

# SLICE KNOTS IN $S^3$

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## Introduction

A KNOT  $K$  in  $S^3$  is said to be slice if it is the boundary of a properly and smoothly embedded 2-disk  $\Delta$  in  $D^4$ . Let  $F \subset S^3$  be a Seifert surface for  $K$  with Seifert pairing  $\theta: H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$ . Let  $L$  denote the double branched cover of  $S^3$  along  $K$ . Define  $\varepsilon: H_1(F) \rightarrow H^1(F)$  by  $\varepsilon(x)(y) = \theta(x, y) + \theta(y, x)$ . Let  $A \subset H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$  denote the kernel of  $\varepsilon \otimes id_{\mathbb{Q}/\mathbb{Z}}$ . We will give an isomorphism of  $A$  with  $H^1(L, \mathbb{Q}/\mathbb{Z})$ . This isomorphism is natural up to sign. A. J. Casson and C. McA. Gordon have defined an invariant  $\tau(K, \chi)$  of elements  $\chi \in H^1(L, \mathbb{Q}/\mathbb{Z})$ . Since  $\tau(K, \chi) = \tau(K, -\chi)$ , we can view  $\tau$  as defined on  $A$ . Let  $A' \subset A$  be the subset of elements of  $A$  with prime power order.

The following theorem combines the results of Casson and Gordon [3], [4], [5], [7] (Section 13) with the earlier work of J. Levine [11] in a nontrivial way.

**THEOREM (0.1).** *If  $K$  is slice then there is a direct summand  $H$  of  $H_1(F)$  such that*

- 1)  $2 \text{ rank } H = \text{rank } H_1(F)$ .
- 2)  $\theta(H \times H) = 0$ .
- 3) For all  $\chi \in A' \cap H \otimes \mathbb{Q}/\mathbb{Z}$ ,  $\tau(K, \chi) = 0$ .

In Section 1, we will give a proof of this for homology slice knots in homology 3-spheres. In Section 2, we will use this theorem to define a homomorphism of the knot cobordism group to a certain Witt group  $\Gamma'$ . The cobordism class of a knot  $K$  goes to zero in  $\Gamma'$  if and only if  $K$  satisfies the conclusion of Theorem (0.1). We also discuss the relation of  $\Gamma'$  to previous Witt groups defined by Levine and Casson-Gordon:  $G_-$  and  $\Lambda'$ .

In Section 3, we discuss  $\tau(K, \chi)$  and show how to estimate  $\tau(K, \chi)$  (actually  $\sigma_1 \tau(K, \chi)$ ) in terms of curves lying on a Seifert surface. More precisely, if  $x \in H_1(F)$  is primitive let  $C_x$  denote the collection of knots in  $S^3$  obtained by representing  $x$  by a simple closed curve  $\gamma$  on  $F$  and then viewing  $\gamma$  in  $S^3$ . Our estimates together with (0.1) give the following corollary.

**COROLLARY (0.2).** *If  $K$  is slice then there is a direct summand  $H$  of*

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$H_1(F)$  such that

- 1)  $2 \operatorname{rank} H = \operatorname{rank} H_1(F)$ .
- 2)  $\theta(H \times H) = 0$ .
- 3) If  $x \in H$  is primitive, and  $x \otimes s/m \in A'$  for some  $0 < s < m$ , then for all  $J \in C_x$ , we have

$$|\sigma_{(s/m)}(J)| \leq \operatorname{genus}(F).$$

Here  $\sigma_{(s/m)}$  is an ordinary knot signature. Steve Kaplan has independently found a result similar to (0.2). His bound on  $|\sigma_{(s/m)}(J)|$  is  $2(\operatorname{genus} F) - 1$ . He uses the approach of [4], while we follow [3] more closely. When  $F$  has genus one, our result coincides with his. Note that in the genus-one case,  $C_x$  is a singleton. We denote its element  $J_x$ . Also  $\sigma_{(s/m)}$  of a knot is always even. This yields:

**COROLLARY (0.3).** *Let  $K$  be a slice knot which possesses a genus-one Seifert surface  $F$ . Then there exists a primitive  $x \in H_1(F)$  such that  $\theta(x, x) = 0$  and  $\sigma_{(s/m)}(J_x) = 0$  for all  $0 < s < m$  where  $m$  is any prime power dividing  $\sqrt{|\det(\theta + \theta^T)|}$ .*

Corollary (0.3) lends itself to an interesting interpretation. Namely if  $J_x$  is slice then one can use the smooth disk in  $D^4$  with boundary  $J_x$  to do ambient surgery on  $F$  and obtain a slice disk for  $K$ . This is essentially Levine's program (which works in higher dimensions) for showing knots with metabolic Seifert pairings are slice. On the other hand, if  $J_x$  is slice,  $\sigma_{s/m}(J_x) = 0$  for all  $0 < s < m$  and  $m$  a prime power. If a genus-one Seifert surface has a metabolic Seifert pairing, then there are exactly two (up to sign) primitive classes with square zero. Let  $J_1$  and  $J_2$  be the associated knots. If  $F$  fails to satisfy the conclusion of (0.3), then neither  $J_1$  nor  $J_2$  can be slice, and Levine's program cannot be carried out. Thus the Casson-Gordon invariants seem to be secondary obstructions to carrying out the program.

In Section 3, we also calculate  $\tau(K, \chi)$  exactly in the case of genus-one knots. It turns out that the cobordism class of such a knot maps to zero in  $\Gamma'$  if and only if the conclusion of (0.3) is satisfied.

Finally in section 4 we give an example of a non-slice knot which is detected by  $\Gamma'$  but not by  $G_-$  or  $\Lambda'$ .

I wish to thank R. A. Litherland for pointing out an error in the proof of Theorem (1.1) in an earlier version of this paper. At that time I knew another correct proof for (0.1), given in [13]. However, this method could not be used to prove the homology slice version (1.1) below. Thus, I have modified the original proof in a way suggested by Litherland.

## Section 1

Suppose more generally we have  $K$  a knot in a homology sphere  $S$ . By the same transversality argument as is used in  $S^3$ , we can find a Seifert

surface  $F \subset S$ . Let  $\theta$  be the Seifert pairing, and define  $\varepsilon, A, A'$  as in the Introduction. Let  $L$  be the double branched cover of  $S$  along  $K$ . There are several ways to give the identification of  $A$  with  $H^1(L, \mathbb{Q}/\mathbb{Z})$ . We begin with a method which relates well to the proof of Theorem (1.1). We discuss some other methods following the proof of (1.1).

Let  $X$  denote  $S$  slit along  $F$ , i.e., the complement of the interior of  $B$ , a bicollar neighborhood of  $F$ .  $L$  can be constructed by gluing together two copies, say  $X_0$  and  $X_1$  of  $X$  along  $\partial X_0$  and  $\partial X_1$  appropriately. The deck transformation  $T$  acts on  $L$  by switching  $X_0$  and  $X_1$ . Let  $F_+$  (respectively  $F_-$ ) denote the lift of  $F$  in  $L$  which corresponds to the copy of  $F$  in  $\partial X_0$  lying just above (respectively below)  $F$ .

As is well known,  $H_1(L)$  is an odd torsion group, see [3] Lemma 2, for instance. Since  $L/T = S$  and  $H_1(S) = 0$ , a standard transfer argument shows  $T$  cannot fix any nonzero element in  $H_1(L)$ . Therefore  $x + T(x) = 0$  for all  $x$  in  $H_1(L)$ . In other words, the action of  $T$  on  $H_1(L)$  is multiplication by  $-1$ . It follows that the inclusion induces a surjection of  $H_1(X_0)$  onto  $H_1(L)$ . Thus a homomorphism  $\chi \in H^1(W, \mathbb{Q}/\mathbb{Z})$  is determined by its restriction  $\chi|_{X_0}$ . Thus restriction gives an injective map  $H^1(L, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(X_0, \mathbb{Q}/\mathbb{Z})$ .

