Inventiones math. 47, 53–69 (1978)

Inventiones mathematicae

On the Signature of a Link

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0. Introduction

In [4], Goeritz described how a quadratic form could be obtained from a regular projection of a knot, and showed that some of the algebraic invariants of this form are invariants of the knot. The signature, however, is not. Later, in [20], Trotter introduced a different notion of quadratic form, defined in terms of an orientable surface spanning the knot, and showed that the signature of this form (now known as the signature of the knot) is a knot invariant. Now Goeritz's form has a natural interpretation in terms of a spanning surface obtained from the given knot projection (see [17] and §2), and the non-invariance of its signature can be regarded as a consequence of the fact that this surface need not be orientable. In the present paper, we show how to define a quadratic form using any spanning surface, which simultaneously generalizes the forms of Goeritz and Trotter, and are able to relate the signature of this form to the signature of the knot. In particular, we obtain a simple algorithm for calculating the signature $\sigma(k)$ of a knot k from any regular projection of k, which expresses $\sigma(k)$ as the signature of the corresponding Goeritz matrix plus a certain 'correction term'.[1] Since the Goeritz matrix is often considerably smaller than any Seifert matrix, (for instance, any torus knot of type $(2, 2n+1)$ has a 1×1 Goeritz matrix), this should be of interest to the practical knot theorist, so for convenience we summarize the algorithm now. First colour the projection of k in chequerboard fashion. To each double point D assign an incidence number $\eta(D) = \pm 1$ as shown in Figure 1.

Let G be the associated symmetric integral matrix of Goeritz (see [4] or §1 below). Divide the double points into two types as shown in Figure 2. (For this we must orient k, but the result is independent of the choice of orientation. Also, it does not matter which strand passes over.)

Define $\mu = \sum \eta(D)$, summed over all double points D of type II. (Note that if the shaded surface is orientable, then there are no double points of type II). The result is then

[1] For another algorithmic approach to the computation of the signature, see Conway's article [3].

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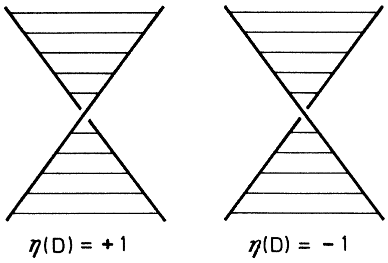

Fig. 1

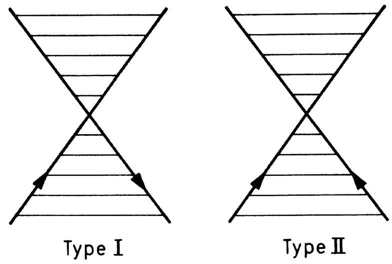

Fig. 2

Theorem 6 [2]. $\sigma(k) = \operatorname{sign} G - \mu$.

We remark that, having somehow arrived at this function $\operatorname{sign} G - \mu$, the fact that it is an invariant of k can be verified directly by showing that it is unchanged by any of the Reidemeister moves on a knot projection [1], [15]. Indeed, the hard part of this was done by Goeritz, viz. describing the effect of these moves on G; keeping track of μ is comparatively simple. It is of some historical interest that, in this sense, the signature of a knot could have made its appearance thirty years before it did.

As the title suggests, everything we do works equally well for links as for knots, but, hoping to clarify the exposition, we treat knots first, in §§1–4, and indicate later, in §5, the modifications necessary for the general case. §1 contains a review of the Goeritz matrix. In §2, we defined our quadratic form $\mathscr{G}_V$ for any surface V spanning the knot or link. In §3, we define an 'Euler number' $e(V)$ for the spanning surface V. Then, by interpreting $\mathscr{G}_V$ as the intersection form of a certain 2-fold branched covering of the 4-ball (this was done for orientable V by Kauffman and Taylor [8]), and using the G-signature theorem, we show that $\operatorname{sign} \mathscr{G}_V + \frac{1}{2} e(V)$ is independent of the choice of V. By taking V to be orientable, we can relate this invariant to the signature of the knot or link. The algorithm described above is derived in §4. (See §5, Theorems 6′ and 6″, for the case of links.) In §6 we outline an alternative approach, avoiding the use of covering spaces, which involves a generalization to arbitrary spanning surfaces of the notion of S-equivalence. As a final illustration of the usefulness of non-orientable spanning surfaces, we give, in §7, a quick proof of a result of Shinohara [18] on the signature of knots contained in knotted solid tori.

Note on Conventions. We have reversed the convention of Goeritz regarding which regions of a knot projection are white and which black; this means that the spanning surface relevant to the discussion is now indicated by the *black* regions. (This convention agrees with Seifert [17].) Also, we have not assigned incidence number zero to double points at which a white region is incident with itself; to compensate for this we have adopted a definition of G which renders the incidence numbers of such points irrelevant. The advantage of this is that it eliminates separate consideration of these points when interpreting μ.

We work throughout in the smooth category.

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2 We are informed that an equivalent formula is known to A. Marin

To avoid repetition, we have used the word 'surface' (except where it appears in the phrase 'closed surface') to mean 'compact surface without closed components'.

Knots, links, and surfaces are unoriented unless otherwise specified, but the 3-sphere, the 4-ball, and, in general, all manifolds of dimensions 3 and 4, will be oriented.

1. The Quadratic Form of a Knot (after Goeritz)

For the convenience of the reader we shall give here the definition of Goeritz's matrix [4]. Let $k \subset S^3$ be a knot, and K (the image of) a regular projection of k onto the plane $R^2 \subset R^3 = S^3 - \{\infty\}$. Colour the regions of $R^2 - K$ alternately black and white. Denote the white regions by $X_0, X_1, \ldots, X_n$. Assign an incidence number $\eta(D) = \pm 1$ to each double point D as in §0. For $0 \leqq i, j \leqq n$ define

$$g_{ij} = \begin{cases} -\sum \eta(D) \text{ summed over double points } D \text{ incident to } X_i \text{ and } X_j, \text{ if } i \neq j \\ -\sum_{k=0,\ldots,n,\, k\neq i} g_{ik} \quad \text{if } i = j. \end{cases}$$

Let $G' = (g_{ij})$, $i,j = 0, \ldots, n$. Then the Goeritz matrix $G = G(K)$ associated to K is the $n \times n$ symmetric integer matrix obtained from G' by deleting the 0-th row and column; i.e. $G(K) = (g_{ij})$, $i,j = 1, \ldots, n$. (We shall use the notation $G(K)$ despite the fact that $G(K)$ depends on more than just K, namely, the ordering of the white regions, and, in particular, the choice of X_0. As regards the latter, the convention of Goeritz is to take X_0 to be the unbounded region; any other choice then corresponds simply to changing the projection by re-siting the point at infinity.)

