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THE FIRST EXOTIC CLASS OF *BF*

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INTRODUCTION

FOLLOWING G. W. Whitehead, [14; p. 233], let $G(n)$ denote the space of maps of degree 1 of $S^n \to S^n$ and let $F(n) \subset G(n)$ be the subspace of those maps which are base point preserving. We have a natural fibration $F(n) \to G(n) \to S^n$.

Both $F(n)$ and $G(n)$ are associative H-spaces with multiplication given by composition and by [4], they have classifying spaces which we denote by $BF(n)$ and $BG(n)$ respectively. $BG(n)$ is the classifying space for n-sphere fibrings [11]. The suspension of maps induces natural inclusions $F(n) \subset F(n+1)$ and $G(n) \subset G(n+1)$ which preserve the multiplication. Let $F = \bigcup_n F(n)$, $G = \bigcup_n G(n)$. Then F and G are of the same weak homotopy type. Similarly one defines BG and BF which are also of the same weak homotype type.

Let $SO(n+1)$ be the special orthogonal group of R^{n+1}. We have a natural inclusion $SO(n+1) \subset G(n)$. If we identify $SO(n)$ with the subgroup of those rotations in $SO(n+1)$ which leave the base point fixed, we obtain a natural inclusion $SO(n) \subset F(n)$. This map induces in the limit $SO \subset F$ and a map of classifying spaces $f : BSO \to BF$.

BSO is the classifying space for stable orientable real vector bundles, and similarly BF is the classifying space for stable orientable sphere fibrings. Given a stable sphere fibring ξ over X with classifying map $\chi : X \to BF$, we say that ξ admits a stable orientable orthogonal structure if χ lifts to a map $\bar{\chi} : X \to BSO$.

With any coefficients, the elements of $H^*(BF)$ are called characteristic classes for orientable sphere fibrings by analogy with the characteristic classes for orientable vector bundles which come from $H^*(BSO)$. In particular, for p an odd prime, we have defined Wu classes $q_i \in H^{ir}(BF; Z_p)$, where $r = 2(p-1)$, as follows: Let ξ be an orientable $(n-1)$-sphere fibring over X. Denote the Thom complex of ξ by $T(\xi)$ and by T, the Thom isomorphism

$$T : H^q(X) \to H^{q+n}(T(\xi)).$$

The Wu class $q_i(\xi)$ is given by $T^{-1}\mathscr{P}^i T(1)$ where $\mathscr{P}^i$ is the Steenrod power [7; p. 120]. We denote by q_i the universal Wu class defined by taking ξ to be the universal $(n-1)$-sphere fibring over $BG(n-1)$, assuming n much bigger than i.

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Consider $f^* : H^*(BF; Z_p) \to H^*(BSO; Z_p)$. We know $f^*q_i \neq 0$ but that $H^n(BSO; Z_p) = 0$ if $n \not\equiv 0 \bmod 4$ [7; p. 84]. Milnor [9] shows that $\beta q_i \neq 0$, $i < p$ where β is the Bockstein coboundary mod p, and that $\{\beta q_i, 1 \leqq i < p\}$ form a set of generators for the ideal $\operatorname{Ker} f^*$ in dimensions $\leqq pr - 2$. More generally any class in the kernel of f^* is an obstruction to a stable orientable orthogonal structure. Those which are not given in terms of Bockstein coboundaries of Wu classes we refer to as *exotic* classes.

In this note, we prove that in fact $H^{pr-1}(BF; Z_p) \approx Z_p$ with generator a class e_1 which can be obtained from a "twisted" secondary cohomology operation acting in the Thom space of BF. Since $pr - 1 \not\equiv 0 \bmod 4$, $f^*e_1 = 0$. It is the first exotic class.

If $p = 2$, then $H^3(BF; Z_2) \cong Z_2 + Z_2$ where a generator of one of the summands projects onto the Stiefel–Whitney class w_3 and the generator of the other summand, also denoted by e_1, is produced in a similar fashion to the class e_1 mod p. Thus $e_1 \in \operatorname{Ker} f^*$ and in fact the class e_1 mod 2 is the first obstruction to a cross-section of $f: BSO \to BF$.

Using the class e_1 with Z_2 coefficients we show there is a sphere fibring $S^2 \to E_0 \to S^3$ such that E_0 satisfies Poincaré duality with integer coefficients, but E_0 is not of the homotopy type of a PL-manifold. We wish to express our thanks to W. Browder for pointing out this example. (C.T.C. Wall has informed us that he has an example of a 5-dimensional space which satisfies integral Poincaré duality but is not even cobordant to a PL-manifold. He also uses the class e_1.)

We also acknowledge helpful conversations with B. Sanderson and especially with J. Milnor.

§1. FUNCTIONAL COHOMOLOGY OPERATIONS

In this section we review the concept of functional cohomology operations in a slightly more general framework than that considered by Steenrod in [12].