We wish to determine the image. If  $\psi \in H^1(X_0, \mathbb{Q}/\mathbb{Z})$  extends to  $\chi \in H^1(L, \mathbb{Q}/\mathbb{Z})$  then  $\chi|_{X_1} = -\chi|_{X_0} T = -\psi T$ . Thus  $\psi$  extends iff  $\psi$  agrees with  $-\psi T$  along  $\partial X_0 = \partial X_1$ . However  $T$  acts on  $\partial X_0 \approx F_+ \cup F_-$  by interchanging  $F_+$  and  $F_-$ .  $T$  maps  $\psi|_{X_0} + \psi T|_{X_0}$  to its negative. Thus  $\psi$  extends to  $H_1(W)$  iff  $\psi|_{F_+} + \psi T|_{F_+} = 0$ . Thus we have an exact sequence

$$0 \rightarrow H^1(L, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(X_0, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(F_+, \mathbb{Q}/\mathbb{Z})$$

We have isomorphisms (with  $\mathbb{Q}/\mathbb{Z}$  coefficients understood)  $H^1(X) \approx H^2(S, X) \approx H^2(B, \partial B) \approx H_1(B) \approx H_1(F)$  given by, respectively, the coboundary, excision, Lefschetz duality, and a homotopy equivalence. Using the identifications of  $X_0$  with  $X$  and  $F_+$  with  $F$ , we have an exact sequence

$$0 \rightarrow H^1(L, \mathbb{Q}/\mathbb{Z}) \rightarrow H_1(F, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(F, \mathbb{Q}/\mathbb{Z}).$$

We wish to identify the last map with  $\varepsilon \otimes id_{\mathbb{Q}/\mathbb{Z}}$ . The universal coefficient theorem gives isomorphisms:  $H_1(F) \otimes \mathbb{Q}/\mathbb{Z} \approx H_1(F, \mathbb{Q}/\mathbb{Z})$  and  $H^1(F, \mathbb{Q}/\mathbb{Z}) \approx H^1(F) \otimes \mathbb{Q}/\mathbb{Z}$ . If  $J$  is a curve on  $F$  representing  $x \in H_1(F)$ , then  $x \otimes s/d$  corresponds under the above isomorphisms to the element of  $H^2(S, X, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(H_2(S, X), \mathbb{Q}/\mathbb{Z})$  which assigns to a 2-cycle  $a$ , the value  $(s/d)(a \circ J)$ . Here  $\circ$  denotes intersection number. Therefore  $x \otimes s/d$  maps to the element  $\psi \in H^1(X_0, \mathbb{Q}/\mathbb{Z})$  which assigns to a 1-cycle  $b$ ,  $s/d$  times the linking number of  $b$  and  $J$ . Moreover  $\psi|_{F_+} + \psi T|_{F_+} \in H^1(F_+, \mathbb{Q}/\mathbb{Z})$  is then  $s/d \varepsilon(x)$ . Our identification of  $H^1(L, \mathbb{Q}/\mathbb{Z})$  with  $A$  depends on a choice of a component of the cover of  $X$  to be  $X_0$ . If we make a different choice we only change the isomorphism by a sign.

We say  $K$  in  $S$  is homology slice if there is some homology 4-ball  $D$  with  $\partial D = S$  and a smooth properly embedded 2-disk  $\Delta$  in  $D$  with boundary  $K$ .

**THEOREM (1.1).** *If  $K$  is homology slice, then the conclusion of Theorem (0.1) holds.*

*Proof.* Since  $D - \Delta$  is a homology circle, a standard transversality argument produces an embedded, oriented 3-manifold  $R \subset D$  with  $\partial R = F \cup \Delta$ . Let  $H \subset H_1(F)$  be the inverse image of the torsion subgroup of  $H_1(R)$  under the map induced by inclusion. Since  $F \cup \Delta$  is the boundary of  $R$ , another standard argument (using Lefschetz duality) shows  $2 \text{ rank } H$  equals  $\text{rank } H_1(F)$ . Curves on  $F$  representing elements in  $H$  rationally bound surfaces in  $R$ . These surfaces can be used to calculate linking numbers in  $S$ . One sees  $\theta(H \times H) = 0$ . We have just recapitulated Levine's argument in this dimension [11].

Now let  $W$  be the double branched cover of  $D$  along the slice disk  $\Delta$  for  $K$ . Let  $i$  denote the inclusion of  $L$  in  $W$ , and  $T$  the covering transformation. By the argument in [3], Lemma 2, or using Smith theory,  $H_1(W)$  is an odd torsion group. Since  $H_1(D) = 0$ ,  $T$  must act as multiplication by  $-1$  on  $H_1(W)$  (by the same argument given for  $H_1(L)$ ).

Let  $Y$  denote the complement of the interior of a bicollar neighborhood of  $R$  in  $D$ .  $W$  then can be built from two copies  $Y_0$  and  $Y_1$  of  $Y$ . We have an exact sequence for  $W$  analogous to that given for  $L$  above. The first map, for instance, is given by (with  $\mathbb{Q}/\mathbb{Z}$  coefficients understood)

$$\begin{aligned} H^1(W) \rightarrow H^1(Y_0) \approx H^1(Y) \approx H^2(D, Y) \approx H^2(R \times I, R \times S^0 \cup \Delta \times I) \\ \approx H_2(R \times I, F \times I) \approx H_2(R, F). \end{aligned}$$

Moreover, it fits into a commutative diagram (due to Litherland)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(W, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_2(R, F, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^1(R, \mathbb{Q}/\mathbb{Z}) \\ & & \downarrow i^* & & \downarrow \partial & & \downarrow \\ 0 & \longrightarrow & H^1(L, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_1(F, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\epsilon \otimes id} & H^1(F, \mathbb{Q}/\mathbb{Z}) \\ & & & & \downarrow j^* & & \\ & & & & H_1(R, \mathbb{Q}/\mathbb{Z}) & & \end{array}$$

The first vertical map is given by restriction. Now  $H \otimes \mathbb{Q}/\mathbb{Z}$  is the kernel of  $j_*$  and thus the image of  $\partial$ . Therefore the image of  $i^*$  is a subgroup of  $A \cap H \otimes \mathbb{Q}/\mathbb{Z}$ .

Let  $\{x_1, \dots, x_g\}$  be a basis for  $H$  extended to a basis  $\{x_1, \dots, x_g, y_1, \dots, y_g\}$  for  $H_1(F)$ . Let  $\{x_1^\#, \dots, y_g^\#\}$  be the dual basis with

respect to  $\langle \rangle$  the nonsingular intersection pairing on  $H_1(F)$ . Thus  $\{y_1^\#, \dots, y_g^\#\}$  forms a basis for  $H^\perp$  the subspace that annihilates  $H$  under  $\langle \rangle$ . Since  $\langle x, y \rangle = \theta(x, y) - \theta(y, x)$ ,  $H \subset H^\perp$ . Since they have the same rank,  $H = H^\perp$ . Thus  $\{x_1, \dots, x_g, x_1^\#, \dots, x_g^\#\}$  is a basis for  $H_1(F)$ . With respect to this basis  $\theta$  is given by a matrix of the form

$$\begin{bmatrix} 0 & C+I \\ C^T & E \end{bmatrix}$$

where  $E = E^T$ . Let  $\{z_1, \dots, z_{2g}\}$  be the dual basis for  $H^1(F)$ . With respect to these bases  $\varepsilon$  is given by

$$\begin{bmatrix} 0 & 2C+I \\ 2C^T+I & 2E \end{bmatrix}.$$

Let  $m$  denote  $|\det(2C+I)|$  then using the above matrix representation of  $\varepsilon$  we see  $|A| = m^2$  and  $|A \cap H \otimes \mathbb{Q}/\mathbb{Z}| = m$ . Since  $A = H^1(L, \mathbb{Q}/\mathbb{Z})$ , we have  $|H_1(L)| = m^2$ . By Lemma 3 of [3], the image of  $H_1(L)$  in  $H_1(W)$  has order  $m$  thus  $|\text{image } i^*| = m$ . Since  $\text{image } i^*$  is a subgroup of  $A \cap H \otimes \mathbb{Q}/\mathbb{Z}$  and they both have the same order, we can conclude they are equal. Thus if  $\chi \in A \cap H \otimes \mathbb{Q}/\mathbb{Z}$ ,  $\chi$  extends to an element of  $H^1(W, \mathbb{Q}/\mathbb{Z})$ . By Theorem 2 of [3],  $\tau(K, \chi) = 0$ .  $\square$

We now give some alternative descriptions of the identification of  $A$  with  $H^1(L, \mathbb{Q}/\mathbb{Z})$ . Let  $V$  denote a matrix for  $\theta$  with respect to some basis for  $H_1(F)$ . Rolfsen, 8.D.1 (page 212) [12], shows that  $V + V^T$  (a matrix for  $\varepsilon$ ) is a presentation matrix for  $H_1(L)$ . Using the above basis for  $H_1(F)$ , we can write an element in  $A$  as a vector with  $\mathbb{Q}/\mathbb{Z}$  coefficients which, when multiplied by  $V + V^T$ , becomes zero. The corresponding  $\chi$  assigns to the  $i$ th generator for  $H_1(L)$  the  $i$ th entry in this vector. The kernel condition insures that all the relations map to zero.

