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Duality and Finite Spaces

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Abstract. Using that finite topological spaces are just finite orders, we develop a duality theory for sheaves of Abelian groups over finite spaces following closely Grothendieck's duality theory for coherent sheaves over proper schemes. Since the geometric realization of a finite space is a polyhedron, we relate this duality with the duality theory for Abelian sheaves over polyhedra.

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0. Introduction

Finite topological spaces have increasing importance in mathematics. Since they are just finite posets, they correspond with finite distributive lattices; hence their combinatorial importance is clear, as well as in lattice theory. Moreover, each finite space X has a geometric realization $|X|$ which is a triangulated polyhedron and so we get any polyhedron [1, 12]. There is a canonical map $|X| \to X$ and it is a homotopical equivalence [11], so that finite spaces include the homotopical study of triangulated polyhedra and show why classical algebraic topology is essentially of combinatorial nature. Finite spaces have been used in studies on foundations of Homotopy Theory [7]. On the other hand, coherent spaces (spectrums of distributive lattices) are just projective limits of finite spaces and therefore finite spaces provide a cornerstone for the development of a dimension theory for arbitrary topological spaces and locales [9, 13]. Finally, the topological space underlying any Noetherian n-dimensional scheme is a projective limit of n-dimensional finite spaces.

Finite spaces provide a unifying tool in mathematics, relating in a systematic way different areas where combinatorial ideas play a role.

In this paper we develop a duality theory for sheaves of Abelian groups on finite spaces, following closely Grothendieck's duality theory for coherent sheaves on proper schemes ([4]). Since the geometric realization $|X|$ of any finite space X is a polyhedron, we have Verdier's duality theory for sheaves of Abelian groups on $|X|$. We prove that the canonical map $|X| \to X$ induces a quasi-isomorphism between their respective dualizing complexes.

In this paper any topological space is assumed to be T_0.

1. Finite Topological Spaces

DEFINITION 1.1. A *finite space* is a T_0-topological space with a finite number of points, i.e. with a finite number of closed subsets.

It is well-known [10] that a finite space is the same as a finite poset (short for partially ordered set). If X is a finite poset, then we consider a topology on X by letting all hereditary subsets (subsets Z of X such that any point preceding a point of Z belongs to Z) be the closed sets. This topology determines the order, since we have $x \leqslant y$ just when x lies in the closure of y. Conversely, any T_0-topology on a finite set defines an order on it: $x \leqslant y$ if and only if x is in the closure of y. This order determines the given topology because, the set being finite, it is determined once we know the closure of each point. We shall always identify finite spaces with finite posets, using each time the most convenient term for a clear understanding. It is obvious that continuous maps correspond with order-preserving maps ($x \leqslant y$ implies $f(x) \leqslant f(y)$).

1.1. HOMOLOGY

Now, to fix ideas and notation, we shall briefly recall some well-known facts about the homology of finite posets [3].

If X is a finite poset, we denote by $C_n(X, \mathbb{Z})$ the Abelian free group generated by sequences $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n$ of points of X. We define morphisms $d\colon C_n(X, \mathbb{Z}) \to C_{n-1}(X, \mathbb{Z})$ for $n \geqslant 1$ by defining them on the generators:

$$d(x_0 \leqslant \cdots \leqslant x_n) = \sum_{0 \leqslant i \leqslant n} (-1)^i (x_0 \leqslant \cdots \leqslant x_{i-1} \leqslant x_{i+1} \leqslant \cdots \leqslant x_n).$$

It is not difficult to show that $d^2 = 0$, hence we have defined a chain complex $C.(X, \mathbb{Z})$ which is called the *chain complex* of X. Given an Abelian group G, the chain complex of X with coefficients G is defined to be $C.(X, G) = C.(X, \mathbb{Z}) \otimes G$.