First we need some definitions. Let A be a Hopf algebra over the field Z_p and M an algebra over A. We may form the *split extension algebra*† $A(M)$ of M by A. As a vector space $A(M) = M \otimes A$. The multiplication is given by the following diagram:

$$M \otimes A \otimes M \otimes A \xrightarrow{1 \otimes \psi_A \otimes 1 \otimes 1} M \otimes A \otimes A \otimes M \otimes A \xrightarrow{1 \otimes 1 \otimes t \otimes 1} \tag{1.1}$$
$$M \otimes A \otimes M \otimes A \otimes A \xrightarrow{1 \otimes \lambda \otimes 1} M \otimes M \otimes A \otimes A \xrightarrow{\varphi_M \otimes \varphi_A} M \otimes A$$

where ψ_A is the diagonal in A, t is the permutation of factors, λ is the left A-module structure of M and φ_M, φ_A are the multiplications in M and A respectively.

If N is an M-module over A, then we make N into an $A(M)$-module by the rule:

$$(m \otimes \theta)n = m \cdot \theta(n). \tag{1.2}$$

† This name was suggested to us by N. E. Stasheff.

The algebra $A(M)$ has been introduced by Massey and Peterson in [5], where it is called the semi-tensor product of M and A. Also J. P. Meyer in [6] considers the algebra $A(M)$ for purposes very similar to ours.

Let now X be a space and A the mod p Steenrod algebra. The cohomology ring $H^*(X)$ of X with Z_p-coefficients is an A-module, so we may form $A(H^*(X))$ which we will denote simply by $A(X)$. Let $f: Y \to X$ be a mapping, then f induces in a natural way a ring homomorphism $f^*: A(X) \to A(Y)$. More generally, if (X, X_1) is a pair, the cohomology sequence of (X, X_1) is an $A(X)$-sequence and if $f:(Y, Y_1) \to (X, X_1)$ is a map of pairs, then the induced mapping in the cohomology sequences is compatible with $f^*: A(X) \to A(Y)$.

Suppose that $f: Y \to X$ and that $u \in H^q(X)$ and $\theta \in A(X)$ are such that $\theta u = 0, f^*u = 0$, where θ is a homogeneous element of degree r. Following Steenrod [12], we can define a functional operation $\theta_f u$. It is a coset of $H^{q+r-1}(Y) \bmod f^*H^{q+r-1}(X) + (f^*\theta)H^{q-1}(Y)$. Using the relative mapping cylinder we can define θ_f in the relative case, just as in [12; p. 974].

In the same way as in [12; (15.7)] and [12; (15.8)] one has the following two naturality properties, which we state for convenience in the relative case:

PROPOSITION 1.3. *Suppose we are given*:

$$(Z, Z_1) \xrightarrow{g} (Y, Y_1) \xrightarrow{h} (X, X_1)$$

$\theta \in A(X)$, $u \in H^*(X, X_1)$, *then if* $\theta_h(u)$ *is defined*, $\theta_{hg}(u)$ *is defined and*

$$g^*\theta_h(u) = \theta_{hg}(u).$$

PROPOSITION 1.4. *Suppose we are given*:

$$(Z, Z_1) \xrightarrow{g} (Y, Y_1) \xrightarrow{h} (X, X_1)$$

$\theta \in A(X)$, $u \in H^*(X, X_1)$. *Then if* $\theta_{hg}(u)$ *is defined*, $(h^*\theta)_g(h^*u)$ *is defined and*

$$\theta_{hg}(u) = (h^*\theta)_g(h^*u)$$

modulo the indeterminacy of the right hand side.

We have a natural inclusion $i: A \to A(X)$, given by $i(\alpha) = 1 \otimes \alpha$, for $\alpha \in A$' $1 \in H^0(X; Z_p)$ the unit of the ring $H^*(X; Z_p)$, when X is connected.

PROPOSITION 1.5. *Let* $f:(Y, Y_1) \to (X, X_1)$ *be a map*, $\theta \in A(X)$, $u \in H^*(X, X_1)$, *then if* $\theta = i(\alpha)$ *and* θ_f *is defined, we have* $\alpha_f u$ *defined and* $\theta_f u = \alpha_f u$.

The proof of (1.5) follows immediately from the construction.

The algebra $A(X)$ turns out to be a natural algebra to consider when one studies sphere fibrings.

