MORSE-NOVIKOV THEORY, HEEGAARD SPLITTINGS AND CLOSED ORBITS OF GRADIENT FLOWS

HIROSHI GODA, HIROSHI MATSUDA, AND ANDREI PAJITNOV

ABSTRACT. The works of Donaldson [2] and Mark [14] make the structure of the Seiberg-Witten invariant of 3-manifolds clear. It corresponds to certain torsion type invariants counting flow lines and closed orbits of a gradient flow of a circle-valued Morse map on a 3-manifold. We study these invariants using the Morse-Novikov theory and Heegaard splitting for sutured manifolds, and make detailed computations for knot complements.

1. INTRODUCTION

Let $K \subset S^3$ be an oriented knot, put $C_K = S^3 - K$. The canonical cohomology class $\xi \in H^1(C_K) = [C_K, S^1]$ can be represented by a Morse map $f : C_K \to S^1$. In this paper we study the dynamics of the gradient flow of f.

Milnor pointed out in [16] a relationship between the Reidemeister torsion and dynamical zeta functions. His theorem applies to fibred knots, that is to the case when we can choose the map f without critical points. The theorem implies in particular that the Alexander polynomial of any fibred knot in S^3 is essentially the same as the Lefschetz zeta function of the monodromy map of the fibration f. The periodic points of the monodromy map correspond to the closed orbits of the gradient flow of the fibration $C_K \to S^1$; thus Milnor's theorem establishes a relation between the dynamics of this gradient flow and and the Alexander polynomial of the knot.

When the knot K is not fibred, the Morse map f necessarily has critical points. The Milnor's formula is no more valid, however it can be generalized to this case at the cost of adding a correction term, as it was discovered by Hutchings and Lee ([11], [12]). This correction term is essentially the torsion of the Novikov complex associated with the circle-valued Morse map f (see [18], [20]). This complex is an analog of the Morse complex for the circle-valued case, and is obtained through counting the flow lines of the gradient joining the critical points of the map.

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The torsion of the Novikov complex and the Lefschetz zeta function are in general very difficult to compute due to the complexity of the transversal gradient flows used in the construction of the Novikov complex. In the paper [14], Mark introduced a new class of gradient flows for circle-valued Morse maps (*symmetric flow*), which are not transversal but, somewhat unexpected, the Morse-Novikov theory can be extended to this case. He used these flows to give a yet another proof of the Meng-Taubes theorem (see the original paper of Meng and Taubes [15] and the later works of Turaev [24] and Donaldson [2] for alternative proofs of the theorem).

The symmetric flows have a simple geometric structure allowing to carry over to this setting a large part of the Morse-Novikov theory, and on the other hand to perform explicit computations with these flows. This is the main aim of the present paper. We begin by studying the geometric properties of symmetric gradients (we work actually with a slightly wider class of vector fields called *half-transversal gradients*), and establish the basic theorem of the Morse-Novikov theory for this class of flows. This theorem is valid in a more general context than the Mark's results, and we believe that our proof is simpler.

Then we proceed to detailed study of the geometry of the Morse map f. In the case when f is a fibration the first return map from a regular fiber to itself is a diffeomorphism, called *the monodromy of the fibration*; this is the basic notion which helps to understand the dynamics of the gradient flow. We generalize this notion to the case when f has critical points. Our monodromy is a diffeomorphism of two surfaces constructed from a Heegaard splitting for the complement of a knot [6] (we recall the basic notions of the theory of Heegaard splittings in Section 5). This diffeomorphism depends on the choice of the gradient, however it can be efficiently computed in particular cases, which leads to the computation of the Lefschetz zeta function of certain symmetric gradients for the twist knots and the pretzel knot of type $(5, 5, 5)$. The monodromy enables us also to compute the determinant of the boundary operator in the Novikov complex for the case of these knots (the so-called *Novikov torsion*).

The dynamics of the gradient flows of circle-valued Morse maps are closely related to the Seiberg-Witten invariants of 3-manifolds. Meng and Taubes [15] showed that the Seiberg-Witten invariant of any closed 3-manifold M with $b_1(M) \geq 1$ can be identified with the Milnor torsion. Fintushel and Stern [3] proved that for any knot K in S^3 the Seiberg-Witten invariant of the manifold $M \times S^1$, where M is the result of the zero-surgery on K, equals the Alexander polynomial of K multiplied by a certain standard factor. In [2], Donaldson gives a new proof of the Meng-Taubes theorem by applying the ideas from Topological Quantum Field Theory. These TQFTs were used by Mark to prove a conjecture of Hutchings-Lee concerning the relation of the

Seiberg-Witten invariants with the Novikov torsion. Some results in this paper have been announced in [10].

2. Half-transversal flows

Let $f : M \to S^1$ be a Morse function on a closed manifold M. The dynamics of the gradient flow of f is best understood when f does not have critical points. In this case we choose a regular surface for f, and the dynamics of the gradient flow is determined by the first return map of this surface to itself. This map is called *the monodromy* of the gradient flow. If f has critical points the situation is much more complicated since for every transversal f-gradient the first return map is not everywhere defined. It turns out however that in the case of 3-dimensional manifolds there is an important class of non-transversal gradient flows for which the first return map determines a self-diffeomorphism of the level surface. We will first give a definition of the corresponding class of gradient flows on cobordisms.

Let Y be a 3-dimensional cobordism; denote $\partial_- Y$, $\partial_+ Y$ the lower, respectively the upper boundary of Y. Let $\psi : Y \to [a, b]$ be a Morse map without critical points of indices 0 and 3. The subset U_1 of all points x in the upper boundary $\partial_+ Y$ such that the $(-v)$-trajectory starting at x reaches the lower boundary $\partial_- Y$ is open in $\partial_+ Y$ and the gradient descent determines a diffeomorphism $(-v)^{\leadsto} : U_1 \xrightarrow{\approx} U_0$ of U_1 onto an open subset $U_0 \subset \partial_- Y$.

For two critical points p, q of f we call a *flow line of v from q to p* an integral curve γ of v such that

$$\lim_{t \to -\infty} \gamma(t) = q, \qquad \lim_{t \to \infty} \gamma(t) = p.$$

We shall identify two flow lines of v which are obtained from each other by a reparameterization.

Definition 2.1. A ψ-gradient v is called a *smooth descent gradient* if

(i) the number of critical points of index 1 is equal to the number of critical points of index 2, and they can be arranged in two sequences

$$S_1(\psi) = \{p_1, \ldots, p_k\}, \quad S_2(\psi) = \{q_1, \ldots, q_k\}$$

in such a way that for every i there are two flow lines of v joining q_i with p_i and these $2k$ flow lines are the only flow lines of v.

(ii) the map $(-v)^{\leadsto} : U_1 \to U_0$ can be extended to a C^∞ map $g : \partial_+ Y \to \partial_- Y$. *

* It seems to us that the point i) actually follows from ii), but we can not prove it at present.

Now let us return to circle-valued Morse maps. Let $f : M \to S^1$ be such a map, where M is a 3-dimensional closed manifold and v be an f-gradient. Cutting M along a regular surface S of f we obtain a cobordism Y, a Morse function $\psi : Y \to [0, 1]$ and a ψ-gradient $\bar{v} = v|Y$.

Definition 2.2. The f-gradient v is called *half-transversal* if there is a regular level surface S such that $\bar{v} = v \mid Y$ is a smooth descent gradient of $\psi = f \mid Y$ and we have the following transversality condition for stable and unstable manifolds:

$$\mathcal{W}^{st}(q) \pitchfork \mathcal{W}^{un}(p) \tag{2.1}$$

for every critical points p, q of f with $\operatorname{ind} q = 2, \operatorname{ind} p = 1$.

It is not difficult to show that the subset of all half-transversal gradients is dense in the set of smooth descent gradients.

Definition 2.3. Let v be a half-transversal gradient for a Morse function $f : M \to S^1$ and S be the corresponding level surface of f. The first return map for $(-v)$ determines a diffeomorphism of S to itself which will be called *the monodromy* of the flow generated by v, and denoted by g.

The notion of half-transversal gradient, introduced above originates from the paper of T. Mark [14] where the class of *symmetric flows* was introduced. In our terminology Mark's symmetric gradient on a cobordism Y is a smooth descent gradient with the following additional restriction: there is an involution $I : Y \to Y$ swapping the lower and upper boundaries of Y and such that $I_*(v) = -v$ and $\psi \circ I$ equals $-\psi$ up to an additive constant. We do not know if the class of smooth descent gradients is really wider than Mark's class of symmetric gradients. However the existence of the involution I seems restrictive and we prefer to work with more general notion of smooth descent gradients.

