SOME ASPECTS OF CLASSICAL KNOT THEORY

by

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0. Introduction

Man's fascination with knots has a long history, but they do not appear to have been considered from the mathematical point of view until the 19th century. Even then, the unavailability of appropriate methods meant that initial progress was, in a sense, slow, and at the beginning of the present century rigorous proofs had still not appeared. The arrival of algebraic-topological methods soon changed this, however, and the subject is now a highly-developed one, drawing on both algebra and geometry, and providing an opportunity for interplay between them.

The aim of the present article is to survey some topics in this theory of knotted circles in the 3-sphere. Completeness has not been attempted, nor is it necessarily the case that the topics chosen for discussion and the results mentioned are those that the author considers the most important: non-mathematical factors also contributed to the form of the article.

For additional information on knot theory we would recommend the survey article of Fox [43], and the books of Neuwirth [112] and Rolfsen [128]. Reidemeister's book [125] is also still of interest. As far as problems are concerned, see [44], [112], [113], [75], as well as the present volume. Again, we have by no means tried to include a complete bibliography, although we hope that credit for ideas has been given where it is due. For a more extensive list of early references, see [26].

In the absence of evidence to the contrary, we shall be working in the smooth category (probably), and homology will be with integer coefficients.

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Contents

1. Enumeration

It seems that the first mathematician to consider knots was Gauss, whose interest in them began at an early age [31, p. 222]. Unfortunately, he himself wrote little on the subject [49, V, p. 605; VIII, pp. 271-286], despite the fact that he regarded the analysis of knotting and linking as one of the central tasks of the 'geometria situs' foreseen by Leibniz [49, V, p. 605]. His student Listing, however, devoted a considerable part of his monograph [88] to knots, and in particular made some attempt to describe a notation for knot diagrams.

A more successful attack, inspired by Lord Kelvin's theory of vortex atoms, was launched in the 1860's(1) by the Scottish physicist Tait. His first papers on knots were published in 1876-77 (see [145]). Later, with the help of the 'polyhedral diagrams' of the Reverend Kirkman, Tait and Little (the latter had done some earlier work [90]) made considerable progress on the enumeration ('census') problem, so that by 1900 there were in existence tables of prime knots up to 10 crossings and alternating prime knots of 11 crossings [91], [92], [93], [145].

Essentially nothing was done by way of extending these tables until about 1960, when Conway invented a new and more efficient notation which enabled him to list all (prime) knots up to 11 crossings and all links up to 10 crossings [19], (revealing, in particular, some omissions in the 19th century tables).

There are two main aspects of this kind of enumeration: completeness and non-redundancy. One wants to know (i.e. prove) that one has listed all knots up to a given crossing number, and also that the knots listed are distinct. The former belongs to combinatorial mathematics, and although a proof of completeness throughout the range of the existing tables would no doubt be long and tedious, it is not hard to envisage how such a proof would go. Indeed, implicit in the compilation of the tables is the possession of at least the outline of such a proof. Although some omissions in Conway's tables have recently been brought to

(1) see Maxwell's letter of 1867 quoted in [77, p. 106]

light by Perko (see [117] and references therein), it seems safe to assume that essentially all knots up to 11 crossings have now been listed. (The author understands that we may soon see a proof of completeness in this range.)

As regards the question of non-redundancy, methods for proving that two knot diagrams represent different knots became available only with the advent of algebraic topology, and as a consequence the compilers of the early tables, as they themselves were aware, had to rely on purely empirical evidence that their listed knots were distinct.

Proofs of the existence of non-trivial knots, based on the fundamental group, were known at least as early as 1906 (see [146]), but not until 1927 was there any systematic attempt to establish the non-redundancy of the tables. Then, Alexander and Briggs [3], using the torsion numbers of the first homology of the 2- and 3-fold branched cyclic covers, distinguished all the tabled knots up to 8 crossings and all except 3 pairs up to 9 crossings. (Alexander had pointed out in 1920 (see [3]) that any topological invariant of the k-fold branched cyclic cover of a knot, in particular the Betti and torsion numbers, will be an invariant of the knot, an observation which was made independently by Reidemeister [122].) The Alexander polynomial, introduced in [2], also suffices to distinguish all knots up to 8 crossings, and all except 6 pairs up to 9 crossings. For each of the 3 remaining 9-crossing pairs not distinguished by Alexander and Briggs, the two knots in question have isomorphic $\mathbb{Z}[t,t^{-1}]$-module structures in their infinite cyclic covers, so new methods are necessary to distinguish them. This was done by Reidemeister, by means of the mutual linking numbers of the branch curves in certain (irregular) p-fold dihedral covers, and, more recently, Perko has used these linking invariants, in branched covers associated with representations on dihedral groups and the symmetric group on 4 letters, to distinguish all tabled knots up to 10 crossings [115].

It would now appear that the number of prime knots with crossing number ≤ 10 is 249, as tabulated below.

crossing number	3	4	5	6	7	8	9	10
number of prime knots	1	1	2	3	7	21	49	165

(See [3] for pictures of knots up to 9 crossings, and [115] for those with 10 crossings.) There are 550 11-crossing knots now known [117], and although there is a good chance that these might be all, the task of proving them distinct is a formidable one that has not yet been completed. Indeed, as intimated in [117] (which contains some partial results), invariants more delicate than those which suffice up to 10 crossings are now required.

2. The Group

The knot problem becomes discretized when looked at from the point of view of combinatorial topology. It is noted in [30], for example, that it can be formulated entirely in terms of arithmetic. However, this kind of 'reduction' seems to be of no practical value, nor does it seem to have any theoretical consequences (for decidability, for example). There are also many natural numerical invariants of a knot which may be defined, such as the minimal number of crossing points in any projection of the knot, the minimal number of crossing-point changes required to unknot the knot (the 'gordian number' [160]), the maximal euler characteristic of a spanning surface (orientable or not), and so on (see [125, pp. 16-17]). But these tend to be hard to compute.

The first successful algebraic invariant to be attached to a knot was the fundamental group of its complement, (the group of the knot), and presentations of certain knot groups appear fairly early in the literature (see [146]). General methods for writing down a presentation of the knot group from a knot projection were given by Wirtinger (unpublished (?) ;see [125, III, §9])and Dehn [27]. Actually it was soon recognized [28] that a knot contains (at least a priori) more information than just its group, as we now explain. Let $K \subset S^3$ be our given (smooth) knot, and let X be its exterior, that is, the closure of the complement of a tubular neighbourhood N of K. (The exterior and the complement are equivalent invariants: clearly the exterior determines the complement, and the converse follows from [33].) Choosing orientations for S^3 and K determines a longitude-meridian pair $\lambda, \mu \in \pi_1(X)$ in the usual way (λ and μ are represented by oriented curves ℓ and m on ∂X which intersect (transversely) only at the base-point, where ℓ is homologous to K in N and null-homologous in X, and m is null-homologous in N and inherits its orientation from that of K and S^3). If two (oriented) knots $K_1, K_2 \subset S^3$ are equivalent in the strongest possible sense that there is an orientation-preserving homeomorphism of S^3 (or, equivalently, an isotopy) taking K_1 to K_2, preserving their orientations, then there is an isomorphism $\pi_1(X) \xrightarrow{\cong} \pi_1(X_2)$ taking (λ_1, μ_1) to (λ_2, μ_2). If we ignore the orientations of K_1 and K_2 in our definition of equivalence, then we have an isomorphism $\pi_1(X_1) \xrightarrow{\cong} \pi_1(X_2)$ taking (λ_1, μ_1) to either (λ_2, μ_2) or $(\lambda_2^{-1}, \mu_2^{-1})$. If, in addition, we ignore the orientation of S^3, then our isomorphism merely takes λ_1 to $\lambda_2^{\pm 1}$ and μ_1 to $\mu_2^{\pm 1}$. Using this additional peripheral information, Dehn [28] proved for example that the trefoil is not isotopic to its mirror-image, a fact which had long been 'known' empirically. (Incidentally the knot tables list only one representative from each class under the weakest equivalence, leaving the amphicheirality and (much harder) invertibility questions to be decided separately [19], [115], [118].)

The natural question arises as to what extent the peripheral structure is determined by the group alone. Thus Dehn asks [28, p. 413] whether every automorphism of a knot group preserves the peripheral structure, and in [2, p, 275] Alexander suggests that 'many, if not all, of the topological properties of a knot are reflected in its group.' In 1933, however, Seifert showed [135], using linking invariants of their cyclic branched covers, that the granny knot and the reef (or square) knot, although they have isomorphic groups, are inequivalent, even ignoring orientations. (Although there seems to be an implicit assumption to the contrary in [38], where an alternative proof is given, it follows from Seifert's proof that in fact the two knots have non-homeomorphic complements. Fox's proof does show, however, that there is no isomorphism between the groups of the two knots preserving the peripheral structure.)

Despite such examples, the group is still a powerful invariant. It was shown by Dehn [27], for example, (modulo his 'lemma', which was introduced specifically for this purpose) that the only knot with group $\mathbb{Z}$ is the unknot. This finally became a theorem in 1956 when Dehn's lemma was established by Papakyriakopoulos [114]. At the same time, Papakyriakopoulos also proved the first version of the sphere theorem, and as a consequence, the asphericity of knots, that is, the fact that the complement of a knot is a $K(\pi,1)$. It follows that the group of a knot determines the homotopy type of its complement.

The role of the peripheral structure was finally completely clarified by Waldhausen's work [155] on irreducible, sufficiently large, 3-manifolds (this work in turn being based on earlier ideas of Haken). Specializing to the case that concerns us here, Waldhausen showed that if K_1 and K_2 are knots with exteriors X_1, X_2, then any homotopy equivalence of pairs $(X_1,\partial X_1) \to (X_2,\partial X_2)$ is homotopic to a homeomorphism. This implies, for example, that knots (under the strongest form of equivalence, which takes both the ambient orientation and that of the knot into account), are classified by (isomorphism classes of) their associated triples $(\pi K,\lambda,\mu)$. We may remark that it is a purely algebraic exercise to pass from such a classifying triple to a classifying group [20]. Other, more complicated, but more geometric, ways of nailing down the peripheral structure within a single group are given in [140], [163] and [37]. (The classifying groups obtained there are, respectively, the free product of the groups of two cables about $K \# K_0$ (where K_0 is, say, the figure eight knot), the group of the double of K, and the group of the (p,q)-cable of K where $|p| \geq 3$ and $|q| \geq 2$.)

The situation may to some extent be summarized by the following diagram, where, for simplicity, $\sim$ now denotes the weak form of knot equivalence which disregards orientations, (and P_i denotes the peripheral subgroup $\pi_1(\partial X_i)$).

$$\begin{array}{ccccc}
K_1 \sim K_2 \Rightarrow X_1 \cong X_2 & \Leftrightarrow & (X_1,\partial X_1) \simeq (X_2,\partial X_2) & \Rightarrow & X_1 \simeq X_2 \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
(\pi K_1,\lambda_1,\mu_1) \cong (\pi K_2,\lambda_2^{\pm 1},\mu_2^{\pm 1}) & \Rightarrow & (\pi K_1,P_1) \cong (\pi K_2,P_2) & \Rightarrow & \pi K_1 \cong \pi K_2
\end{array}$$

The two upward implications on the right are consequences of asphericity.

The question of the reversibility of the implications on the left, that is, whether a knot is determined by its complement, was rasied by Tietze in 1908 [146], and is still unsettled. It is related to the following question, asked by Bing and Martin [9]:

Question (P). If a tubular neighbourhood of a non-trivial knot K in S^3 is removed and sewn back differently, is the resulting 3-manifold ever simply-connected? (Here, 'differently' has to be interpreted in the obvious way.)

This may be broken down into the following 2 questions:

(1) Do we ever get a fake 3-sphere?

(2) Do we ever get S^3?

One may further ask

(3) If 'yes' in (2), do we get the same knot?

Knots are determined by their complement if and only if an affirmative answer to (2) is always accompanied by an affirmative answer to (3). There is much evidence that the answer to Question (P) is negative. In particular, it is known that this is the case for torus knots [134], composite knots [9], [53], doubled knots [9], [53], most cable knots [53], [139], knots in knotted solid tori with winding number ≥ 3 [89], and many others; (see [75] for additional references). (One says that these knots 'have Property P'.) Also, Thurston has recently shown (unpublished) that if K has a hyperbolic structure (more precisely, the complement of K has a complete Riemannian metric with constant negative sectional curvature and finite volume), then all except possibly finitely many resewings of the tubular neighbourhood of K yield non-simply-connected manifolds. (The existence of a hyperbolic structure is equivalent to the group-theoretic condition that every free abelian subgroup of πK of rank 2 be conjugate to the peripheral subgroup P, and this in turn is satisfied if and only if K has no companions and is not a torus knot.) A proof that all knots have Property P, however, (or even a proof of a negative answer to either (1) or (2)), seems beyond the scope of existing techniques. Question (3) may be easier. (Indeed it follows from the finiteness theorem of Thurston mentioned above that if K is a hyperbolic knot, and some non-trivial resewing of a tubular neighbourhood of K gives S^3, then the new knot is at least not isotopic to K. For, if the resewing in question is the one which 'kills' $\mu\lambda^n$, say, $n \neq 0$, then the new knot's being isotopic to K would imply the existence of a self-homeomorphism h of the exterior X of K taking $\lambda \mapsto \lambda^{\varepsilon}$, $\mu \mapsto \mu^{\varepsilon}\lambda^{\varepsilon n}$, where $\varepsilon = \pm 1$. Since h^r would then take $\mu \mapsto (\mu\lambda^{rn})^{\pm 1}$, the resewing corresponding to $\mu\lambda^{rn}$ would yield S^3, for all r, contradicting the finiteness statement.)