Another way this presentation arises is as follows. Let  $\tilde{D}$  denote the branched cover of  $D$  along  $F$  pushed slightly into  $D$  [8], [1]. Then  $\tilde{D}$  may be obtained by adding 2-handles according to a framed link in the boundary of some homology ball ( $D^4$  if  $D = D^4$ ). The linking matrix for this framed link is  $V + V^T$ . The meridians for the components can be identified with Rolfsen's generators. One may use  $\tilde{D}$  to understand the linking form  $l$  on  $H_1(L)$ . Let  $c$  denote the correlation isomorphism  $H_1(L) \rightarrow H^1(L, \mathbb{Q}/\mathbb{Z})$  given by  $c(u) = l(\cdot, u)$ . Define a linking form  $\beta$  on  $H^1(L, \mathbb{Q}/\mathbb{Z}) \approx A$  by  $\beta(cu, cv) = -l(u, v)$ . Then one can show

$$\beta(x \otimes r, y \otimes s) = \hat{r}\hat{s}[\theta(x, y) + \theta(y, x)] \bmod 1$$

where  $\hat{r}$  and  $\hat{s}$  are rational numbers that reduce mod 1 to  $r$  and  $s$ .

This means given two elements of  $A$  written as vectors with  $\mathbb{Q}/\mathbb{Z}$

coefficients  $a$  and  $b$ , pick any two vectors  $\hat{a}$  and  $\hat{b}$  with rational coefficients which reduce mod 1 to  $a$  and  $b$ , then multiply  $\hat{a}^T(V + V^T)\hat{b}$  out and reduce mod 1.

## Section 2

One may form a knot cobordism group  $C$ . Elements of  $C$  are equivalence classes of knots  $K$  in  $S^3$ . Two knots  $K_0$  and  $K_1$  are equivalent if there is a smooth proper embedding  $f: S^1 \times I \rightarrow S^3 \times I$  with  $f(S^1 \times 0) = K_0 \subset S^3 \times \{0\}$ ,  $f(S^1 \times 1) = K_1 \subset S^3 \times \{1\}$ . The group operation is given by the connected sum of knots. Inverses are given by taking mirror images and reversing string orientation. This group was first studied in 1957 by Fox and Milnor [2].

One can define a related group  $\mathcal{K}$  as follows. One forms a semigroup of isomorphism classes of pairs  $(S, K)$  where  $S$  is a oriented homology 3-sphere and  $K$  is an oriented knot in  $S$ . We say  $(S, K)$  is null cobordant if there is some homology cobordism  $W^4$  of  $S$  to some other homology 3-sphere and there is a smooth  $D^2 \subset W$  with  $\partial D^2 = K \subset S$ . If both  $(S_0, K_0) \# (S_1, K_1)$  and  $(S_1, K_1)$  are null cobordant then so is  $(S_0, K_0)$ . Let  $-(S, K)$  be obtained from  $(S, K)$  by reversing the orientations of  $S$  and  $K$ . We say  $(S_0, K_0) \sim (S_1, K_1)$  if  $(S_0, K_0) \# -(S_1, K_1)$  is null cobordant. This is an equivalence relation.  $\mathcal{K}$  is then equivalence classes of pairs with addition given by the connected sum. There is a natural morphism  $C \rightarrow \mathcal{K}$ .

In 1968, Levine [11] defined a group  $G_-$  and a epimorphism  $C \rightarrow G_-$  as part of his study of codimension-2 knot cobordism. See Kervaire's article [9] for an excellent discussion. We briefly outline the definition of  $G_-$ . Elements of  $G_-$  are equivalence classes of pairs  $(V, \theta)$  where  $V$  is a finitely generated free  $\mathbb{Z}$  module and  $\theta$  is a bilinear pairing with the property that  $\theta - \theta^T$  is unimodular.  $(V, \theta) \sim (W, \theta')$  if  $(V, \theta) \oplus (W, -\theta')$  is metabolic (i.e. admits a half dimensional summand on which the form vanishes). The map from  $C$  to  $G_-$  assigns to the class of  $K$  the class of  $(H_1(F), \theta)$ . This clearly factors through  $\mathcal{K}$ .

Casson and Gordon have also defined homomorphism from  $\mathcal{K}$  to a different Witt group  $\Lambda'$  [5].  $\Lambda'$  is defined to be equivalence classes of triples  $(A, \beta, \tau)$  where  $A$  is a finite abelian group,  $\beta: A \times A \rightarrow \mathbb{Q}/\mathbb{Z}$  is bilinear and nonsingular (in the sense that the induced map  $A \rightarrow \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$  is an isomorphism) and  $\tau: A \rightarrow W(\mathbb{C}(t), J) \otimes \mathbb{Q}$  is a map of sets. Let  $A' \subset A$  be the subset of elements of prime power order.

One defines  $(A_0, \beta_0, \tau_0) \oplus (A_1, \beta_1, \tau_1) = (A_0 \oplus A_1, \beta_0 \oplus \beta_1, \tau_0 \oplus \tau_1)$  where  $\beta_0 \oplus \beta_1(x_0 \oplus x_1, y_0 \oplus y_1) = \beta_0(x_0, y_0) + \beta_1(x_1, y_1)$  and  $\tau_0 \oplus \tau_1(x_0 + x_1) = \tau(x_0) + \tau(x_1)$ .  $(A, \beta, \tau)$  is called metabolic if there exists  $G \subset A$  such

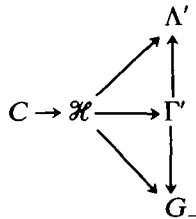
that

$$1) |G|^2 = |A| \quad 2) \beta(G \times G) = 0 \quad 3) \tau(G \cap A') = 0$$

One defines  $-(A, \beta, \tau) = (A, -\beta, -\tau)$  and says  $(A_0, \beta_0, \tau_0) \sim (A_1, \beta_1, \tau_1)$  if  $(A_0, \beta_0, \tau_0) \oplus -(A_1, \beta_1, \tau_1)$  is hyperbolic. One proves by a formally parallel proof to Kervaire's for  $G_-$  that  $\sim$  is an equivalence relation.

The homomorphism of Casson and Gordon sends the class of  $(K, S)$  to the class of  $(H^1(L, \mathbb{Q}/\mathbb{Z}), \beta, \tau(K, ))$ . There is an analogous group  $\Lambda$  defined identically except condition 3) above reads  $\tau(G) = 0$ .

We will next define another group  $\Gamma'$  which combines  $G_-$  and  $\Lambda'$  in a nontrivial way. In fact one has a commutative diagram.



In Section 4, we will give an example of an element in  $C$  whose images in  $\Lambda'$  and  $G_-$  are zero but whose image in  $\Gamma'$  is nonzero.