Now we will define the Novikov complex and the Lefschetz zeta function for half-transversal gradient flows. The usual procedure of counting flow lines yields the Novikov incidence coefficient

$$N(q_i, p_j; v) = \sum_{k \in \mathbf{N}} n_k(q_i, p_j; v) t^k \qquad \in \mathbf{Z}[[t]]$$

where

$$n_k(q_i, p_j; v) = \sum_{\gamma \in \Gamma_k(q_i, p_j; v)} \varepsilon(\gamma)$$

(here $\Gamma_k(q_i, p_j; v)$ stands for the set of all flow line of $(-v)$ joining q_i with p_j and $\varepsilon(\gamma)$ is the sign attributed to each flow line with respect to the choice of orientations of the 2-dimensional stable manifolds). The Novikov incidence coefficients form a square matrix D with entries in $\mathbf{Z}[[t]]$. The chain complex

$$0 \longleftarrow \mathcal{N}_1^- \xleftarrow{D} \mathcal{N}_2^- \longleftarrow 0 \tag{2.2}$$

where $\mathcal{N}_i^-$ is the free $\mathbf{Z}[[t]]$-module generated by critical points of f of index i is called the *positive Novikov complex* of the pair (f, v) and denoted by $\mathcal{N}_*^-(f, v)$ or simply $\mathcal{N}_*^-$ if no confusion is possible. The chain complex

$$0 \longleftarrow \mathcal{N}_1 \xleftarrow{D} \mathcal{N}_2 \longleftarrow 0 \tag{2.3}$$

where $\mathcal{N}_i$ is the free $\mathbf{Z}((t))$-module generated by critical points of f of index i is called the *Novikov complex* of the pair (f, v) and denoted by $\mathcal{N}_*(f, v)$ or simply $\mathcal{N}_*$ if no confusion is possible. The first of the two chain complexes above is more convenient in computations, however only the homotopy type of the second one is a homotopy invariant of the map $f : M \to S^1$ (see Theorem 3.1).

Definition 2.4. The element $\det D \in \mathbf{Z}[[t]]$ is called *the Novikov torsion* of the pair (f, v) and denoted by $\tau(f, v)$ or τ_g.

Proceeding to the Lefschetz zeta functions, we will need to impose one more restriction on the gradient flow.

Definition 2.5. Let $f : M \to S^1$ be a Morse function on a closed manifold M and v an f-gradient. We say that v is *of finite dynamics* if for every $n \in \mathbf{Z}$ the set of all closed orbits γ satisfying $f_*([\gamma]) = n \in H_*(S^1)$ (where $[\gamma] \in H_1(M)$ is the homology class of γ) is finite.

For a half-transversal f-gradient of finite dynamics we can define the *dynamical Lefschetz zeta function of* $(-v)$:

$$\zeta_{-v}(t) = \exp\Big(\sum_{\gamma} \frac{\varepsilon(\gamma)}{m(\gamma)} t^{m(\gamma)}\Big)$$

where the sum is extended over the set of all closed orbits γ of $(-v)$, $\varepsilon(\gamma)$ is the Poincaré index of γ, and $m(\gamma)$ is the multiplicity of γ. It is clear that ζ_{-v} is equal to the Lefschetz zeta function of the diffeomorphism g:

$$\zeta_g(t) = \exp\Big(\sum_{n \geq 1} \frac{L(g^n)}{n} t^n\Big) \tag{2.4}$$

where $L(g^n)$ is the graded trace of the homomorphism induced by g in the homology.

Let us now define the class of gradient flows with which we will be working in this paper.

Definition 2.6. Let M be a three-dimensional closed manifold, and $f : M \to S^1$ a Morse function without critical points of indices 0 or 3; let v be a half-transversal f-gradient of finite dynamics. We say that (f, v) is a *regular Morse pair.*

We will also work with Morse functions $f : M \to S^1$ on manifolds with boundary. The definition of the regular Morse pair (f, v) is carried over to this setting in an obvious way, with the following modifications:

(1) The restriction $f \mid \partial M : \partial M \to S^1$ is required to be a fibration whose monodromy is isotopic to identity.
(2) The gradient vector field v is required to be tangent to ∂M. Such gradient is called *a gradient of finite dynamics* if for every $n \in \mathbf{Z}$ the set of all closed orbits γ satisfying $f_*([\gamma]) = n$ is finite.

For a regular Morse pair (f, v) on a 3-dimensional manifold with boundary we define the Novikov complex $\mathcal{N}_*(f, v)$ and the Lefschetz zeta function $\zeta_{-v} \in \mathbf{Z}[[t]]$, which counts the closed orbits of $(-v)$ not belonging to the boundary ∂M.

3. The Novikov complex and the zeta function of half-transversal flows

The attractive feature of half-transversal flows is that the Novikov boundary operators and the Lefschetz zeta function of the gradient flow are accessible here through calculations with homotopical quantities associated with the monodromy. Let M be a closed 3-manifold and (f, v) a regular Morse pair on M. Let $\overline{M}$ denote the infinite cyclic covering of M corresponding to f and $\Delta_*(\overline{M})$ denote the simplicial chain complex of $\overline{M}$. Set $\Lambda = \mathbf{Z}[t, t^{-1}]$ and $\widehat{\Lambda} = \mathbf{Z}[[t]][t^{-1}] = \mathbf{Z}((t))$. Both $\mathcal{N}_*(f, v)$ and $\widehat{\Delta}_*(\overline{M}) = \Delta_*(\overline{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$ are based free finitely generated chain complexes over $\widehat{\Lambda}$. The next theorem asserts in particular that there is a chain equivalence between them. A usual procedure allows to associate to each such equivalence its *torsion*, which is an element in

$$\mathrm{Wh}(\widehat{\Lambda}) = K_1(\widehat{\Lambda})/U$$

where U is the subgroup of all elements of the form $\pm t^n$. The group $\mathrm{Wh}(\widehat{\Lambda})$ is easily identified with the multiplicative group of all power series in $\mathbf{Z}[[t]]$ with first coefficient equal to 1 (see [20] Chapter 13, §4 for details), so we shall consider the torsions as power series with coefficients in $\mathbf{Z}[[t]]$. The next theorem is the main aim of this section.

Theorem 3.1. *Let M be a closed 3-manifold and (f, v) a regular Morse pair on M. There is a chain homotopy equivalence*

$$\phi : \mathcal{N}_*(f, v) \to \Delta_*(\overline{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$$

such that

$$\tau(\phi) = \zeta_{-v}.$$

Observe that this theorem implies the isomorphism

$$H_*(\mathcal{N}_*(f, v)) \approx H_*(\overline{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}.$$

Let us first outline the proof. Lift $f : M \to S^1$ to a Morse function $F : \overline{M} \to \mathbf{R}$. The regular level surface $S \subset M$ (see Definition 2.2) lifts to a regular level surface of F which will be denoted by the same letter S. Denote by S^- the part of $\overline{M}$ lying below S with respect to the function F. We will construct a certain chain complex $\mathcal{Z}_*$ which is free over $\mathbf{Z}[t]$ and computes the homology of S^-. Then we construct an embedding

$$\mathcal{N}_*(f, v) \hookrightarrow \widehat{\mathcal{Z}}_* = \mathcal{Z}_* \underset{P}{\otimes} \widehat{P}, \quad \text{where} \quad P = \mathbf{Z}[t],\ \widehat{P} = \mathbf{Z}[[t]],$$

such that the quotient complex is acyclic and its torsion is equal to the Lefschetz zeta function of $-v$. The schema of the argument resembles that of the papers [12] and [19], however the present case is in a sense simpler, due to a very particular nature of the half-transversal flows.

Proceeding to details, let us first return to the cobordism Y obtained from M by cutting along S. We have naturally arising diffeomorphisms $\psi_+ : \partial_+ Y \to S, \quad \psi_- : \partial_- Y \to S$. Put

$$c_i = \mathcal{W}^{un}(p_i, v) \cap \partial_+ Y.$$

Replacing Y by a diffeomorphic cobordism if necessary, we can always assume that the circles c_i, $1 \le i \le k$ are standardly embedded in $\partial_+ Y$ as shown in Figure 1. They are therefore a part of the standard cellular decomposition of $\partial_+ Y$ which consists of m disjoint circles c_i, and m circles d_i having a common point A. For a subset $X \subset \partial_+ Y$ we denote TX the track of X, that is,

$$TX = \{\gamma(x, t; -v) \mid t \ge 0 \text{ and } x \in X\}.$$

We will now define a filtration $\mathcal{E}^i$ in the cobordism Y. The term $\mathcal{E}^0$ of the filtration contains two points: A and tA. The term $\mathcal{E}^1$ contains $\mathcal{E}^0$ and the following subsets: the circles d_i, c_i, the track TA of the point A, the circles

$$\gamma_i = \mathcal{W}^{un}(q_i) \cup \mathcal{W}^{st}(p_i) \quad \text{for} \quad i \le i \le k$$

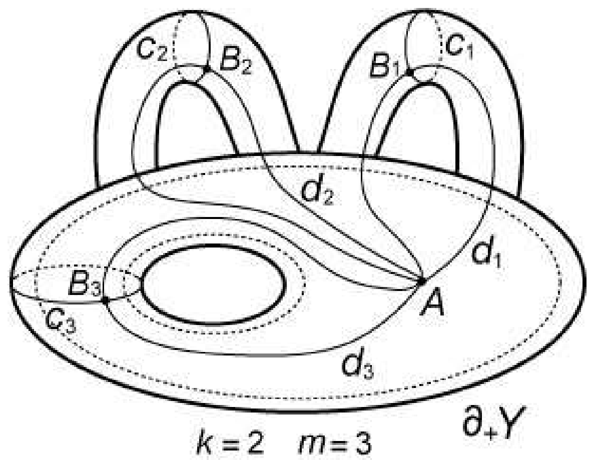

FIGURE 1.