Returning to our diagram of implications, the example of the reef and granny shows that the horizontal implications on the right are not reversible. On the other hand, Johannson [66], [67] and Feustel [36] have shown that if $\pi K_1 \cong \pi K_2$, and X_1 contains no essential annuli, then $X_1 \cong X_2$. Now the only knots whose exteriors contain essential annuli are composite knots and cable knots. The cable knots with unknotted core are just the torus knots, and they are known to be determined by their group [14]. So let K be a non-trivial knot, and let $K_{p,q}$ denote the (p,q)-cable about K, that is, a curve on the boundary ∂N of a tubular neighbourhood N of K, homologous in ∂N to $p[m]+q[\ell]$. (Here, p and q are coprime integers with $|q| \geq 2$, and (ℓ, m) is a longitude-meridian pair on ∂N.) Feustel-Whitten [37] have shown that if $|p| \geq 3$, then $\pi K_{p,q}$ determines $K_{p,q}$. So prime knot complements are known to be determined by their group except possibly for cable knots $K_{p,q}$ with $|p| \leq 2$.

The problem concerning these remaining cable knots turns out to be related to the general question of whether knots are determined by their complement. More precisely, suppose there exist inequivalent knots K_1, K_2 with homeomorphic exteriors X_1, X_2. The homeomorphism $X_1 \to X_2$ must take m_1 to a curve homologous in ∂X_2 to $\pm[m_2] + n[\ell_2]$, for some $n \neq 0$. Then Hempel (unpublished) and Simon [141] show that if there is such a counterexample, with $|n| \neq 1, 2$, or 4, then there exist cable knots of type $(\pm 1, \pm n/2)$ (n even), or $(\pm 2, \pm n)$ (n odd), with isomorphic groups, whose complements are not homeomorphic.

In the other direction, it can be shown (see [37]) that if all knots have Property P, (or even if the answer to Question (2) above is negative), then prime knots are determined by their group.

As regards composite knots, Feustel-Whitten have also shown [37] that if K_1 is composite, and $\pi K_1 \cong \pi K_2$, then the prime factors of K_2 are precisely those of K_1, up to orientations.

To summarize, the question of whether a knot is determined by its group factors naturally into two questions: (A) does the group determine the complement? and (B) does the complement determine the knot? (B) is unsettled, although the expected answer is 'yes'. The answer to (A) is 'no', but may be 'yes' for prime knots; the unsettled cases of this are related to (B). Thus it may be that the failure of knots to be determined by their group is solely due to the phenomenon which arises by changing the (ambient and intrinsic) orientations of the prime factors of a composite knot.

3. Abelian Invariants

The exterior of a knot K has the homology of a circle (as can be seen, for example, by Alexander duality), and as a consequence, once we have chosen orientations for S^3 and K, there is a canonical epimorphism from $\pi_1(X)$ to the cyclic group C_k of order k, for each k, $1 \leq k \leq \infty$ This defines a canonical normal subgroup of index k in $\pi_1(X)$, or, the geometric equivalent, a regular covering space X_k of X with group of covering translations isomorphic to C_k. Although the homology of X is itself uninteresting, this is not always true of these covering spaces, and the derivation of tractable, 'abelian', knot invariants from this point of view has occupied a central place in the development of the subject.

The homology of the X_k can be viewed on at least the following levels (throughout, we shall take coefficients in some commutative Noetherian ring R, with $R = \mathbb{Z}$, $\mathbb{Z}/p$, or $\mathbb{Q}$ being uppermost in our minds).

(1) the R-module structure of $H_1(X_k;R)$

(2) the module structure of $H_1(X_k;R)$ over the group ring $R[C_k]$.

If R is an integral domain, and $Q(\ )$ denotes field of fractions, we also have

(3) for $k < \infty$, the product structure given by the linking pairing $T_1(X_k;R) \times T_1(X_k;R) \to Q(R)/R$ on the R-torsion subgroup of $H_1(X_k;R)$. ($R = \mathbb{Z}$ is really the only case of interest here.)

(4) the product structure given by the Blanchfield pairing (see §7) $H_1(X_\infty;R) \times H_1(X_\infty;R) \to Q(R[C_\infty])/R[C_\infty]$.

We may remark here that, for $k < \infty$, it is traditional to work with the corresponding branched cyclic covering M_k, rather than with the unbranched covering X_k. Since M_k is a closed 3-manifold, and for other reasons too (see §5), this is perhaps more natural. However, the two are essentially equivalent from the present point of view, as it is not hard to show that $H_1(X_k;R) \cong H_1(M_k;R) \oplus R$, as $R[C_k]$-modules, the module structure on R being induced by the trivial action of C_k.

Apart from the obvious relationships between the above considerations (1)-(4), we have that the $R[C_\infty]$-module $H_1(X_\infty;R)$ determines the $R[C_k]$-module $H_1(X_k;R)$, $1 \leq k < \infty$, (see §5), and the Blanchfield pairing on $H_1(X_\infty;R)$ determines the linking pairing on $T_1(X_k;R)$, $1 \leq k < \infty$.

4. The Infinite Cyclic Cover

Let us first consider the $R[C_\infty]$-module $H_1(X_\infty;R)$. If t denotes the canonical multiplicative generator of C_∞, (determined by the orientations of S^3 and K), we may identify $R[C_\infty]$ with the Laurent polynomial ring $\Pi = R[t,t^{-1}]$. Since R is Noetherian, Π is also, by the Hilbert basis theorem. Furthermore, since X is a finite complex, the chain modules $C_q(X_\infty;R)$ are finitely-generated (free) Π-modules, and hence $H_1(X_\infty;R)$ is a finitely-generated Π-module.

The following argument of Milnor [96] establishes the crucial property that $t-1\colon H_1(X_\infty;R) \to H_1(X_\infty;R)$ is surjective. (Since $H_1(X_\infty;R)$ is finitely-generated and Π is Noetherian, it follows that $t-1$ is also injective.) The short exact sequence of chain complexes

$$0 \to C_*(X_\infty;R) \xrightarrow{t-1} C_*(X_\infty;R) \to C_*(X;R) \to 0$$

gives rise to a homology exact sequence which ends up with

$$\begin{array}{ccccccccc} H_1(X_\infty;R) & \xrightarrow{t-1} & H_1(X_\infty;R) & \to & H_1(X;R) & \to & H_0(X_\infty;R) & \xrightarrow{t-1} & H_0(X_\infty;R) \\ & & & & \wr\| & & \wr\| & & \wr\| \\ & & & & R & \longrightarrow & R & \xrightarrow{0} & R \end{array}$$

This proves the assertion.

A consequence of this (see [85]) is that $H_1(X_\infty;R)$ is a Π-torsion-module.

Now suppose R is a field. Then Π is a principal ideal domain, and hence $H_1(X_\infty;R)$ decomposes as a direct sum of cyclic Π-modules

$$\Pi/(\pi_1) \oplus \Pi/(\pi_2) \oplus \ldots \oplus \Pi/(\pi_n) \ ,$$

where the ideals (π_i) satisfy $(\pi_i) \subset (\pi_{i+1})$, $1 \leq i < n$, (and are then uniquely determined). The Π-module $H_1(X_\infty;R)$ is thus completely described by this sequence of ideals $(\pi_1) \subset (\pi_2) \subset \ldots \subset (\pi_n)$. Furthermore, since $H_1(X_\infty;R)$ is a Π-torsion module, no (π_i) is zero. (In the present case, i.e. R a field, the fact that $H_1(X_\infty;R)$ is Π-torsion actually follows immediately from the direct sum decomposition of $H_1(X_\infty;R)$ and the divisibility by $t-1$.)

To determine the R-vector space structure of $H_1(X_\infty;R)$, let $(\Delta) = (\pi_1\pi_2 \ldots \pi_n)$ be the order ideal of $H_1(X_\infty;R)$. We may suppose for convenience that Δ is normalized so that it contains no negative powers of t and has non-zero constant coefficient. Then

$$\dim H_1(X_\infty;R) = \deg \Delta \ ,$$

and Δ is just the characteristic polynomial of the automorphism t. We shall see later (§7) that $\deg \Delta$ is always even.

Taking $R = \mathbb{Q}$ in particular, we have a complete description of the $\Gamma = \mathbb{Q}[t,t^{-1}]$-module $H_1(X_\infty;\mathbb{Q})$ by a sequence of non-zero ideals $(\gamma_1) \subset (\gamma_2) \subset \ldots \subset (\gamma_n)$. The picture over $\Lambda = \mathbb{Z}[t,t^{-1}]$ is not quite so clear, as Λ is not a principal ideal domain, but one can define some invariants. Thus there are the <u>elementary ideals</u> $E_1 \subset E_2 \subset \ldots$, where E_i is defined to be the ideal in Λ generated by the determinants of all the $(n-i+1) \times (n-i+1)$ submatrices of any $m \times n$ presentation matrix for the module [164, pp. 117-121]. (We may suppose $m \geq n$ without loss of generality, and we put $E_i = \Lambda$ if $i > n$.) Even these are fairly intractable, but since Λ is a unique factorization domain, each E_i is contained in a unique minimal principal ideal (Δ_i). One thus obtains a sequence of elements $\Delta_1, \Delta_2, \ldots, \Delta_n$ of Λ, each determined up to multiplication by a unit (the only units of Λ are $\pm t^r$, $r \in \mathbb{Z}$), such that $\Delta_{i+1} | \Delta_i$, $1 \leq i < n$. Suitably normalized, Δ_i is called the i^{th} <u>Alexander polynomial</u> of the knot, $\Delta_1 = \Delta$ being called simply the <u>Alexander polynomial</u>. Equivalently, one can consider the elements λ_i defined by $\lambda_i = \Delta_i / \Delta_{i+1}$; λ_i is the i^{th} <u>Alexander invariant</u>. These definitions are essentially contained in Alexander's paper [2].

The surjectivity of $t-1: H_1(X_\infty) \to H_1(X_\infty)$ can be expressed by saying that, regarding $\mathbb{Z}$ as a Λ-module via the augmentation homomorphism $\varepsilon: \Lambda \to \mathbb{Z}$, $H_1(X_\infty) \otimes_\Lambda \mathbb{Z} = 0$. It follows that $\varepsilon(E_i) = \mathbb{Z}$, and hence $\varepsilon(\Delta_i) = \Delta_i(1) = \pm 1$. It seems most natural (see §8) to normalize Δ_i so that it is a polynomial in t such that $\Delta_i(0) \neq 0$ and $\Delta_i(1) = 1$. From this it is not too hard to show that if the elements γ_i of Γ which describe the direct sum decomposition of $H_1(X_\infty;\mathbb{Q})$ are normalized so as to be polynomials with integer coefficients with g.c.d. 1, such that $\gamma_i(0) \neq 0$ and $\gamma_i(1) > 0$, then $\lambda_i = \gamma_i$, $1 \leq i \leq n$. It thus transpires that in the presence of the integral information $H_1(X) \cong \mathbb{Z}$, the Alexander polynomials are essentially rational invariants.

In view of the last remark, it is no surprise that the Alexander polynomials do not in general determine the elementary ideals. For example, the knot 9_{46} in the Alexander-Briggs table and the stevedore's knot (6_1) have modules $H_1(X_\infty)$ which are, respectively, $\Lambda/(2-t) \oplus \Lambda/(2t-1)$ and $\Lambda/(2-5t+2t^2)$. In both cases, $H_1(X_\infty;\mathbb{Q})$ is the cyclic Γ-module $\Gamma/(2-5t+2t^2)$. However, for the stevedore's knot, $E_2 = \Lambda$, whereas for 9_{46}, $E_2 = (2-t, 2t-1) \neq \Lambda$ (map Λ onto $\mathbb{Z}$ by $t \mapsto -1$; the image of $(2-t, 2t-1)$ is $3\mathbb{Z}$).