Let $\xi = (E, p, X)$ be an orientable $(n-1)$-sphere fibring. We denote by $T(\xi)$ or $T(E)$, the Thom complex of ξ and by T the Thom isomorphism,

$$T: H^q(X) \to H^{q+n}(T(\xi)).$$

Associated with ξ one can define a mapping

$$\xi^* : A \to A(X) \tag{1.6}$$

as follows. If $\theta \in A$ and $\psi(\theta) = \Sigma\, \theta_i \otimes \theta_i'$ under the diagonal, then set:

$$\xi^*(\theta) = \sum T^{-1}\theta_i T(1) \otimes \theta_i'. \tag{1.7}$$

Let $C^*(\xi) \subset H^*(X)$ be the characteristic ring of ξ, i.e. the minimal subalgebra of $H^*(X)$ closed under A generated by $T^{-1}\theta T(1)$ for $\theta \in A$, and let $A(X, \xi)$ be the split extension algebra of $C^*(\xi)$ by A.

PROPOSITION 1.8. *$\xi^* : A \to A(X)$ is a ring homomorphism. In fact* $\operatorname{Im} \xi^* \subset A(X, \xi)$. *Moreover for $\theta \in A$ we have*

$$\theta T = T\xi^*(\theta).$$

We omit the proof since it is straightforward.

§2. SECONDARY COHOMOLOGY OPERATIONS

In [1], Adams associates with every relation in A, a family of secondary cohomology operations, such that any two operations in the family differ by a stable primary operation. By enlarging the natural indeterminacy so as to contain all the stable primary operations one thus obtains a "unique" secondary operation. In this section we construct the analogous notion of a unique secondary operation associated to a relation in $A(X)$.

Let $X \supset X_1 \supset X_2$, then $H^*(X_1, X_2)$ is an $A(X)$-module. Let $\sum_{k=1}^{m} \alpha_k \beta_k = 0$ be a relation in $A(X)$ of degree $r + 1$. Following Adem [2; p. 98] we represent the above relation by the composition $\alpha\beta = 0$. Let $u \in H^q(X_1, X_2)$ be such that $\beta u = 0$, i.e. $\beta_k u = 0$ for $k = 1, \dots, m$. We define a coset $\{\alpha, \beta, u\}$ of $H^{q+r}(X_1, X_2)$ modulo a subgroup $Q^{q+r}(\alpha; X_1, X_2)$ as follows. Let $d : X_1 \to X_1 \times X_1$ be the diagonal map, then d induces a map of pairs $\Delta : (X_1, X_2) \to X_1 \times (X_1, X_2)$. Let $K = K(Z_p, q)$ and k a point of K. Let $f : (X_1, X_2) \to (K, k)$ be such that $f^*\gamma = u$, where γ is the fundamental class of K. Let $i : X_1 \to X$ be the inclusion, and $g : (X_1, X_2) \to X \times (K, k)$ be given by $g = (i \times f)\Delta$. Then

$$g^*(1 \times \gamma) = u.$$

Let $\pi : X \times K \to X$ be the projection, which induces $\pi^* : A(X) \to A(X \times K)$. Then $\pi^*(\alpha) = \alpha \times 1$, $\pi^*(\beta) = \beta \times 1$, so $g^*(\beta \times 1)\cdot(1 \times \gamma) = \beta u = 0$ and $(\alpha \times 1)\cdot(\beta \times 1)(1 \times \gamma) = 0$. Therefore as in §1 we may define

$$\alpha_g\beta(1 \times \gamma) \in H^{q+r}(X_1, X_2)$$

modulo $\Sigma\, \alpha_k H^{q+r-t_k}(X_1, X_2) + g^*H^{q+r}(X \times (K, k))$ where $t_k = \deg \alpha_k$. We now describe $\operatorname{Im} g^*$. In fact, if we let $[A(X)u]$ denote the graded subalgebra of $H^*(X_1, X_2)$ generated by elements of the form θu, where $\theta \in A(X)$, we claim that $g^*H^{q+r}(X \times (K, k)) \subset [A(X)u]$. This follows readily from the results of Cartan on the cohomology of K. With the above notation, define

(2.1) $$\{\alpha, \beta, u\} \in H^{q+r}(X_1, X_2)$$

mod $Q^{q+r}(\alpha; X_1, X_2) = \Sigma\, \alpha_k H^{q+r-t_k}(X_1, X_2) + [A(X)u]$ to be $\alpha_g\beta(1 \times \gamma)$. Then $\{\alpha, \beta, u\}$ satisfies the following naturality conditions.

PROPOSITION 2.2. *Let $X \supset X_1 \supset X_2$ and $Y \supset Y_1 \supset Y_2$ be triples and $f: (Y, Y_1\ Y_2) \to (X, X_1, X_2)$ be a map. Suppose that $\{\alpha, \beta, u\}$ is defined, then $\{f^*\alpha, f^*\beta, f^*u\}$ is defined and*

$$f^*\{\alpha, \beta, u\} = \{f^*\alpha, f^*\beta, f^*u\}.$$

Proof. This follows from the definition of $\{\alpha, \beta, u\}$ and (1.3).