It follows from [4] that matrices $G(K_1)$ and $G(K_2)$ obtained from any two projections K_1 and K_2 of k are related by a finite sequence of certain moves. The list of moves was subsequently reduced by Kneser and Puppe [10] to the following two.

$$G \mapsto RGR^T, \quad R \text{ integral and unimodular} \tag{1}$$

$$G \mapsto \left[\begin{array}{c|c} G & 0 \\ \hline 0 & \pm 1 \end{array}\right]. \tag{2}$$

In particular, the absolute value of the determinant of G is an invariant of the knot k. Goeritz also showed that the Minkowski units C_p of G at odd primes p are invariants of k. However, the signature sign G is clearly not invariant under move (2), and it is this shortcoming that we seek to rectify. (The Minkowski unit C_2, which is determined by the C_p for odd p together with the signature and the determinant, is also not an invariant of k [4].)

2. Spanning Surfaces

Let $V \subset S^3$ be a surface (orientable or not, but in any case unoriented) with ∂V the knot k. We define a bilinear form $\mathscr{G}_V$: $H_1(V) \times H_1(V) \to \mathbb{Z}$ as follows. Suppose α, $\beta \in H_1(V)$ are represented by 1-cycles a, b. Then $2b$ can be pushed off V into $S^3 - V$, obtaining τb, say. (If b preserves orientation, we push off one copy of b to each side of V.) Define $\mathscr{G}_V(\alpha, \beta)$ to be the linking number of a and τb. More formally, let N be a closed tubular neighbourhood of V; N is the total space of an I-bundle over V with projection π: $N \to V$, say. Let $\tilde{V}$ be the corresponding ∂I-bundle. Then $\pi | \tilde{V}$: $\tilde{V} \to V$ is the orientation double cover if V is non-orientable, or the trivial double cover otherwise. Let τ: $H_1(V) \to H_1(\tilde{V})$ be the transfer map. Since V and $\tilde{V}$ are disjoint subsets of S^3, linking number defines a pairing

$$Lk\colon\ H_1(V) \times H_1(\tilde{V}) \to \mathbb{Z}.$$

We define

$$\mathscr{G}_V(\alpha, \beta) = Lk(\alpha, \tau \beta) \qquad (\alpha, \beta \in H_1(V)).$$

Remarks. (1) If V is orientable, this coincides with the quadratic form defined by Trotter in [20]. For in this case, let i_+, i_-: $V \to \tilde{V}$ be the two sections of $\tilde{V} \to V$; then $\tau \beta = i_{+*} \beta + i_{-*} \beta$, so

$$\begin{aligned}\mathscr{G}_V(\alpha, \beta) &= Lk(\alpha, i_{+*} \beta) + Lk(\alpha, i_{-*} \beta)\\ &= Lk(\alpha, i_{+*} \beta) + Lk(\beta, i_{+*} \alpha).\end{aligned}$$

(2) It will follow from Theorem 3 below that $\mathscr{G}_V$ is symmetric: a direct proof will be given in §6.

We have remarked above that $\mathscr{G}_V$ generalizes Trotter's quadratic form; we now show that it also generalizes Goeritz's. Let K be a regular projection of the knot k, and adopt the notation of §1. Associated to K is a spanning surface $V = V(K)$ of k, namely that indicated by the black regions. A little more precisely, we may assume that k coincides with K except in a neighbourhood of each double point. Then $V(K)$ is built up out of discs and bands. Each disc lies in $S^2 = R^2 \cup \{\infty\}$ and is a closed black region less a small neighbourhood of each adjacent double point. Each double point gives a small, half-twisted band (see Fig. 3).

We can embed a graph Γ in $V \cap S^2$ as a deformation retract of V; Γ has one node in each disc of V, and one edge running across each band (see Fig. 4). Let the complementary regions of Γ in S^2 be $Y_0, \ldots, Y_n$; we number these so that $X_i \subset Y_i$. Each Y_i inherits an orientation from S^2; the homology classes $\alpha_i = [\partial Y_i]$ generate $H_1(\Gamma) \cong H_1(V)$ subject to the single relation $\sum_{i=0}^{n} \alpha_i = 0$.

Theorem 1. *With the above notation,*

$$\mathscr{G}_{V(K)}(\alpha_i, \alpha_j) = g_{ij}.$$

Consequently, $\mathscr{G}_{V(K)}$ has matrix $G(K)$ with respect to the basis $\{\alpha_i | 1 \leqq i \leqq n\}$ of $H_1(V)$.

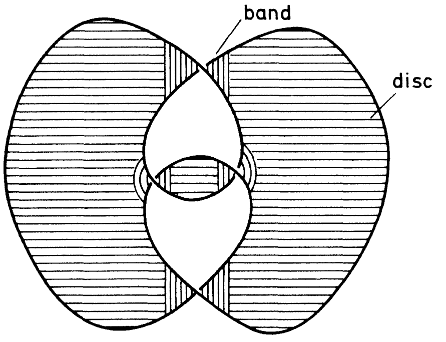

Fig. 3

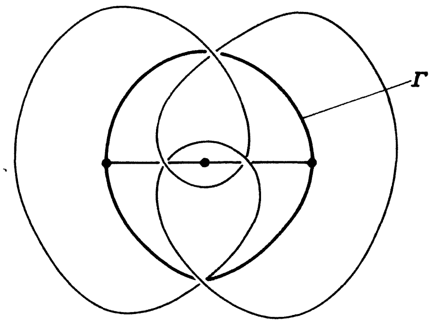

Fig. 4

Proof. Because $\sum_{i=0}^{n} \alpha_i = 0$ and $\sum_{j=0}^{n} g_{ij} = 0$, it suffices to prove the asserted equality for $i \neq j$.