$\Gamma'$  is defined to be equivalence classes of triples  $(V, \theta, \tau)$  where  $V$  is a finitely generated free  $\mathbb{Z}$ -module,  $\theta$  is a bilinear pairing  $\theta: V \times V \rightarrow \mathbb{Z}$  such that  $\theta - \theta^T$  is nonsingular, and  $\tau$  is a function from  $A$  to  $W(\mathbb{C}(t), J) \otimes \mathbb{Q}$ . Here  $A$  is an abelian group associated to  $(V, \theta)$  as follows:  $A = \ker \varepsilon \otimes id_{\mathbb{Q}/\mathbb{Z}} \subset V \otimes \mathbb{Q}/\mathbb{Z}$ , where  $\varepsilon: V \rightarrow \text{Hom}(V, \mathbb{Z})$  is given by  $\varepsilon(x)y = \theta(x, y) + \theta(y, x)$ . We think of a triple  $(V, \theta, \tau)$  as a single algebraic object.

One defines  $(V_0, \theta_0, \tau_0) \oplus (V_1, \theta_1, \tau_1)$  to be  $(V_0 \oplus V_1, \theta_0 \oplus \theta_1, \tau_0 \oplus \tau_1)$  where  $\theta_0 \oplus \theta_1$  and  $\tau_0 \oplus \tau_1$  are defined as in the definitions of  $G_-$  and  $\Lambda'$ .  $(V, \theta, \tau)$  is called metabolic if there exists a direct summand  $H$  of  $V$  such that 1)  $2 \text{ rank } H = \text{rank } V$ ; 2)  $\theta(H \times H) = 0$ ; and 3)  $\tau(A' \cap H \otimes \mathbb{Q}/\mathbb{Z}) = 0$ . One defines  $-(V, \theta, \tau) = (V, -\theta, -\tau)$  and  $(V_0, \theta_0, \tau_0) \sim (V_1, \theta_1, \tau_1)$  if  $(V_0, \theta_0, \tau_0) \oplus -(V_1, \theta_1, \tau_1)$  is hyperbolic. We remark that one may define a similar group  $\Gamma$  in the same way except condition 3) above should read  $\tau(A \cap H \otimes \mathbb{Q}/\mathbb{Z}) = 0$ .

The cancellation lemma below shows that  $\sim$  is an equivalence relation.  $\Gamma'$  is then  $\sim$  classes of triples  $(V, \theta, \tau)$ . The homomorphism from  $\mathcal{K}$  to  $\Gamma'$  sends the class of  $K$  to  $(H_1(F), \theta, \tau(K, ))$ . Theorem (1.1) and Proposition (3.2) allow one to see that this indeed gives a homomorphism. One finally has homomorphisms from  $\Gamma'$  to  $G_-$ , respectively  $\Lambda'$  sending  $(V, \theta, \tau)$  to  $(V, \theta)$ , respectively  $(A, \beta, \gamma)$ . Where  $A \subset V \otimes \mathbb{Q}/\mathbb{Z}$  is as above and  $\beta(x \otimes r, y \otimes s) = +\hat{r}\hat{s}(\theta(x, y) + \theta(y, x)) \bmod 1$ . Here  $\hat{r}, \hat{s}$  are any elements of  $V \otimes \mathbb{Q}$  which reduce to  $r$  and  $s$ .

**CANCELLATION LEMMA.** *If  $(W, \psi, \lambda)$  is metabolic and  $(V, \theta, \tau) \oplus (W, \psi, \lambda)$  is metabolic, then  $(V, \theta, \tau)$  is metabolic.*

*Proof.* (Again based on Kervaire's proof for  $G_-$ ). Let  $L \subset V \oplus W$  and  $K \subset W$  be the direct summands with half the rank which make the triples metabolic. Let  $H \subset V$  be the smallest direct summand which includes  $p(L \cap V \oplus K)$  where  $p: V \oplus W \rightarrow V$  is the projection. Kervaire shows that  $2 \text{ rank } H = \text{rank } V$  and that  $\theta(H \times H) = 0$ .

We only need to show that  $\tau(H \otimes \mathbb{Q}/\mathbb{Z} \cap A') = 0$ . Let  $B' \subset B \subset W \otimes \mathbb{Q}/\mathbb{Z}$  denote the analogous subsets to  $A$  and  $A'$ . Thus  $\lambda$  is defined on  $B$ , and  $\tau \oplus \lambda$  is defined on  $A \oplus B$ . We are given that  $\lambda(K \otimes \mathbb{Q}/\mathbb{Z} \cap B') = 0$  and  $\tau \oplus \lambda(L \otimes \mathbb{Q}/\mathbb{Z} \cap (A \oplus B')) = 0$ . Let  $p$  also denote the projection  $p: A \oplus B \rightarrow A$ . Define  $\tilde{H}$  as  $p(L \otimes \mathbb{Q}/\mathbb{Z} \cap (V \otimes \mathbb{Q}/\mathbb{Z}) \oplus (K \otimes \mathbb{Q}/\mathbb{Z}) \cap A \oplus B)$ .

Let  $\tilde{H}' \subset \tilde{H}$  be the subset of elements of prime power order. If  $h \in \tilde{H}'$ , then  $h = v \otimes b$  and  $l \otimes a = (v \otimes b) \oplus (k \otimes c)$  where  $l \in L$ ,  $v \in V$ ,  $k \in K$ , and  $a, b, c \in \mathbb{Q}/\mathbb{Z}$ . Moreover  $q'b = 0$  for some prime  $q$ . Write the denominator of  $c$  as  $q^s d$  where  $d \in \mathbb{Z}$  and  $d \not\equiv 0 \pmod{q}$ . By the Chinese Remainder Theorem, there is an  $n \in \mathbb{Z}$  such that  $dn \equiv 1 \pmod{q^s}$ . We have  $dnh = v \otimes دنب = v \otimes b = h$ ,  $dn(l \otimes a) = h \oplus (nk \otimes dc)$  and  $dc$  has prime power order. So  $\tau(h) = 0$ . Therefore  $\tau(\tilde{H}') = 0$ .

It is clear that  $\tilde{H} \subset H \otimes \mathbb{Q}/\mathbb{Z} \cap A$ . It is easy to show that  $|H \otimes \mathbb{Q}/\mathbb{Z} \cap A|^2 = |A|$  (see the proof of (1.1)). On the other hand  $|\tilde{H}|^2 = |A|$ . This follows from the proof of the cancellation law for  $\Lambda$  which in turn is modelled directly after Kervaire's proof for  $G_-$ . It follows  $\tilde{H} = H \otimes \mathbb{Q}/\mathbb{Z} \cap A$  and we are done.

### Section 3

Recall  $\tau(K, \chi) \in W(\mathbb{C}(t), J) \otimes \mathbb{Q}$ , [7]. Here  $\mathbb{C}(t)$  denotes the field of rational functions over  $\mathbb{C}$  in the variable  $t$  and  $J$  is the involution on  $\mathbb{C}(t)$  that conjugates the complex numbers and sends  $t$  to  $t^{-1}$ .  $W(\mathbb{C}(t), J)$  is the associated Witt group of finite dimensional hermitian inner product spaces. The signature is an isomorphism from  $W(\mathbb{R})$  the Witt group over  $\mathbb{R}$  to  $\mathbb{Z}$ . There is a natural map  $W(\mathbb{R}) \rightarrow W(\mathbb{C}(t), J)$ . Together these yield a homomorphism  $\rho: \mathbb{Q} \rightarrow W(\mathbb{C}(t), J) \otimes \mathbb{Q}$ . There is also a homomorphism  $\sigma_1: W(\mathbb{C}(t), J) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ , [3]. It is easy to see that  $\sigma_1 \circ \rho$  is the identity.

Given a closed 3-manifold  $N$  and a map  $\varphi: H_1(N) \rightarrow C_m \oplus C_\infty$ , one may define an invariant  $\tau(N, \varphi) \in W(\mathbb{C}(t), J) \otimes \mathbb{Q}$  of the associated cover, in the way  $\tau(K, \chi)$  is defined in [7].