PROPOSITION 2.3. *If $X \supset X_1 \supset X_2$ and X is acyclic, then $\{\alpha, \beta, u\}$ contains $\Phi(u)$, where Φ is any secondary operation associated with the relation $\alpha\beta = 0$ in A.*

Proof. The result follows from (1.5) and the first paragraph of this section, together with the second formula of Peterson–Stein as given in [2; (5.2)].

§3. A SPECIAL SECONDARY COHOMOLOGY OPERATION

In this section we work with Z_p coefficients, p odd.

Consider as in §1, the algebra $A(X)$ associated with the space X. Let $r = 2(p - 1)$. Let $\varphi(x)$ be the element of $A(X)$ given by

$$\varphi(x) = 1 \otimes \mathscr{P}^1 + x \otimes 1.$$

where $x \in H^r(X)$ and $\mathscr{P}^1$ is the Steenrod power, which is also of degree r.

PROPOSITION 3.2. *For any space X and any class $x \in H^r(X)$, we have: $\varphi(x)^p = 0$ is a relation in $A(X)$.*

Proof. By naturality it suffices to prove that $\varphi(\gamma)^p = 0$, where $\gamma \in H^r(K(Z_p, r))$ is the fundamental class. To this end, consider $BSO(m)$ with $m > r$. Let ξ_m be the universal m-plane bundle over $BSO(m)$ and let q_1 be its first Wu class, i.e. $q_1 = T^{-1}\mathscr{P}^1T(1)$ where T is the Thom isomorphism associated with ξ_m. As in (1.6) we have that ξ_m induces $\xi_m^*: A \to A(BSO(m))$ with $\xi_m^*(\mathscr{P}^1) = \varphi(q_1)$. Now $(\mathscr{P}^1)^p = 0$ is an Adem relation and since ξ_m^* is a ring homomorphism (1.8), $\varphi(q_1)^p = 0$. We need now the following.

LEMMA 3.3. *Let $f: BSO(m) \to K(Z_p, r)$ be such that $f^*\gamma = q_1$. Then in dimensions $\leqq pr$,* Ker f^* *is generated by those elements in the Cartan basis containing a Bockstein.*

Proof. First recall that $H^*(BSO(m); Z)$ has only 2-torsion and hence all the Bocksteins in $H^*(BSO(m)) = H^*(BSO(m); Z_p)$ are zero. Therefore any element in the Cartan basis containing a Bockstein lies in Ker f^*.

Now in dimensions $\leqq pr$, $H^*(K(Z_p, r))$ is a free commutative algebra generated by $\mathscr{P}^k\gamma, \beta\mathscr{P}^k\gamma, \beta\mathscr{P}^k\beta\gamma$, where $k < p$, with the single relation $\mathscr{P}^{p-1}\gamma = \gamma^p$. Therefore it suffices to show that $\{\mathscr{P}^k q_1, 0 \leqq k \leqq p - 2\}$ generate a polynomial subalgebra of $H^*(BSO(m))$. Recall that $H^*(BSO(m))$ is a polynomial algebra generated by the Pontrjagin classes mod p and that the Wu classes q_k generate a polynomial subalgebra [7]. Wu formulae give

$$\mathscr{P}^k q_1 = (k + 1)q_{k+1} + Q_k$$

where Q_k is a polynomial in the classes $q_1, \dots, q_k$. The above formula shows immediately that the algebra generated by $q_1, \ldots, \mathscr{P}^{p-2}q_1$ is contained in the algebra generated by $q_1, \ldots, q_{p-1}$. An easy induction, using the above formula, gives the converse and thus the $\{\mathscr{P}^k q_1, 0 \leqq k \leqq p-2\}$ generate a polynomial subalgebra.

To complete the proof of (3.2) we observe that $\varphi(\gamma)$ does not lie in the kernel of $f^* : A(K(Z_p, r)) \to A(BSO(m))$. Also $\varphi(\gamma)^p$ does not contain Bocksteins in its expression. However, $f^*\varphi(\gamma)^p = \varphi(q_1)^p = 0$ and therefore $\varphi(\gamma)^p = 0$ follows from (3.3).

Using the fact that $\varphi(x)^p = 0$ is a universal relation, i.e. valid in $A(X)$, for all X, we may now construct a secondary cohomology operation associated with this relation, as in §2.

Given classes $x_1, \ldots, x_k \in H^*(X)$, we denote by $A[x_1, \ldots, x_k]$ the minimal subalgebra of $H^*(X)$ containing $x_1, \ldots, x_k$ and closed under the action of A. We will denote by ψ any secondary operation in the sense of Adams [1], associated with the relation $(\mathscr{P}^1)^p = 0$.