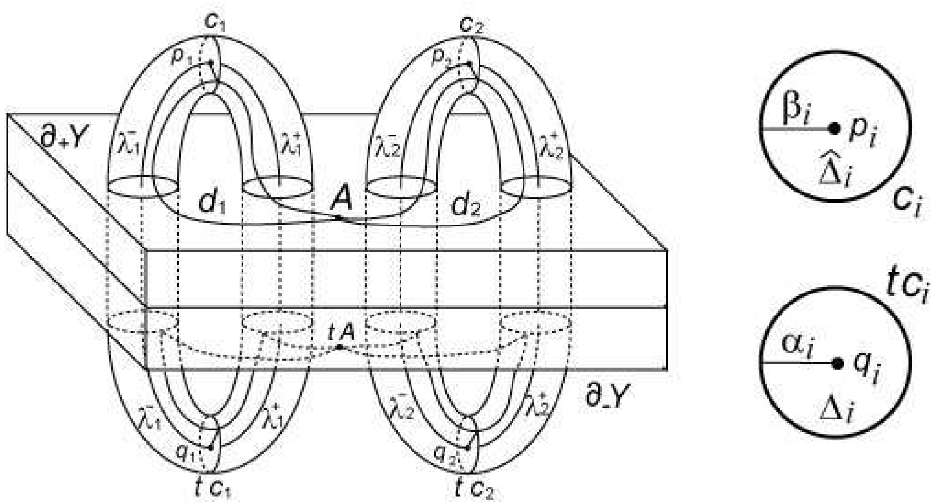

FIGURE 2.

the arcs α_i, β_i as shown in Figure 2, the circles Ic_i, $Id_i \subset \partial_- Y$. The term $\mathcal{E}^2$ contains $\mathcal{E}^1$ and the following subsets: the boundary ∂Y of Y, the stable manifolds of the critical points of index 2 and the unstable manifolds of the critical points of index 1, and the closure of the tracks of c_i and d_i. The term $\mathcal{E}^3$ is the whole Y. It is not difficult to see that $\mathcal{E}^i$ is a cellular filtration of Y, that is, the homology of the quotient $\mathcal{E}^i/\mathcal{E}^{i-1}$ does not vanish only in degree i.

Now we shall use this filtration to explore the homotopy type of the covering $\overline{M}$. The natural map $Y \to M$ lifts to an embedding of Y to $\overline{M}$ whose image will be identified with Y. The covering $\overline{M}$ is the union of the images of Y under the action of $\mathbf{Z}$:

$$\overline{M} = \bigcup_{n\in\mathbf{Z}} t^n Y$$

where t is the downward generator of $\mathbf{Z}$, so that $F(tx) = F(x) - 1$ for every $x \in \overline{M}$. The neighbor copies t^nY and $t^{n+1}Y$ are intersecting by $\partial_- t^nY = t^n\partial_- Y = t^{n+1}\partial_+ Y = \partial_+ t^{n+1}Y$. Recall from Section 2 that the gradient descent determines a diffeomorphism $g : \partial_+ Y \to \partial_- Y$. We endow $\partial_- Y$ with the cellular decomposition induced from $\partial_+ Y$ by g. Let h be any cellular approximation of the map $\psi_+ \circ \psi_-^{-1} : \partial_- Y \to \partial_+ Y$. Then $\overline{M}$ has the homotopy type of the space

$$N = \Big(\bigsqcup_{n\in\mathbf{Z}} t^n Y\Big)\Big/\mathcal{R}$$

where the equivalence relation $\mathcal{R}$ identifies $\partial_- t^nY \approx \partial_- Y$ with $\partial_+ t^{n+1}Y \approx \partial_+ Y$ via the map $h : \partial_- Y \to \partial_+ Y$. The space N has a natural free action of $\mathbf{Z}$ and we have a homotopy equivalence $\overline{M} \to N$ respecting this action. Put

$$N^- = \Big(\bigsqcup_{n\in\mathbf{N}} t^n Y\Big)\Big/\mathcal{R}.$$

We will now use the filtration $\mathcal{E}$ of Y to construct a filtration of N^-. Put

$$\mathcal{F}^i = \bigcup_{n\in\mathbf{N}} t^n \mathcal{E}^i.$$

The filtration $\mathcal{S}_*(\mathcal{F}^i)$ of the singular chain complex $\mathcal{S}_*(N^-)$ of N^- is cellular and the homology

$$H_i(\mathcal{F}^i/\mathcal{F}^{i-1})$$

is a free P-module. Now we will describe the generators of this module. We denote the stable manifold of p_i by $D(p_i; v)$. The set $D(p_i; v) \setminus \{p_i\}$ consists of two arcs, their closures will be denoted by λ_i^+, λ_i^- (the signs correspond to the chosen orientations). Put $\lambda_i = \lambda_i^+ \cup \lambda_i^-$. Let β_i be an arc in Δ_i joining p_i and $B_i = c_i \cap d_i$. Similarly let α_i be an arc joining tA with tB_i. Let d_i' be the part of d_i between A and B_i and denote by χ_i^+ the following composition of arcs

$$\chi_i^+ = d_i' \cdot \beta_i \cdot \lambda_i^+ \cdot \alpha_i \cdot (td_i')^{-1} \quad \text{where } 1 \le i \le k.$$

Similarly, set

$$\chi_i^- = d_i' \cdot \beta_i \cdot \lambda_i^- \cdot \alpha_i \cdot (td_i')^{-1} \quad \text{where } 1 \le i \le k.$$

The fundamental class of $\partial_+ Y$ modulo the union of c_i and d_i is denoted by ω_2. The fundamental class of Y modulo the subspace $\mathcal{E}^2$ is denoted by ω_3. Here is the list of the free generators of $\mathcal{Z}_r = H_r(\mathcal{F}^r/\mathcal{F}^{r-1})$: as a $\mathbf{Z}[t]$-module:

$$\begin{array}{lll}
r=0: & A & \\
r=1: & c_i,\, d_i & \text{for } 1 \le i \le m = \text{ genus}(\partial_+ Y), \\
 & \chi_i^+,\, \chi_i^- & \text{for } 1 \le i \le k. \\
r=2: & \omega_2, & \\
 & \widehat{\Delta}_i,\, \Delta_i,\, Td_i & \text{for } 1 \le i \le k, \text{ and} \\
 & Tc_i,\, Td_i & \text{for } k+1 \le i \le m. \\
r=3: & \omega_3 = T\omega_2. &
\end{array}$$

Here $\widehat{\Delta}_i$ is the unstable manifold of p_i in Y; we have $\partial\widehat{\Delta}_i = c_i$, and similarly for Δ_i. (By a certain abuse of notations we use the same symbol c_i for the cycle and its geometric support; similar convention holds for the other notations.) Now we shall describe the boundary operators in the adjoining complex

$$\partial_r : \mathcal{Z}_r \to \mathcal{Z}_{r-1} :$$

$\partial_1 : \mathcal{Z}_1 \to \mathcal{Z}_0$:

$$\partial(c_i) = 0 = \partial(d_i), \quad \partial(\chi_i^+) = \partial(\chi_i^-) = \partial(TA) = A - th(A).$$

$\partial_2 : \mathcal{Z}_2 \to \mathcal{Z}_1$:

$$\left.\begin{aligned}
\partial(\widehat{\Delta}_i) &= -c_i \\
\partial(\Delta_i) &= th(c_i) \\
\partial(Td_i) &= d_i + \lambda_i - th(d_i)
\end{aligned}\right\} \text{ for } 1 \le i \le k,$$

$$\left.\begin{aligned}
\partial(Tc_i) &= c_i - th(c_i) \\
\partial(Td_i) &= d_i - th(d_i)
\end{aligned}\right\} \text{ for } k+1 \le i \le m, \text{ and}$$

$$\partial(\omega_2) = 0.$$

$\partial_3 : \mathcal{Z}_3 \to \mathcal{Z}_2$:

$$\partial(\omega_3) = \omega_2 - th(\omega_2).$$

The chain complex $\mathcal{Z}_*$ is chain equivalent to the simplicial chain complex of N^-. Any chain equivalence

$$\xi : \mathcal{Z}_* \to \Delta_*(N^-)$$

has a well-defined torsion $\tau(\xi) \in \mathrm{Wh}(\mathbf{Z}[t]) = K_1(\mathbf{Z}[t])/\{\pm 1\}$. This last group vanishes (by the Bass-Heller-Swan theorem), therefore $\tau(\xi) = 0$, and the torsion of the chain equivalence

$$\widehat{\xi} : \widehat{\mathcal{Z}}_* = \mathcal{Z}_* \underset{\mathbf{Z}}{\otimes} \mathbf{Z}[[t]] \to \Delta_*(N^-) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}[[t]]$$

in the group $K_1(\mathbf{Z}[[t]])/\{\pm 1\}$ vanishes. To prove our theorem it suffices therefore to construct a chain equivalence

$$\mathcal{N}_*^- = \mathcal{N}_*^-(f, v) \xrightarrow{\sigma} \widehat{\mathcal{Z}}_*$$

such that $\tau(\sigma) = \zeta_{-v}$. We will embed $\mathcal{N}_*^-$ to $\widehat{\mathcal{Z}}_* = \mathcal{Z}_* \underset{\mathbf{Z}[t]}{\otimes} |\mathbf{Z}[[t]]$ and compute its quotient complex. Let us first observe that the Novikov complex for our half-transversal flow can be expressed in terms of the monodromy g or its homotopy substitute h:

$$\partial q_i = \sum N(q_i, p_j) p_j, \qquad \text{where } N(q_i, p_j) = \sum_{k \in \mathbf{N}} t^k \langle h^k(c_i), c_j \rangle$$

where $\langle \cdot, \cdot \rangle$ stands for the pairing in $H_1(\partial_+ Y)$. Now let us make a simple change of basis † in $\mathcal{Z}_*$ replacing $\widehat{\Delta}_i$ by the element $\widehat{\Delta}_i - \Delta$ which will be denoted by Tc_i (in order to stress the analogy with the tracks of the circles d_i). Extending the map T by linearity to a homomorphism $H_1(\partial_+ Y) \to \mathcal{Z}_2$ it is easy to check the following formula:

$$\partial(T\mu) = \mu - th(\mu) + \sum_j \langle \mu, c_j \rangle \lambda_j. \tag{3.1}$$