Again, the elementary ideals do not in general determine the Λ-module $H_1(X_\infty)$ (see [47]). Further invariants which have been studied include ideals in certain Dedekind domains, ideal classes, and Hermitian forms over certain rings of algebraic integers [47], [84]. A complete classification has not yet been found.

An important property of the Λ-module $H_1(X_\infty)$ is that it has deficiency 0. (Since $E_1 \neq 0$, any presentation of $H_1(X_\infty)$ must have at least as many relations as generators, so deficiency 0 just means that there is a presentation with the same number of generators and relations.) This may be seen by interpreting $H_1(X_\infty)$ as the abelianized commutator subgroup of the group π of K, and noting that π has a presentation of deficiency 1, for example, either the Wirtinger or Dehn presentation. (Since $H_1(\pi) \cong \mathbb{Z}$, it follows that the deficiency of π is 1.) It is also a consequence of duality (see §7), or, again, follows from the description of $H_1(X_\infty)$ in terms of a Seifert matrix (see §8; this is also related to duality). Deficiency 0 implies that the first elementary ideal E_1 is principal, i.e. $E_1 = (\Delta)$.

Returning briefly to rational coefficients, note that, up to multiplication by a rational unit, γ_1 is the minimal polynomial of the automorphism t of $H_1(X_\infty;\mathbb{Q})$, in other words, the annihilator of $H_1(X_\infty;\mathbb{Q})$ is $(\gamma_1) = (\lambda_1)$. Over Λ, it follows from general considerations, (see [164, p. 123], for example), that E_1 annihilates $H_1(X_\infty)$. Crowell [25] has shown that in fact the annihilator of $H_1(X_\infty)$ is precisely the principal ideal (λ_1) of Λ.

Turning to the abelian group structure of $H_1(X_\infty)$, this seems hard to describe in general, but we do have the result of Crowell [24] that $H_1(X_\infty)$ is always $\mathbb{Z}$-torsion-free. The crucial facts are, firstly, that $H_1(X_\infty)$ has deficiency 0 as a Λ-module, and, secondly, that the Alexander polynomial is primitive (i.e. g.c.d. of coefficients is 1; this follows from $\varepsilon(\Delta) = 1$). Here is the proof. Let A be a square presentation matrix for $H_1(X_\infty)$ over Λ. It must be shown that for any integer q, $A\underline{x} \equiv 0 \pmod q$ implies $\underline{x} \equiv 0 \pmod q$. But $(\operatorname{adj} A)(A\underline{x}) = (\det A)\underline{x} = \Delta\underline{x}$, and since Δ is primitive, this implies the result.

If $x_1, \ldots, x_n$ generate $H_1(X_\infty)$ as a Λ-module, then $\{t^j x_i : 1 \leq i \leq n, -\infty < j < \infty\}$ generate $H_1(X_\infty)$ over $\mathbb{Z}$. Since $\Delta x_i = 0$, $1 \leq i \leq n$, we see that if the constant coefficient of Δ (and hence, by the symmetry of Δ (see §7), the leading coefficient also) is ± 1, then $H_1(X_\infty)$ is finitely-generated over $\mathbb{Z}$, and is therefore free abelian of rank $\deg \Delta$. The converse is also true. For these and other results on the abelian group structure of $H_1(X_\infty)$, see [24], (also [121]).

5. The Finite Cyclic Covers

To relate $H_1(X_k;R)$ to $H_1(X_\infty;R)$, consider the short exact sequence of chain complexes

$$0 \to C_*(X_\infty;R) \xrightarrow{t^k-1} C_*(X_\infty;R) \to C_*(X_k;R) \to 0 .$$

As before, this gives rise to an exact sequence

$$H_1(X_\infty;R) \xrightarrow{t^k-1} H_1(X_\infty;R) \to H_1(X_k;R) \to R \to 0 .$$

If we give R the trivial Π-action, and $H_1(X_k;R)$ the Π-module structure induced by the canonical covering translation, this is an exact sequence of Π-modules. From this and the fact that $H_1(X_k;R) \cong H_1(M_k;R) \oplus R$ (with the trivial Π-action on R), it follows that, as Π- or $R[C_k]$-modules,

$$H_1(M_k;R) \cong \operatorname{coker}(t^k-1)^{(2)} .$$

This relation between $H_1(M_k;R)$ and $H_1(X_\infty;R)$ can be conveniently expressed in matrix terms. Let $B(t)$ be any presentation matrix for $H_1(X_\infty;R)$ over Π, with respect to generators $x_1, \ldots, x_n$, say. Then $\operatorname{coker}(t^k-1)$ is generated

(2) Throughout this section, it is understood that this refers to the action on $H_1(X_\infty;R)$.

over R by the images of $\{t^j x_i: 1 \leq i \leq n,\ 0 \leq j < k\}$, and with respect to these generators, is presented by the matrix $B(T)$ obtained from $B(t)$ by replacing a typical entry $\sum a_r t^r$ by $\sum a_r T^r$, where T is the $k \times k$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(see [52], [41], [112]).

Over certain coefficient rings, information can be extracted in other ways. For example, over $\mathbb{Z}/p$ (p prime), $(t^{p^r}-1) = (t-1)^{p^r}$ is an automorphism of $H_1(X_\infty; \mathbb{Z}/p)$; hence $H_1(M_{p^r}; \mathbb{Z}/p) = 0$. In particular, M_{p^r} is a $\mathbb{Q}$-homology sphere.

Again, if R is any field, from the direct sum decomposition

$$H_1(X_\infty;R) \cong \bigoplus_{i=1}^{n} \Pi/(\pi_i)$$

we obtain a similar decomposition

$$H_1(M_k;R) \cong \bigoplus_{i=1}^{n} \Pi/(\pi_i, t^k-1) \ .$$

Taking $R = \mathbb{C}$, we have the following further simplification pointed out by Sumners [144]. Applying $-\otimes\mathbb{C}$ to the decomposition $\bigoplus_{i=1}^{n} \Gamma/(\lambda_i)$ of $H_1(X_\infty;\mathbb{Q})$, and writing $\Psi = \mathbb{C}[C_\infty]$, we get $H_1(X_\infty;\mathbb{C}) \cong \bigoplus_{i=1}^{n} \Psi/(\lambda_i)$. Over Ψ, however, each $\Psi/(\lambda_i)$ decomposes as a direct sum $\oplus\, \Psi/((t-\alpha)^{e(\alpha)})$ over all distinct roots α of λ_i. Since $((t-\alpha)^{e(\alpha)}, t^k-1) = (t-\alpha)$ or Ψ according as α is or is not a k^{th} root of 1, we see that

$$\dim_{\mathbb{C}} H_1(M_k;\mathbb{C}) = \sum_{i=1}^{n} \ell_i$$

where ℓ_i is the number of distinct roots of λ_i which are k^{th} roots of 1. This result was first obtained by Goeritz [52], by explicitly diagonalizing $B(T)$ over $\mathbb{C}$.

Note that (as was pointed out in [52]), $H_1(M_k;\mathbb{C})$, or equivalently, the first Betti number of M_k, does not just depend on the Alexander polynomial $\Delta=\lambda_1, \ldots, \lambda_n$. The order of $H_1(M_k)$, however, does. Indeed, using Goeritz's diagonalization it may be shown that

$$\text{order } H_1(M_k) = |\det B(T)| = \left|\prod_{i=1}^{k} \Delta(\omega^i)\right|, \quad \text{where } \omega = e^{\frac{2\pi i}{k}} .$$

(This was first observed by Fox [41]; the proof given there, however, needs some modification.)

The behaviour of $H_1(M_k)$ as a function of k is sometimes quite interesting. For example, if k is odd, then $H_1(M_k)$ is always of the form $G \oplus G$ [119], [54]. Other results, in particular, necessary and sufficient conditions for $H_1(M_k)$ to be periodic in k, are given in [55].

We shall mention Seifert's work on branched cyclic covers [136], [137] in §8.

6. The Group Again

Let π be the group of a knot K. Since covering spaces of the exterior X of K correspond to subgroups of π, much of the material discussed in §§3-5 can be expressed in purely group-theoretic terms. Thus $\pi_1(X_\infty)$ is just the commutator subgroup π' of π, so $H_1(X_\infty)$ is isomorphic to π'/π''. The Λ-module structure of $H_1(X_\infty)$ can also be described group-theoretically: let $z \in \pi$ be any element which maps to the chosen generator t of C_∞; then the action of t on $H_1(X_\infty)$ corresponds to conjugation by z on π'/π''. Hence, given some presentation of π, it will be possible to derive a Λ-module presentation for π'/π''. If the presentation of π is in turn obtained in some way from a projection of K, we will then have a recipe for computing the Λ-module π'/π'' from a knot diagram. The algorithms described by Alexander [2] and Reidemeister [125, II, §14] are of this kind, based respectively on the Dehn and Wirtinger presentations of the knot group.

Similarly, for $1 \leq k < \infty$, $\pi_1(X_k)$ is isomorphic to the kernel π_k of the canonical epimorphism $\pi \to C_k$, so $H_1(X_k)$ can be identified with π_k/π_k'. Given a presentation of π, a presentation of π_k may be written down (using the Reidemeister-Schreier algorithm, for example), and hence a presentation (over $\mathbb{Z}$) of $H_1(X_k)$. (If one prefers to work with $H_1(M_k)$, then the branching relation must also be added, but as mentioned in §3, the difference between $H_1(X_k)$ and $H_1(M_k)$ is easy to take account of.) Thus again one can give a recipe for writing down a presentation matrix for (say) $H_1(M_k)$ in terms of a projection of the knot. This is done in [3] and [8]. (See also [125].)

Yet another algorithm for writing down a presentation of the Λ-module π'/π'' from a presentation of π is given by the free differential calculus of Fox [39], [40], [41] (see also [23], [26]), which we now briefly describe.

Let $P = (x_1, \ldots, x_n : r_1, \ldots, r_m)$ be a presentation of some group G. Corresponding to P, there is an obvious space X with $\pi_1(X) \cong G$, namely the finite 2-complex consisting of a single 0-cell p, n 1-cells, which we shall call $x_1, \ldots, x_n$, and m 2-cells, $D_1, \ldots, D_m$ (with base-points on their boundaries), the attaching map of D_i being r_i, $1 \leq i \leq m$. Now let H be some quotient of G, and $\tilde{X} \to X$ the regular covering with group of covering translations isomorphic to H. The cell structure of X lifts to a cell structure for $\tilde{X}$; choose a 0-cell $\tilde{p}$ lying over p, let $\tilde{x}_j$ be the unique lift of x_j which starts at $\tilde{p}$, and $\tilde{D}_i$ the unique lift of D_i such that $\partial\tilde{D}_i$ is the lift of r_i which starts at $\tilde{p}$. Then $C_0(\tilde{X})$, $C_1(\tilde{X})$, $C_2(\tilde{X})$ are the free $\mathbb{Z}[H]$-modules on $\{\tilde{p}\}$, $\{\tilde{x}_j : 1 \leq j \leq n\}$, and $\{\tilde{D}_i : 1 \leq i \leq m\}$ respectively.

The free differential calculus is a convenient tool for describing the boundary homomorphism $\partial_2 : C_2(\tilde{X}) \to C_1(\tilde{X})$, and consequently the $\mathbb{Z}[H]$-module $H_1(\tilde{X})$. (Since the latter can be described solely in terms of the group, we could use any space with $\pi_1(X) \cong G$; in particular, the result will be independent of the presentation P.)

Let F be the free group on $x_1, \ldots, x_n$, and $\varphi : \mathbb{Z}[F] \to \mathbb{Z}[G]$ the homomorphism induced by the epimorphism $F \to G$ corresponding to the presentation P. Let $\alpha : \mathbb{Z}[G] \to \mathbb{Z}[H]$ be the quotient homomorphism. For each j, $1 \leq j \leq n$, there is a unique $\mathbb{Z}$-linear function

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F] \to \mathbb{Z}[F]$$

such that

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

and
$$\frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j} .$$

If w is any word in the x_j's, regarded as a loop in X based at p, w lifts to a unique path $\tilde{w}$ starting at $\tilde{p}$. It may then be readily verified (for example, by induction on the length of w) that, as a 1-chain in $\tilde{X}$,

$$\tilde{w} = \sum_{j=1}^{n} \alpha\varphi(\frac{\partial w}{\partial x_j})\tilde{x}_j .$$

In particular, with respect to the $\mathbb{Z}[H]$-bases $\{\tilde{D}_i : 1 \leq i \leq m\}$, $\{\tilde{x}_j : 1 \leq j \leq n\}$, $\partial_2 : C_2(\tilde{X}) \to C_1(\tilde{X})$ is given by the $m \times n$ matrix

$$(\alpha\varphi(\frac{\partial r_i}{\partial x_j})) .$$

One also sees that $\partial_1 : C_1(\tilde{X}) \to C_0(\tilde{X})$ is given by

$$\partial_1(\tilde{x}_j) = (\alpha\varphi(x_j)-1)\tilde{p} .$$

The short exact sequence

$$0 \to \ker \partial_1 \to C_1(\tilde{X}) \to \operatorname{im} \partial_1 \to 0$$

gives, after factoring out by $\operatorname{im} \partial_2$, the short exact sequence (of $\mathbb{Z}[H]$-modules)

$$0 \to H_1(\tilde{X}) \to \operatorname{coker} \partial_2 \to \operatorname{im} \partial_1 \to 0 .$$

Since $\operatorname{coker} \partial_2$ is presented by the 'Jacobian' matrix described above, and since we know $\operatorname{im} \partial_1$, we can extract information about $H_1(\tilde{X})$.