We have

$$\mathcal{G}_{V(K)}(\alpha_i, \alpha_j) = Lk(\alpha_i, \tau\alpha_j) = Y_i \cdot \tau\alpha_j.$$

Now, over the discs of V, $\tilde{V}$ is contained in two parallel copies of S^2, so $\tau\alpha_j$ only intersects S^2 close to double points; in particular it only intersects Y_i close to double points incident to both X_j and X_i. Each such double point contributes one point of $Y_i \cdot \tau\alpha_j$, with sign $+\eta(D)$. (See Fig. 5.)

Hence, $Y_i \cdot \tau\alpha_j = g_{ij}$, as required. □

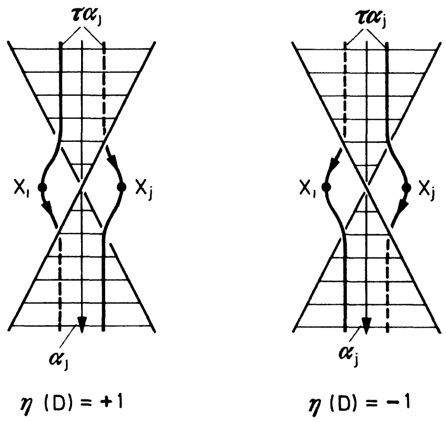

Fig. 5

3. Branched Covering Spaces

Suppose F is a (possibly non-orientable) surface properly embedded in D^4 with ∂F a knot k. Then, by a standard duality argument, $H_1(D^4 - F; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by a meridian of F. Hence we can form the 2-fold branched cover of

D^4 with branch set F; we will denote this by M_F. Since F has a 1-dimensional spine, its normal 1-sphere bundle in D^4 has a section F'. (Note: we shall not distinguish between a section and its image.) Let $k' = \partial F'$, and $e(F) = -Lk(k, k')$. (k' is to be oriented similarly to k; $e(F)$ is then independent of the choice of orientation of k.) We might call $e(F)$ the *Euler number* of F; it is the normal Euler number of a closed surface obtained by capping off k with an orientable surface in S^3 and pushing into int D^4. Using σ to denote the signature of a 4-manifold, we have

Theorem 2. *Suppose F_1 and F_2 are properly embedded surfaces in D^4 with $\partial F_1 = \partial F_2$ a knot. Then*

$$\sigma(M_{F_1}) + \tfrac{1}{2}e(F_1) = \sigma(M_{F_2}) + \tfrac{1}{2}e(F_2).$$

Proof. Consider the pair $(S^4, E) = (D^4, F_1) \cup_\partial (-D^4, F_2)$. This has a 2-fold branched cover $M_E = M_{F_1} \cup_\partial (-M_{F_2})$. By Novikov additivity,

$$\sigma(M_E) = \sigma(M_{F_1}) - \sigma(M_{F_2}),$$

while by the G-signature theorem ([2]; for an elementary proof in the case of an involution see [6])

$$\sigma(M_E) = 2\,\sigma(S^4) - \tfrac{1}{2}e(E) = -\tfrac{1}{2}e(E)$$

where $e(E)$ denotes the normal Euler number of the closed surface $E \subset S^4$ (see [21]).

Now, $e(E)$ may be interpreted as follows. Take a section E' of the normal bundle of E which is transverse to E. At each point of $E \cap E'$ choose a local orientation of E. This determines a local orientation of E', and so an incidence number ± 1 for the intersection point; this is independent of the orientation choice. Then $e(E)$ is the sum of these incidence numbers over all points of $E \cap E'$.

It follows easily that $e(E) = e(F_1) - e(F_2)$, which completes the proof. □

Now suppose V is a spanning surface in S^3 for the knot k. Push int V into int D^4 in the obvious way to obtain a properly embedded surface $\hat{V} \subset D^4$ with $\partial \hat{V} = k$. The following result is already well-known in the case that V is orientable. (See, for example, [5], [8].)

Theorem 3. *With the above notation,*

$$(H_2(M_{\hat{V}}), \cdot) \cong (H_1(V), \mathcal{G}_V),$$

where $\cdot$ denotes the intersection form on $H_2(M_{\hat{V}})$.

Proof. We can construct $M_{\hat{V}}$ by gluing together two copies of the manifold obtained by cutting open D^4 along the trace of the isotopy which pushed int V into int D^4. But this manifold is homeomorphic to D^4; the part exposed by the cut corresponds to a tubular neighbourhood N of V in S^3. Thus $M_{\hat{V}}$ may be described as follows. Let ι: $N \to N$ be the involution given by reflection of the fibre. Take two copies D^4_1 and D^4_2 of D^4, and let

$$M_{\hat{V}} = D^4_1 \cup D^4_2 / x \in N \subset D^4_1 \sim \iota(x) \in N \subset D^4_2.$$

Now consider the Mayer-Vietoris sequence for $M_{\hat{V}}$:

$$0=H_2(D_1^4)\oplus H_2(D_2^4)\to H_2(M_{\hat{V}})\xrightarrow{\varphi} H_1(N)\to H_1(D_1^4)\oplus H_1(D_2^4)=0.$$

The inverse S of the isomorphism φ may be described as follows. If a is a 1-cycle in N, let

$$S\,a=(\text{cone on } a \text{ in } D_1^4)-(\text{cone on } \iota(a) \text{ in } D_2^4).$$

Then $S([a])=[S\,a]$. Hence if a, b are disjoint 1-cycles in N,

$$S([a])\cdot S([b])=Lk(a,b)+Lk(\iota(a),\iota(b)).$$

Finally, let $i_*\colon H_1(V)\to H_1(N)$ be the isomorphism induced by inclusion, and let $\alpha, \beta\in H_1(V)$. Represent α, β by 1-cycles a and b in V. Then a and τb are disjoint 1-cycles in N, and τb is homologous to $2b$ in N. Moreover, $\iota(a)=a$, $\iota(\tau b)=\tau b$. Hence

$$\begin{aligned} S\,i_*\alpha\cdot S\,i_*\beta &=\tfrac{1}{2}S([a])\cdot S([\tau b]) \\ &=\tfrac{1}{2}(Lk(a,\tau b)+Lk(\iota(a),\iota(\tau b))) \\ &=Lk(a,\tau b) \\ &=\mathcal{G}_V(\alpha,\beta). \end{aligned}$$