The n-th *homology group* of X with coefficients G is the n-th homology group of the chain complex $C.(X, G)$ and it is denoted by $H_n(X, G)$. It is clear that $C.(X, G)$ and $H_n(X, G)$ are both natural in the variables X and G.

Let X, Y be two finite posets. Then the set $\mathrm{Hom}(X, Y)$ of all order-preserving maps $X \to Y$ has a canonical order:

$$f \leqslant g \quad \text{if and only if } f(x) \leqslant g(x) \quad \text{for any point } x \text{ of } X$$

so that $\mathrm{Hom}(X, Y)$ is also a finite space. We say that two continuous maps are *homotopic* when they are in the same connected component of $\mathrm{Hom}(X, Y)$. It is clear that any finite poset with a unique minimal point (or maximal point) is a *contractible* space: it has the homotopy type of a point.

Moreover, the morphism $f_*\colon H_n(X, G) \to H_n(Y, G)$ only depends on the homotopy class of the continuous map $f\colon X \to Y$.

1.2. COHOMOLOGY

Any finite space is, by definition, a topological space. Hence, we have at our disposal the cohomology theory of Abelian sheaves on arbitrary topological spaces due to Grothendieck [6]. Now we recall the most basic facts about it [5, 6]:

A *sheaf* F of Abelian groups on a topological space X assigns to every open subset $U \subseteq X$ an Abelian group $F(U)$ and to every inclusion map $U \subseteq V$ a morphism $r_{UV}: F(V) \to F(U)$ so that

(1) $F(\phi) = 0$.
(2) $r_{UW} = r_{UV} r_{VW}$ whenever $U \subseteq V \subseteq W$.
(3) Given an open cover $\{U_i\}$ of an open subset U and a family of elements $s_i \in F(U_i)$ such that for each pair (i, j) we haver $r(s_i) = r(s_j)$ in $F(U_i \cap U_j)$, there exists a unique $s \in F(U)$ with $s_i = r(s)$ for all index i.

As U varies over all open neighbourhoods of a fixed point x of X, the collection $F(U)$ is a direct system of Abelian groups and the *stalk* of X at x is defined to be

$$F_x = \varinjlim_{x \in U} F(U).$$

A sequence of sheaves of Abelian groups is exact if and only if it is exact on stalks.

Given a topological space X, the cohomology functors $H^p(X, -)$ are defined to be the right derived functors of the additive functor $F \rightsquigarrow F(X)$. This makes sense because $F \rightsquigarrow F(X)$ is left exact and the category of sheaves of Abelian groups on X has enough injectives [6]. These cohomology groups have the general properties of derived functors (long exact cohomology sequence, spectral sequences, etc.). For any sheaf of Abelian groups F on X, the group $H^p(X, F)$ is said to be the p-th *cohomology group* of X with coefficients F.

Let $f: X \to Y$ be a continuous map of topological spaces. For any sheaf F of Abelian groups on X, the *direct image* f_*F is defined by $(f_*F)(V) = F(f^{-1}(V))$ for any open set $V \subseteq Y$. If G is a sheaf of Abelian groups on Y, we may define the *inverse image* f^*G since the functor f_* has a left adjoint f^*:

$$\mathrm{Hom}_X(f^*G, F) = \mathrm{Hom}_Y(G, f_*F).$$

It is well-known that we have $(f^*G)_x = G_{f(x)}$ for any point $x \in X$.

The higher direct image functors R^pf_* are defined to be the right derived functors of the direct image functor f_*.

When f is the projection of X onto a point, we have $F(X) = f_*F$, and f^*G is said to be the *constant sheaf* G on X. Hence, in this case $R^pf_*(F) = H^p(X, F)$.