In fact, specializing to the knot situation, with $G = \pi$ and $H = C_\infty$, it is not hard to prove that $\operatorname{im} \partial_1 \cong \Lambda$. The above sequence therefore splits, showing that the matrix $(\alpha\varphi(\frac{\partial r_i}{\partial x_j}))$ is a presentation matrix for the Λ-module $(\pi'/\pi'') \oplus \Lambda$.

7. Duality

The modules $H_1(X_\infty;R)$ have additional properties derived from duality. These are somewhat deeper, and the history reflects this. For example, the fact that $\Delta_i(1) = 1$ was proved by Alexander in [2], whereas the symmetry property $\Delta(t) = t^{\deg \Delta}\Delta(t^{-1})$ was first proved by Seifert [136], (the explanation given by Reidemeister in [125, p. 40], in terms of the group, seems to be insufficient), and not fully explained as a duality property until Blanchfield [12]. We now briefly discuss this duality, following Levine [85].

The chain module $C_q = C_q(X_\infty, \partial X_\infty;R)$ is a free Π-module on the q-simplices in $X - \partial X$ of some triangulation of X. Let $C'_q = C'_q(X_\infty;R)$ be the chains on the lifts of the q-simplices of the dual triangulation of X. There is then a non-singular pairing (see [95])

$$\langle\ ,\ \rangle: C_q \times C'_{3-q} \to \Pi$$

defined by

$$\langle c, c' \rangle = \sum_{i=-\infty}^{\infty} (c \cdot t^i c')t^i ,$$

where $\cdot$ denotes ordinary intersection number. This pairing is sesquilinear with respect to the conjugation $^-$ of Π induced by $t \longmapsto t^{-1}$. It induces a duality isomorphism

$$\overline{H_q(X_\infty, \partial X_\infty; R)} \cong H^{3-q}(\mathrm{Hom}_\Pi(C'_*, \Pi)) ,$$

where — denotes the conjugate module in which the action of $\pi \in \Pi$ is defined by $a \longmapsto \bar{\pi}a$. We are mainly interested in the case $q = 1$. Let us then note that since $H_1(\partial X_\infty; R)$ is generated by the boundary of the lift of a Seifert surface, $H_1(\partial X_\infty; R) \to H_1(X_\infty; R)$ is zero, and hence $H_1(X_\infty; R) \cong H_1(X_\infty, \partial X_\infty; R)$.

Now suppose R is a field, so that Π is a principal ideal domain. Then, by the universal coefficient theorem and the fact that $H_2(X_\infty; R)$ is Π-torsion, (the surjectivity of $t-1$ on $H_2(X_\infty; R)$ follows in the same way as for $H_1(X_\infty; R)$), we get

$$\overline{H_1(X_\infty; R)} \cong \mathrm{Ext}_\Pi(H_1(X_\infty; R), \Pi) .$$

Since $H_1(X_\infty; R)$ is also Π-torsion, we finally obtain the fundamental duality isomorphism

$$\overline{H_1(X_\infty; R)} \cong H_1(X_\infty; R) .$$

In particular, taking $R = \mathbb{Q}$, this implies the familiar duality property of the Alexander polynomials

$$(\Delta_i) = (\bar{\Delta}_i) , \quad \text{i.e.} \quad \Delta_i(t) = t^{\deg \Delta_i} \Delta_i(t^{-1}) .$$

(Note that this, and the fact that $\Delta_i(1) = 1$, implies that $\deg \Delta_i$ is even.)

Now consider the case $R = \mathbb{Z}$. Levine [85] shows that, since Λ has global dimension 2, the universal coefficient spectral sequence still gives us an isomorphism

$$\overline{H_1(X_\infty)} \cong \mathrm{Ext}_\Lambda(H_1(X_\infty), \Lambda) .$$

It follows from this, incidentally, that $H_1(X_\infty)$ is $\mathbb{Z}$-torsion-free. (Here is the argument; see [85, p. 9]. For any positive integer m, the short exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{m} \Lambda \longrightarrow \Lambda/m\Lambda \longrightarrow 0$$

gives rise to an exact sequence

$$\mathrm{Hom}_\Lambda(H_1(X_\infty),\Lambda/m\Lambda) \longrightarrow \mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda) \xrightarrow{m} \mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda) .$$

But $H_1(X_\infty)$ is annihilated by Δ, which is primitive since $\varepsilon(\Delta) = 1$, and multiplication by a primitive on $\Lambda/m\Lambda$ is injective, by the Gauss lemma. Hence $\mathrm{Hom}_\Lambda(H_1(X_\infty),\Lambda/m\Lambda) = 0$ and multiplication by m on $\mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda)$ is injective.)

It is interesting to note that over Λ, however, we no longer necessarily have the strong duality statement $\overline{H_1(X_\infty)} \cong H_1(X_\infty)$. Failure of this may sometimes be detected, for example, by the ideal class invariant described in [47].

Returning to arbitrary (Noetherian) coefficients R, here is a slightly different interpretation of duality. Since $H_1(X_\infty;R)$ is Π-torsion, $\langle\ ,\ \rangle$ induces a form

$$\beta_R: H_1(X_\infty;R) \times H_1(X_\infty;R) \longrightarrow Q(\Pi)/\Pi ,$$

where $Q(\Pi)$ denotes the field of fractions of Π. The definition of β_R is as follows. (Note the analogy with the $\mathbb{Q}/\mathbb{Z}$-valued linking form on the torsion subgroup of the first homology of an oriented 3-manifold.) Let $c \in C_1$, $d \in C_1'$ be representative cycles for elements $x,y \in H_1(X_\infty;R) \cong H_1(X_\infty,\partial X_\infty;R)$. Since $H_1(X_\infty;R)$ is Π-torsion, there exists $c' \in C_2'$ such that $\partial c' = \pi d$ for some nonzero $\pi \in \Pi$. Define

$$\beta_R(x,y) = \frac{\langle c,c'\rangle}{\pi} .$$

This form β_R is sesquilinear and Hermitian, and is called the <u>Blanchfield pairing</u> (over R) of the knot. (See [12].)

Consider the case when R is a field. Since the adjoint to β_R is the composition

$$\overline{H_1(X_\infty;R)} \cong \mathrm{Ext}_\Pi(H_1(X_\infty;R),\Pi) \cong \mathrm{Hom}_\Pi(H_1(X_\infty;R),Q(\Pi)/\Pi) ,$$

β_R is non-singular. Here, the first isomorphism comes from duality and universal coefficients, and the second from the short exact sequence

$$0 \longrightarrow \Pi \longrightarrow Q(\Pi) \longrightarrow Q(\Pi)/\Pi \longrightarrow 0 ,$$

using the fact that $H_1(X_\infty;R)$ is Π-torsion.

It turns out that this is also true when $R = \mathbb{Z}$ (see [12], [85]), that is, $\beta = \beta_{\mathbb{Z}}$ induces an isomorphism

$$\overline{H_1(X_\infty)} \cong \mathrm{Hom}_\Lambda(H_1(X_\infty),Q(\Lambda)/\Lambda) .$$

As regards the classification of Blanchfield pairings, the case $R = \mathbb{Q}$ has been done, as follows. In [152], Trotter defines a function $\chi\colon Q(\Gamma)/\Gamma \longrightarrow \mathbb{Q}$ such that

$$\chi\beta_{\mathbb{Q}} : H_1(X_\infty;\mathbb{Q}) \times H_1(X_\infty;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

is non-singular, skew-symmetric, has $t\colon H_1(X_\infty;\mathbb{Q}) \longrightarrow H_1(X_\infty;\mathbb{Q})$ as an isometry, and has the property that the isomorphism class of the pair $(\chi\beta_{\mathbb{Q}}, t)$ determines the isometry class of $\beta_{\mathbb{Q}}$. But pairs consisting of a non-singular ε-symmetric $(\varepsilon = \pm 1)$ bilinear form on a finite dimensional R-vector space, together with an isometry, have been classified (see [98]). The rational Blanchfield pairings are thereby also classified.

When $R = \mathbb{Z}$, a complete classification has not yet been achieved. (See [84], [152], for partial results; also §9.)

8. The Seifert Form

So far, we have tried to keep the discussion as 'co-ordinate-free' as possible All the cyclic covers of a knot K, however, can be constructed with the aid of any orientable surface spanning K, and it is illuminating to examine the properties discussed above from this point of view. Indeed, this is the way in which many of them were first discovered.

It was an early observation that colouring alternately black and white the regions created by a general position projection of a knot K onto the plane defines two surfaces in S^3 bounded by K. However, it may happen that neither of these is orientable. (It is not hard to show that only the trivial projection of the trivial knot has both surfaces orientable.) The first proof that every knot does bound an orientable surface is given in [48]. Later, Seifert [136] gave another proof, and used such a spanning surface, now called a Seifert surface for K, to derive many important invariants and their properties.

One might proceed as follows. Let the knot K have tubular neighbourhood N and exterior X. Then $N \cong S^1 \times D^2$ and $\partial X = \partial N \cong S^1 \times S^1$. The composition $H_2(N,\partial N) \overset{\text{excision}}{\cong} H_2(S^3,X) \overset{\partial}{\cong} H_1(X)$ shows that $H_1(X) \cong \mathbb{Z}$ is generated by the image of the meridian element $\mu = [* \times S^1] \in H_1(\partial X)$. Let λ be a generator for $\ker(H_1(\partial X) \longrightarrow H_1(X))$. We may choose a trivialization of the normal bundle N so that $\lambda = [S^1 \times *] \in H_1(\partial X)$. Projection $\partial X \longrightarrow S^1$ onto the second factor now extends to $p: X \longrightarrow S^1$. Making p transverse regular, $\mathrm{rel}(p|\partial X)$, to a point in S^1 then gives a bicollared surface $F \subset X$ such that $\partial F (= F \cap \partial X)$ and K together bound an annulus in N. We may assume that F is connected. Cutting X along this Seifert surface F gives a manifold Y whose boundary contains two copies $F^{\pm}$ of F. Taking a countable infinity $\{Y_i\}$ of copies of Y and identifying F^-_{i+1} with F^+_i in the obvious way, for all i, we obtain the infinite cyclic cover X_∞ of X, on which the group C_∞ of covering translations acts by taking each Y_i to Y_{i+1}.

It turns out that all the algebraic information about $H_1(X_\infty)$ discussed above is contained in the Seifert form of F, that is, the bilinear form

$$\alpha: H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

defined by

$$\alpha([w],[z]) = \mathrm{lk}(w^+, z) \ ,$$

where w, z are 1-cycles in F, w^+ is the cycle obtained by translating w off F in the positive normal direction, and lk denotes linking number in S^3. This form has the property that $\alpha - \alpha^T$ (^{T} denotes transpose) is just the intersection form on $H_1(F)$. In particular, $\det(\alpha-\alpha^T) = 1$. Choosing some basis for $H_1(F)$, we get a $2h \times 2h$ <u>Seifert matrix</u> A representing α, where $h = \text{genus } F$.

A Mayer-Vietoris argument on $X_\infty = \bigcup_{i=-\infty}^{\infty} Y_i$ shows [80] that $H_1(X_\infty)$ is presented as a Λ-module by the matrix $tA - A^T$. In particular, up to a unit of Λ, the Alexander polynomial $\Delta = \det(tA-A^T)$. Since putting $t = 1$ gives the unimodular matrix $A - A^T$ (3), the properties $\varepsilon(E_i) = \mathbb{Z}$, $\varepsilon(\Delta_i) = 1$, of the elementary ideals and Alexander polynomials are immediate.