As before let  $L$  be the double branched cover of  $S^3$  along  $K$ . Let  $M$  be the result of 0-framed surgery on  $L$  along  $\tilde{K}$ , the lift of  $K$  in  $M$ . There is a natural isomorphism  $H_1(M) = H_1(L) \oplus \mathbb{Z}$ . Given  $\chi: H_1(L) \rightarrow \mathbb{Q}/\mathbb{Z}$ ,  $\chi$  will map into some cyclic group  $C_m$ . Define  $\chi^+: H_1(M) \rightarrow C_m \oplus C_\infty$  by



$\chi^+(x, n) = (\chi(x), t^n)$ . Then  $\tau(K, \chi) = \tau(M, \chi^+)$ . Define  $\bar{\chi} \in H^1(M, \mathbb{Q}/\mathbb{Z})$  by  $\bar{\chi}(x, n) = \chi(x)$ . Since the deck transformation  $T$ , is a diffeomorphism of  $L$ , fixing  $\bar{K}$ , and acting by  $-1$  on  $H_1(L)$ ,  $\tau(K, \chi) = \tau(K, -\chi)$ .

Given  $\varphi \in H^1(N, \mathbb{Q}/\mathbb{Z})$ , define  $\eta(N, \varphi) = \dim H_1^1(N, \mathbb{C})$ . Here we use the notation of [7] for twisted homology groups. Also define  $\sigma(N, \varphi) \in \mathbb{Q}$  as in [7]. These invariants are also discussed in [6] in slightly different guise. The relation is given in [3]. The proof of Theorem 3 in [3] shows that

$$|\sigma_1(\tau(K, \chi)) - \sigma(M, \bar{\chi})| \leq \eta(M, \bar{\chi}) \quad (3.1)$$

Let  $K_0$  and  $K_1$  be two knots and  $K_2 = K_0 \# K_1$ . For each knot  $K_i$ , one has as above  $L_i, M_i, \bar{K}_i$ , and  $A_i = H^1(L_i, \mathbb{Q}/\mathbb{Z})$ . We have  $L_2 = L_0 \# L_1$ ,  $\bar{K}_2 = \bar{K}_0 \# \bar{K}_1$  and  $A_2 = A_0 \oplus A_1$ .

PROPOSITION (3.2). *If  $\chi_0 \in A_0$  and  $\chi_1 \in A_1$ , then  $\tau(K_0 \# K_1, \chi_0 \oplus \chi_1) = \tau(K_0, \chi_0) + \tau(K_1, \chi_1)$ .*

*Proof.* Let  $W$  be  $M_0 \cup M_1 \times I$  together with a 1-handle joining the 2 components and a 2-handle attached as indicated in Figure (1a). Claim:  $\partial_+ W = M_2$ . To see this slide a 2-handle to get a new description of  $\partial_+ W$  (b). Then trade the 2-handle for a 1-handle (c) yielding yet another description of  $\partial_+ W$ . Finally remove the pair of cancelling 1 and 2 handles (d). The result is  $M_2$ . We have  $\partial W = M_2 \cup -M_0 \cup -M_1$ . (Here the minus signs indicate reversed orientation). Moreover the  $C_m \times C_\infty$  covers of  $M_0, M_1, M_2$  given by  $\chi_0^+, \chi_1^+$  and  $(\chi_0 + \chi_1)^+$  extend to a  $C_m \times C_\infty$  cover of  $W$ .

To calculate  $H_\star^1(W, M_0 \cup M_1, \mathbb{C}(t))$  we have a chain complex,

$$0 \longrightarrow C_2 = \mathbb{C}(t) \xrightarrow{\partial_2} C_1 = \mathbb{C}(t) \longrightarrow 0$$

The map  $\partial_2$  can be calculated by reading off the intersections of the attaching circle of the 2-handle with the belt 2-sphere of the 1-handle weighted with elements of  $C_m \oplus C_\infty$ . In this case  $\partial_2$  is multiplication by  $1-t$  and so  $H_\star^1(W, M_0 \cup M_1, \mathbb{C}(t)) = 0$ . Thus the inclusion induces an isomorphism  $H_\star^1(M_0 \cup M_1, \mathbb{C}(t)) \rightarrow H_\star^1(W, \mathbb{C}(t))$  and the intersection pairing on  $H_2^1(W, \mathbb{C}(t))$  is identically zero. One can also show  $\text{Sign}(W) = 0$ . The result follows.  $\square$

PROPOSITION (3.3). *Let  $W = N \times I \bigcup_{i=1}^n h_i^2$  where the  $h_i^2$  denote 2-handles attached along curves  $\gamma_i$  in a closed 3-manifold  $N$ . Orient  $W$  so that  $\partial W = -N \cup P$ . Let  $\chi \in H^1(N, \mathbb{Q}/\mathbb{Z})$  such that  $\chi[\gamma_i] = 0$  and  $\chi \neq 0$ .  $\chi$  extends uniquely to  $H_1(W)$  and thus defines  $\chi \in H^1(P, \mathbb{Q}/\mathbb{Z})$ . Then*

$$|\sigma(P, \chi) - \sigma(N, \chi) + \text{Sign } W| + |\eta(P, \chi) - \eta(N, \chi)| \leq n$$

with equality mod 2.

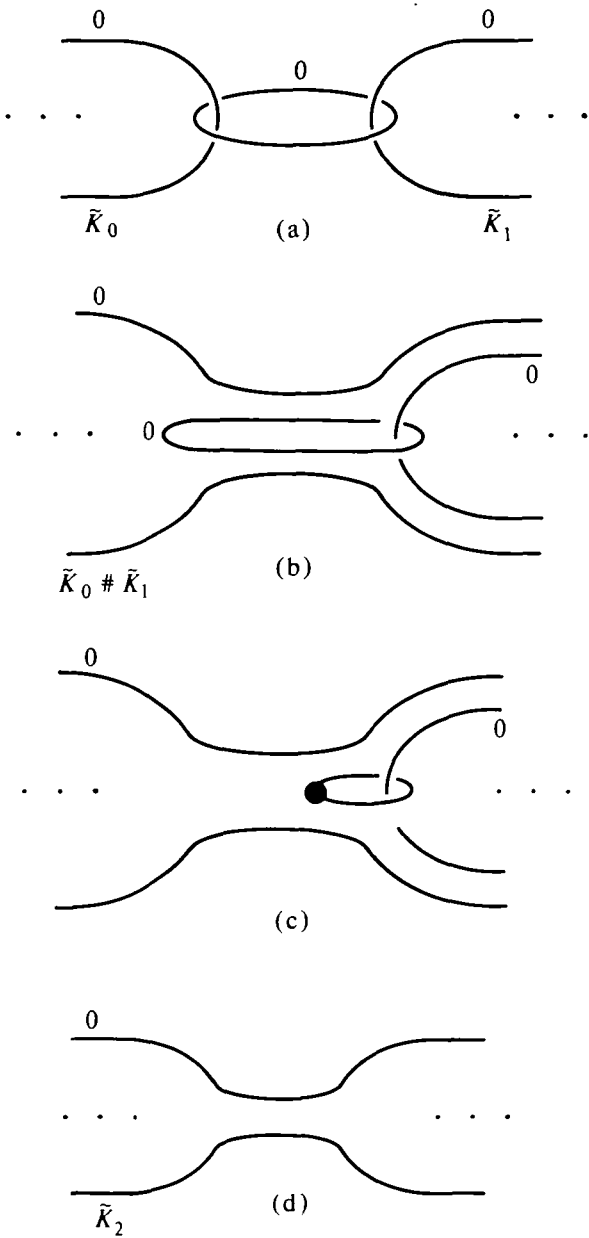


FIG. 1

*Proof.* Here we write  $H_\star^i(\ )$  for  $H_\star^i(\ , \mathbb{C})$ . Since  $\psi \neq 0$ ,  $H_3^i(N) = 0$ . Let  $A_i = h_i \cap N \times I$ , then  $H_\star^i(A_i) = H_\star^i(S^1)$ . The Mayer Vietoris sequence for  $W$  as the above union shows  $H_3^i(W) = 0$ . Another part gives

$$0 \rightarrow H_2^i(N) \rightarrow H_2^i(\tilde{W}) \rightarrow H_1^i(\bigcup A_i) \rightarrow H_1^i(N) \leftarrow H_1^i(W) \rightarrow 0$$

By Poincare duality  $H_2^i(N) = H_1^i(N)$ , thus  $\dim H_2^i(W) = \dim H_1^i(W) + n$ .