These results hold for arbitrary topological spaces, but we are interested in a very particular case: when X is a finite space. If X is finite, each point x has a minimal neighbourhood U_x, so that $F_x = F(U_x)$. Hence, a sheaf F of Abelian groups on a finite poset X is just a family of Abelian groups $\{F_x\}$, $x \in X$, and morphisms $r_{yx}: F_x \to F_y$, $x \leqslant y$, such that $r_{zy} r_{yx} = r_{zx}$ whenever $x \leqslant y \leqslant z$ (cf. the linear representations of X, [15]). For example, the constant sheaf G is just $F_x = G$ for any point x and $r_{xy} = \mathrm{id}_G$ for any pair $x \leqslant y$.

When X is a finite space, the cohomology groups $H^p(X, F)$ have the following combinatorial interpretation:

Let F be a sheaf of Abelian groups on a finite poset X. Then we denote by $C^n(X, F)$ the Abelian group

$$C^n(X, F) = \prod_{x_0 \leqslant \cdots \leqslant x_n} F_{x_n}$$

and we define morphisms $d\colon C^n(X, F) \to C^{n+1}(X, F)$ by the following formula:

$$\begin{aligned}&(da)(x_0 \leqslant \cdots \leqslant x_{n+1})\\&\quad= \sum_{0 \leqslant i \leqslant n} (-1)^i a(x_0 \leqslant \cdots \hat{x}_i \cdots \leqslant x_{n+1}) + (-1)^{n+1} \bar{a}(x_0 \leqslant \cdots \leqslant x_n)\end{aligned}$$

where $\bar{a}(x_0 \leqslant \cdots \leqslant x_n)$ is the image of $a(x_0 \leqslant \cdots \leqslant x_n)$ under the morphism, $r\colon F_{x_n} \to F_{x_{n+1}}$. It is easy to check that $0 = d^2$, so that we have a cochain complex $C^\cdot(X, F)$ (see [3]). Now, we have:

THEOREM 1.2 [8]. $H^p(X, F) = H^p[C^\cdot(X, F)]$.

COROLLARY 1.3. *If G is an Abelian group, the groups $H^p(X, G)$ are isomorphic to the cohomology groups of the complex* $\mathrm{Hom}_{\mathbb{Z}}(C_\cdot(X, \mathbb{Z}), G)$.

COROLLARY 1.4. *Constant sheaves over contractible finite spaces are acyclic.*

Finally, we shall need the following result:

DEFINITION 1.5. We define the *dimension* of a finite poset to be the supremum of all integers n such that there exists a properly ascending chain $x_0 < x_1 < \cdots < x_n$ of points of X.

THEOREM 1.6 [6]. *The cohomology groups $H^p(X, F)$ vanish when p is greater than the dimension of the finite space X.*

2. Duality

Now we develop a duality theory for sheaves of Abelian groups on finite spaces, following closely Grothendieck's work [4] for proper maps between schemes and Verdier's work [14] for classical topological spaces.

If F is a sheaf of Abelian groups over a finite space X of dimension n, we denote by $C^\cdot(F)$ the n-th truncation of the canonical flasque resolution of F [5]:

$$C^\cdot(F) = \mathbf{C}^0(F) \xrightarrow{d_1} \mathbf{C}^1(F) \cdots \mathbf{C}^{n-2}(F) \xrightarrow{d_{n-1}} \mathbf{C}^{n-1}(F) \to \mathrm{Coker}\ d_{n-1}.$$

If $\mathscr{K}^\cdot$ is a complex of sheaves over X, we denote by $C^\cdot(\mathscr{K}^\cdot)$ the singly graded complex associated to the double complex $\{C^p(\mathscr{K}^q)\}$.

The duality theory for a continuous map $f: X \to S$ between finite spaces is very simple because of the following facts:

(a) The stalk of $R^i f_*(F)$ at any point p of S is $H^i(f^{-1}(U_p), F)$, where U_p is the minimal neighborhood of p in S. Hence, it vanishes when i is greater than the dimension of X (1.5).
(b) The functor f_* commutes with filtered inductive limits, since finite spaces are obviously Noetherian ([5], II3,10.1).
(c) The functorial resolution $C^{\cdot}(F)$ is bounded, f_*-acyclic and commutes with direct sums and filtered inductive limits, because any direct product which appear in the definition of the canonical flasque resolution is finite when X is finite.