The consequences of duality are also easily seen in this setting. For example the conjugate module $\overline{H_1(X_\infty)}$ is presented by the matrix $t^{-1}A - A^T$, which is equivalent to $(tA-A^T)^T$. In particular, $\overline{E}_i = E_i$ and $(\overline{\Delta}_i) = (\Delta_i)$ for all i. Since $\det(tA-A^T) \neq 0$, the presentation of $H_1(X_\infty)$ corresponding to $tA - A^T$ is actually a short free resolution

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_0 \longrightarrow H_1(X_\infty) \longrightarrow 0 \ ,$$

where F_0, F_1 are free Λ-modules of rank $2h$. Hence $\mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda) \cong \mathrm{coker}(\mathrm{Hom}(\varphi, \mathrm{id}))$, and the latter is clearly presented by $(tA-A^T)^T$. So we derive our previous duality statement

(3) This is why it is natural, at least for $i = 1$, to normalize so that $\Delta_i(1) = 1$.

$$\overline{H_1(X_\infty)} \cong \mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda) \ .$$

The Blanchfield pairing $\beta: H_1(X_\infty) \times H_1(X_\infty) \longrightarrow Q(\Lambda)/\Lambda$ is also determined by A; it is given by the matrix $(1-t)(tA-A^T)^{-1}$ [70], [85], [152].

Finally, we mention that the matrix M defined by Murasugi [107] in terms of a knot projection can be shown to be a Seifert matrix for a Seifert surface constructed from the knot projection [138].

Turning to the finite cyclic covers, if we write $B(t) = tA - A^T$, then (see §5) $B(T)$ will be a presentation matrix for $H_1(M_k)$. Now $B(T)$ is $2hk \times 2hk$, but Seifert [136] showed how to reduce it, using the permissible matrix operations, to the $2h \times 2h$ matrix $C^k - (C-I)^k$, where $C = A(A-A^T)^{-1}$. He also showed [137] that (and in what sense) the linking form $T_1(M_k) \times T_1(M_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is determined by the matrix $(C-I)^k(A-A^T)$. (See [150] for a more general formulation.) This can often be used to detect non-amphicheirality.

9. S-Equivalence

The Seifert form α is clearly an invariant of the pair (S^3,F). Hence, allowing for a change of basis of $H_1(F)$, the equivalence class of A under integral congruence $A \mapsto P^TAP$, P invertible over $\mathbb{Z}$, is an invariant of (S^3,F). (If we choose a symplectic basis for $H_1(F)$, A will satisfy $A - A^T = J = \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In [150], such an A is called a standard Seifert matrix. Then every Seifert matrix is congruent to a standard one, and two standard Seifert matrices A, B are congruent if and only if they are symplectically congruent, that is, $B = P^TAP$ where P satisfies $P^TJP = J$.)

Since we may always increase the genus of any Seifert surface F for K by adding a 'hollow handle' to it, it is clear that to get an invariant of the knot we must also allow matrix enlargements of the form

$$A \longmapsto \begin{bmatrix} & A & & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & A & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

(The *'s record the way the handle links F.) The equivalence relation on Seifert matrices generated by congruence and these enlargements is known as S-equivalence. It will also be convenient to call two knots S-equivalent if they have S-equivalent Seifert matrices.

S-equivalence was first introduced, in an algebraic setting, by Trotter [150]. It also appears in [107]. The following remarks show that it is likely to be an important concept. Firstly, any two Seifert matrices for a given knot K are S-equivalent.(4) (Here is an outline of a proof. Let the matrices be associated with Seifert surfaces F_0, F_1 for K, and let these in turn correspond, via transversality, to maps $p_0, p_1 \colon X \longrightarrow S^1$, such that $p_0|\partial X = p_1|\partial X$, where X is the exterior of K. Then p_0, p_1 extend to $p\colon X \times I \longrightarrow S^1$, with $p_t|\partial X = p_0|\partial X$ for all $t \in I$. Transverse regularity gives a connected, orientable 3-manifold $M \subset X \times I$ such that $\partial M = F_0 \cup \partial F_i \times I \cup F_1$. Now choose a handle decomposition of M on F_0 with only 1- and 2-handles, such that the former precede the latter, and such that, regarding M as $F_0 \cup \text{collar} \cup \text{handle} \cup \text{collar} \ldots$, each handle is embedded in a level $X \times \{t\}$, and the collars are compatible with the I factor (see [72]). Then in a level between the 1- and 2-handles, M intersects X in a Seifert surface for K which is obtained from each of F_0, F_1 by adding hollow handles.) Secondly, given a Seifert matrix A for K, it is easy to see that any matrix obtained from A by a sequence of enlargements (and congruences) is also a Seifert matrix for K. (But this is not necessarily true for reductions.)

(4) In [107], it is noted that by examining the effects of the Reidemeister moves on a knot diagram, the S-equivalence class of the Murasugi matrix can be shown to to be an invariant of K.

Thirdly, in higher (odd) dimensions, S-equivalence completely classifies the so-called simple knots [83].

Probably the most important result concerning S-equivalence relates it to the Blanchfield pairing:

Two knots are S-equivalent if and only if their (integral) Blanchfield pairings are isometric.

A purely algebraic proof of this has been given by Trotter [152]. It is also a consequence of some results of Kearton [70] and Levine [83] on higher-dimensional knots. (In [83] it is shown that, for $n \geq 2$, two simple knots of S^{2n-1} in S^{2n+1} are isotopic if and only if they are S-equivalent, and, in [70], that they are isotopic if and only if their Blanchfield pairings are isometric. Since the algebra only depends on $n \pmod 2$, this implies the stated result.)

In [150] it is shown that every Seifert matrix is S-equivalent to a non-singular one, that is, one with $\det A \neq 0$. Since $\Delta = \det(tA-A^T)$, we see that $\det A = \Delta(0)$ is then an invariant of the knot. Also, the Γ-module $H_1(X_\infty;\mathbb{Q})$ is presented by $tI - A^{-1}A^T$, which shows that $\dim H_1(X_\infty;\mathbb{Q}) = 2h$ (if A is $2h \times 2h$), and that the automorphism t is given by the matrix $A^{-1}A^T$.

It is known that S-equivalence of non-singular Seifert matrices is definitely weaker than integral congruence [83], but there are the following partial results in the other direction. Non-singular Seifert matrices determine isometric rational Blanchfield pairings (recall (§7) that these are classified) if and only if they are congruent over $\mathbb{Q}$ [152]. If A and B are S-equivalent non-singular Seifert matrices, so $\det A = \det B = d$, say, then A and B are congruent over $\mathbb{Z}[d^{-1}]$ [150], [83], [152]. (The converse is false [83].) If $|d|$ is prime, then in fact A and B are congruent over $\mathbb{Z}$ [152]. If d is square-free, then A and B are congruent over the p-adic integers $\mathbb{Z}_p$ for all primes p [152].

10. Characterization

The first realization result concerning the invariants we have been discussing is Seifert's proof [136] that a polynomial Δ is the Alexander polynomial of a knot if and only if it satisfies

(i) $\Delta(1) = 1$, and

(ii) $\Delta(t) = t^{\deg \Delta}\Delta(t^{-1})$.

To do this, Seifert actually shows that any integral matrix A such that $A - A^T = J$ can be realized as a Seifert matrix. This is done by taking an orientable surface of the appropriate genus, regarded as a disc with bands, and embedding it in S^3 by twisting and linking the bands so as to realize A as the matrix (with respect to the basis of $H_1(F)$ represented by the cores of the bands) of the Seifert form. It follows (by changing basis) that any matrix A with $\det(A-A^T) = 1$ is a Seifert matrix.

It turns out that in Seifert's realization of the polynomial, the module which arises, i.e. the module presented by $tA - A^T$, is actually the cyclic Λ-module $\Lambda/(\Delta)$. By taking connected sums, it follows that any sequence of polynomials $\lambda_1, \ldots, \lambda_n$ satisfying the (necessary) conditions

(i) $\lambda_i(1) = 1$, $1 \leq i \leq n$

(ii) $\lambda_i(t) = t^{\deg \lambda_i}\lambda_i(t^{-1})$, $1 \leq i \leq n$, and

(iii) $\lambda_{i+1}|\lambda_i$, $1 \leq i < n$

can occur as the Alexander invariants of a knot. (This can be equivalently expressed in terms of the Alexander polynomials.) A different proof is given in [79].

In particular, the Γ-modules which can occur as $H_1(X_\infty;\mathbb{Q})$ for some knot are completely and simply characterized.

Over the integers, we have the following realization result of Levine [85], which brings in the Blanchfield pairing:

Let H be a finitely-generated Λ-module such that $t-1\colon H \to H$ is surjective, and let $\beta\colon H \times H \to Q(\Lambda)/\Lambda$ be a non-singular, sesquilinear, Hermitian pairing. Then β is the Blanchfield pairing of some knot.

To prove this, it is sufficient to show that every such β is given by $(1-t)(tA-A^T)^{-1}$ for some integral matrix A with $\det(A-A^T) = 1$. (Is there a direct algebraic proof of this?) This Levine does by showing that β may be realized as the Blanchfield pairing of some knot of S^{4k+1} in S^{4k+3}, for (any) $k > 0$; a Seifert matrix for this knot is then the desired A.

11. The Quadratic and Other Forms

Because of its historical significance, we shall now make a few remarks about the quadratic form of a knot, although from many points of view this is best discussed in a 4-dimensional setting (see §12).

There are actually two distinct, but related, concepts here. The first is due to Goeritz [51], who associated with a knot diagram an integral quadratic form as follows. Colour the regions of the diagram alternately black and white, the unbounded region being coloured white,[5] and number the other white regions $W_1, \ldots, W_n$. At a crossing point c as shown in Figure 1

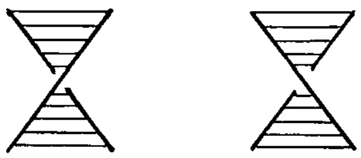

Figure 1

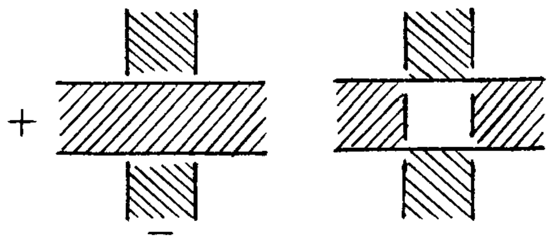

Figure 2

assign 1, -1 respectively if the adjacent white regions are distinct, and 0 otherwise. Call this index $\eta(c)$. Then define the $n \times n$ matrix $G = (g_{ij})$ by

$$g_{ii} = \sum \eta(c) \quad \text{over crossings adjacent to } W_i,$$

$$g_{ij} = -\sum \eta(c) \quad \text{over crossings adjacent to } W_i \text{ and } W_j, \quad i \neq j.$$

It may be verified [51], [76] that the class of G under the equivalence relation generated by (integral) congruence and

$$G \longmapsto \begin{bmatrix} G & 0 \\ 0 & \pm 1 \end{bmatrix}$$

[5] Goeritz chose black, but it turns out that this is psychologically confusing.

is invariant under any of the 3 so-called Reidemeister moves [3], [123] on a knot diagram, and is therefore an invariant of the knot K. In particular, the absolute value of the determinant, and the Minkowski units C_p for odd primes p, are invariants of K, (but C_2 and the signature are not) [51].

In [137], Seifert relates G to the 2-fold branched cover M_2 of K, by observing that the latter can be obtained by cutting S^3 along the spanning surface for K corresponding to the shaded regions of the knot projection and gluing together two copies of the resulting manifold in an appropriate fashion. In particular, he shows that G is a presentation matrix for $H_1(M_2)$, and that the linking form $H_1(M_2) \times H_1(M_2) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is given by $\pm G^{-1}$, the sign depending on the orientation of M_2. (See also §12.) Note that $|\det G| = \text{order } H_1(M_2) = |\Delta(-1)|$ is always odd.

Such linking forms are classified by certain ranks and quadratic characters corresponding to each p-primary component (p an odd prime). See [135], [62]. In [120] (see also [78]) it is shown that these invariants determine the Minkowski units C_p, and, more generally, Kneser-Puppe in [76] show that in fact the linking form completely determines the equivalence class (in the above sense) of the quadratic form.

More recently, Trotter [150] considered the quadratic form given by $A + A^T$, where A is a Seifert matrix for K. (See also [107], which studies $M + M^T$, where M is the Murasugi matrix.) S-equivalence on A induces the equivalence relation on $A + A^T$ generated by congruence and addition of a hyperbolic plane $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is a stronger equivalence than the one discussed previously. Also, it may be shown that if the shaded surface F obtained from a knot projection happens to be orientable, then the corresponding Goeritz matrix coincides with $A + A^T$ for some Seifert matrix A associated with F. Finally, for any Seifert matrix A of K, $A + A^T$ is in the equivalence class of Goeritz matrices of K. This may be seen by isotoping the given Seifert surface, regarded as a disc with bands, so that the bands cross over as shown in Figure 2, where + denotes one side of the surface and - the other. The modification shown in Figure 2 produces an orientable surface obtainable from the indicated knot projection by shading; the corresponding Goeritz matrix will then be $B+B^T$ for some Seifert matrix B, which must be S-equivalent to A. (Here, it is easy to see the S-equivalence directly: join the two bands by a 1-handle at each band-crossing.)

