.5) of [3] may be useful. W has the rational homology of S^1 wedge $4g$ 2-spheres. The $+1$ eigenspace of $H_2(W, \mathbb{Q})$ has dimension $2g$.

Let V denote the double branched cover of D^4 along G. V is obtained from W by adding $G \times D^2$. The Mayer-Vietoris sequence for the -1 eigenspace shows V has the rational homology of a wedge of $2g$ 2-spheres. One has Sign $V = \sigma(K)$ [6]. Since the signature of the intersection pairing on the $+1$ eigenspace of W is zero, one has $\operatorname{Sign} W=\sigma(K)$. W is certainly spin. A Mayer-Vietoris sequence shows the restriction map $H^2(V,\mathbb{Z}_2)\to H^2(W,\mathbb{Z}_2)$ is injective, so V is spin as well.

Since M is the boundary of V, Lemma 2 applies and $\beta=\beta_1\oplus\beta_2$ as above. Moreover, β_2 has a metabolizer consisting of characters that extend to $H_1(V)$. Let χ be such a character, its extension will map $H_1(V)$ onto some cyclic subgroup C_m. If $\chi\in H_1(M)^*$ has order a power of a prime p, we can and do insist that m be a (possibly larger) power of p.

This defines a C_m cover of V and thus of W. X, the infinite cyclic cover of Y, is also an infinite cyclic cover of W. If we pull the C_m cover of W up to X, we obtain $\tilde{X}$ a $C_m\times C_\infty$ cover of W. If we identify $G\times S^1$ properly in ∂W, this cover restricted to $G\times S^1$ is given by a map $\psi: H_1(G\times S^1)=H_1(G)\oplus H_1(S^1)\to C_m\times C_\infty$ which maps $H_1(G)$ to zero in C_∞, $H_1(S^1)$ to zero in C_m and $H_1(S^1)$ isomorphically onto C_∞. Inductively, pick a collection of g disjoint curves ψ_i on G representing a half basis in the kernel of ψ. Attach g round 2-handles ($D^2\times I\times S^1$'s) to W along $\psi_i\times S^1$ in $G\times S^1$ to form U. Note that the boundary of U is obtained by zero framed surgery to M along the lift of K, which we will denote by L. Since the $C_m\times C_\infty$ cover extends uniquely to U and its restriction to L is the cover involved in the definition of $\tau(K,\chi)$, $\tau(K,\chi)$ can be computed in terms of this cover of U.

We can regard U as $W\bigcup_{\partial W} Q$ where Q is obtained from $L\times I$ by attaching g round one-handles along $D^2\times S^0\times S^1$'s that travel around the meridian of K. The intersection form on Q is seen to be identically zero. Thus, $\operatorname{Sign} U=\operatorname{Sign} W=\sigma(K)$. Let $\tau(U)$ denote the image in $W(\mathbb{C}(t),J)$ of the intersection pairing on $H_2^t(U,\mathbb{C}(t))$. Then $\sigma_\lambda(\tau(K,\chi))=\sigma_\lambda(\tau(U))-\operatorname{Sign} U$. On the other hand $|\sigma_\lambda\tau(U)|\leqq\dim H_2^t(U,\mathbb{C}(t))$. To complete the proof, we obtain the required upper bounds for this last term.

The $C_m\times C_\infty$ cover restricted to each round 2-handle is m copies of $D^2\times I\times\mathbb{R}$ attached to $\tilde{W}$ along a $S^1\times I\times\mathbb{R}$. A Mayer-Vietoris sequence shows that the inclusion induces an isomorphism $H_*^t(U,\mathbb{C}(t))\to H_*^t(W,\mathbb{C}(t))$. Since $H_*^t(W,\mathbb{C}(t))$ can be computed from a chain complex whose n^{th} group is the vector space over $\mathbb{C}(t)$ generated by the n cells of W, $e(W)$ can be computed as usual from $\dim H_n^t(W,\mathbb{C}(t))$.

If m is a prime power, we may apply Lemma 2 below. One sees $H_n(\tilde{X},\mathbb{Q})$ is finite dimensional for $n\neq 2$ and thus $H_n^t(W,\mathbb{C}(t))=0$ for $n\neq 2$. Therefore $\dim H_2^t(W,\mathbb{C}(t))=e(W)=4g$.