Let us now make one more simple change of basis, replacing the cycle Δ_i by

$$\widetilde{\Delta}_i = \Delta_i - \sum_{j=1}^{\infty} t^j T(h^j c_i). \tag{3.2}$$

This infinite sum corresponds geometrically to the stable manifold of the critical point p_i. There is however one essential difference between the formula (3.2) and the similar formulas for the case of the transversal flows (see, for example, formula (66) from [19]). The formula (3.2) contains the term $Tc_i = \widehat{\Delta}_i - \Delta_i$ and similar ones which are not strictly speaking the geometric traces of the cells. An easy computation using the formula (3.1) shows that the homomorphism $\sigma : \mathcal{N}_*^- \to \mathcal{Z}_*$ defined by

$$\sigma(p_i) = \lambda_i, \quad \sigma(q_i) = \widetilde{\Delta}_i$$

† A change of basis is called *simple* if the torsion of the transition matrix vanishes in $\mathrm{Wh}(\mathbf{Z}[[t]]) = K_1(\mathbf{Z}[[t]])/\{\pm 1\}$.

is an embedding of chain complexes. The quotient complex Q_* is also easily computed; here is the list of free $\mathbf{Z}[[t]]$-generators for Q_j:

$$\begin{array}{ll} j=0: & A \\ j=1: & TA,\ c_i,\ d_i,\ \chi_i^+ \\ j=2: & Tc_i,\ Td_i,\ Td_i',\ \omega_2 \\ j=3: & T\omega_3 \end{array}$$

We have $\partial(Td_i') = \chi_i^+$ and

$$\partial(z) = 0, \qquad \partial(Tz) = 1 - th(z)$$

for every z from the following list:

$$A,\ c_i,\ d_i,\ \omega_2.$$

After factoring out the chain complex generated by χ_i^+ and Td_i', we obtain the chain complex of the mapping torus of the map h. It is well known that its torsion equals the Lefschetz zeta function of h (see the classical paper of J. Milnor [16]). This completes the proof of Theorem 3.1. □

Remark 3.2. The theorem above is valid also in the case of regular Morse pairs on manifolds with boundary, and the proof is similar.

4. Novikov torsion and the Alexander polynomial for knots

Theorem 3.1 establishes a relation between two natural geometric objects: the homotopy equivalence $\phi : \mathcal{N}_*(f,v) \to \Delta_*(\overline{M}) \underset{\Lambda}{\otimes} \widehat{\Lambda}$ and the Lefschetz zeta function of the flow generated by v. For computational purposes it is convenient to reformulate it in another way. Let (f,v) be a regular Morse pair on a 3-manifold M (with or without boundary). Let $\mathbf{F}$ be a field.

Definition 4.1. We say that (f,v) is $\mathbf{F}$-*acyclic*, if

$$H_*(\overline{M}) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \mathbf{F}((t)) = 0.$$

Put $\mathcal{N}_*(f,v;\mathbf{F}) = \mathcal{N}_*(f,v) \underset{\widehat{\Lambda}}{\otimes} \mathbf{F}((t))$. It follows from Theorem 3.1 that if (f,v) is $\mathbf{F}$-acyclic, then the homology of the complex $\mathcal{N}_*(f,v;\mathbf{F})$ also vanishes. The images of the elements $\tau(f,v), \zeta_{-v}$ in the ring $\mathbf{F}[[t]]$ will be denoted by $\tau^{\mathbf{F}},\ \zeta^{\mathbf{F}}_{-v}$. The $\mathbf{F}$-acyclicity condition implies that the torsion of the chain complex

$$\widehat{\Delta}^{\mathbf{F}}_*(\overline{M}) = \Delta_*(\overline{M}) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \mathbf{F}((t))$$

is well defined as an element of

$$\mathrm{Wh}(\mathbf{F}((t))) \approx K_1(\mathbf{F}((t)))/U,$$

where U is the subgroup of all elements of the form $\pm t^n$. We will denote this torsion by $\tau_M^{\mathbf{F}}$ omitting in the notation the obvious dependence of this element on the homotopy class of f.

Proposition 4.2. *In the assumptions of Theorem 3.1 assume moreover that (f, v) is $\mathbf{F}$-acyclic. Then*

$$\tau^{\mathbf{F}} \cdot \zeta_{-v}^{\mathbf{F}} = \tau_M^{\mathbf{F}}.$$

Proof. Tensoring by $\mathbf{F}((t))$ the chain equivalence ϕ we obtain a chain equivalence

$$\phi^{\mathbf{F}} : \mathcal{N}_*(f, v; \mathbf{F}) \to \widehat{\Delta}^{\mathbf{F}}(\overline{M})$$

of two acyclic complexes. The torsion of such chain equivalence equals the quotient of the torsions of the complexes. □

Let $K \subset S^3$ be an oriented knot, $M = S^3 \setminus \mathrm{Int}\, N(K)$, and $\mathbf{F} = \mathbf{Q}$. Let (f, v) be a regular Morse pair on M such that the homotopy class $[f] \in H^1(M) \approx [M, S^1] \approx \mathbf{Z}$ is the positive generator of this group. The condition of $\mathbf{Q}$-acyclicity is fulfilled here, so the above proposition is valid. It is well known that in this case the torsion τ_M equals the Alexander polynomial divided by $(1 - t)$ and we obtain the following corollary:

Corollary 4.3. *Let K be a knot in S^3, let $M = S^3 \setminus \mathrm{Int}\, N(K)$ and (f, v) be a regular Morse pair on M. Let τ be the Novikov torsion of (f, v). Then*

$$\tau \cdot \zeta_{-v} = \frac{\Delta_K}{1 - t}$$

where Δ_K stands for the Alexander polynomial of the knot K.

5. Heegaard splitting for sutured manifolds

The notion of a sutured manifold was introduced by Gabai [4]. See also [22]. In this section, we recall the notations and define Heegaard splitting for the sutured manifolds [6].

Definition 5.1. A *sutured manifold* (X, R_+, R_-) is a compact oriented 3-manifold X with ∂X decomposed into the union along the boundary of two connected surfaces $\tilde{R}_+$ and $\tilde{R}_-$ oriented so that $\partial \tilde{R}_+ = \partial \tilde{R}_- = \gamma$ and $\partial X = \tilde{R}_+ \cup \tilde{R}_-$. Let $A(\gamma)$ denote a collection of disjoint annuli comprising a regular neighborhood γ, and define $R_\pm = \tilde{R}_\pm - \mathrm{Int}\, A(\gamma)$. Thus $\partial X = R_+ \cup R_- \cup A(\gamma)$. We regard R_+ as the set of components of $\partial X - \mathrm{Int}\, A(\gamma)$ whose normal vectors point out of X, and R_- as those whose normal vectors point into X. The symbol will denote R_+ or R_- respectively

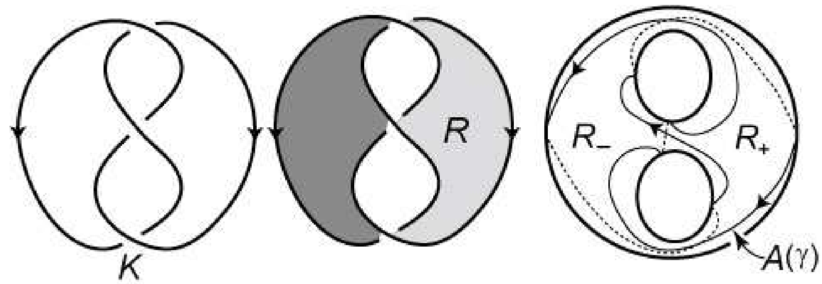

FIGURE 3.

while $R(\gamma)$ denotes $R_+ \cup R_-$. If $\partial\tilde{R}_+ = \partial\tilde{R}_- = \emptyset$, each component of $\tilde{R}_\pm = R_\pm$ is a closed surface.

Let L be a non-split oriented link in a homology 3-sphere, and $\bar{R}$ a Seifert surface of L. Set $R = \bar{R} \cap E(L)$ $(E(L) = \mathrm{cl}(S^3 - N(L)))$. Let P be a regular neighborhood of R in $E(L)$, then P forms $R \times [-1,1]$ where $R = R \times \{0\}$. We denote by $\acute{R}_+$ ($\acute{R}_-$ resp.) $R \times \{1\}$ ($R \times \{-1\}$ resp.), then $(P, \acute{R}_+, \acute{R}_-)$ may be regarded as a sutured manifold. We call $(P, \acute{R}_+, \acute{R}_-)$ a *product sutured manifold for* R. Further, let $X = \mathrm{cl}(E(L) - P)$, and $R_\pm = \acute{R}_\mp$, then we may also regard (X, R_+, R_-) as a sutured manifold. We call (X, R_+, R_-) the *complementary sutured manifold for* R. In this paper, we call this the *sutured manifold for* R for short.