Thus $S\,i_*$ is the required isomorphism

$$(H_1(V),\mathcal{G}_V)\xrightarrow{\cong}(H_2(M_{\hat{V}}),\cdot). \quad \square$$

Now let k^V be a parallel copy of k missing V, and define $e(V)=-Lk(k,k^V)$. One easily proves

Lemma 4. $e(V)=e(\hat{V})$. $\square$

Corollary 5. *If V is a spanning surface for the knot k, then*

$$\sigma(k)=\operatorname{sign}\mathcal{G}_V+\tfrac{1}{2}e(V).$$

Proof. By Theorems 2 and 3 and Lemma 4, the right hand side of the asserted equation is independent of the spanning surface V. But if V is orientable, $\operatorname{sign}\mathcal{G}_V$ is by definition $\sigma(k)$, while $e(V)=0$. $\square$

We digress to note the following consequence of Theorem 3. Let M be a compact 4-manifold with $H_1(M)=0$, and suppose the intersection form on $H_2(M)$ is given, with respect to some basis, by the integral matrix A. By Lefschetz duality,

$$H_2(M,\partial M)\cong H^2(M)\cong \operatorname{Hom}(H_2(M),\mathbb{Z})=H_2(M)^*,$$

say, and the exact sequence of the pair $(M,\partial M)$ gives an exact sequence

$$H_2(M)\xrightarrow{A} H_2(M)^*\to H_1(\partial M)\to 0,$$

where the left-hand map is represented by the matrix A with respect to our basis for $H_2(M)$ and the corresponding dual basis for $H_2(M)^*$. It follows that A is a relation matrix for $H_1(\partial M)$.

Now suppose $H_1(\partial M;\mathbb{Q})=0$, and consider the linking form λ: $H_1(\partial M)\times H_1(\partial M)\to\mathbb{Q}/\mathbb{Z}$. Suppose α, $\beta\in H_1(\partial M)$ are represented by 1-cycles a, b. Let $\Delta\in\mathbb{Z}-\{0\}$ be an annihilator of $H_1(\partial M)$; for example, we could take $\Delta=\det A$. Then $\Delta a=\partial u$, $\Delta b=\partial v$, for some 2-chains u, v in ∂M, and by definition

$$\lambda(\alpha,\beta)=\frac{u\cap b}{\Delta}\quad(\mathrm{mod}\,1),$$

where $\cap$ denotes intersection number. We also have $a=\partial x$, $b=\partial y$ for some 2-chains x, y in M. Then $\Delta x-u$ and $\Delta y-v$ represent classes X, Y, say, $\in H_2(M)$. It is easy to see geometrically that

$$X\cdot Y=\Delta x\cap\Delta y-u\cap\Delta b,$$

and hence

$$\lambda(\alpha,\beta)=-\frac{X\cdot Y}{\Delta^2}\quad(\mathrm{mod}\,1).$$

Note that the map $H_2(M)\to H_2(M,\partial M)$ takes $X\mapsto[\Delta x]$, $Y\mapsto[\Delta y]$. Let $[x]$, $[y]$ correspond, under the isomorphism $H_2(M,\partial M)\cong H_2(M)^*$, to $\hat{X}$, $\hat{Y}\in H_2(M)^*$. Then (regarding X, Y, $\hat{X}$, $\hat{Y}$ as column vectors with respect to our bases of $H_2(M)$ and $H_2(M)^*$), we have $AX=\Delta\hat{X}$, $AY=\Delta\hat{Y}$. Therefore, recalling that A is invertible over $\mathbb{Q}$,

$$\frac{X\cdot Y}{\Delta^2}=\frac{X^tAY}{\Delta^2}=\frac{\Delta^2\hat{X}^tA^{-1}AA^{-1}\hat{Y}}{\Delta^2}=\hat{X}^tA^{-1}\hat{Y}.$$

Thus $\lambda(\alpha,\beta)=-\hat{X}^tA^{-1}\hat{Y}(\mathrm{mod}\,1)$. Also, the map $H_2(M)^*\to H_1(\partial M)$ takes $\hat{X}\mapsto\alpha$, $\hat{Y}\mapsto\beta$. In other words, λ is represented by the matrix $-A^{-1}$ with respect to the images in $H_1(\partial M)$ of our basis for $H_2(M)^*$.

Hence we recover the results of Seifert [17], that if k is a knot with 2-fold branched cover N, and G is any Goeritz matrix for k, then G is a relation matrix for $H_1(N)$, and the linking form on $H_1(N)$ is given by $\pm G^{-1}$ (the sign depending on the choice of orientation of N).

4. The Algorithm

Let K be a regular projection of the knot k, and recall from §0 the definition of $\mu=\mu(K)$: a crossing point is of type I or II according as the string orientations are or are not consistent with respect to the black regions. (See Fig. 2.) Then $\mu(K)$ is the sum of the incidence numbers of double points of type II.

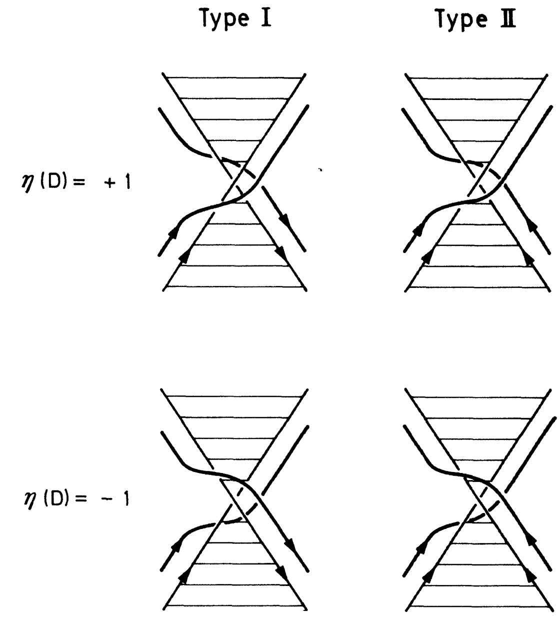

Fig. 6

Theorem 6. *For any regular projection K of the knot k, with associated Goeritz matrix G,*

$$\sigma(k) = \operatorname{sign} G - \mu(K).$$

The theorem is immediate from Corollary 5, Theorem 1, and

Lemma 7. $\mu(K) = -\frac{1}{2} e(V(K))$.