If G is a ribbon surface, then Y and its covers can be built without 3-handles. Thus, $H_3^t(W,\mathbb{C}(t))=0$ and $H_1^t(\partial W,\mathbb{C}(t))$ maps onto $H_1^t(W,\mathbb{C}(t))$. Now the ∂W is $M-K$ union $G\times S^1$ and the $C_m\times C_\infty$ cover of $G\times S^1$ is $\tilde{G}\times\mathbb{R}$ attached along some copies of $S^1\times\mathbb{R}$ where $\tilde{G}$ is a C_m cover of G. Therefore, a Mayer-Vietoris sequence shows that $\dim H_1^t(\partial W,\mathbb{C}(t))=\mu(K,\chi)$. Therefore, $\dim H_1^t(W,\mathbb{C}(t))\leqq\mu(K,\chi)$ and $\dim H_2^t(W,\mathbb{C}(t))\leqq 4g+\mu(K,\chi)$. □

The following lemma is a slight modification of Lemma 4 of [1].

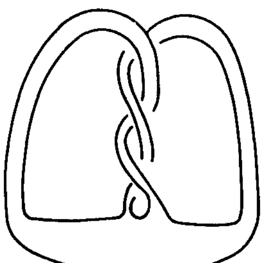
Lemma 2. *Let X be a connected infinite cyclic cover of a finite complex Y and $\tilde{X}$ a p^r cyclic cover of X, for p a prime. If $H_k(Y)=0$, then $H_k(\tilde{X},\mathbb{Q})$ is finite dimensional. If $H_1(Y)=\mathbb{Z}$, then $H_1(\tilde{X},\mathbb{Q})$ is finite dimensional.*

Proof. First one shows that $H_k(X, \mathbb{Z}_p)$ is finite if $H_k(Y)=0$ and $H_1(X, \mathbb{Z}_p)$ is finite if $H_1(Y)=\mathbb{Z}$ using Milnor's exact sequence for the homology of infinite cyclic covers. See the proof of assertion 5 in [7]. Next using a sequence of Smith homology groups (1.2) [3], one can show by induction that $\dim H_j(\tilde{X}, \mathbb{Z}_p) \leq p^r \dim H_j(X, \mathbb{Z}_p)$. As in the proof of Lemma 4 [1], one can show that $\tilde{X}$ is the infinite cyclic cover of a finite complex. An application of Lemma 6 of [1] completes the proof of the lemma. □

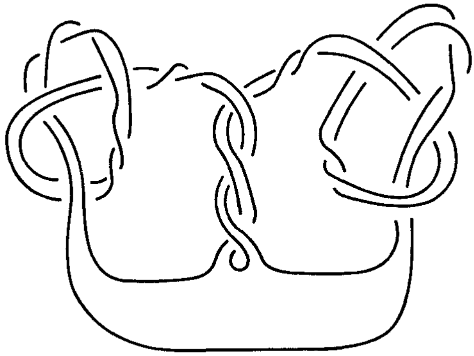
Remark. Let M_d denote the d-fold cyclic branched cover of S^3 along K where d is a prime power. Then one can prove the analog of Theorem 1 for these covers by the same argument with slight modifications. The correct statement is obtained by substituting: M_d for M, rank $2(d-1)g$ for rank $2g$, $\sum_{s=1}^{d-1} \sigma_{s/d}(K)$ for $\sigma(K)$ (see [3] p. 363 for $\sigma_{s/d}$), and $2dg$ for $4g$.

§3. Examples

Consider the evident genus one Seifert surface for the knot J_0 in Fig. 1. Let J_n denote the new knot obtained after we have tied n trefoils in each band with



J_0



zero twist. J_1 is also indicated. J_0 is slice and J_n is algebraically slice (in fact J_n is algebraically doubly null-concordant). For $n>0$, let Q_n be the connected sum of n copies of J_n.

Corollary 1. $g_s(Q_n)=g(Q_n)=n$.