From Trotter's form additional invariants may be extracted, notably the signature (therefore referred to as the signature of the knot, $\sigma(K)$), and hence also the Minkowski unit C_2. Also, recall that any non-singular Seifert matrix for K is $2h \times 2h$, where $2h = \deg \Delta$. Hence it follows from Witt's cancellation theorem that over any local ring in which 2 is invertible, the forms $A+A^T$ coming from non-singular Seifert matrices A are all congruent. In particular, this holds for the p-adic integers $\mathbb{Z}_p$, p odd, and $\mathbb{R}$, and hence (since $A+A^T$ is even, see [62]) the genus of $A+A^T$ is an invariant of K [150].

The forms of both Goeritz and Trotter are generalized in [58], where it is shown how a quadratic form may be defined for any spanning surface. The signature of such a form is related to the signature $\sigma(K)$ of the knot. In particular, the correction term needed to obtain $\sigma(K)$ from the signature of a Goeritz matrix can be simply described in terms of the given knot projection.

By symmetrizing A to $A+A^T$, we obtained the signature. Other signatures may be obtained, by Hermitianizing A in other ways. Precisely, let ξ be a complex number, and consider the Hermitian matrix $A(\xi) = (1-\bar{\xi})A + (1-\xi)A^T$. (We may suppose without loss of generality that $\xi \in S^1$, that is, $|\xi| = 1$.) Then S-equivalence on A induces the equivalence relation on $A(\xi)$ generated by congruence (by integral matrices) and addition of $\begin{bmatrix} 0 & 1-\bar{\xi} \\ 1-\xi & 0 \end{bmatrix}$. In particular, the signature of $A(\xi)$ depends only on K, and therefore defines a function $\sigma_K: S^1 \longrightarrow \mathbb{Z}$. Since $A(\xi) = (\xi-1)(\bar{\xi}A-A^T)$, σ_K is continuous away from the roots of the Alexander polynomial $\Delta = \det(tA-A^T)$. These signatures $\sigma_K(\xi)$ are (essentially) those considered by Levine in [81]. For certain roots of unity ξ, they were introduced earlier by Tristram [148]. We shall see later (§12) that for ξ a root of unity, $\sigma_K(\xi)$ actually has a natural geometric interpretation.

Another approach to these signatures is the following. Milnor [96] and Erle [34] show that, over any field R, the (skew-symmetric) cup product pairing

$$H^q(X_\infty, \partial X_\infty; R) \times H^{2-q}(X_\infty, \partial X_\infty; R) \to H^2(X_\infty, \partial X_\infty; R) \cong R$$

is non-singular. Taking $q = 1$ and setting

$$\langle x, y\rangle = x \cup (ty) + y \cup (tx)$$

then defines a non-singular, R-valued, symmetric bilinear form $\langle\ ,\ \rangle$ on $H^1(X_\infty, \partial X_\infty; R)$. With respect to an appropriate basis, $\langle\ ,\ \rangle$ is given by $A + A^T$, where A is a non-singular Seifert matrix, and thus coincides with Trotter's quadratic form (tensored with R). (See [34] for details.)

We remark that the non-singularity of the above cup product pairing can be interpreted as a Poincaré duality in X_∞ of formal dimension 2. However, this non-singularity definitely fails over $\mathbb{Z}$; for example, $H^1(X_\infty, \partial X_\infty)$ $(\cong H^1(X_\infty))$ is often zero.

Taking $R = \mathbb{R}$, let λ be a symmetric, irreducible factor of the Alexander polynomial, so $\lambda = (t-\xi)(t-\bar{\xi})$ where $\xi = e^{i\theta}$, say. Milnor [96] then defines $\sigma_\theta(K)$ to be the signature of the restriction of $\langle\ ,\ \rangle$ to the λ-primary component. The signature of the knot $\sigma(K)$ is the sum of all the $\sigma_\theta(K)$.

These signatures $\sigma_\theta(K)$ turn out to be equivalent to the signature function σ_K; Matumoto has shown [94] that $\sigma_\theta(K)$ is just the jump in σ_K at $e^{i\theta}$.

12. Some 4-Dimensional Aspects

It is enlightening to consider the branched cyclic covers from a 4-dimensional point of view. The basic construction is the following. Pushing the interior of a Seifert surface F for K in S^3 into the interior of the 4-ball B^4 gives a properly embedded surface $\hat{F} \subset B^4$ with $\partial\hat{F} = K$. For $1 \leq k < \infty$, we then have $M_k = \partial V_k$, where M_k, V_k is the k-fold branched cyclic cover of (S^3, K), $(B^4, \hat{F})$ respectively.

Let us first consider the case $k = 2$. In S^3, choose a thickening $F \times [-1,1]$ of $F \times 0$. Then V_2 may be constructed by taking two copies of B^4, and identifying (x,t) in one copy with $(x,-t)$ in the other, for all $x \in F$, $t \in [-1,1]$, (and then smoothing). The canonical covering translation just interchanges the copies of B^4. A Mayer-Vietoris argument shows that $H_2(V_2) \cong H_1(F)$, and that if A is the Seifert matrix associated with some basis of $H_1(F)$, then the intersection form on $H_2(V_2)$, with respect to the corresponding basis, is given by the matrix $A + A^T$ (see, for example, [69]).

Actually this works even if F is non-orientable. A thickening of F will now be a twisted $[-1,1]$-bundle over F, but we may still carry out the above construction using the local product structure. The intersection form on $H_2(V_2)$ can again be described in terms of F; in particular, if F arises from the shaded regions of a knot diagram, then the intersection form is given by the Goeritz matrix G [58].

By duality we have the exact sequence

$$H_2(V_2) \xrightarrow{\varphi} \mathrm{Hom}(H_2(V_2), \mathbb{Z}) \longrightarrow H_1(M_2) \longrightarrow 0$$

where φ is adjoint to the intersection form. Thus, if the latter is given by a matrix B, say, then φ will be represented by B with respect to dual bases. It is then clear that B is a presentation matrix for $H_1(M_2)$. It also follows, using $\det B \neq 0$, that the linking form on $H_1(M_2)$ is given by $-B^{-1}$. This recovers the results of Seifert [137] on the 2-fold branched cover.

Now let us consider the higher order branched covers; here, F must be orientable. As before, the intersection form on $H_2(V_k)$ may be described in terms of the Seifert form of F. In particular, one may write down a presentation matrix B for $H_1(M_k)$ in terms of a Seifert matrix, and again, if $H_1(M_k)$ is finite (as will always be the case if k is a prime-power, for example), the linking form on $H_1(M_k)$ will be given by $-B^{-1}$.

Using the cyclic group action, one may derive finer information. The intersection form on $H_2(V_k)$ extends naturally to a Hermitian form on $H_2(V_k;\mathbb{C})$, with respect to which the automorphism τ of $H_2(V_k;\mathbb{C})$, induced by the canonical covering translation, is an isometry. Let $\omega = e^{\frac{2\pi i}{k}}$. Then $H_2(V_k;\mathbb{C})$ decomposes as an orthogonal direct sum $E_0 \oplus E_1 \oplus \ldots \oplus E_{k-1}$, where E_r is the ω^r-eigenspace of τ. Let $\sigma_r(V_k)$ be the signature of the restriction of our Hermitian form to E_r. It then turns out that

$$\sigma_r(V_k) = \operatorname{sign}((1-\omega^{-r})A + (1-\omega^r)A^T), \qquad 0 \leq r < k$$

where A is a Seifert matrix for F. (See [32], [154], [18]). These signatures $\sigma_r(V_k) = \sigma_K(\omega^r)$, $0 < r < k$, are the <u>k-signatures</u> of the knot K. In particular, $\sigma_1(V_2)$ is just the signature of V_2.

We saw earlier that $\sigma_K(\xi)$ depends only on K. Here, rather more is true. We could construct V_k with $\partial V_k = M_k$ using any (orientable) surface $F \subset B^4$ with $\partial(B^4, F) = (S^3, K)$. Then $\sigma_r(V_k)$ is independent of F. To see this, we shall use the G-signature theorem [6]; (for an elementary proof for semi-free actions in dimension 4, which is all that is needed here, see [57]). Recall that the τ^s-signatures $\operatorname{sign}(\tau^s, V_k)$ are defined as follows. We have $H_2(V_k; \mathbb{C}) = H^+ \oplus H^- \oplus H^0$, where the Hermitian 'intersection' form is $\pm$-definite on $H^{\pm}$ and zero on H^0. Then

$$\operatorname{sign}(\tau^s, V_k) = \operatorname{trace}(\tau^s | H^+) - \operatorname{trace}(\tau^s | H^-) .$$

By similarly decomposing each eigenspace $E_r = E_r^+ \oplus E_r^- \oplus E_r^0$, we may take $H^+ = E_0^+ \oplus E_1^+ \oplus \ldots \oplus E_{k-1}^+$, etc., which (recalling that $\sigma_0(V_k) = 0$) shows that

$$\operatorname{sign}(\tau^s, V_k) = \sum_{r=1}^{k-1} \omega^{rs} \sigma_r(V_k) , \qquad 0 < s < k .$$

Inverting, we obtain

$$\sigma_r(V_k) = \frac{1}{k} \sum_{s=1}^{k-1} (\omega^{-rs} - 1) \operatorname{sign}(\tau^s, V_k) , \qquad 0 < r < k .$$

Now suppose V_k, V_k' arise from surfaces $F, F' \subset B^4$ with $\partial F = \partial F' = K$. Let $W = V_k \cup -V_k'$, identified along M_k. Since the fixed-point set $F \cup -F'$ of the C_k-action on W has trivial normal bundle, the G-signature theorem gives $\operatorname{sign}(\tau^s, W) = 0$, $0 < s < k$, and therefore, by Novikov additivity,

$$\operatorname{sign}(\tau^s, V_k) = \operatorname{sign}(\tau^s, V_k') , \qquad 0 < s < k .$$

Hence

$$\sigma_r(V_k) = \sigma_r(V_k') , \qquad 0 < r < k ,$$

as required.

A variation of the above proof in the case $k = 2$ allows one to compute the signature of a knot K from an arbitrary (not necessarily orientable) surface F in B^4 with $\partial F = K$ [58]. In particular, this leads to the relation between the signature of K and the signature of any Goeritz matrix for K which was alluded to in §11.

13. Concordance

Two knots K_0, K_1 in S^3 (everything oriented) are said to be concordant if there is a smooth, oriented, submanifold T of $S^3 \times I$, homeomorphic to $S^1 \times I$, such that $T \cap S^3 \times 0 = K_0$, $T \cap S^3 \times 1 = -K_1$. This concept was introduced by Fox and Milnor [46]. Concordance is an equivalence relation, and the equivalence classes form an abelian group C_1 under connected sum, the zero element 0 being represented by the unknot, and the inverse of $[K]$ being represented by the inverted mirror-image of K. A knot represents 0 in C_1 if and only if it is slice, that is, bounds a smooth 2-disc in B^4. This knot concordance group C_1 has not yet been computed; indeed our comparative lack of knowledge about its structure is a central example of our present ignorance concerning 4-manifolds in general.

Historically, the first necessary condition to be established for a knot to be slice (see [46]) was that the Alexander polynomial must satisfy $\Delta \sim \lambda\bar{\lambda}$, for some $\lambda \in \Lambda$. This is enough to show that C_1 is not finitely-generated. Later, the concordance invariance of the signature was proved [107]. (Murasugi works entirely with his matrix M, but as we remarked earlier, this is a particular Seifert matrix.) This implies the existence of elements of infinite order in C_1. The Miwkowski units are also concordance invariants [106], as are the p-signatures [148] and the signatures $\sigma_\theta(K)$ [96].

This information is all subsumed under the invariance of the 'Witt class' of the Seifert form, which we shall discuss soon, but we pause briefly to consider the signature function $\sigma_K: S^1 \longrightarrow \mathbb{Z}$ of §11, as a direct approach to this is possible via branched covering spaces.