Proof. Let $V = V(K)$. We can take the push-off k^V of k which misses V to lie in the plane R^2 except in a neighbourhood of the double points. Figure 6 shows k^V close to a double point D in each of the four possible cases for D. If we calculate $Lk(k, k^V)$ by counting the number of times k^V passes under k from right to left, we see that D contributes $2\eta(D)$ if it is of type II, 0 if of type I. Hence $Lk/k, k^V)$ $= 2\mu(K)$; i.e.

$$-e(V) = 2\mu(K). \quad \square$$

5. Links

We now describe how the theory has to be modified to take care of links.

Everything in §1 remains valid, except that the following additional move on the Goeritz matrix must be permitted. (See [11]; also §6.)

$$G \mapsto \left[\begin{array}{c|c} G & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{3}$$

§2 carries over exactly as stated.

Regarding §3, the following comments are in order. Let F be a surface properly embedded in D^4, with ∂F the link $l = k_1 \cup \cdots \cup k_m$, say. Then $H_1(D^4 - F; \mathbb{Z}_2) \cong \mathbb{Z}_2^{\beta_0(F)}$, generated by meridians of F, and we can form the 2-fold branched cover M_F of D^4 with branch set (the whole of) F. Since F has no closed components, its normal 1-sphere bundle has a section F', with $\partial F' = l' = k'_1 \cup \cdots \cup k'_m$, say. We then define

$$e(F) = -\sum_{i=1}^{m} Lk(k_i, k'_i),$$

where each pair k_i, k'_i is oriented compatibly but otherwise arbitrarily. Again, this can be interpreted as a normal Euler number, as follows. Attach 2-handles to D^4 along tubular neighbourhoods of the components of l, using the 0-framing for each component, giving a 4-manifold W, say. The union of F with the cores of the 2-handles is a closed surface E, and the normal Euler number of E in W is just $e(F)$. In particular, we see that $e(F)$ is independent of the choice of section F'. The definition of $e(V)$ for a surface V in S^3 extends analogously to the case where V has several boundary components:

$$e(V) = -\sum_{i=1}^{m} Lk(k_i, k_i^V).$$

Theorems 2 and 3 and Lemma 4 carry over exactly as stated, and as a consequence we have

Corollary 8. *If V is a spanning surface for the link l, then*

$$\operatorname{sign} \mathcal{G}_V + \tfrac{1}{2} e(V)$$

depends only on l. □

From our present point of view, in which non-orientable spanning surfaces are just as good as orientable ones, and (hence) links are unoriented, the invariant of Corollary 8 is the natural generalization to links of the signature of a knot. To relate this invariant to the classical signature of a link, we must now orient our links. More precisely, we consider a link together with an orientation of each of its components, modulo simultaneous reversal of these orientations. Let $\bar{l}$ be such a *semi-oriented* link, with underlying (unoriented) link l. One then defines the signature $\sigma(\bar{l})$ of $\bar{l}$ to be $\operatorname{sign} \mathcal{G}_V$ for any orientable spanning surface V of l which has a semi-orientation inducing the specified semi-orientation on $\bar{l}$. (Actually the original definition, due to Murasugi [13], was described in terms of a regular projection, in a purely combinatorial fashion, but this can be interpreted in terms of a certain orientable spanning surface associated with the projection (see [7, 19]).)

Now let V be any surface in S^3 whose boundary is the semi-oriented link $\bar{l} = \bar{k}_1 \cup \cdots \cup \bar{k}_m$. ($V$ need not be orientable, and even if it is, we do not assume that it is compatible with the semi-orientation of $\bar{l}$.) Define

$$\bar{e}(V) = -\sum_{i,j=1}^{m} Lk(\bar{k}_i, \bar{k}_j^V).$$

Then

$$\bar{e}(V) = -\sum_{i=1}^{m} Lk(\bar{k}_i, \bar{k}_i^V) - 2\sum_{1 \leq i < j \leq m} Lk(\bar{k}_i, \bar{k}_j) = e(V) - 2\lambda(\bar{l})$$

where $\lambda(\bar{l})$ is the *total linking number* of $\bar{l}$ (see [13]).

We then have the following analogue of Corollary 5, which identifies the invariant of Corollary 8 with the invariant $\xi(l) = \sigma(\bar{l}) + \lambda(\bar{l})$ of Murasugi [13]. (Note that this gives another proof of Theorem 1 of [13], that $\sigma(\bar{l}) + \lambda(\bar{l})$ is independent of the orientation; this proof is closely related to that of Kauffman and Taylor [8].)

Corollary 5′. *If V is any spanning surface for the link l, then* $\xi(l) = \operatorname{sign} \mathcal{G}_V + \frac{1}{2} e(V)$.

Proof. The right hand side is independent of V by Corollary 8. So choose V orientable; then a semi-orientation of V will detrmine a semi-oriented link $\bar{l}$, say. Since V is orientable, $\bar{e}(V) = 0$, hence $\frac{1}{2} e(V) = \lambda(\bar{l})$, while, by definition, $\operatorname{sign} \mathcal{G}_V = \sigma(\bar{l})$. □

Corollary 5′ can be equivalently stated in terms of the signature:

Corollary 5″. *If V is any spanning surface for the link l, then*

$$\sigma(\bar{l}) = \operatorname{sign} \mathcal{G}_V + \tfrac{1}{2} \bar{e}(V)$$

for any semi-oriented link $\bar{l}$ with underlying link l. □

We can now obtain two different (but of course closely related) generalizations of Theorem 6, to give algorithms for computing $\xi(l)$ and $\sigma(\bar{l})$ respectively from a link projection. (Since $\lambda(\bar{l})$ is easy to compute, any one of $\xi(l)$, $\sigma(\bar{l})$ can be readily obtained from the other. We discuss them both, however, since each has some claim to being the more natural generalization of the signature of a knot. Thus, on the one hand, $\xi(l)$ requires no mention of orientations, while on the other, the algorithm for $\sigma(\bar{l})$ involves the most natural generalization of the knot projection function μ.)