Example 5.2. Let K be the trefoil knot in the 3-sphere S^3 and R the genus 1 Seifert surface as illustrated in Figure 3. The (complementary) sutured manifold for R is homeomorphic to the manifold in the righthandside of the figure. (Note that the 'outside' of the genus 2 surface is the complementary sutured manifold.)

Definition 5.3. A *compression body* W is a connected 3-manifold obtained from a compact surface ∂_-W by attaching 1-handles to $\partial_-W \times \{1\} \subset \partial_-W \times [0,1]$. Dually, a compression body is obtained from a connected surface ∂_+W by attaching 2-handles to $\partial_+W \times \{1\} \subset \partial_+W \times [0,1]$ and 3-handles to any spheres thereby created. If $W = \partial_+W \times [0,1]$, W is called a *trivial* compression body.

We collapse a compression body W, so that we may obtain $\partial_-W \cup$ (arcs), where the arcs correspond to cores of the attaching 1-handles. We say the family of arcs the *spine* of W. We denote by $h(W)$ the number of the attaching 1-handles of W.

Definition 5.4. A pair (W, W') is a Heegaard splitting for a sutured manifold (X, R_+, R_-) if :

(i) both W and W' are compression bodies;
(ii) $W \cup W' = X$;

(iii) $W \cap W' = \partial_+ W = \partial_+ W', \partial_- W = R_+$ and $\partial_- W' = R_-$.

If $\gamma \neq \emptyset$, then $\partial_- W$ and $\partial_- W'$ have boundaries so that $\partial(\partial_- W) \times [0,1] \cup \partial(\partial_- W') \times [0,1] = A(\gamma)$ and $\partial(\partial_+ W) = \partial(\partial_+ W') = \gamma$. This case are treated in [6] and [7]. See also [8] for the concrete examples. We should note that if R_+ is homeomorphic to R_-, we have $h(W) = h(W')$.

Remark 5.5. This Heegaard splitting corresponds to a circle-valued Morse map $M \to S^1$ for a closed orientable 3-manifold M with $b_1(M) > 0$ or the complement of a non-split link in a homology 3-sphere M. In both cases, we suppose that we have a compact surface R as a representative of $H_1(M)$. Then, we obtain the sutured manifold (X, R_+, R_-) from M by cutting along R. So, we have a Heegaard splitting (W, W') of (X, R_+, R_-) as above. See [9] and [21] for the detail.

Definition 5.6. Suppose that R_+ is homeomorphic to R_-. Set $h(X, R_+, R_-) = \min\{h(W)(= h(W')) \mid (W, W') \text{ is a Heegaard splitting for } (X, R_+, R_-)\}$. We call it the *handle number* of (X, R_+, R_-). The *Morse-Novikov number* $\mathcal{MN}$ of (M, R) or (X, R_+, R_-) is the minimal possible number of the critical points of the corresponding Morse map.

Remark 5.7. By Corollary 2.8 in [9], we may see that $\mathcal{MN}(M, R) = 2 \times h(X, R_+, R_-)$.

Definition 5.8. Suppose that (W, W') is a Heegaard splitting of a sutured manifold (X, R_+, R_-), and let λ be a properly embedded arc in W' parallel to an arc in $\partial_+ W'$. Here "parallel" means that there is an embedded disk D in W' whose boundary is the union of λ and an arc in $\partial_+ W'$. Now add a neighborhood of λ to W and delete it from W'. This adds a 1-handle to W (whose core is λ) and also adds a 1-handle to W' (whose cocore is a disk in D). Thus we have again the Heegaard splitting $(\widehat{W}, \widehat{W'})$ of (X, R_+, R_-) where the genus of $\widehat{W}$ ($\widehat{W'}$ resp.) is one greater than W (W' resp.). This process is called a *stabilization* of (W, W').

We may regard a compression body W as a sutured manifold (W, R_+, R_-), that is, we may suppose $\partial_+ W = R_+$ and $\partial_- W = R_-$. A compression body W has a natural Heegaard splitting: A surface S parallel to $\partial_+ W$ splits W into two compression bodies, at least one of them is trivial. Call this the *trivial splitting* of W. A splitting is called *standard* if it is obtained from the trivial splitting by stabilization. In [23], Scharlemann and Thompson proved the next theorem:

Theorem 5.9 ([23])**.** *Every Heegaard splitting of a compression body* (W, R_+, R_-) *with* $\gamma = \emptyset$ *is standard.*

Remark 5.10. In [23], two types of trivial splittings, called 'type 1 and 2', are treated. Here we have only to consider the 'type 1' trivial splitting.

This theorem induces the following theorem. The idea is due to Lei [13].

Theorem 5.11. *Any two Heegaard splittings of the the same sutured manifold with $\gamma = \emptyset$ have a common stabilization.*

Proof. Let (W, W') and (V, V') be Heegaard splitting of a sutured manifold (X, R_+, R_-) with $\gamma = \emptyset$ such that $\partial_- W = R_+$ and $\partial_- V' = R_-$. Let λ_W and $\lambda_{V'}$ be the spines of W and V'. Then, the standard general position argument allows that $N(\partial_- W \cup \lambda_W) \cap N(\partial_- V' \cup \lambda_{V'}) = \emptyset$. We denote by X the sutured manifold with $R_+ = \partial_+ W$ and $R_- = \partial_+ V'$, and let S be a Heegaard splitting surface for X. Then S is also a Heegaard splitting surface for (X, R_+, R_-). Moreover, S becomes a Heegaard splitting surface for the compression bodies $W' = X -$ Int $N(\partial_- W \cup \lambda_W)$ and $V = X -$ Int $N(\partial_- V' \cup \lambda_{V'})$. Hence the Heegaard splitting surface S is a stabilization of both (W, W') and (V, V') by Theorem 5.9. $\square$

As in Remark 5.5, if there is a circle-valued Morse map $f : M \to S^1$, we have a Heegaard splitting (W, W') of the sutured manifold (X, R_+, R_-). We also say that (W, W') is a Heegaard splitting of M or Y. Let $\lambda_W = \cup_i \lambda_W^i$ ($\lambda_{W'} = \cup_i \lambda_{W'}^i$ resp.) be the set of spines of W (W' resp.).

Definition 5.12. A family $(W, W', \lambda_W, \lambda_{W'})$ is called a *symmetric Heegaard splitting* of M if it satisfies the following conditions:

(i) (W, W') is a Heegaard splitting of M;
(ii) there is one to one correspondence between the arcs λ_W^i and $\lambda_{W'}^i$ ($i = 1, \ldots, k$). Further, $\partial \lambda_W^i = \partial \lambda_{W'}^i$ for each i.

Remark 5.13. For a half-transversal gradient flow, we can construct a symmetric Heegaard splitting so that $\cup_i (\lambda_W^i \cup \lambda_{W'}^i)$ are the circles of the half-transversal flow. Conversely, for every symmetric Heegaard splitting $\mathcal{H}$, there is a homeomorphism φ of Y such that $\varphi(\mathcal{H})$ is obtained from a half-transversal gradient flow.

6. Counting closed orbits

In this section, we establish a method to count closed orbits using the idea described in the previous sections.

Let R be compact connected manifold, $g : R \to R$ be a continuous map. Assume that g has only finite number of the critical points. The *Lefschetz number* is defined as follow:

$$L(g) = \sum_{i=1}^{\ell} \mathrm{ind}(x_i),$$

where $\mathrm{ind}(x_i)$ is the index of the fixed point x_i (see [1]). Let G_i be the endomorphism of the homology group $H_i(R)$ induced by g. Then the Lefschetz fixed point theorem asserts the following:

$$L(g) = \sum_i (-1)^i \mathrm{trace}(G_i : H_i(R) \to H_i(R)). \tag{6.1}$$

Let K be a fibred knot in the 3-sphere S^3. Then K has a Seifert surface R and the complement of K is the fiber bundle over S^1 with fiber R. Let $(P, \acute{R}_+, \acute{R}_-)$ be the product sutured manifold for R, and (X, R_+, R_-) the complementary sutured manifold for R. Then (X, R_+, R_-) has also product sutured manifold structure.

The monodromy g induces the transformation matrix $G_i : H_i(R) \to H_i(R)$. We call G_1 the *monodromy matrix* of the fibred knot K. Concretely, we can have a presentation of G_1 as follows. Let $c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_m$ be symplectic basis of $H_1(R)$, where m is the genus of R. (See e.g. [17].) We suppose that $c_i \cdot d_i = 1$ here. Push them off along the normal vector of R, and put them on $\acute{R}_+$ and $\acute{R}_-$. Then we may see that they are basis of $H_1(\acute{R}_+)$ and $H_1(\acute{R}_-)$. Since $R_\pm = \acute{R}_\mp$, we may denote the basis of $H_1(R_+)$ ($H_1(R_-)$ resp.) by $c_1^+, \ldots, c_m^+, d_1^+, \ldots, d_m^+$ ($c_1^-, \ldots, c_m^-, d_1^-, \ldots, d_m^-$ resp.). By using the product structure of (X, R_+, R_-), we push further $c_1^-, \ldots, c_m^-, d_1^-, \ldots, d_m^-$ into R_+, and denote their images in $H_1(R_+)$ by $c_1', \ldots, c_m', d_1', \ldots, d_m'$. Then,

$$\begin{pmatrix} c_1' \\ c_2' \\ \cdot \\ \cdot \\ \cdot \\ d_m' \end{pmatrix} = G_1 \begin{pmatrix} c_1^+ \\ c_2^+ \\ \cdot \\ \cdot \\ \cdot \\ d_m^+ \end{pmatrix}.$$

We show an example here.