We have then,

THEOREM 3.4. *Let (X_1, X_2) be a pair, $x \in H^r(X_1)$, $u \in H^q(X_1, X_2)$ be classes such that $\varphi(x)u = 0$. Then we can define a coset $\Phi(x, u)$ of $H^{q+pr-1}(X_1, X_2)$ modulo $A[u, xu, \ldots, x^{p-1}u]^{q+pr-1} + \varphi(x)^{p-1}H^{q+p-1}(X_1, X_2)$ which is natural, i.e. if $f:(Y_1, Y_2) \to (X_1, X_2)$ is a map of pairs, then $\Phi(f^*x, f^*u)$ is defined and $\Phi(f^*x, f^*u) = f^*\Phi(x, u)$. Moreover if $x = 0$ we have $\Phi(0, u) = \psi(u)$ modulo $(\mathscr{P}^1)^{p-1}H^{q+p-1}(X_1, X_2) + A[u]^{q+pr-1}$.*

Proof. Let $X = K(Z_p, r)$ and let $i : X_1 \to X$ be such that $i^*\gamma_1 = x$, where γ_1 is the fundamental class of X. We may assume that i is an inclusion. Then consider $X \supset X_1 \supset X_2$. We have $\varphi(\gamma_1) \in A(X)$ satisfies $\varphi(\gamma_1)^p = 0$ and $\varphi(\gamma_1)u = 0$. Thus we are in the situation of §2 and we have defined the coset $\{\varphi(\gamma_1)^{p-1}, \varphi(\gamma^1), u\}$ of $H^{q+pr-1}(X_1, X_2)$ modulo the sub-group $\varphi(x)^{p-1}H^{q+p-1}(X_1, X_2) + [A(X)u]^{q+pr-1}$. However it is easily seen that $[A(X)u]^{q+pr-1} = A[u, xu, \ldots, x^{p-1}u]^{q+pr-1}$. Therefore if we define $\Phi(x,u) = \{\varphi(\gamma_1)^{p-1}, \varphi(\gamma_1), u\}$, the properties asserted in (3.4) follow from (2.2) and (2.3).

§4. THE CLASS e_1 FOR p ODD

In this section we apply (3.4) to sphere fibrings to obtain the class $e_1 \in H^{pr-1}(BF; Z_p)$ mentioned in the introduction.

Let ξ be the universal $(n-1)$-sphere fibring over $BG(n-1)$, the classifying space for such fibrings, where n is large ($n > pr$). Let E_0 be the total space of ξ and E the mapping cylinder of $p : E_0 \to BG(n-1)$. Then we have the Thom isomorphism

$$T : H^q(BG(n-1)) \to H^{q+n}(E, E_0)$$

given by cup product with the Thom class $U \in H^n(E, E_0)$. Set $q_1 = q_1(\xi) = T^{-1}\mathscr{P}^1 U$. Then

$$\varphi(-q_1)U = \mathscr{P}^1 U - q_1 U = 0 \tag{4.1}$$

and we have

LEMMA 4.2. *$\Phi(-q_1, U)$ is defined with zero indeterminacy, thus*

$$\Phi(-q, U) \in H^{n+pr-1}(E, E_0).$$

Proof. Using (4.1) and (3.4) we see that $\Phi(-q_1, U)$ is defined. We need to see that it has zero indeterminacy. From [9], we have $H^{r-1}(BG(n-1)) = 0$ and therefore $H^{n+r-1}(E, E_0) = 0$. The assumption $n > pr$ implies that $A[U, q_1 U, \dots, q_1^{p-1} U]^{n+pr-1}$ contains no terms in U^2. Thus a typical element of this group is of the form $\alpha_0 U + \alpha_1(q_1 U) + \cdots + \alpha_{p-1}(q_1^{p-1} U)$, where $\alpha_k \in A_{p(r-k)-1}$ for $k = 0, 1, \dots, p-1$, and A_t is the subspace of A of homogeneous elements of degree t. Now it is easy to verify that $A_{p(r-k)-1} = 0$ for $k = 0, 1, \dots, p-1$.

For n sufficiently large, we have $H^{pr-1}(BF) \cong H^{pr-1}(BG(n-1))$ and we define $e_1 \in H^{pr-1}(BF)$ to be the class which corresponds to $T^{-1}\varphi(-q_1, U)$ under this isomorphism.

THEOREM 4.3. *$H^{pr-1}(BF) \cong Z_p$ with generator e_1.*

We first show that $e_1 \neq 0$. In fact we now construct a sphere fibring ξ over a sphere with $e_1(\xi) \neq 0$.