First consider $\xi(l)$. We need to be able to calculate $e(V(L))$ for the shaded surface $V(L)$ corresponding to a projection L of l. To this end, attach a number $\zeta(D)$ to each double point D of L as follows: if a single component of l crosses itself at D, let $\zeta(D) = 0$ or 1 according as D is of type I or II; if two distinct components cross at D, let $\zeta(D) = \frac{1}{2}$. Define $\nu(L) = \sum \zeta(D)\,\eta(D)$ summed over all double points D of L. Note that $\nu(L)$ is an invariant of the unoriented diagram L, and coincides with $\mu(L)$ when l is a knot.

One proves easily

Lemma 7′. $\nu(L) = -\frac{1}{2} e(V(L))$. □

And hence,

Theorem 6′. *For any regular projection L of the link l, with associated Goeritz matrix G,*

$$\xi(l) = \operatorname{sign} G - \nu(L). \quad \square$$

Now let $\bar{l}$ be a semi-oriented link, and let $\bar{L}$ be some (correspondingly semi-oriented) regular projection of $\bar{l}$. Define $\mu(\bar{L})=\sum\eta(D)$, summed over all double points D of type II of $\bar{L}$. (Note that a semi-orientation is necessary to make the distinction between points of type I and points of type II meaningful.)

Then

Lemma 7″. $\mu(\bar{L})=-\frac{1}{2}\bar{e}(V(\bar{L}))$. □

And hence

Theorem 6″. *For any (semi-oriented) regular projection* $\bar{L}$ *of the semi-oriented link* $\bar{l}$, *with associated Goeritz matrix* G,

$$\sigma(\bar{l})=\operatorname{sign} G-\mu(\bar{L}). \quad \square$$

6. A Down-to-Earth Approach

We have developed the properties of our quadratic form by paralleling the approach of Kauffman and Taylor [8] in the case of an orientable surface. A development along more traditional lines, avoiding the use of covering spaces, can also be given; we now outline this.

The following result is, as previously remarked, a consequence of Theorem 3; we give here a direct proof.

Proposition 9. *Let* V *be a surface in* S^3. *Then*

$$\mathscr{G}_V\colon\ H_1(V)\times H_1(V)\to\mathbb{Z}$$

is symmetric.

Proof. We adopt the notation of §2.

Orient $\tilde{V}$ so that a positive normal points out of N, and let $i_+, i_-\colon \tilde{V}\to S^3-\tilde{V}$ be given by translation in the positive and negative normal direction, respectively. Let α, $\beta\in H_1(V)$. Then

$$\begin{aligned}\mathscr{G}_V(\alpha,\beta)&=Lk(\alpha,\tau\beta)\\&=Lk(\alpha,i_{+*}\tau\beta)\\&=\tfrac{1}{2}Lk(\tau\alpha,i_{+*}\tau\beta)\end{aligned}$$

so $\mathscr{G}_V(\alpha,\beta)-\mathscr{G}_V(\beta,\alpha)=\frac{1}{2}Lk(\tau\alpha,i_{+*}\tau\beta-i_{-*}\tau\beta)=\frac{1}{2}\tau\alpha\cdot\tau\beta$. But each point of $\alpha\cap\beta$ gives rise to two oppositely signed points of $\tau\alpha\cap\tau\beta$, so $\tau\alpha\cdot\tau\beta=0$. □

Suppose V_1, V_2 are two surfaces in S^3 with $\partial V_1=\partial V_2$. Suppose further there is a 3-ball $B^3=B^1\times B^2\subset S^3-\partial V_i$ such that

$$V_1\cap B^3=\partial B^1\times B^2$$

$$V_2\cap B^3=B^1\times\partial B^2$$

and

$$\operatorname{cl}(V_1-B^3)=\operatorname{cl}(V_2-B^3).$$

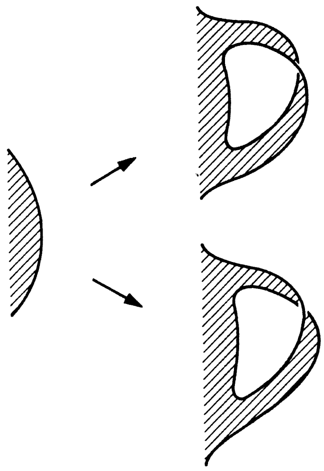

Fig. 7

In this situation we say that V_2 *comes from* V_1 *by a* 1-*handle move.*

(Here, and in the proof of Theorem 11 below, we leave corner-smoothing to the reader.)

We say that two surfaces in S^3 are S^*-*equivalent* if they are related by a finite sequence of the following moves and their inverses

(A) Ambient isotopy

(B) A 1-handle move

(C) Addition of a small, $\frac{1}{2}$-twisted handle (see Fig. 7).

The boundaries of two S^*-equivalent surfaces are necessarily of the same link type.

(The terminology arises as follows. The relation of S-equivalence of Seifert matrices ([20, 14]) has a natural geometric analogue, namely the equivalence relation on orientable surfaces generated by the moves (A), (B) and $(B)^{-1}$ (through orientable surfaces only). It seems reasonable to call this S-equivalence also, whence S^*-equivalence.)

Proposition 10. *If* V_0, V_1 *are* S^*-*equivalent surfaces in* S^3, *then*

$$\operatorname{sign}\mathscr{G}_{V_0}+\tfrac{1}{2}e(V_0)=\operatorname{sign}\mathscr{G}_{V_1}+\tfrac{1}{2}e(V_1).$$

Proof. That $\operatorname{sign}\mathscr{G}_V$ is invariant under (B) is proved just as in the orientable case ([14, 20]); clearly $e(V)$ is also invariant.