Example 6.1. Let K be the trefoil knot and R the Seifert surface as shown in Figure 3. Set c and d as generators of R illustrated in Figure 4. The upper right-hand figure in Figure 4 shows that the sutured manifold (X, R_+, R_-) for R with $c^\pm, d^\pm \subset R_\pm$. This (complementary) sutured manifold X is a product sutured manifold, that is, X is homeomorphic to $R \times [0,1]$ where $R_- = R \times \{0\}$ and $R_+ = R \times \{1\}$. Then we can consider a 'flow' φ_s ($s \in [0,1]$) using this product structure such that $\varphi_s(a) = a \times \{s\} \subset R \times \{s\}$ for a subset a in R_-. $\varphi_s(c^-)$ and $\varphi_t(d^-)$ $(s, t \in (0,1), (s \neq t))$

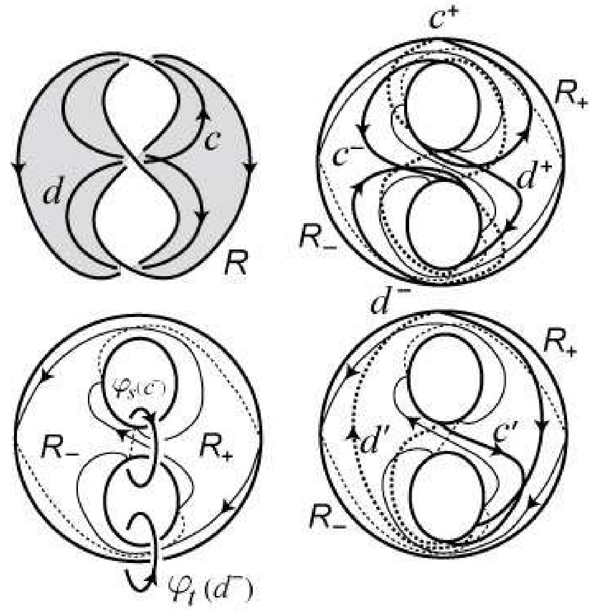

FIGURE 4.

are depicted in the lower left-hand figure in Figure 4, and the lower right-hand figure shows $\varphi_1(c^-)$ and $\varphi_1(d^-)$, denoted by c' and d'. Therefore we can observe that

$$\begin{pmatrix} c' \\ d' \end{pmatrix} = \begin{pmatrix} d^+ \\ -c^+ + d^+ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c^+ \\ d^+ \end{pmatrix}.$$

Thus we have

$$G_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

In this case, we can observe that $\operatorname{trace}(G_0 : H_0(R) \to H_0(R)) = 1$ and $G_2 = 0$. From (2.4) and (6.1), we have :

$$\begin{aligned}
\zeta_g(t) &= \exp\big(\sum_{k=1}^{\infty} \frac{t^k}{k}(1 - \operatorname{trace} G_1^k)\big) \\
&= \exp\big(\log(1-t)^{-1} + \operatorname{trace}(\log(I - t \cdot G_1))\big) \qquad (\,|t| < 1\,) \\
&= \frac{\det(I - t \cdot G_1)}{1-t} \\
&= \frac{1 - t + t^2}{1-t}.
\end{aligned}$$

Here I is the unit matrix. Note that the Alexander polynomial of the trefoil knot is $1 - t + t^2$. In general, if a knot K is fibred, the numerator $\det(I - t \cdot G_1)$ equals the

Alexander polynomial of K. Therefore we have the following well-known theorem. See [16] for example.

Theorem 6.2 ([16]). *Let K be a fibred knot in S^3, and we denote by g the monodromy of K. Then,*

$$\zeta_g(t) = \frac{\Delta_K(t)}{1-t}.$$

Here $\Delta_K(t)$ is the Alexander polynomial of K.

Now let us consider the case of non-fibred.

Let M be a compact orientable 3-manifold with $b_1(M) > 0$. Let $f : M \to S^1$ be a Morse map, and R a regular level surface for f. We obtain a sutured manifold (X, R_+, R_-) from M cutting along R. As pointed out in Remark 5.5 and Definition 5.12, there is a symmetric Heegaard splitting $(W, W', \lambda_W, \lambda_{W'})$ corresponding to f. Set $k = h(W)(= h(W'))$ the number of the attaching 1-handles of W.

According to Definition 2.3, the monodromy g induces the transformation matrix $G_1 : H_1(S) \to H_1(S)$, which can be obtained as follows. We denote the symplectic basis of $H_1(\partial_+ W)$ ($H_1(\partial_+ W')$ resp.) by $c^+_1, \dots, c^+_k, c^+_{k+1}, \dots, c^+_m$, and $d^+_1, \dots, d^+_k, d^+_{k+1}, \dots, d^+_m$ ($c^-_1, \dots, c^-_k, c^-_{k+1}, \dots, c^-_m$, and $d^-_1, \dots, d^-_k, d^-_{k+1}, \dots, d^-_m$ resp.). Here c^+_j and d^+_j (c^-_j and d^-_j resp.) $(j = 1, \dots, k)$ are derived from the attaching 1-handles of W (W' resp.), namely, c^+_j (c^-_j resp.) $(j = 1, \dots, k)$ is a cocore of the attaching 1-handle of W (W' resp.) and d^+_j (d^-_j resp.) $(j = 1, \dots, k)$ is a 'longitude' corresponding to c^+_j (c^-_j resp.), so that $c^+_j \cdot d^+_\ell = \delta_{j\ell} = c^-_j \cdot d^-_j$ $(j, \ell = 1, \dots, m)$. cf. Figure 1. As in the case of a fibred knot, the generators c^+_j, d^+_j and c^-_j, d^-_j $(j = k+1, \dots, m)$ are obtained from corresponding generators of $H_1(R)$ using the half transversal flow associated with f, see Figure 5. Let $c'_1, \dots, c'_m$, $d'_1, \dots, d'_m$ be the images of $c^-_1, \dots, c^-_m$, $d^-_1, \dots, d^-_m$ in $H_1(\partial_+ W)$. Then we may describe:

$$\begin{aligned}&\begin{pmatrix} c'_1 & \cdots & c'_k & c'_{k+1} & \cdots & c'_m & d'_1 & \cdots & d'_k & d'_{k+1} & \cdots & d'_m \end{pmatrix}^T \\ &= G_1 \begin{pmatrix} c^+_1 & \cdots & c^+_k & c^+_{k+1} & \cdots & c^+_m & d^+_1 & \cdots & d^+_k & d^+_{k+1} & \cdots & d^+_m \end{pmatrix}^T.\end{aligned}$$

We call G_1 the *monodromy matrix*. For $n \geq 1$, we have:

$$\begin{aligned}&\Big(g^n_*(c^+_1) \cdots g^n_*(c^+_k)\, g^n_*(c^+_{k+1}) \cdots g^n_*(c^+_m)\, g^n_*(d^+_1) \cdots g^n_*(d^+_k)\, g^n_*(d^+_{k+1}) \cdots g^n_*(d^+_m) \Big)^T \\ &= G^n_1 \begin{pmatrix} c^+_1 & \cdots & c^+_k & c^+_{k+1} & \cdots & c^+_m & d^+_1 & \cdots & d^+_k & d^+_{k+1} & \cdots & d^+_m \end{pmatrix}^T\end{aligned}$$

Here $(\cdot)^T$ stands for the transposition of a matrix.

The monodromy g is an orientation preserving diffeomorphism between surfaces, then $G_1 \in Sp(2m, \mathbb{Z})$, in particular $\det G_1{=}1$. Further R is a closed or once punctured

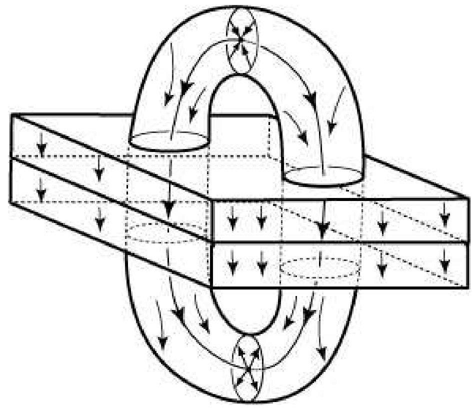

FIGURE 5.

surface in our setting. If R is closed, then $\mathrm{trace}(G_0) = \mathrm{trace}(G_2) = 1$. So, if $|t|$ is sufficiently small,

$$\begin{aligned}\zeta_g(t) &= \exp\Big(\sum_{k=1}^{\infty} \frac{t^k}{k}(2 - \mathrm{trace}\ G_1^k)\Big)\\ &= \exp\Big(\log(1-t)^{-2} + \mathrm{trace}(\log(I - t\cdot G_1))\Big)\\ &= \frac{\det(I - t\cdot G_1)}{(1-t)^2}\end{aligned}$$

If R is a once punctured surface, we have:

$$\zeta_g(t) = \frac{\det(I - t\cdot G_1)}{1-t}$$

by the same argument, if $|t|$ is sufficiently small. Here I stands for the identity matrix.

7. Counting flow lines

In this section, we consider counting gradient flow lines from critical points of index 2 to those of index 1, which are obtained from a circle-valued Morse map $M \to S^1$, according to Section 2.