DUALITY THEOREM 2.1. *Let $f: X \to S$ be a continuous map between finite spaces and let $I^{\cdot}$ be an injective resolution of the constant sheaf $\mathbb{Z}$ on S. There is a complex $D_f^{\cdot}$ of injective sheaves over X such that, for every complex $\mathscr{K}^{\cdot}$ of sheaves of Abelian groups over X we have a functorial isomorphism*

$$\mathrm{Hom}^{\cdot}(f_*(C^{\cdot}(\mathscr{K}^{\cdot}))\, I^{\cdot}) = \mathrm{Hom}^{\cdot}(\mathscr{K}^{\cdot}, D_f^{\cdot})$$

where Hom$^{\cdot}$ *denotes the complex of morphisms. Moreover, $D_f^{\cdot}$ is bounded below.*

Proof. Fix two integer numbers p, q and let us consider the contravariant functor which assigns to each sheaf F of Albelian groups over X the Abelian group $\mathrm{Hom}(f_*(C^p(F), I^q))$. This functor is exact (because $C^p(F)$ is f_*-acyclic and I^q is an injective sheaf) and it takes inductive limits into projective limits; hence it is representable [4]. So there is an injective sheaf $D^{-p,q}$ on X and a natural isomorphism

$$\mathrm{Hom}(f_*(C^p(F), I^q)) = \mathrm{Hom}(F, D^{-p,q}).$$

Now let $D_f^d = \oplus_{i+j=d} D^{i,j}$. For each integer number n we have

$$\begin{aligned}
\mathrm{Hom}^n(f_*(C^{\cdot}(\mathscr{K}^{\cdot})), I^{\cdot}) &= \prod_i \mathrm{Hom}\Big(\bigoplus_{p+q=i} f_*(C^p(\mathscr{K}^q)), I^{i+n}\Big)\\
&= \prod_i \prod_{p+q=i} \mathrm{Hom}(f_*(C^p(\mathscr{K}^q)), I^{i+n})\\
&= \prod_q \prod_{p+q=i} \mathrm{Hom}(\mathscr{K}^q, D^{-p,i+n})\\
&= \prod_q \mathrm{Hom}\Big(\mathscr{K}^q, \bigoplus_{p+q=i} D^{-p,i+n}\Big)\\
&= \mathrm{Hom}^n(\mathscr{K}^{\cdot}, D_f^{\cdot})
\end{aligned}$$

and this is a functorial isomorphism. Hence there is a differential $d: D_f^{\cdot} \to D_f^{\cdot}$ of degree 1 such that

$$\mathrm{Hom}^{\cdot}(f_*(C^{\cdot}(\mathscr{K}^{\cdot})), I^{\cdot}) = \mathrm{Hom}^{\cdot}(\mathscr{K}^{\cdot}, D_f^{\cdot})$$

is an isomorphism of complexes.

Remarks. If the dimension of X is n, then $D_f^p = 0$ for $p < -n$.

Let P be the one point space and let $f\colon X \to P$. We define the *dualizing complex* $D_X^{\cdot}$ of X to be $D_f^{\cdot}$.

The complex $D_f^{\cdot}$ depends on the given resolution $I^{\cdot}$. If we choose different injective resolutions of the constant sheaf $\mathbb{Z}$, then we get quasi-isomorphic complexes. Hence the complex $D_f^{\cdot}$ is well-defined up to quasi-isomorphisms.