The effect of (C) on $\mathscr{G}_V$ is to take orthogonal sum with the form $[\varepsilon]$, where $\varepsilon = \pm 1$ depending on the sense of the twist in the band; since 2ε is subtracted from $e(V)$, the result follows. □

(In this way we see that (the matrices of) the quadratic forms of S^*-equivalent surfaces are related by the moves (1) and (2) of §1, provided the boundary is a knot. For a link, we must include the move

$$G \mapsto \left[\begin{array}{c|c} G & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3}$$

corresponding to a 1-handle move connecting two components of a disconnected surface. This is also necessary for the Goeritz matrix of a link, as was observed by Kyle [11].)

Proposition 10 and the following theorem now imply that for any spanning surface V of a link l, $\operatorname{sign}\mathscr{G}_V+\frac{1}{2}e(V)$ depends only on l.

Theorem 11. *If V_0, V_1 are surfaces in S^3 with $\partial V_0=\partial V_1$, then V_0 and V_1 are S^*-equivalent.*

We outline two proofs of this, one 3-dimensional, the other 4-dimensional. Let $l=\partial V_0=\partial V_1$.

Proof I. First we can use move (C) (and (A)) to ensure that $V_0\cap V_1$ consists of l together with simple closed curves in $\operatorname{int}V_0\cap\operatorname{int}V_1$. Using (B), we may also assume that V_0 and V_1 are connected. Let M be the closure of a component of the complement in S^3 of a regular neighbourhood of $V_0\cup V_1$. Note that M is a manifold, but that if M' is the closure of the corresponding component of $S^3-(V_0\cup V_1)$, then M' may fail to be a manifold along some curves of $\operatorname{int}V_0\cap\operatorname{int}V_1$; we call such a curve *bad*. Regard M as a cobordism between the parts of ∂M coming from V_0, V_1 respectively, and take a handle decomposition of M on one end. We may assume there are only 1- and 2-handles, and that the former precede the latter. Let W be the surface between the 1- and 2-handles. If V_i' is obtained from V_i by deleting the part of V_i corresponding to ∂M and attaching W in the obvious way, then V_i' is obtainable from V_i by a sequence of moves (B) (and (A)), $i=0,1$. If ∂M accounts for all of $V_0\cup V_1$, we are done. If not, there must be some curve $c\subset\operatorname{int}V_0\cap\operatorname{int}V_1$ which is good for M. (Otherwise, any arc on V_i, $i=0,1$, starting at a point coming from ∂M must finish at another such point. Since V_i is connected, all points of $V_0\cup V_1$ would then come from ∂M.) Pushing one copy of W off the other we get a surface V_1'', ambient isotopic to V_1', such that each curve of $\operatorname{int}V_0\cap\operatorname{int}V_1$ gives rise to at most one curve of $\operatorname{int}V_0'\cap\operatorname{int}V_1''$. Moreover, by suitably choosing the direction in which to push W, we can arrange that c is eliminated. (We could not necessarily do this if c were bad.) Thus $\operatorname{int}V_0'\cap\operatorname{int}V_1''$ has fewer components than $\operatorname{int}V_0\cap\operatorname{int}V_1$, and the proof is completed by induction. □

Remark. For the case of orientable surfaces, an outline of the above approach is given by Rice in [16]. However, this does not seem to take into account the possibility of the existence of bad curves (which may occur even in the orientable case).

Proof II. For $i=0,1$, let $l^{(i)}$ be a push-off of l missing V_i; by use of move (C) (and (A)) we may assume $Lk(k_j,k_j^{(0)})=Lk(k_j,k_j^{(1)})$ for each component k_j of l. Hence we can take $l^{(0)}=l^{(1)}=l'$, say. Let V_i' be a section of the normal bundle of V_i in S^3, with $\partial V_i'=l'$. Consider S^4 as

$$D^4\bigcup_{S^3\times\{0\}}S^3\times[0,1]\bigcup_{S^3\times\{1\}}-D^4$$

and let $E\subset S^4$ be the closed surface

$$V_0\times\{0\}\bigcup_{l\times\{0\}}l\times[0,1]\bigcup_{l\times\{1\}}V_1\times\{1\}.$$

The sections V_0', V_1' define a section E' of the normal 1-sphere bundle of E, showing that $e(E)=0$.

The following steps will complete the proof.

(1) E bounds a 3-manifold in S^4.

(2) E bounds a 3-manifold M in $S^3\times[0,1]$, with $M\cap S^3\times\{i\}=V_i$ for $i=0,1$.

(3) Take a handle decomposition of M on V_0, with each handle in a level $S^3\times\{t_i\}$, $t_i\in(0,1)$, and with $M\cap S^3\times(t_i,t_{i+1})$ a vertical collar on $\partial(M\cap S^3\times[0,t_i])\cap S^3\times\{t_i\}$. (See [9].)

(4) Eliminate 0- and 3-handles from the decomposition of (3), and order the remaining 1- and 2-handles so that the former precede the latter. Then the intersection of M with a level between the 1- and 2-handles is a spanning surface obtained from V_i by 1-handle moves, for $i=0,1$.

Of these, only (1) needs further explanation.

Actually it is true that if E is any closed surface in S^4 with $e(E)=0$, then E bounds a 3-manifold in S^4. To see this, let T be a tubular neighbourhood of E, and $X=S^4-\operatorname{int}T$. Starting with any section E' of ∂T, it may be modified to produce a section E'' such that $Lk_{\mathbb{Z}_2}(\alpha,E'')=0$ for all $\alpha\in H_1(E)$. By duality and excision this implies that E'' bounds homologically mod 2 in X. Next, we find a map $f\colon \partial T\to RP^\infty$ transverse regular along E'' to a codimension one copy of RP^∞. The fact that E'' bounds mod 2 in X is sufficient to allow the extension of f to $X\to RP^\infty$, which gives a 3-manifold in X with boundary E''. □

Remark. In our situation, if l is a knot, we may take E'' to be the E' defined by V_0' and V_1', but if l has more than one component, this E' may have to be modified.