In our setting, there are only critical points of index 1 and 2, we can observe the torsion $\tau_g(t)$ of the chain complex ((2.3) $0 \longleftarrow \mathcal{N}_1 \overset{D}{\longleftarrow} \mathcal{N}_2 \longleftarrow 0$) as follows.

As in the previous sections, we consider only a monodromy matrix which is obtained from a symmetric Heegaard splitting and a half-transversal flow. The Novikov module $\mathcal{N}_1$ ($\mathcal{N}_2$ resp.) of the pair (f, v) is generated by $S_1(f) = \{p_1, \ldots, p_k\}$ ($S_2(f) = \{q_1, \ldots, q_k\}$ resp.), i.e., the center points of the disk bounded c_i (tc_i resp.) ($i = 1, 2, \ldots, k$). See Figure 1 and 2. Therefore the $i \times (m+j)$th-component of the matrix G_1 stands for the algebraic number of the flow lines between q_i and p_j ($1 \le i, j \le k$).

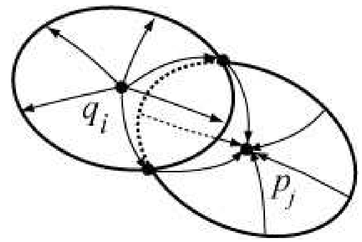

FIGURE 6.

See Figure 6 for the schematic image. Let $D_{ij}^{(n)}$ be the $i \times (m+j)$th-component of G_1^n, $(1 \leq i, j \leq k)$. Then we have:

Definition 7.1. We define

$$\tau_g(t) = \det(D_{ij}(t)), \quad \text{where } D_{ij}(t) = \sum_{n=1}^{\infty} (D_{ij}^{(n)} \cdot t^{n-1}),\ 1 \leq i, j \leq k.$$

If M has no critical points, i.e., M is the fibre bundle over S^1 with fibre R, then $\tau_g(t)$ is defined to be 1.

By taking $|t|$ sufficiently small, we have:

$$\sum_{k=1}^{\infty} G_1^n \cdot t^{n-1} = G_1(I - t \cdot G_1)^{-1}.$$

Therefore, $D_{ij}(t)$ is the $i \times (m+j)$th-component of $G_1(I - t \cdot G_1)^{-1}$, $(1 \leq i, j \leq k)$.

We present the concrete examples for $\tau_g(t)$ in Section 8.

8. EXAMPLES

In this section, we consider twist knots $\mathcal{K}_{2n-1}\,(n = 1, 2, 3, \ldots)$. Note that the Alexander polynomial of $\mathcal{K}_{2n-1}$ is $-n + (2n-1)t - nt^2$. A twist knot has a genus one Seifert surface R_n as illustrated in Figure 7. The twist knot $\mathcal{K}_1$ is the trefoil knot, then it is fibred and treated in Example 6.1. So, we assume that $n \geq 2$.

Let X_n be the complement of the knot $\mathcal{K}_{2n-1}$.

Lemma 8.1. *$\mathcal{MN}(X_n, R_n) = 2$ for any $n\,(n = 2, 3, \ldots)$.*

Proof. Let λ and λ' be arcs whose boundaries are in R_n as illustrated in Figure 8, and (X_n, R_+, R_-) the sutured manifold for R_n. Note that $\partial\lambda = \partial\lambda'$, and R_+ (R_- resp.) intersects λ (λ' resp.) transversely in one point. Then the regular neighborhood of $R_+ \cup \lambda$ and $R_- \cup \lambda'$ in X_n are compression bodies. Therefore we have only to show that the sutured manifold $\mathrm{cl}((X_n, R_+, R_-) - (N(R_+ \cup \lambda) \cup N(R_- \cup \lambda')))$, denoted by

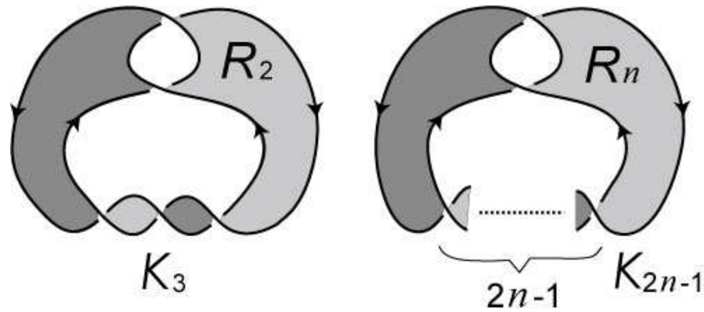

FIGURE 7.

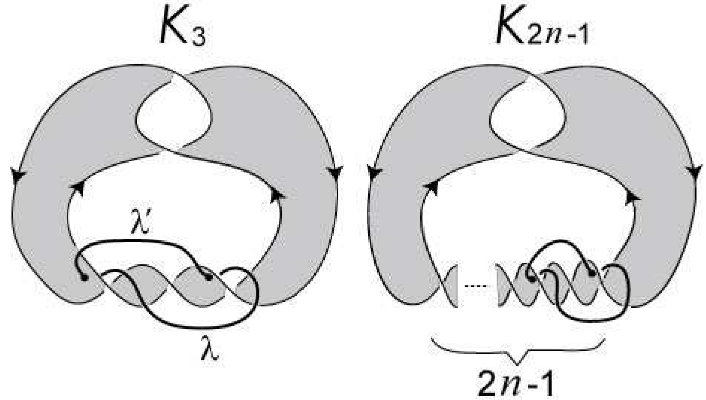

FIGURE 8.

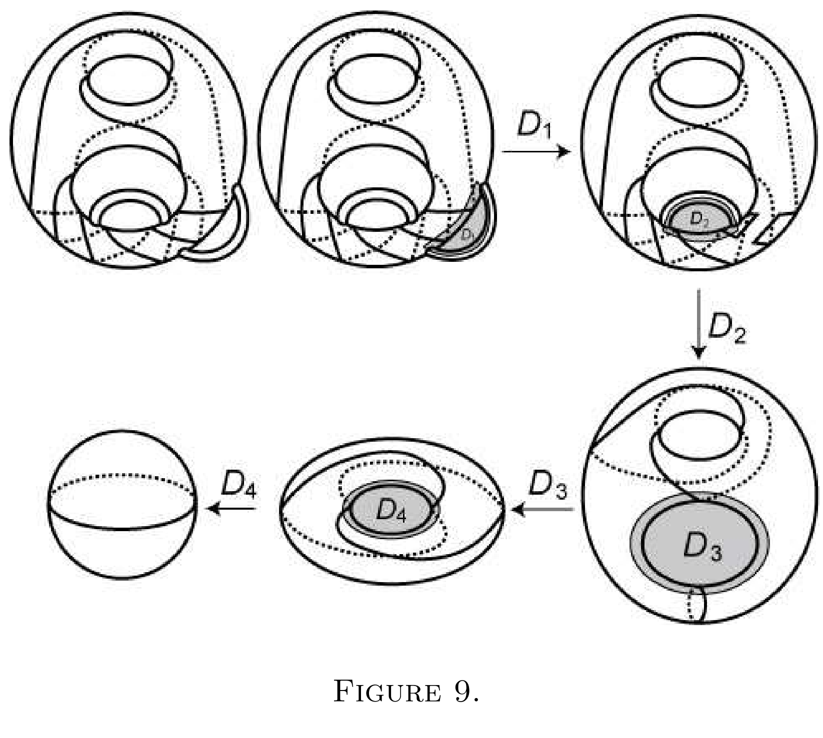

FIGURE 9.

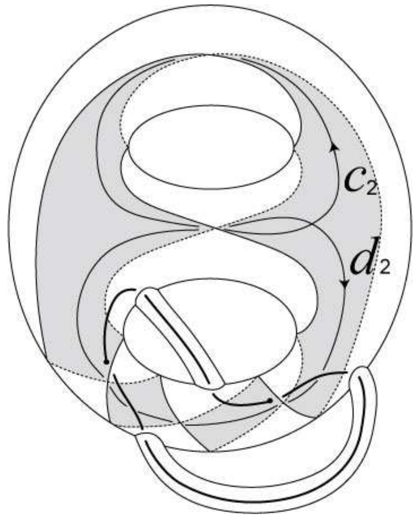

FIGURE 10.

$(\breve{X}_n, \breve{R}_+, \breve{R}_-)$, is a product sutured manifold. We consider the case of $\mathcal{K}_5$ $(n = 3)$ since the other cases can be seen by the same method.