Let us recall [11] how given $f: S^r \to G(n-1)$ one constructs a sphere fibring $S^{n-1} \to E_0 \xrightarrow{p} S^{r+1}$ associated with f. Let $g: S^r \times S^{n-1} \to S^{n-1}$ be the adjoint map of f. Then $E_0 = (e^{r+1} \times S^{n-1}) \cup_g S^{n-1}$ and p is induced by $e^{r+1} \times S^{n-1} \xrightarrow{u} e^{r+1} \xrightarrow{v} S^{r+1}$ where u is projection onto the first factor and v shrinks the boundary of e^{r+1} to a point. Let E be the mapping cylinder of p. Then $T(E) = E/E_0$ is the Thom complex and is of the form $e^{r+n+1} \cup_h S^n$. In the case of orthogonal sphere bundles the class of h in $\pi_{n+r}(S^n)$ coincides with the image of f under G. W. Whitehead's J-homomorphism, [8; Lemma 1]. A corresponding homomorphism

$$J': \pi_r(G(n-1)) \to \pi_{r+n}(S^n) \tag{4.4}$$

is readily defined from the suspension $G(n-1) \to F(n)$ and the Hurewicz isomorphism $\pi_r(F(n)) \approx \pi_{r+n}(S^n)$. If $f: S^r \to G(n-1)$, then $J'[f]$ is represented by g where S^{n+r} is represented as a quotient of $I \times S^r \times S^{n-1}$ and $g(t, x, y) = (t, f(x)y)$. From the explicit constructions it is not hard to verify the analogue of [8; Lemma 1].

PROPOSITION 4.5. *The attaching map h in $T(E)$ is given by $[h] = J'[f]$.*

PROPOSITION 4.6. *If $r < 2n-4$, then*

$$J': \pi_r(G(n-1)) \to \pi_{n+r}(S^n)$$

is an isomorphism.

This follows from comparing the EHP sequence with the homotopy sequence of the fibring $F(n-1) \to G(n-1) \to S^{n-1}$.

For $n > r$, we may identify $\pi_{n+r}(S^n)$ with the stable group π_r^s. Toda in [13; (4.15)] defines an element $\beta_1 \in \pi_{pr-2}^S$. Let $\bar\beta_1 \in \pi_{pr-2}(G(n-1))$ be such that $J'(\bar\beta_1) = \beta_1$. Recall that by ψ we denote a secondary cohomology operation based on the relation $(\mathscr{P}^1)^p = 0$.

PROPOSITION 4.7. *Let (S^{n+1}, E_0, S^{pr-1}) be the sphere fibring associated with $\bar\beta_1$. Let $T(E)$ be the Thom complex of this fibring and $U \in H^n(T(E))$ be a generator. Then $\psi(U)$ is defined, has zero indeterminacy and is a generator of $H^{n+pr-1}(T(E))$.*

Proof. By (4.5) and (4.6), it follows that $T(E) = S^n \cup e^{n+pr-1}$ where the attaching map of the top cell represents β_1. Proposition (4.7) is a consequence of the following lemma, due essentially to Toda.

LEMMA 4.8. *Let* $X = S^n \cup_g e^{n+pr-1}$, *where* $[g] = \beta_1$. *Then for* u *a generator of* $H^n(X)$, $\psi(u)$ *is defined and* $\psi(u) \neq 0$.

Proof. From [13; (4.10)], there exists a CW-complex

$$L(M, k) = S^M \cup e^{M+r} \cup e^{M+2r} \cup \cdots \cup e^{M+kr}, \qquad k \leqq p-1,$$

such that if $x \in H^M(L(M, k))$ is a generator, then $\mathscr{P}^i x \neq 0$ for $0 \leqq i \leqq k \leqq p-1$. If n is large, let $q: S^{n+2p-3} \to S^n$ represent a generator of the p-primary component. Then q extends to a map $G: L(n+2p-3, p-2) \to S^n$ such that if $h: S^{n+pr-2} \to L(n+2p-3, p-2)$ is the attaching map for the cell e^{n+pr-1} of $L(n+2p-3, p-1)$, then $Gh = g$. Then we obtain a mapping $F: L(n+2p-3, p-1) \to X$ which is topological in the top cell and such that $\mathscr{P}^1_F u = x$, where $\mathscr{P}^1_F$ is the functional operation. Then using the first formula of Peterson–Stein as given in [2; (5.1)], we have

$$F^*\psi(u) = \mathscr{P}^{p-1}(\mathscr{P}^1_F u) = \mathscr{P}^{p-1}x \neq 0$$

but since F^* is an isomorphism in the top dimension, (4.8) follows.

COROLLARY 4.9. *Let* $\xi = (S^{n-1}, E_0, S^{pr-1})$ *be the sphere fibring associated with* $\bar{\beta}_1$. *Then* $e_1(\xi) \in H^{pr-1}(S^{pr-1})$ *is a generator.*

Proof. In the notation of (4.7), $T^{-1}(\psi(u)) \in H^{pr-1}(S^{pr-1})$ is a generator. By (3.4), $\psi(u) = \Phi(0, u)$ and by naturality of the Thom isomorphism and of the operation Φ, together with the definition of e_1, the corollary follows.