Let  $\sigma$  and  $\eta$  denote the signature and nullity of the quadratic form on  $H_2^i(W)$ . Then

$$|\sigma| + \eta \leq n + \dim H_1^i(W). \quad (1)$$

with equality mod 2. Since a matrix representing the form will also represent the map  $H_2^i(W) \rightarrow H_2^i(W, N \cup P)$  we have

$$0 \rightarrow \mathbb{C}^n \rightarrow H_1^i(P) \oplus H_1^i(N) \rightarrow H_1^i(W) \rightarrow H_1^i(W, N \cup P)$$

By Lefschetz duality  $H_1^i(W, N \cup P) = 0$ . Thus we have

$$\dim H_1^i(W) = \eta(N, \chi) + \eta(P, \chi) - \eta \quad (2)$$

Since  $H_2^i(N)$  injects into  $H_2^i(W)$  and the intersection pairing vanishes on the image,  $\eta \geq \dim H_2^i(N) = \eta(N, \chi)$ . We may also view  $W$  as  $P \times I$  union 2-handles. Thus  $\eta \geq \eta(P, \chi)$  and

$$2\eta - \eta(N, \chi) - \eta(P, \chi) \geq |\eta(N, \chi) - \eta(P, \chi)|. \quad (3)$$

Equality mod 2 holds trivially. Using (1), (2), and (3) one has

$$|\sigma| + |\eta(N, \chi) - \eta(P, \chi)| \leq n$$

with equality mod 2. Finally by definition

$$\sigma(P, \chi) - \sigma(N, \chi) = \sigma - \text{Sign } W.$$

For  $0 < s < m$ , define  $\sigma_{s/m}(K) = \text{Sign}((1 - w^s)\theta + (1 - w^{-s})\theta')$   $\eta_{s/m}(K) = \text{nullity}((1 - w^s)\theta + (1 - w^{-s})\theta')$  where  $w = e^{2\pi i/m}$ . Recall if  $m$  is a prime power, then  $\eta_{s/m}(K) = 0$ , [6] (3.1).

**THEOREM (3.4).** *If  $\chi = x \otimes s/m \in A \subset H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$ ,  $0 < s < m$ ,  $x \in H_1(F)$  is primitive and  $J \in C_x$ , then*

$$\left| \sigma_1(\tau(K, \chi)) - 2\sigma_{s/m}(J) - \frac{4(m-s)s}{m^2} \theta(x, x) + \sigma_1(K) \right| \leq \beta_1(F) + 2\eta_{s/m}(J).$$

*Proof.* We can view  $F$  as a disk with  $\beta_1(F)$  twisted knotted bands attached where the first has  $J$  tied in it and represents  $x$ . By [1],  $L$  is given by surgery on a framed link with  $\beta_1(F)$  components. The linking matrix is the matrix of  $\theta + \theta^T$  with respect to the basis of  $H_1(F)$  given by the bands. Let  $J^*$  denote the knot obtained by reversing the string orientation on  $J$ . It is easily seen that  $J$  and  $J^*$  have the same signatures and nullities. The

first component of the link is  $J\#J^*$ .  $\chi$  takes the value  $s/m$  on the meridian of this component and zero on the meridians of the other components.

Let  $(N, \chi)$  be the 3-manifold given by  $2\theta(x, x)$ -framed surgery along  $J\#J^*$ . By [6] (3.6),

$$\begin{aligned}\sigma(N, \chi) &= 2\sigma_{s/m}(J) - \text{Sign}(\theta(x, x)) + \frac{4(m-s)s}{m^2} \theta(x, x) \\ \eta(N, \chi) &= 2\eta_{s/m}(J).\end{aligned}$$

Now  $L$  is obtained from  $N$  by  $(\beta_1(F) - 1)$  surgeries and  $M$  is obtained from  $L$  by one more surgery. Thus  $M$  is obtained from  $N$  by  $\beta_1(F)$  surgeries. This gives us a  $W$  as in (3.3).  $M$  plays the role of  $P$ . Moreover  $\text{Sign } W = \sigma_1(K) - \text{Sign}(\theta(x, x))$ . So by (3.3),

$$\begin{aligned}\left| \sigma(M, \chi) - 2\sigma_{s/m}(J) - \frac{4(m-s)s}{m^2} \theta(x, x) + \sigma_1(K) \right| \\ + |\eta(M, \chi) - \eta_{s/m}(J)| \leq \beta_1(F).\end{aligned}$$

This, the formula (3.1), and the triangle inequality give the result.

*Remark.* Given a particular choice of  $\chi \in A$ , there is a great deal of choice in applying Theorem (3.4). First there is the choice of  $x$  and then  $J \in C_x$ . The estimates of  $\sigma_1(\tau(K, \chi))$  obtained can vary widely. In particular, a judicious choice of two estimates taken together can give a sharp estimate for  $\sigma_1(\tau(K, \chi))$ . For example let  $K$  be the connected sum of  $n$  trefoils. Take  $F$  to be the connected sum of  $n$  genus one Seifert surfaces for the trefoil. There is an  $x \in H_1(F)$  coming from the Seifert surface of the first trefoil such that  $\theta(x, x) = -3$  and  $\chi = x \otimes \frac{1}{3} \in A$ .  $C_x$  contains both the unknot and the connected sum of  $n-1$  trefoils. This yields

$$|\sigma_1 \tau(K, \chi) + \frac{8}{3} - 2n| \leq 2n$$

and

$$|\sigma_1 \tau(K, \chi) + 2(2n-2) + \frac{8}{3} - 2n| \leq 2n$$

Taken together this yields

$$-\frac{8}{3} \leq \sigma_1 \tau(K, \chi) \leq \frac{4}{3}$$

In fact by (3.5) below  $\sigma_1 \tau(K, \chi) = -\frac{2}{3}$ . The inequalities above then read  $|2n-2| \leq 2n$ . This shows that the inequality in (3.4) cannot really be sharpened much.

**THEOREM (3.5).** *If  $g(F) = 1$ , and  $\chi = x \otimes s/m$ , where  $0 < s < m$ ,  $m$  is a prime power and where  $x$  is primitive, then*

$$\tau(K, \chi) = \rho \left( 2\sigma_{s/m}(J_x) + \frac{4(m-s)s}{m^2} \theta(x, x) - \sigma_1(K) \right).$$

*Proof.* View  $F$  as a disk with two twisted knotted bands with one representing the class  $x$  with  $J_x$  tied in it. Let  $F'$  be formed by tying  $-J_x$  in this band and putting  $-\theta(x, x)$  more twists in it. Let  $K' = \partial F'$  and in general use prime to denote objects associated to  $K'$ .

Note that  $K'$  is slice. Moreover the 3-manifold  $R$  in the proof of (1.1) can be taken to be  $F' \times I \cup h^2$  where  $h^2$  is a 2-handle attached along a curve representing  $x$ . So by the proof of (1.1),  $\tau(K', \chi') = 0$ . We will now compare  $\tau(K, \chi)$  to  $\tau(K', \chi')$ . This idea is due to Steve Kaplan.