Let D_1 be the product disk in $(\breve{X}_3, \breve{R}_+, \breve{R}_-)$ as illustrated in Figure 9 (shaded part), that is, the disk D_1 is properly embedded disk in $\breve{X}_3$ such that $\partial D_1 \cap \breve{R}_+$ ($\partial D_1 \cap \breve{R}_- resp.$) is an arc properly embedded in $\breve{R}_+$ ($\breve{R}_-$ resp.). We decompose $\breve{X}_3$ along D_1 and connect the suture naturally, then we obtain a new sutured manifold $(\breve{X}_3^1, \breve{R}_+^1, \breve{R}_-^1)$. This decomposition is called a product decomposition [5]. Similarly, we decompose $(\breve{X}_3^1, \breve{R}_+^1, \breve{R}_-^1)$ along the product disk D_2, then we have a sutured manifold $(\breve{X}_3^2, \breve{R}_+^2, \breve{R}_-^2)$. See Figure 9. Thus we have a sequence of the product decompositions:

$$(\breve{X}_3, \breve{R}_+, \breve{R}_-) \stackrel{D_1}{\to} (\breve{X}_3^1, \breve{R}_+^1, \breve{R}_-^1) \stackrel{D_2}{\to} (\breve{X}_3^2, \breve{R}_+^2, \breve{R}_-^2) \stackrel{D_3}{\to} (\breve{X}_3^3, \breve{R}_+^3, \breve{R}_-^3) \stackrel{D_4}{\to} (\breve{X}_3^4, \breve{R}_+^4, \breve{R}_-^4),$$

where $\breve{X}_3^4$ is homeomorphic to the 3-ball and both $\breve{R}_+^4$ and $\breve{R}_-^4$ are disks. This shows that $(\breve{X}_3, \breve{R}_+, \breve{R}_-)$ is a product sutured manifold by [5]. By the same argument, we have that $(\breve{X}_n, \breve{R}_+, \breve{R}_-)$ is a product sutured manifold. This completes the proof. $\square$

We denote by (W_n, W_n') the Heegaard splitting of (X_n, R_+, R_-), which is obtained in the proof of Lemma 8.1.

Lemma 8.2. *The Heegaard splitting (W_n, W_n') is symmetric.*

Proof. Since $\partial\lambda = \partial\lambda'$ and $(\breve{X}_n, \breve{R}_+, \breve{R}_-)$ is a product sutured manifold, we have this lemma. $\square$

For the simplicity, we discuss the case of $\mathcal{K}_3$ $(n = 2)$ in the next lemma. The general case can be obtained by the same method.

Lemma 8.3. *The Heegaard splitting (W_2, W_2') induces a monodromy matrix presented by*

$$G_1 = \begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Moreover, we have

$$\zeta_g(t) = (1-t)^3 \text{ and } \tau_g(t) = \frac{-2+3t-2t^2}{(1-t)^4}.$$

Proof. We take a basis c_2, d_2 of $H_1(R)$ as illustrated in Figure 10, then we have a basis c_2^+, d_2^+ of $H_1(R_+)$ (c_2^-, d_2^- of $H_1(R_-)$ resp.) as in the upper right-hand figure (lower left-hand figure resp.) in Figure 11. Note the positions of λ, λ' and c_2, d_2 in Figure 10. Let $(\breve{X}_2, \breve{R}_+, \breve{R}_-)$ be the sutured manifold $\mathrm{cl}(X_2, R_+, R_-) - (N(R_+ \cup \lambda) \cup N(R_- \cup \lambda'))$ as in the proof of Lemma 8.1. Here we see that c_1^+, c_2^+, d_1^+, $d_2^+ \subset \breve{R}_+$ and c_1^-, c_2^-, d_1^-, $d_2^- \subset \breve{R}_-$. Since $(\breve{X}_2, \breve{R}_+, \breve{R}_-)$ is a product sutured manifold, we can move c_1^-, c_2^-, d_1^-, d_2^- by a free homotopy from $\breve{R}_-$ to $\breve{R}_+$. We denote their images by c_1', c_2', d_1', d_2'. Then we can see that they sit as in the lower right-hand figure in Figure 11. Hence we have: $c_1' = c_1^+ + c_2^+ - 2d_1^+ - d_2^+$, $c_2' = c_2^+ - d_1^+$, $d_1' = d_1^+$, $d_2' = -c_2^+ + d_2^+$. Therefore we have the monodromy matrix G_1 in the statement of this lemma, and we have

$$\zeta_g(t) = \frac{\det(I - t \cdot G_1)}{1-t} = \frac{(1-t)^4}{1-t} = (1-t)^3.$$

Note that the convergence radius is 1.

On the other hand,

$$G_1(I - t \cdot G_1)^{-1} = \begin{pmatrix} \frac{1}{(1-t)} & \frac{1}{(1-t)^3} & \frac{-2+3t-2t^2}{(1-t)^4} & \frac{-1}{(1-t)^2} \\ 0 & \frac{1}{1-t} & \frac{-1}{(1-t)^2} & 0 \\ 0 & 0 & \frac{1}{1-t} & 0 \\ 0 & \frac{-1}{(1-t)^2} & \frac{t}{(1-t)^3} & \frac{1}{1-t} \end{pmatrix}.$$

Thus we have $\tau_g(t) = \dfrac{-2+3t-2t^2}{(1-t)^4}$. □

By the same argument, we have:

Proposition 8.4. *The Heegaard splitting (W_n, W_n') induces a monodromy matrix presented by*

$$\begin{pmatrix} 1 & 1 & -n & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

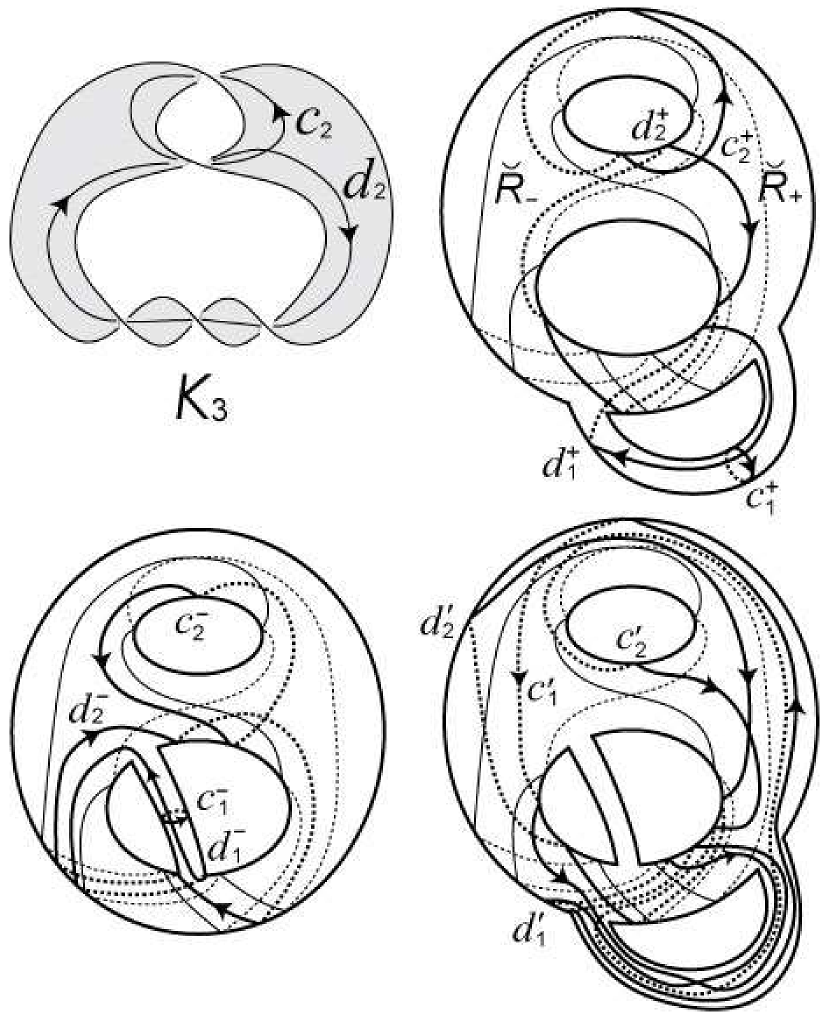

FIGURE 11.

Moreover, we have

$$\zeta_g(t) = (1-t)^3 \text{ and } \tau_g(t) = \frac{-n + (2n-1)t - nt^2}{(1-t)^4}.$$

Example 8.5. Let K be the pretzel knot of type $(5, 5, 5)$ and we consider the symmetric Heegaard splitting associated with Figure 12. Then,

$$G_1 = \begin{pmatrix} 1 & 0 & 1 & -5 & -2 & 0 \\ 0 & 1 & 0 & -3 & -5 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

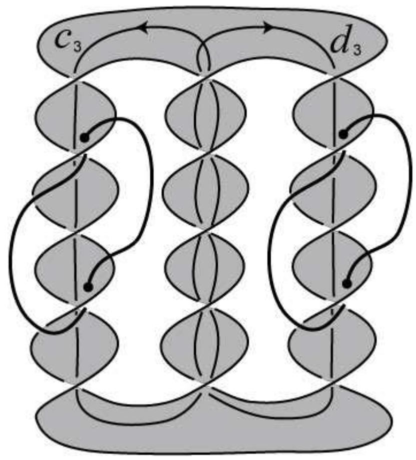

FIGURE 12.

Thus we have $\zeta_g(t) = (1-t)^5$. Further,

$$D_{11}(t) = D_{22}(t) = \frac{-5}{(1-t)^2},\ D_{12}(t) = \frac{-2+3t}{(1-t)^3},\ D_{21}(t) = \frac{-3+2t}{(1-t)^3},$$

$$\tau_g(t) = \frac{19-37t+19t^2}{(1-t)^6}.$$

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Department of Mathematics, Tokyo University of Agriculture and Technology, 2-24-16 Naka-cho, Koganei, Tokyo 184-8588, Japan

E-mail address: goda@cc.tuat.ac.jp

Department of Mathematics, Graduate school of Science, Hiroshima University, Hiroshima 739-8526, Japan

E-mail address: matsuda@math.sci.hiroshima-u.ac.jp

Laboratoire de Mathématiques Jean-Leray UMR 6629, Université de Nantes, Faculté des Sciences, 2, rue de la Houssinière, 44072, Nantes, Cedex, France

E-mail address: pajitnov@math.univ-nantes.fr