7. An Application

Let k be a knot, and l a link contained in an unknotted solid torus $T\subset S^3$. Let $f\colon T\to S^3$ be an embedding such that $f(T)$ is a tubular neighbourhood of k. Suppose also that f is faithful, that is, the image of a longitude of T (a simple closed curve on ∂T homologous in T to the core of T and null-homologous in $S^3-\operatorname{int}T$) is a longitude of $f(T)$. Let l^* denote the link $f(l)$, and let $n\in\mathbb{Z}_2$ be the mod 2 homology class $[l]\in H_1(T;\mathbb{Z}_2)$.

Theorem 12. $\xi(l^*)=\begin{cases}\xi(l), & \text{if } n=0\\ \xi(l)+\sigma(k), & \text{if } n=1.\end{cases}$

Remarks. (1) Taking l to be a knot, we recover the result of Shinohara [18, Theorem 9].

(2) Our method can also handle more general situations, in which we start with a link and replace some of its components by links lying in tubular neighbourhoods of these components. But since it does not seem possible to give a concise general statement, we omit the details.

(3) The second author has proved results analogous to those alluded to in (2) above for the Tristram-Levine signatures, using the interpretation of these as eigenspace signatures of higher-order branched covers. For the case of a knot, see [12].

Proof of Theorem 12. First consider the case $n=0$. Then l bounds a surface W, say, in T, and as spanning surface for l^* we may take $f(W)$. Since f is faithful, $Lk(C_1, C_2)=Lk(f(C_1), f(C_2))$ for any two disjoint 1-cycles C_1, C_2 in T. Hence $e(f(W))=e(W)$, and $\mathscr{G}_{f(W)}\cong\mathscr{G}_W$, which implies $\xi(l^*)=\xi(l)$.

Now suppose $n=1$. In this case, there is a surface W_0 in T whose boundary is the union of l and a longitude J of T. Let W be the spanning surface for l obtained from W_0 by capping off J with the obvious 2-disc in $S^3-\operatorname{int} T$. From an orientable spanning surface for k, one obtains an orientable surface V properly embedded in $S^3-\operatorname{int} f(T)$ whose boundary is the longitude $f(J)$ of $f(T)$. Then $V^*=V\cup_{f(J)} f(W_0)$ is a spanning surface for l^*. Since f is faithful, $e(V^*)=e(W)$. From a Mayer-Vietoris exact sequence, we obtain

$$H_1(V^*)\cong H_1(V)\oplus(H_1(f(W_0))/[f(J)])\cong H_1(V)\oplus H_1(W).$$

Moreover, since no 1-cycle in $f(T)$ links any 1-cycle in V, and since f is faithful, $\mathscr{G}_{V^*}$ is isomorphic to the orthogonal direct sum $\mathscr{G}_V\oplus\mathscr{G}_W$. Hence

$$\xi(l^*)=\operatorname{sign}\mathscr{G}_{V^*}+\tfrac{1}{2}e(V^*)=\operatorname{sign}\mathscr{G}_V+\operatorname{sign}\mathscr{G}_W+\tfrac{1}{2}e(W).$$

But $e(V)=0$ since V is orientable. Hence $\xi(l^*)=\xi(l)+\sigma(k)$. □

References

1. Alexander, J.W., Briggs, G.B.: On types of knotted curves. Ann. of Math. (2) **28**, 562–586 (1926/27)
2. Atiyah, M.F., Singer, I.M.: The index of elliptic operators III. Ann. of Math. (2) **87**, 546–604 (1968)
3. Conway, J.H.: An enumeration of knots and links, and some of their algebraic properties. Computational Problems in Abstract Algebra, pp. 329–358. Oxford and New York: Pergamon Press 1969
4. Goeritz, L.: Knoten und quadratische Formen. Math. Z. **36**, 647–654 (1933)
5. González-Acuña, F.: On homology spheres. Thesis, Princeton University 1969
6. Jänich, K., Ossa, E.: On the signature of an involution. Topology **8**, 27–30 (1969)
7. Kauffman, L.: Cyclic branched covers, $O(n)$-actions, and hypersurface singularities. Thesis, Princeton University 1972
8. Kauffman, L., Taylor, L.: Signature of links. Trans. Amer. Math. Soc. **216**, 351–365 (1976)
9. Kearton, C., Lickorish, W.B.R.: Piecewise Linear critical levels and collapsing. Trans. Amer. Math. Soc. **170**, 415–424 (1972)
10. Kneser, M., Puppe, D.: Quadratische Formen und Verschlingungsinvarianten von Knoten. Math. Z. **58**, 376–384 (1953)
11. Kyle, R.H.: Branched covering spaces and the quadratic forms of links. Ann. of Math. (2) **59**, 539–548 (1954)
12. Litherland, R.A.: Signatures of iterated torus knots. To appear in the Proceedings of the 1977 Sussex Geometric Topology Conference
13. Murasugi, K.: On the signature of links. Topology **9**, 283–298 (1970)
14. Murasugi, K.: On a certain numerical invariant of link types. Trans. Amer. Math. Soc. **117**, 387–422 (1965)

15. Reidemeister, K.: Elementare Begründung der Knotentheorie. Abh. Math. Sem. Univ. Hamburg **5**, 24–32 (1927)
16. Rice, P.M.: Equivalence of Alexander matrices. Math. Ann. **193**, 65–75 (1971)
17. Seifert, H.: Die Verschlingungsinvarianten der zyklischen Knotenüberlagerungen. Abh. Math. Sem. Univ. Hamburg **11**, 84–101 (1935)
18. Shinohara, Y.: On the signature of knots and links. Trans. Amer. Math. Soc. **156**, 273–285 (1971)
19. Shinohara, Y.: On the signature of knots and links. Dissertation, Tallahassee: Florida State University 1969
20. Trotter, H.F.: Homology of group systems with applications to knot theory. Ann. of Math. (2) **76**, 464–498 (1962)
21. Whitney, H.: On the topology of differentiable manifolds. Lectures in topology, the University of Michigan conference of 1940, (R.L. Wilder and W.L. Ayres, ed.). Ann Arbor: The University of Michigan press 1941

Received February 1, 1978