Let  $N$  be the 3-manifold obtained by doing  $\theta(x, x)$ -framed surgery to  $S^3$  along  $J_x$ . Let  $\varphi: H_1(N) \rightarrow C_m$  map the meridian of  $J_x$  to  $e^{2\pi i s/m}$  and  $\varphi^+: H_1(N) \rightarrow C_m \times C_\infty$  map the meridian to  $(e^{2\pi i s/m}, 0)$ . It is clear that  $\tau(N, \varphi^+) = \rho(\sigma(N, \varphi))$ . By [6] (3.6),

$$\sigma(N, \varphi) = \sigma_{s/m}(J_x) - \text{Sign}[\theta(x, x)] + \frac{2(m-s)s}{m^2} \theta(x, x).$$

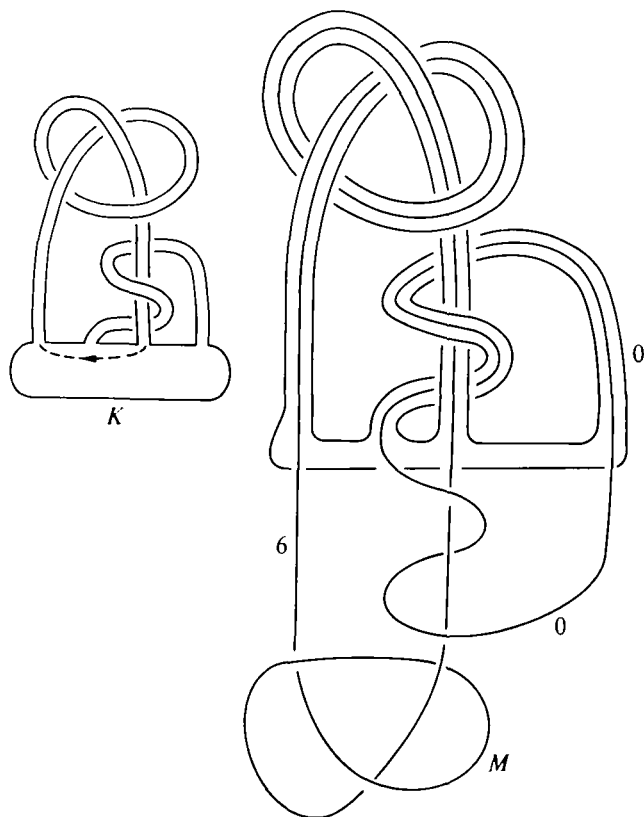


FIG. 2

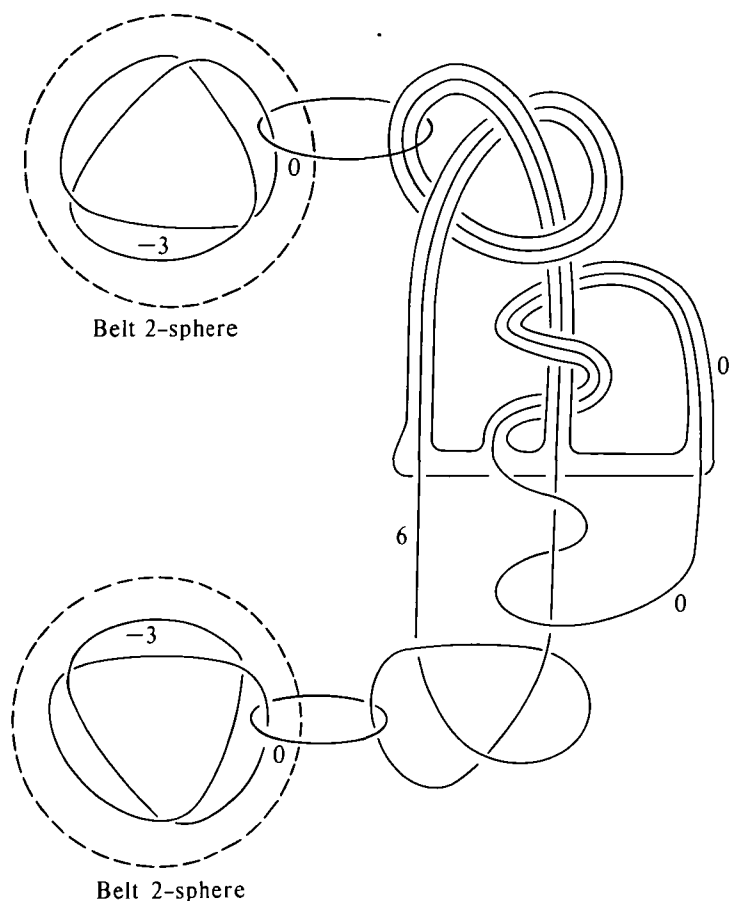


FIG. 3

Let  $N^*$  denote the result of  $\theta(x, x)$  framed surgery to  $S^3$  along  $J_x^*$ , the reverse of  $J_x$ . Then  $\tau(N^*, \varphi^+) = \tau(N, \varphi^+)$ .

We build a 4-manifold  $W$  with boundary  $M' \cup N \cup N^* \cup -M$  such that the  $C_m \times C_\infty$  covers given by  $\chi^+$ ,  $\chi^{++}$ , and  $\varphi^+$  extend to a cover of  $W$ .  $W$  is formed by attaching two 1-handles to  $(M \cup -N \cup -N^*) \times I$  joining the separate components and then attaching two 2-handles in a certain way along  $M \# -N \# -N^*$ .

This is best described by example. Figure 2 illustrates  $K$  and  $x$  and also a surgery description of  $M$  obtained using [1]. In Figure 3 we show  $M \# -N \# -N^*$  together with the belt 2-spheres of the 1-handles and the attaching circles for our 2-handles. If we ignore the belt 2-spheres, this is a surgery description of  $\partial_+ W$ . Figure 4 shows another surgery description

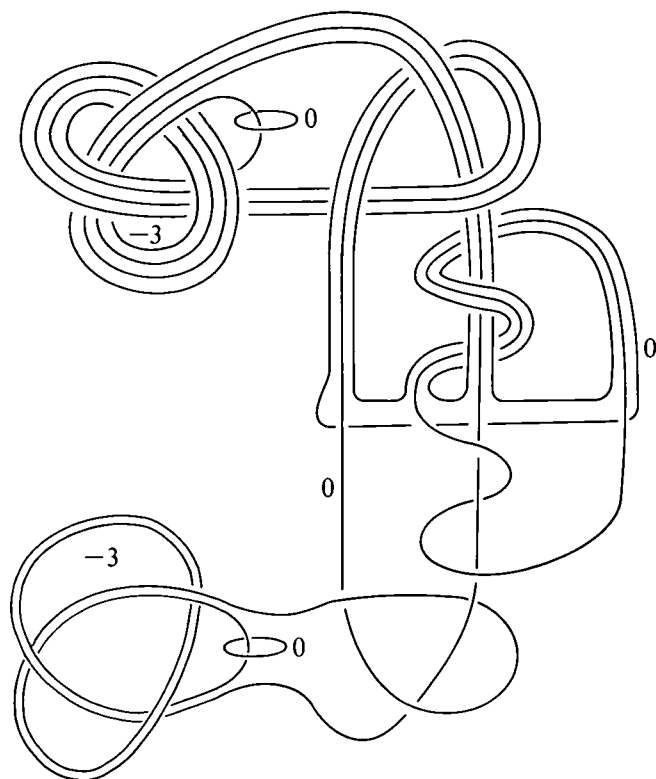


FIG. 4

obtained by sliding 2-handles [10]. Finally we can erase the two trefoils with framing-3 and the simply linking meridians with framing 0 as these surgeries cancel each other, and recognize  $M'$ .

By Novikov additivity,

$$\text{Sign } W = \sigma_{\text{link}}(K') - (\sigma_{\text{link}}(K) - 2 \text{Sign}(\theta(x, x))).$$

One then shows, as in the proof of (3.2), that the intersection pairing on  $H_2^1(W, \mathbb{C}(t))$  is identically zero. Thus

$$\tau(K', \chi') = \tau(K, \chi) - 2\tau(N, \varphi^+) - \rho(\text{Sign } W).$$

This completes the proof.

The results of this section hold equally well for knots in homology 3-spheres. The proofs require only minor modification.