Note. The above argument may be applied to any bounded below complex $I^{\cdot}$ of injective sheaves on S, so that we get a bounded below complex $f^!(I^{\cdot})$ of injective sheaves on X such that $\mathrm{Hom}^{\cdot}(\mathscr{K}^{\cdot}, f^!(I^{\cdot})) = \mathrm{Hom}^{\cdot}(f_*(C^{\cdot}(\mathscr{K}^{\cdot})), I^{\cdot})$. Hence we have a functor $f^!\colon D^+(S) \to D^+(X)$, between the respective derived categories of bounded below complexes of sheaves of Abelian groups, which is a right-adjoint of the functor $\mathbf{R}F_*\colon D(X) \to D(S)$:

$$\mathbf{R}\,\mathrm{Hom}^{\cdot}(\mathscr{K}^{\cdot}, f^!(L^{\cdot})) = \mathbf{R}\,\mathrm{Hom}^{\cdot}(\mathbf{R}f_*(\mathscr{K}^{\cdot}), L^{\cdot}).$$

COROLLARY 2.2. *If F is a sheaf of Abelian groups on X, then there is a split exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H^{i+1}(X, F), \mathbb{Z}) \to \mathrm{Ext}^{-i}(F, D_X^{\cdot}) \to \mathrm{Hom}(H^i(X, F), \mathbb{Z}) \to 0.$$

Proof. If $I^{\cdot}$ denotes an injective resolution of the group $\mathbb{Z}$, then there is a split exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H^{i+1}(X, F), \mathbb{Z}) \to H^{-i}[\mathrm{Hom}^{\cdot}(\Gamma(C^{\cdot}(F)), I^{\cdot})] \to \mathrm{Hom}(H^i(X, F), \mathbb{Z}) \to 0.$$

Now, by the Duality Theorem we have $\mathrm{Hom}^{\cdot}(\Gamma(C^{\cdot}(F), I^{\cdot}) = \mathrm{Hom}^{\cdot}(F, D_X^{\cdot})$ and $H^{-i}[\mathrm{Hom}^{\cdot}(F, D_X^{\cdot})]$ is just $\mathrm{Ext}^{-i}(F, D_X^{\cdot})$ because $D_X^{\cdot}$ is a complex of injective sheaves.

THEOREM 2.3. *The homology groups of a finite space are just the hypercohomology groups of its dualizing complex*:

$$H_i(X, \mathbb{Z}) = \mathbb{H}^{-i}(X, D_X^{\cdot}).$$

Proof. Let $I^{\cdot}$ be an injective resolution of the group $\mathbb{Z}$. Then

$$\Gamma(X, D_X^{\cdot}) = \mathrm{Hom}^{\cdot}(\mathbb{Z}, D_X^{\cdot}) = \mathrm{Hom}^{\cdot}(\Gamma(X, C^{\cdot}(\mathbb{Z})), I^{\cdot})$$

and, by (1.8), we have that the natural morphisms

$$\Gamma(X, C^{\cdot}(\mathbb{Z})) \to C^{\cdot}(X, C^{\cdot}(\mathbb{Z})) \leftarrow C^{\cdot}(X, \mathbb{Z})$$

are quasi-isomorphisms (recall that $C^{\cdot}(X, G)$ denotes the cochain complex with coefficients G). Therefore, the complex $\Gamma(X, D_X^{\cdot})$ is quasi-isomorphic to $\mathrm{Hom}^{\cdot}(C^{\cdot}(X, \mathbb{Z}), I^{\cdot})$ and this complex is quasi-isomorphic to $\mathrm{Hom}^{\cdot}(C^{\cdot}(X, \mathbb{Z}), \mathbb{Z})$ since $C^{\cdot}(X, \mathbb{Z})$ is a complex of free Abelian groups.

Now, it is clear that $\mathrm{Hom}^{\cdot}(C^{\cdot}(X, \mathbb{Z})) = C_{\cdot}(X, \mathbb{Z})$ and we conclude that $\mathbb{H}^{-i}(X, D_X^{\cdot}) = H^{-i}[\mathrm{Hom}^{\cdot}(C^{\cdot}(X, \mathbb{Z}), \mathbb{Z})]$ is $H_i(X, \mathbb{Z})$.