Milnor, in [9], shows implicitly that $H^{pr-1}(BF; Z_p) \cong 0$ or Z_p. Since e_1 is non-zero in the special case above, it is non-zero in the universal example and therefore $H^{pr-1}(BF; Z_p) \cong Z_p$.

§5. THE CLASS e_1 mod 2

In this section we indicate the necessary modifications that are needed to define the class $e_1 \in H^3(BF; Z_2)$ and give an example.

Let $\rho: H^*(X; Z) \to H^*(X; Z_2)$ be the reduction mod 2 homomorphism, and let A be the Steenrod algebra over Z_2. If $x \in H^2(X; Z_2)$, let $\theta(x) \in A(X)$ be defined by $\theta(x) = Sq^2 \otimes 1 + 1 \otimes x$. It is easy to verify that for any $u \in \operatorname{Im} \rho$, $(\theta(x))^2 u = 0$. Then as in §3 we may define $\Theta(x, u)$ for classes $u \in \operatorname{Im} \rho$ which satisfy $\theta(x)u = 0$. In fact if $u \in H^q(X; Z_2)$ then $\Theta(x, u) \in H^{q+3}(X; Z_2)$ modulo $A[u, xu]^{q+3} + \theta(x)H^{q+1}(X; Z_2)$, and similarly to (3.4) we have, $\Theta(0, u) = \varphi_{11}(u)$, where φ_{11} is the secondary cohomology operation based on the relation $Sq^2Sq^2 = 0$, which is valid for reduction mod 2 of integral classes.

In particular consider $BG(n-1)$ and $T(\xi)$, the Thom complex of the canonical $(n-1)$-sphere fibring ξ over $BG(n-1)$. Let $U \in H^n(T(\xi); Z_2)$ be its Thom class and, as usual, let $w_k \in H^k(BG(n-1); Z_2)$ be defined by $w_k = T^{-1}Sq^k U$. Then by definition, $\theta(w_2)U = 0$. Therefore $\Theta(w_2, U)$ is defined and lies in $H^{n+3}(T(\xi); Z_2) \bmod A[U, w_2 U]^{n+3} + \theta(w_2)H^{n+1}(T(\xi); Z_2)$. Now $H^1(BG(n-1); Z_2) = 0$, and the indeterminacy reduces to $(Sq^1 w_2)U = w_3 U$. So that if $\bar{e}_1 = T^{-1}\Theta(w_2, U)$, then $\bar{e}_1 \in H^3(BG(n-1); Z_2)/(w_3)$.

Now to show $\bar{e}_1 \neq 0$, we have by (4.6) that for $n > 3$, $\pi_2(G(n-1)) \approx \pi_{n+2}(S^n) \approx Z_2$. In fact, this remains true for $n = 3$ as can readily be checked using the fact that $\pi_4(S^2) \approx \pi_5(S^3)$, by suspension. Let $S^{n-1} \to E_0 \to S^3$ be the sphere fibring associated with the non-trivial class. The Thom complex $T(E)$ of this fibring is of the form $S^n \cup e^{n+3}$, where the attaching map of e^{n+3} is then the generator of $\pi_{n+2}(S^n)$, by (4.5). However this element is detected by φ_{11} (see [2; p. 104]).

Define $e_1 \in H^3(BG(n-1); Z_2)$ to be a representative of the coset $\bar{e}_1$, and identify it with its corresponding class in $H^3(BF; Z_2)$.

THEOREM 5.1. *$H^3(BF; Z_2) \approx Z_2 + Z_2$ with generators w_3 and e_1.*

Proof. $\pi_2(G(n-1)) \approx \pi_{n+2}(S^n) \approx Z_2$ for $n \geqq 3$, so a quick glance at the Postnikov system shows H^3 has at most four elements.

Let $PL(n)$ be the group of piecewise linear homeomorphisms of S^{n-1} into itself and $SPL(n)$ the subgroup of those homeomorphisms of degree 1. We have an inclusion $SPL(n) \subset G(n-1)$ which induces a map of the stable classifying spaces $BSPL \to BG$. A stable sphere fibring ξ over X with classifying map $\chi : X \to BG$ has a stable SPL-structure if χ lifts to $\bar{\chi} : X \to BSPL$.

In [15] it is shown that $H^3(BSPL; Z_2) \cong H^3(BSO; Z_2)$ and hence e_1 is an obstruction to an SPL-structure; in fact it is the first obstruction. From the above, the sphere fibrings $S^{n-1} \to E_0 \to S^3$ do not admit SPL-structures. Notice that they are the only non-trivial sphere fibrings over S^3.