#### Section 4

When now give the promised example of a knot  $K$  whose image in  $\Gamma'$  is nonzero but which maps to zero in  $G_-$  and  $\Lambda'$ . Figure 5 is a picture of our

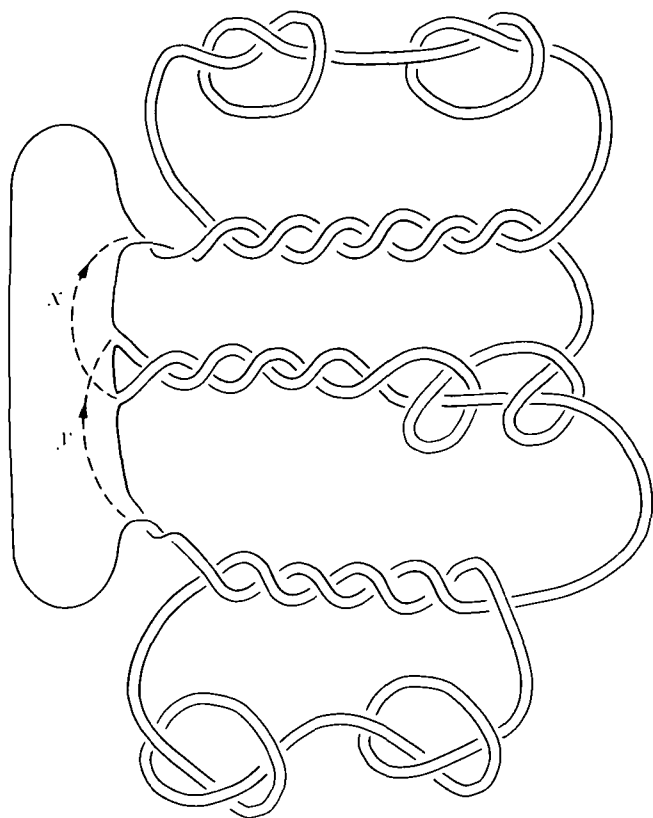


FIG. 5

knot  $K$ . A genus 1 Seifert surface  $F$  is clearly visible. With respect to the basis  $\{x, y\}$ , the Seifert matrix for  $F$  is

$$\begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}.$$

Thus  $K$  has a null bordant Seifert matrix and so is zero in  $G_-$ .  $A \subset H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$  is generated by  $x \otimes \frac{1}{9}$  and  $y \otimes \frac{1}{9}$ . (This is denoted  $A = \langle x \otimes \frac{1}{9}, y \otimes \frac{1}{9} \rangle$ ) and is isomorphic to  $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ . There are exactly three subgroups  $G$  such that  $|G| = 9$  and  $\beta(G \times G) = 0$ . They are  $G_1 = \langle x \otimes \frac{1}{9} \rangle \approx \mathbb{Z}_9$ ,  $G_2 = \langle y \otimes \frac{1}{9} \rangle \approx \mathbb{Z}_9$  and  $G_3 = \langle x \otimes \frac{1}{3}, y \otimes \frac{1}{3} \rangle \approx \mathbb{Z}_3 \oplus \mathbb{Z}_3$ . We need to evaluate  $\tau$  on each of these subgroups.

Let  $K(n, m)$  denote the  $(n, m)$  torus knot and  $\bar{J}$  the mirror image of  $J$ . Note that  $J_x = \bar{K}(2, 7) \# K(2, 3) \# K(2, 3)$  and  $J_y = \bar{J}_x$ . Using (5.1) of [6],  $\sigma_3(J_x) = -\sigma_3(J_y) = 2$  and  $\sigma_1(J_x) = -\sigma_1(J_y) = 0$ . Of course for any knot  $\sigma_1 = \sigma_3$ . By Theorem (3.5),  $\tau(x \otimes \frac{4}{9}) = -\tau(y \otimes \frac{4}{9}) = \rho(2)$ . Therefore  $\tau(G_1)$



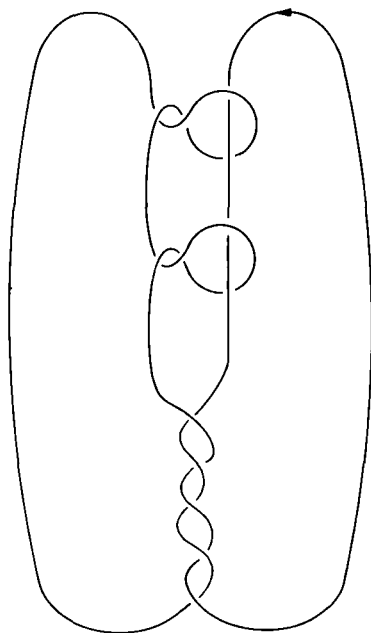


FIG. 6

and  $\tau(G_2)$  are nonzero. Also we have

$$\tau(x \otimes \tfrac{1}{3}) = \tau(x \otimes \tfrac{2}{3}) = \tau(y \otimes \tfrac{1}{3}) = \tau(y \otimes \tfrac{2}{3}) = 0.$$

Now  $J_{x+y} = J_x \# J_y \# K(2, 3) \# K(2, 3)$  so  $\sigma_3(J_{x+y}) = -4$ . By (3.5),  $\tau((x+y) \otimes \tfrac{1}{3}) = \tau((x+y) \otimes \tfrac{2}{3}) = 0$ . Finally  $J_{x-y} = J_x \# J_y \# K_0$  where  $K_0$  is shown in Figure 6.  $K_0$  is drawn with two extra crossings so that a relatively simple genus 4 Seifert surface is readily apparent. A straightforward calculation shows  $\sigma_3(K_0) = 4$ . Thus  $\tau((x-y) \otimes \tfrac{1}{3}) = \tau((x-y) \otimes \tfrac{2}{3}) = 0$ . Thus  $\tau(G_3) = 0$ , and  $K$  maps to zero in  $\Lambda'$ .

Finally there are only two direct summands  $H$  of  $H_1(F)$  of rank 1 such that  $\theta(H \times H) = 0$ . They are  $H_1 = \langle x \rangle$  and  $H_2 = \langle y \rangle$ . Since  $G_i = H_i \otimes Q/Z \cap A$  and  $\tau(G_1)$  and  $\tau(G_2)$  are nonzero, we conclude that  $K$  does not map to zero in  $\Gamma'$ .

## REFERENCES

1. S. Akbulut and R. Kirby, 'Branched Covers of Surfaces in 4-manifolds', *Math. Ann.* 252 (1980), 111-131.
2. R. H. Fox and J. W. Milnor, 'Singularities of 2-spheres in 4-space and equivalence of knots'. *Bull. Amer. Math. Soc.* 63 (1957), 406.
3. A. J. Casson and C. McA. Gordon, 'Cobordism of classical knots, printed notes', Orsay, 1975.

4. A. J. Casson and C. McA. Gordon, 'On slice knots in dimension three', *Proc. Symp. in Pure Math.* XXX, (1978), part two, 39–53.
5. Lecture by A. J. Casson, July, 1978 in Cambridge, England.
6. P. M. Gilmer, 'Configurations of surfaces in 4-manifolds', *Trans. A.M.S.* 264 (1981) 353–380.
7. C. McA. Gordon, 'Some Aspects of Classical Knot Theory', *Knot Theory*, Proc. Plans-sur-bex, Switzerland, 1977. *Lecture Notes in Math.* 685, Springer Verlag, (1978), 1–60.
8. L. Kauffman, 'Branched coverings, open books and knot periodicity', *Topology* 13 (1974), 143–160.
9. M. A. Kervaire, 'Knot cobordism in codimension two', *Manifolds—Amsterdam 1970*, *Lecture Notes in Math.* 197, Springer Verlag, (1971), 83–105.
10. R. Kirby, 'A calculus for framed links in  $S^3$ ', *Inventiones Math.* 45 (1978), 173–207.
11. J. Levine, 'Knot cobordism groups in codimension two', *Comment. Math. Helv.* 44 (1969), 229–244.
12. D. Rolfsen, *Knots and Links*, Mathematical Lecture Series 7, Publish or Perish, Inc., Berkeley, CA (1976).
13. P. M. Gilmer, *On the Ribbon Genus of Knots*, Proc. of the Eleventh Annual U.S.L. Mathematics Conference, October 1980, Research Series No. 52, Dept. of Publications, USL Lafayette, LA (1981), 63–78.

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