Table 2. The variance of the values of r about 0 is 0·142, which is close to the value 0·182 given by Bartlett's formula.

Table 2. *Correlation between two series of twenty-five readings*

$\|r\|$	Frequency	$\|r\|$	Frequency	$\|r\|$	Frequency
0·00–0·04	8	0·30–0·34	19	0·60–0·64	4
0·05–0·09	16	0·35–0·39	16	0·65–0·69	4
0·10–0·14	14	0·40–0·44	8	0·70–0·74	5
0·15–0·19	9	0·45–0·49	7	0·75–0·79	1
0·20–0·24	9	0·50–0·54	7	0·80–0·84	0
0·25–0·29	8	0·55–0·59	8	0·85–0·89	1

I am greatly indebted to Mr F. J. Anscombe for drawing my attention to the work of Bartlett and of Orcutt & James, as well as for aid in translating the dialect of the meteorologist into that of the statistician.

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On the inversion of circulant matrices

By I. J. GOOD

1. In a recent paper concerning trend elimination (Quenouille, 1949) it is found necessary to invert *matrices of the circulant form*

$$\mathbf{A} = \{a_{j-i}\} \quad (i = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots, n-1),$$

where the suffixes are reduced mod n. We shall call a matrix of this form a 'circulix' since its determinant is known as a 'circulant'. The object of the present note is to describe an easy method for inverting circulices. The method depends on simple formulae which give the eigenvalues of $\mathbf{A}$ in terms of the elements of $\mathbf{A}$ and vice versa. $\mathbf{A}^{-1}$ is, as a matter of fact, the circulix whose eigenvalues are the reciprocals of those of $\mathbf{A}$.

2. It is well known* and easily proved that the circulant $|\mathbf{A}|$ is given by the formula

$$|\mathbf{A}| = \prod_{s=0}^{n-1} \sum_{r=0}^{n-1} a_r \omega^{rs},$$

where $\omega = \exp(2\pi i/n)$, $i = \sqrt{(-1)}$. Now $|\mathbf{A} - \lambda\mathbf{I}|$ is also a circulant, so the eigenvalues of $\mathbf{A}$ are

$$e_s(\mathbf{A}) = \sum_r a_r \omega^{rs} \quad (s = 0, 1, \ldots, n-1).$$

This is the 'simple formula' for the eigenvalues, mentioned above. If $\mathbf{A}$ is real and symmetrical,† the case of interest to Quenouille, then the eigenvalues are real and can be computed, without using complex numbers, from one of the formulae

$$e_s(\mathbf{A}) = \begin{cases} a_0 + 2\sum\limits_{r=1}^{\frac{1}{2}n-1} a_r \cos\dfrac{2\pi rs}{n} + (-1)^s a_{\frac{1}{2}n} & (n \text{ even}), \\ a_0 + 2\sum\limits_{r=1}^{\frac{1}{2}(n-1)} a_r \cos\dfrac{2\pi rs}{n} & (n \text{ odd}). \end{cases}$$

In this case $e_{n-s}(\mathbf{A}) = e_s(\mathbf{A})$.

* See, for example, *Encyc. Math. Wiss.* **1**, 1, p. 43.

† So that, a_r is real and $a_{n-r} = a_r$ $(r = 0, 1, \ldots, n-1)$.

3. It is also known* and easily proved that

$$a_s = \frac{1}{n}\sum_{r=0}^{n-1} e_r(\mathbf{A})\,\omega^{-rs} \quad (s = 0, 1, \ldots, n-1).$$

Thus the elements of A are determined by the eigenvalues in almost the same way as the eigenvalues are determined by the elements.

4. If A and B are circulices (of order n) then so are $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{AB}$ and any polynomial in A. Since

$$\mathbf{A}^{-1} = \sum_{m=0}^{\infty} (\mathbf{I}-\mathbf{A})^m,$$

when the elements of $\mathbf{I}-\mathbf{A}$ are sufficiently small, it follows that $\mathbf{A}^{-1}$ is also a circulix, even if the elements of $\mathbf{I}-\mathbf{A}$ are not small. (This can be more clumsily proved without limiting operations.) The eigenvalues of $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{AB}$, $\mathbf{A}^{-1}$ are respectively

$$e_s(\mathbf{A})+e_s(\mathbf{B}),\quad e_s(\mathbf{A})-e_s(\mathbf{B}),\quad e_s(\mathbf{A})\,e_s(\mathbf{B}),\quad \{e_s(\mathbf{A})\}^{-1}.$$

The first three of these assertions† follow at once from § 2, while the fourth is deducible from the third by putting $\mathbf{B} = \mathbf{A}^{-1}$. (It may be observed in passing that $\mathbf{AB} = \mathbf{BA}$.)

Therefore in order to invert a circulix, A, we may proceed as follows:

(i) Calculate the eigenvalues of A by using § 2.

(ii) Calculate the reciprocals of these eigenvalues.

(iii) This gives the eigenvalues of $\mathbf{A}^{-1}$ and the elements of $\mathbf{A}^{-1}$ may then be calculated by using § 3.

If A is symmetrical and real, then there is a considerable saving of calculation at stage (i), as pointed out in § 2. Similarly, there is a considerable saving at stage (iii), since $e_s(\mathbf{A}^{-1})$ is real and $e_{n-s}(\mathbf{A}^{-1}) = e_s(\mathbf{A}^{-1})$.

5. In the application A is symmetrical and real. Actually it is always possible to reduce the inversion to this case, even if A is complex. For $\mathbf{A}\overline{\mathbf{A}}' = \mathbf{B}$ is always a real symmetrical circulix, where $\overline{\mathbf{A}}'$ is the complex conjugate of the transpose of A. Therefore $\mathbf{B}^{-1}$ can be evaluated by the simplified calculation and $\mathbf{A}^{-1}$ can be deduced from the formula $\mathbf{A}^{-1} = \overline{\mathbf{A}}'\mathbf{B}^{-1}$. Each matrix product here is a circulix product and involves only n scalar products of vectors.

6. If, after stage (i), it is found that one of the eigenvalues is very small, it will become probable that the computation is 'badly conditioned'. Since the eigenvalues are linear in the a_r it is easy to calculate the effect of slight variations in the a_r. The worst possible case would arise if variations in the a_r, small enough to be within the range of experimental error, were sufficient to cause an eigenvalue to vanish. In this case A would be singular or nearly singular. Fortunately this type of case can be detected without finishing the whole calculation.

7. Owing to rounding-off errors in the course of the calculation, the result may not be sufficiently accurate. The result, say C, will at any rate be a reasonable approximation to $\mathbf{A}^{-1}$. A very much better approximation may be obtained by calculating $2\mathbf{C}-\mathbf{AC}^2$, as in a known iterative method (for example, Hotelling (1943)) for inverting any symmetrical matrix. The method, being iterative, is self-checking.

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The effect of non-normality on the z-test, when used to compare the variances in two populations

BY D. J. FINCH, *University College, London*

1. Using the k-statistic technique, F. N. David (1949) has derived approximations to the moments of z when the two estimates of variance involved are based on independent samples from two populations. The only assumptions made regarding the functional form of the two populations are that, for each, the cumulants exist up to any order desired. From a knowledge of these approximate moments, David has discussed the effect of skewness on the distribution of the z criterion, Geary (1947) having previously

* See, for example, H. Weyl (1931, p. 34), where the analogy with Fourier transforms is pointed out.

† The third assertion may also be deduced from the fact that the eigenvectors of a circulix are independent of the circulix, being the columns of the matrix $\{\omega^{rs}\}$.