Recall (§12) that for $\xi = e^{\frac{2\pi r i}{k}}$ a k^{th} root of 1, $\sigma_K(\xi) = \sigma_r(V_k)$, the signature of the restriction to the ξ-eigenspace of the intersection form on the k-fold branched cyclic cover V_k. Now suppose $(S^3 \times I, T)$ is a concordance, between knots K_0 and K_1, say, and let W_k be its k-fold branched cyclic cover. If k is a prime-power, then (as in §5), $H_*(W_k;\mathbb{Q}) \cong H_*(S^3 \times I;\mathbb{Q})$; in particular, $H_2(W_k;\mathbb{Q}) = 0$. Hence, by Novikov additivity of the eigenspace signatures, $\sigma_{K_0}(\xi) = \sigma_{K_1}(\xi)$. (In particular, the p-signatures of Tristram are concordance invariants.) Since the roots of 1 of prime-power order are certainly dense in S^1, and since σ_K is continuous except at finitely many points in S^1, it follows that $\sigma_{K_0} = \sigma_{K_1}$ almost everywhere. Hence if we define $\tau_K: S^1 \longrightarrow \mathbb{Z}$ by taking the average of the one-sided limits of σ_K at each point, we see that τ_K is a concordance invariant. This is equivalent to the concordance invariance of the $\sigma_\theta(K)$'s, proved in [96] (see §11). Compare also [81, p. 242]. (Note that τ_K takes values in $\mathbb{Z}$, since if ξ is not a root of the Alexander polynomial of K, $A(\xi)$ (see §11) is non-singular; hence $\sigma_K(\xi) \equiv \operatorname{rank} A(\xi) \pmod 2$ is even. Also, Matumoto has shown [94] that if the 1st Alexander invariant (or minimal polynomial) λ_1 has no repeated roots, then $\tau_K = \sigma_K$.)

We now turn to the Seifert form. (The treatment which follows is that of Levine [81], [82].) Let K be a slice knot, so $(S^3,K) = \partial(B^4,D)$ for some smooth 2-disc D. A tubular neighbourhood of D in B^4 may be identified with $D \times D^2$; let $V = B^4 - D \times \operatorname{int} D^2$ be the exterior of D. Then $\partial V = X \cup D \times S^1$, where X is exterior of K, and projection $D \times S^1 \longrightarrow S^1$ extends to a map $p\colon V \longrightarrow S^1$. Transverse regularity gives a bicollared 3-manifold $M \subset B^4$ with $\partial M = F \cup D$ where F is a Seifert surface for K in S^3. By duality and universal coefficients, the homology exact sequence of $(M, \partial M)$ gives an exact sequence

$$H_1(M,\partial M)^* \xrightarrow{j^*} H_1(M)^* \longrightarrow H_1(\partial M) \xrightarrow{i} H_1(M) \xrightarrow{j} H_1(M,\partial M) ,$$

where $*$ denotes $\mathrm{Hom}(\ ,\mathbb{Z})$. Thus $\mathrm{coker}\, j^* \cong \ker i$, and $\ker j = \mathrm{im}\, i$. But $\ker j$ and $\mathrm{coker}\, j^*$ have the same rank; hence $\mathrm{rank}(\ker i) = \frac{1}{2} \mathrm{rank}\, H_1(\partial M)$. Moreover the Seifert form α on $H_1(F) \cong H_1(\partial M)$ vanishes on $\ker i$. (If w, z are 1-cycles in F representing elements in $\ker i$, there are 2-chains u, v in M with $\partial u = w$, $\partial v = z$. Then u^+, obtained by pushing u off M in the positive normal direction, has $\partial u^+ = w^+$ and is disjoint from v. Hence $\mathrm{lk}(w^+, z) = 0$.)

Recall that Seifert forms can be characterized algebraically as just those bilinear forms $\alpha\colon H \times H \longrightarrow \mathbb{Z}$, H a finitely-generated, free abelian group, such that $\det(\alpha - \alpha^T) = 1$. (This implies that H has even rank.) Write $\alpha \sim \beta$ if the orthogonal sum $\alpha \oplus (-\beta)$ vanishes on a subgroup (and hence on a direct summand) of half the total rank. This is an equivalence relation on Seifert forms, and the equivalence classes form a 'Witt group' $W_S(\mathbb{Z})$ under $\oplus$. Since connected sum of knots induces orthogonal sum of Seifert forms, the discussion in the previous paragraph shows that there is an epimorphism

$$\psi\colon C_1 \longrightarrow W_S(\mathbb{Z}) \quad .$$

The first step in the computation of $W_S(\mathbb{Z})$ is to pass to the rationals. Thus one defines $W_S(\mathbb{Q})$ to be the analogous Witt group of finite-dimensional bilinear forms α over $\mathbb{Q}$ with $\det((\alpha - \alpha^T)(\alpha + \alpha^T)) \neq 0$. The natural map $W_S(\mathbb{Z}) \longrightarrow W_S(\mathbb{Q})$ is injective.

The problem of computing $W_S(\mathbb{Q})$ can be translated into a more standard one by symmetrizing, as follows. Consider pairs $(\langle\ ,\ \rangle, t)$ consisting of a finite-dimensional non-singular symmetric bilinear form $\langle\ ,\ \rangle$ over $\mathbb{Q}$ together with an isometry t, such that ± 1 is not an eigenvalue of t. The classes of such <u>isometric structures</u>, under the equivalence relation obtained by factoring out forms with a t-invariant subspace of half the total dimension on which $\langle\ ,\ \rangle$ vanishes, form a Witt group $W_0(C_\infty,\mathbb{Q})$ under $\oplus$. An isomorphism

$$W_S(\mathbb{Q}) \longrightarrow W_0(C_\infty,\mathbb{Q})$$

is induced by sending a non-singular matrix A representing a class in $W_S(\mathbb{Q})$ to the class in $W(C_\infty,\mathbb{Q})$ with matrix representatives $(A+A^T, A^{-1}A^T)$. (Every class in $W_S(\mathbb{Q})$ has a non-singular representative.) Note that if A is a non-singular Seifert matrix for a knot K, then $A+A^T$ represents the quadratic form of K, and $A^{-1}A^T$ represents the automorphism $t\colon H_1(X_\infty;\mathbb{Q}) \longrightarrow H_1(X_\infty;\mathbb{Q})$.

A complete set of invariants for $W_0(C_\infty,\mathbb{Q})$ has been given by Levine [82], using results of Milnor [98]. These are defined for each λ-primary component V_λ, where λ is a symmetric, irreducible factor of the characteristic polynomial of t, and are: the exponent mod 2 of λ in the characteristic polynomial, the signature of the restriction of $\langle\ ,\ \rangle$ to V_λ (this is the σ_θ of §11), and a Witt class invariant version (analogous to a Minkowski unit) of the Hasse invariant of the restriction of $\langle\ ,\ \rangle$ to V_λ. In particular, $W_0(C_\infty,\mathbb{Q}) \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/4)^\infty \oplus (\mathbb{Z}/2)^\infty$. The image of the injection $W_S(\mathbb{Z}) \longrightarrow W_0(C_\infty,\mathbb{Q})$ is also isomorphic to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/4)^\infty \oplus (\mathbb{Z}/2)^\infty$.

A different but related approach to the computation of $W_S(\mathbb{Z})$ is described by Kervaire in [73]. For further results on the structure of $W_S(\mathbb{Z})$, see [143].

Similar definitions and results hold for knots of S^{4n+1} in S^{4n+3} for $n > 0$; in particular, there is a knot concordance group C_{4n+1} and a homomorphism

$$\psi_{4n+1}\colon C_{4n+1} \longrightarrow W_S(\mathbb{Z}) .$$

Levine has shown that, if $n > 0$, ψ_{4n+1} is an isomorphism [81]. According to Casson-Gordon [17], [18], however, this is not the case for $n=0$. We shall briefly summarize their argument.

Let K be a knot in S^3, and N the closed 3-manifold obtained from S^3 by 0-framed surgery along K. Write N_k for the k-fold cyclic cover of N, $1 \leq k \leq \infty$. Recall that M_k denotes the k-fold branched cyclic cover of K, $1 \leq k < \infty$. Suppose that, for some k, $\chi\colon H_1(M_k) \longrightarrow \mathbb{C}^*$ is a character of order m (that is, the image of χ is C_m, the group of m^{th} roots of 1). Composing χ with the canonical epimorphism $H_1(N_k) \longrightarrow H_1(M_k)$ gives a character χ' of order m on $H_1(N_k)$, inducing an m-fold cyclic covering $\tilde{N}_k \longrightarrow N_k$. Similarly χ induces an m-fold cyclic covering $\tilde{N}_\infty \longrightarrow N_\infty$. Then $\tilde{N}_\infty$ is a regular cover of N_k with group of covering translations $C_m \times C_\infty$. Since $\Omega_3(K(C_m \times C_\infty, 1))$ is finite, there is a regular $C_m \times C_\infty$-covering $\tilde{V}_\infty \longrightarrow V_k$ of compact, oriented 4-manifolds such that

$$\partial \left(\begin{array}{ccc} \tilde{V}_\infty & \longrightarrow & \tilde{V}_k \\ \downarrow & \searrow & \downarrow \\ V_\infty & \longrightarrow & V_k \end{array}\right) = r \left(\begin{array}{ccc} \tilde{N}_\infty & \longrightarrow & \tilde{N}_k \\ \downarrow & \searrow & \downarrow \\ N_\infty & \longrightarrow & N_k \end{array}\right)$$

for some integer $r \neq 0$.

Let $\mathbb{C}(t)$ be the field of rational functions in t with coefficients in $\mathbb{C}$; $\mathbb{C}(t)$ is a $\mathbb{Z}[C_m \times C_\infty] = \mathbb{Z}[C_m][t, t^{-1}]$-module. Write $H_*^t(V_k;\mathbb{C}(t))$ for the twisted homology $H_*(C_*(\tilde{V}_\infty) \otimes_{\mathbb{Z}[C_m \times C_\infty]} \mathbb{C}(t))$. The intersection pairing on the chains of $\tilde{V}_\infty$ (compare §7) induces a form

$$H_2^t(V_k;\mathbb{C}(t)) \times H_2^t(V_k;\mathbb{C}(t)) \longrightarrow \mathbb{C}(t)$$

which is Hermitian with respect to the involution J on $\mathbb{C}(t)$ given by $t \longmapsto t^{-1}$ and complex conjugation. This form therefore defines an element $w(V_k) \in W(\mathbb{C}(t),J)$, the Witt group of finite-dimensional Hermitian forms over $\mathbb{C}(t)$. The ordinary intersection form on $H_2(V_k;\mathbb{Q})$ represents an element of $W(\mathbb{Q})$; let $w_0(V_k)$ be the image of this element in $W(\mathbb{C}(t),J)$. Then define

$$\tau(K,\chi) = \frac{1}{r}\,(w(V_k) - w_0(V_k)) \in W(\mathbb{C}(t),J) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

It can be shown that $\tau(K,\chi)$ is independent of r and V_k.

Now suppose that K is a slice knot, so $(S^3,K)=\partial(B^4,D)$, say. Let W_k be the k-fold branched cyclic cover of (B^4,D), and take k to be a prime-power. Then $\tilde{H}_*(W_k;\mathbb{Q})=0$ (see §5), so, by duality, $H_1(M_k)$ has order ℓ^2, where $G=\ker(H_1(M_k)\longrightarrow H_1(W_k))$ has order ℓ. Note that G has the property, intrinsic to M_k, that the linking form $H_1(M_k)\times H_1(M_k)\longrightarrow \mathbb{Q}/\mathbb{Z}$ vanishes on G. Let V be the closure of the complement of a tubular neighbourhood of D in B^4, and write V_k for the k-fold cyclic cover of V, $1\leq k\leq\infty$. Then $\partial V_k = N_k$.

Let χ be a character of prime-power order m on $H_1(M_k)$, such that $\chi(G)=1$. There is then a character $\bar{\chi}$ on $H_1(W_k)$ such that

$$\begin{array}{ccc} H_1(M_k) & \longrightarrow & H_1(W_k) \\ & \searrow^{\chi'} \quad \swarrow^{\bar{\chi}} & \\ & \mathbb{C}^* & \end{array}$$

commutes. Suppose (but only to simplify the exposition) that $\bar{\chi}$ also has order m. Composing with the canonical epimorphism $H_1(V_k)\longrightarrow H_1(W_k)$, we get a character $\bar{\chi}'$ on $H_1(V_k)$ such that

$$\begin{array}{ccc} H_1(N_k) & \longrightarrow & H_1(V_k) \\ & \searrow^{\chi'} \quad \swarrow^{\bar{\chi}'} & \\ & C_m & \end{array}$$

commutes. We can therefore use V_k to compute $\tau(K,\chi)$. But it can be shown that since V is a homology circle and m is a prime-power, $H_*(\tilde{V}_\infty;\mathbb{Q})$ is finite-dimensional. In particular, $H_2(\tilde{V}_\infty)$ is $\mathbb{Z}[C_\infty]$-torsion. Since $\mathbb{C}(t)$ is flat over $\mathbb{Z}[C_m\times C_\infty]$, it follows that $H_2^t(V_k;\mathbb{C}(t)) = H_2(\tilde{V}_\infty)\otimes_{\mathbb{Z}[C_m\times C_\infty]}\mathbb{C}(t)=0$, and therefore $w(V_k)=0$. Again, since V is a homology circle and k is a prime-power, $H_2(V_k;\mathbb{Q})=0$ (see §5). Hence $w_0(V_k)=0$ also, giving $\tau(K,\chi)=0$.