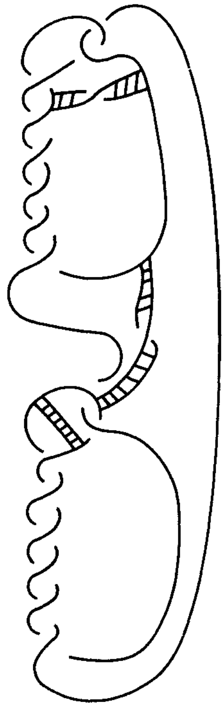
Proof. $g(Q_n)=n$ as genus is additive. Let N denote the double branched cover of S^3 along J_n. Then M, the double branched cover of S^3 along Q_n is the connected sum of n copies of N. $H_1(N)=\mathbb{Z}_3\oplus\mathbb{Z}_3$, thus, $H_1(M)$ is the direct sum of $2n$ copies of $\mathbb{Z}_3$. We assume $g_s(Q_n)<n$ and obtain a contradiction. Since $H_1(M)$ does not possess a presentation (as a group) of rank less than $2n$, there exists some nonzero $\chi\in H_1(M)^*$ with $|\sigma_1\,\tau(Q_n,\chi)|<4n$. Since Q_n is algebraically slice, $\sigma(Q_n)=0$. We can write $\chi=\bigoplus_{i=1}^{n}\chi_i$ where each $\chi_i\in H_1(N)^*$ and some $\chi_i\neq 0$. By [4] (3.2) $\tau(Q_n,\chi)=\sum_{i=1}^{n}\tau(J_n,\chi_i)$. There are eight nonzero $\chi\in H_1(N)^*$. It is easy to check using (3.5) of [4] that for each such χ, $\sigma_1\,\tau(J_n,\chi)\leqq -4n$. □

Our next example uses the linking form in a more essential way. Let K_t denote the t twisted double of the unknot. K_t has genus one and if $t=u(u+1)$, K_t is algebraically slice. Casson and Gordon showed K_t is slice if and only if t is one or two.

Corollary 2. *If* $t>2$ *and* $4t+1$ *is divisible by a prime* $p=3 \bmod 4$, *then* $g_s(K_t\#K_t)=g(K_t\#K_t)=2$.

Proof. Let q denote $4t+1$. The double branched cover is the connected sum of two copies of the lens space $L(q,-2)$. β is thus the direct sum of two copies of a form on $\mathbb{Z}_q$. If we restrict this form to the p-primary component, we must get the form $2A_{p^k}=2B_{p^k}$ (here we adopt Wall's notation [9]). Now suppose β has an even presentation $\langle\ ,\ \rangle$ of rank 2 with signature zero. Let M be a matrix for $\langle\ ,\ \rangle$, then $M=qP$ where P is even, has $\det=\pm 1$ and signature zero. So P must be equivalent to $\begin{bmatrix}0&1\\1&0\end{bmatrix}$. And qP presents the form $\langle 2/q\rangle\oplus\langle -2/q\rangle$. The p-primary component of this form is $A_{p^k}\oplus B_{p^k}$ because -1 is not a square $\bmod p$. Since $A_{p^k}\oplus B_{p^k}\neq 2A_{p^k}$, β does not have such a presentation. Thus, if $g_s(K_t\#K_t)=1$, then there is a nonzero χ such that $\beta(\chi,\chi)=0$ and $|\sigma_1\,\tau(K_t\#K_t,s\chi)|\leqq 4$ for all s. However, calculation shows that this is not the case. □

Remarks. If $t<0$ then $\sigma(K_t)=-2$ and $g_s(K_t\#K_t)=2$. If $t=0$ or 2, K_t is slice, so $g_s(K_t\#K_t)=0$. K_1 is the figure eight knot which is amphicheiral, so $g_s(K_1\#K_1)=0$. The theorem of Casson and Gordon can be used to show $K_3\#K_3$ is not slice. If one makes the indicated ribbon moves in Fig. 2, one obtains the unlink with two components. It follows that $g_s(K_3\#K_3)=g_r(K_3\#K_3)=1$. $K_4\#K_4$ is not slice, but I do not know whether $g_s(K_4\#K_4)$ is one or two.



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Oblatum 14-IV-1980 & 18-IX-1981