3. Geometric Realization

Given a finite poset X, we may define a simplicial complex $\mathbf{C}(X)$ on the vertex set X by letting all finite chains be the faces. The geometric realization of $\mathbf{C}(X)$ will be denoted by $|X|$ and it is said to be the *geometric realization* of X (see [1], [11] and [12]). There is a functorial continuous map

$$g: |X| \to X$$

which sends each point of $|X|$ to the greatest element in its carrier chain. This map is a homotopy equivalence [11].

THEOREM 3.1. *If F is a sheaf of Abelian groups on a finite space X, the inverse image*

$$g^*: H^p(X, F) \to H^p(|X|, g^*F)$$

is an isomorphism for all $p \geqslant 0$.

Proof. It is a well-known result for constant sheaves. So it is also true when $F = \mathbb{Z}_A$, for any closed subset $A \subseteq X$, because $g^{-1}(A)$ is just $|A|$. The exact cohomology sequences associated to the following short exact sequences

$$0 \to \mathbb{Z}_{X-A} \to \mathbb{Z} \to \mathbb{Z}_A \to 0$$

$$0 \to g^*(\mathbb{Z}_{X-A}) \to g^*(\mathbb{Z}) \to g^*(\mathbb{Z}_A) \to 0$$

prove that it is also true when $F = \mathbb{Z}_{X-A}$ (see [5] for a definition of these sheaves). Finally, remark that the theorem is valid for a direct sum whenever it holds for each summand.

Now, let F be an arbitrary sheaf of Abelian groups on X. By ([5], II 2.9.2), F is a quotient of a direct sum of sheaves $\mathbb{Z}_U$, so that F is quasi-isomorphic to a bounded above complex $\mathscr{K}^{\cdot}$ such that the theorem holds for each sheaf $\mathscr{K}^p$. Since we have the following convergent spectral sequences

$$E_1^{p,q} = H^p(X, \mathscr{K}^q) \Rightarrow H^{p+q}(X, F)$$

$$\bar{E}_1^{p,q} = H^p(|X|, g^*(\mathscr{K}^q)) \Rightarrow H^{p+q}(|X|, g^*F)$$

and $g^*: E_1^{p,q} \to \bar{E}_1^{p,q}$ is an isomorphism for any p and q, we get that $g^*: H^n(X, F) \to H^n(|X|, g^*F)$ is an isomorphism.

(3.2) The Duality Theorem 2.2 holds for any continuous map between finite polyhedra (in fact it holds under much more general hypotheses, see [14]). Hence, if P is a finite polyhedron, we have its dualizing complex $D_P^{\cdot}$. Now we compare $D_X^{\cdot}$ and $D_{|X|}^{\cdot}$ when X is a finite space:

THEOREM 3.3. *$D_X^{\cdot}$ and $g_*(D_{|X|}^{\cdot})$ are quasi-isomorphic.*

Proof. Let $I^{\cdot}$ be an injective resolution of the gorup $\mathbb{Z}$. For any sheaf F of Abelian groups on X we have

$$\begin{aligned}\mathrm{Hom}^\cdot(F, g_*(D^\cdot_{|X|})) &= \mathrm{Hom}^\cdot(g^*F, D^\cdot_{|X|})\\ &= \mathrm{Hom}^\cdot(\Gamma(|X|, C^\cdot(g^*F)), I^\cdot)\end{aligned}$$

and, by (3.1), this complex is quasi-isomorphic to

$$\mathrm{Hom}^\cdot(\Gamma(X, C^\cdot(F)), I^\cdot) = \mathrm{Hom}^\cdot(F, D^\cdot_X).$$

Since it holds for an arbitrary sheaf F, we may conclude that $D^\cdot_X$ is quasi-isomorphic to $g_*D^\cdot_{|X|}$.

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