In particular we have

THEOREM 5.2. *Consider the non-trivial sphere fibring $\xi = (S^2, E_0, S^3)$. Then E_0 satisfies Poincaré duality over the integers, but it is not of the homotopy type of an SPL-manifold.*

Proof. It is easy to see that E_0 satisfies Poincaré duality over the integers. From the above, ξ does not have an SPL-structure. Let η be a $(k-1)$-sphere fibring which is an inverse to ξ, i.e. the Whitney join of ξ and η is a product, or equivalently the classifying maps are inverses in $\pi_3(BF)$. Then η does not have an SPL-structure. Moreover, since $w_2(\xi) = 0$, $e_1(\xi) = T^{-1}\varphi_{11}(U)$, where U is the Thom class of $T(\xi)$. By Whitney duality, $w_2(\eta) = 0$ and therefore $e_1(\eta)$ is defined as $e_1(\eta) = T^{-1}\varphi_{11}(V)$, where V is the Thom class of $T(\eta)$. Now, as in [3; (3.2)], one can show that $e_1(\eta) = e_1(\xi)$. Let $\pi : E_0 \to S^3$ and consider the induced fibring $\pi^*(\eta)$. Since π has a cross-section [$\pi_2(F(n-1))$ maps onto $\pi_2(G(n-1))$], $\pi^* : H^3(S^3; Z_2) \cong H^3(E_0; Z_2)$. By naturality $\pi^*(e_1(\eta)) = e_1(\pi^*\eta) \neq 0$, hence $\pi^*\eta$ does not have an SPL-structure.

We now show that $T(\pi^*\eta)$ has top class stably spherical. Let $\xi * \eta$ be the Whitney join of ξ and η and let $E_0(\xi * \eta)$ be its total space. Similarly let $E_0(\pi^*\eta)$ be the total space of $\pi^*\eta$ and define a map $g : E_0(\xi * \eta) \to SE_0(\pi^*\eta)$, where $SE_0(\pi^*\eta)$ is the suspension of $E_0(\pi^*\eta)$, by $g(x, t, y) = (t, x, y)$, Now we have the following homotopy equivalences, $E_0(\xi * \eta) \approx S^3 \times S^{k+2}$ and $SE_0(\pi^*\eta) \approx SE_0 \vee T(\pi^*\eta)$. Furthermore it is easy to see that

g maps the top cells with degree 1. Now $SE_0\,(\xi * \eta) = S^4 \vee S^{k+3} \vee S^{k+6}$ and Sg induces then $g' : S^{k+6} \to ST(\pi^*\eta)$ of degree 1, so $T\,(\pi^*\eta)$ has top class stably spherical. The result (5.2) now follows from a theorem of Spivak [10; Th. A] which states that if X satisfies Poincaré duality, then there exists a sphere fibring ν over X, such that $T(\nu)$ has top class stably spherical and the stable class of ν is unique. If E_0 were of the homotopy type of an SPL-manifold, the stable class of ν would be the stable normal bundle of E_0 and hence would have an SPL-structure. But $\pi^*\eta$ has top class stably-spherical, hence is a representative of ν, but does not have an SPL-structure.

REFERENCES

1. J. F. Adams: On the non-existence of elements of Hopf invariant one, *Ann. Math.* **72** (1960), 20–104.
2. J. Adem: Sobre operaciones cohomologicas secundarias, *Bol. Soc. Mat. Mex.* **7** (1962), 95–110.
3. J. Adem and S. Gitler: Secondary characteristic classes and the immersion problem, *Bol. Soc. Mat. Mex.* (1965) 53–78.
4. A. Dold and R. Lashof: Principal quasifibrations and fibre homotopy equivalence of bundles, *Illinois J. Math.* **3** (1959), 285–305.
5. W. Massey and F. Peterson: The cohomology structure of certain fibre spaces *I*, *Topology* **4** (1965), 47–66.
6. J. P. Meyer: Functional cohomology operations and relations, *Amer. J. Math.* **87** (1965), 649–683.
7. J. Milnor: Lectures on characteristic classes, mimeographed notes, Princeton University, Princeton, N.J.
8. J. Milnor: On the *J*-homomorphism, *Bull. Am. Math. Soc.* **64** (1958), 79–82.
9. J. Milnor: Characteristic classes for spherical fibrations (to appear).
10. M. Spivak: Spaces satisfying Poincaré duality (to appear).
11. J. D. Stasheff: A classification theorem for fibre spaces, *Topology* **2** (1963), 239–246.
12. N. E. Steenrod: Cohomology invariants of mappings, *Ann. Math.* **50** (1949) 954–988.
13. H. Toda: *p*-primary components of homotopy groups, IV, *Mem. Coll. Sci. Kyoto Univ., Ser. A* **32** (1959), 297–332.
14. G. W. Whitehead: A generalization of the Hopf invariant, *Ann. Math.* **51** (1950), 192–237.
15. R. E. Williamson: Cobordism of combinatorial manifolds (to appear).

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