The vanishing of $\tau(K,\chi)$ for certain characters χ is therefore a necessary condition for K to be slice. To utilize this condition, we first define a signature homomorphism

$$\sigma_1 : W(\mathbb{C}(t), J) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q} .$$

It suffices to consider $\varphi \in W(\mathbb{C}(t), J)$; suppose φ has a representative which is given with respect to some basis by the matrix $B(t)$. Then set

$$\sigma_1(\varphi) = \frac{1}{2} \left(\lim_{\theta \to 0^+} \operatorname{sign} B(e^{i\theta}) + \lim_{\theta \to 0^-} \operatorname{sign} B(e^{i\theta}) \right) .$$

It turns out that $\sigma_1(\tau(K, \chi))$ is sometimes related to another invariant, analogous to, but simpler than, τ. The general definition of this goes as follows. Let M be a closed, oriented 3-manifold and χ a character of order m on $H_1(M)$ inducing an m-fold cyclic covering $\tilde{M} \longrightarrow M$. Since $\Omega_3(K(C_m, 1))$ is finite, there exists an m-fold cyclic covering $\tilde{W} \longrightarrow W$ of compact, oriented 4-manifolds with $\partial(\tilde{W} \longrightarrow W) = r(\tilde{M} \longrightarrow M)$ for some integer $r \neq 0$. Writing $H^t_*(W;\mathbb{C})$ for the twisted homology $H_*(C_*(\tilde{W}) \otimes_{\mathbb{Z}[C_m]} \mathbb{C})$, we have a Hermitian intersection form

$$H^t_2(W;\mathbb{C}) \times H^t_2(W;\mathbb{C}) \longrightarrow \mathbb{C} .$$

Let $s(W)$ be the signature of this form, $s_0(W)$ the ordinary signature of W, and define

$$\sigma(M, \chi) = \frac{1}{r} (s(W) - s_0(W)) \in \mathbb{Q} .$$

This is independent of r and W.

Returning to the knot situation, recall our original character χ on $H_1(M_k)$, inducing $\tilde{M}_k \longrightarrow M_k$. It can be shown that if $H_1(\tilde{M}_k;\mathbb{Q}) = 0$, then

$$|\sigma_1(\tau(K, \chi)) - \sigma(M_k, \chi)| \leq 1$$

If, in addition, K is slice, and χ satisfies the conditions described earlier which then imply that $\tau(K,\chi) = 0$, we obtain

$$|\sigma(M_k, \chi)| \leq 1 .$$

Since the invariant $\sigma(M_k,\chi)$ can often be calculated, this is a workable condition. For example, if K is a 2-bridge (or rational) knot, and $k=2$, then M_k is a lens space, and $\sigma(M_k,\chi)$ can be calculated fairly easily using the G-signature theorem. Also, in this case, $\tilde{M}_k$ will always be a rational homology sphere, so K can be slice only if (for suitable χ) $|\sigma(M_k,\chi)| \leq 1$. From this it can be shown that a large number of 2-bridge knots K have $\psi([K]) = 0$ in $W_S(\mathbb{Z})$, but are not slice knots.

14. 3-Manifolds and Knots

In this section and the next we shall discuss some of the functions

$$\{\text{knots}\} \longrightarrow \{\text{3-manifolds}\}$$

which may be defined. Such a function relates knot theory to the general theory of 3-manifolds, and hence by means of it any development in one theory will have consequences for the other. Here, among other things, we shall look at some of the ways in which general results about 3-manifolds have had implications for knot theory. Possible influences in the other direction will be considered in §15.

Probably the most obvious function of the above type is the one which simply associates to a knot its exterior. (This is not known to be injective, but the odds seem good that it is.) Here, as we have already mentioned, Dehn's lemma implied that $\pi K \cong \mathbb{Z}$ only if K is trivial, the sphere theorem implies the asphericity of knots, and Waldhausen's work implies that knots are classified by the triples $(\pi K, \lambda, \mu)$.

We might also mention the fibration theorem of Stallings [142], (see also [111]) which, when applied to knot exteriors, implies that many knots K (in fact, precisely those such that the commutator subgroup of πK is finitely-generated, see [142], [110], [112]), correspond to a 'singular' fibring of S^3 over S^1, in the following sense: $S^3 = \bigcup_{\theta=0}^{2\pi} F_\theta$, where each F_θ is homeomorphic to some compact surface F, $\partial F_\theta = K$ for all $\theta \in S^1$, and $S^3 - K = \bigcup_{\theta=0}^{2\pi} \operatorname{int} F_\theta$ is a fibre bundle over S^1, the fibres being the $\operatorname{int} F_\theta$. In other words, K is the binding of an open book structure on S^3. The fibred knots with finite bundle group are precisely the torus knots; see [165] for a nice description of the fibration in this case.

Thurston's recent (unpublished) work on 3-manifolds implies that a knot K which has no companions and is not a torus knot has an exterior which supports a 'hyperbolic structure'. Also, the decomposition theorem of Johannson [66], [67] and Jaco-Shalen [64] applies to knot exteriors. In particular, using this together with his own work, Thurston has shown that knot groups are residually finite. One hopes and expects that in the near future knot theory will be further enriched by these ideas from hyperbolic geometry.

Another advance in the theory of 3-manifolds which has striking consequences for knot theory is discussed in [158]. There it is indicated how Haken's results on hierarchies of incompressible surfaces in irreducible 3-manifolds, and Hemion's recent solution of the conjugacy problem for the group of isotopy classes of homeomorphisms of a compact, bounded, surface, together imply that the knot problem is algorithmically solvable, or, equivalently, that knots can be classified (i.e. listed, without repetition). Again, the connection is via the exterior of the knot.

Branched covering spaces provide examples of functions $\{\text{knots}\} \longrightarrow \{\text{closed 3-manifolds}\}$, and, as mentioned in §1, invariants of these covers have been used to distinguish knots. Also, by means of such a function, bridge decompositions of the knot are related to Heegaard splittings of the 3-manifold (see [13]), as follows. Let K be a b-bridge knot. Then $(S^3,K) = (B^3_+,A_+) \cup_\partial (B^3_-,A_-)$, where $A_\pm$ is a set of b arcs properly embedded in $B^3_\pm$, and $(B^3_\pm, A_\pm) \cong (B^2,P) \times I$, where P is a set of b points in $\operatorname{int} B^2$. Now let M be some cover of S^3 branched over K. From the bridge decomposition of (S^3,K) one obtains

$M = H_+ \cup_\partial H_-$, say, with $\partial H_\pm$ connected, and $H_\pm \cong \widetilde{B} \times I$, where $\widetilde{B}$ is the corresponding branched cover of (B^2, P). If the projection $M \longrightarrow S^3$ is k-sheeted away from the branch set, and m-sheeted over K, then

$$\chi(\widetilde{B}) = k\,\chi(B^2) - (k-m)\chi(P) = mb - (b-1)k \ .$$

It follows that $H_\pm$ is a solid handlebody of genus $(b-1)k - mb + 1$, giving a Heegaard splitting of M of that genus. For the k-fold branched cyclic cover, the genus is $(k-1)(b-1)$. In particular, for the 2-fold branched cover, we just get $b-1$. In this way, knots of increasing complexity are mapped to 3-manifold decompositions of increasing complexity.

Now it is known that the 2-fold branched covering function is not injective; many examples of pairs of prime knots with the same 2-fold branched cover are described in [11]. It is injective, however, on the set of 2-bridge knots. There, the 2-fold branched cover has genus 1, and is therefore a lens space, and Schubert has proved [132] that this lens space determines the knot. This injectivity already fails for 3-bridge knots [11]. It has been shown by Birman-Hilden [10], however, as a consequence of a rather special feature of the group of isotopy classes of homeomorphisms of a closed surface of genus 2, that if we regard the 2-fold branched covering function as a function {knots} $\longrightarrow$ {equivalence classes of Heegaard splittings of 3-manifolds}, then it _is_ injective on the set of 3-bridge knots.

Finally, in this context we might mention the result of Waldhausen [157], which says that only the unknot has S^3 as its 2-fold branched cover.

15. Knots and 3- and 4-Manifolds

Continuing in the general framework of §14, let us now consider the possibility of using knowledge about knots to give information about 3-manifolds. In particular, functions {knots} $\longrightarrow$ {3-manifolds} which are surjective, or at least have a sizeable image, will be of interest.

Returning to branched covers, Alexander showed [1] that every closed, orientable 3-manifold is a cover of S^3 branched over some link. This has recently been refined (independently) by Hilden [60], Hirsch, and Montesinos [100], who show that every closed, orientable 3-manifold is actually a 3-fold (irregular dihedral) cover of S^3 branched over a knot. This result is best possible in the sense that there are 3-manifolds which are not 2-fold branched covers of S^3. To utilize this function to get invariants of 3-manifolds, it would be helpful to have a purely knot-theoretic description of the equivalence relation on knots which corresponds to homeomorphism of the associated branched covers. Some moves on the knot which leave the branched cover unchanged are known (see [99], [100]), but it has not yet been established whether or not these suffice. (In the same vein, even though the 2-fold branched covering function is not surjective, it would still be interesting to have an intrinsic description of the appropriate equivalence relation on knots.)

Cappell-Shaneson [15] have obtained a formula for the Rohlin μ-invariant of a $\mathbb{Z}/2$-homology sphere M, given as a 3-fold dihedral branched cover of a knot K, which involves (among other things) the classical invariants of K given by the linking numbers of the lifts of K in M [124], [116].

As a concrete example of an application to 3-manifolds of the branched covering space point of view we cite [61], which proves a sharpening of the Hilden-Hirsch-Montesinos theorem, and obtains as a consequence the (known) result that closed, orientable 3-manifolds are parallellizable.

Other interesting ways of constructing 3-manifolds from knots are provided by what is now referred to as Dehn surgery. More precisely, given a knot K and a pair of coprime integers α, β, one can consider the closed, orientable 3-manifold $M(K;\beta/\alpha)$ (we use the 'rational surgery coefficient' notation of [128]), obtained by removing from S^3 a tubular neighbourhood of K and sewing it back so as to identify a meridian on the boundary of the solid torus with a curve on the boundary of the exterior of K homologous to $\alpha[\ell] + \beta[m]$, where (ℓ, m) is a longitude-meridian pair for K. Note that $H_1(M(K;\beta/\alpha)) \cong \mathbb{Z}/|\beta|$. With $|\beta| = 1$, this construction first appeared in [27], where Dehn showed that many non-simply-connected homology spheres, in particular, the dodecahedral space discovered earlier by Poincaré, could be obtained in this way from torus knots. Indeed, the Property P conjecture (see §2) is that if K is non-trivial and $\alpha \neq 0$, then $M(K;1/\alpha)$ is never simply-connected.

It seems likely that the function $M(\_;\beta/\alpha)$ is never injective, although this has only been verified for certain β/α [53], [87]. However, it may not be unreasonable to conjecture that, denoting the unknot by O, and excluding the trivial case $\alpha = 0$, $M(K;\beta/\alpha) \cong M(O;\beta/\alpha)$ only if $K = O$. The case $|\beta| = 1$ is just a weakened form of the Property P conjecture, and the case $\beta = 0$ has also received some attention (under the name 'Property R').

Turning to the question of surjectivity, clearly the most one could hope to obtain in this way is the set of all closed, orientable 3-manifolds M with $H_1(M)$ cyclic. This seems highly unlikely. In particular, it is surely not true that all homology spheres can be obtained by Dehn's original method, although this is apparently rather difficult to prove.

The a priori restriction on the homology disappears if one allows, instead of knots, links with arbitrarily many components, and it is indeed the case that one can now obtain all closed, orientable 3-manifolds. Actually a stronger statement is possible. If L is a framed link in S^3, then (ordinary) framed surgery on S^3 along L gives a 3-manifold $M(L)$, say. Wallace [159] and Lickorish [86] have shown that this function {framed links} $\longrightarrow$ {closed, orientable 3-manifolds} is surjective. Wallace's proof is essentially 4-dimensional; it uses the theorem of Rohlin [126] that 3-dimensional oriented cobordism $\Omega_3 = 0$, together with handlebody techniques. (The argument is: given M, there exists W such that $M = \partial W$; W has a handle decomposition with one 0-handle and no 4-handles. Replace the 1- and 3-handles by 2-handles ('handle trading'), giving W'. The attaching maps of the 2-handles in W' now define a framed link L with $M \cong M(L)$.) Lickorish's proof, on the other hand, is 2-dimensional, in the sense that it is based on the fact that the group of isotopy classes of orientation-preserving homeomorphisms of a closed surface is generated by 'twists'. (This was first proved by Dehn [29].)

Since the trace of the surgery is a 4-manifold bounded by the given 3-manifold, this approach gives another proof that $\Omega_3 = 0$.

The equivalence relation on framed links (in the oriented 3-sphere) which corresponds to (orientation-preserving) homeomorphism of the associated 3-manifolds $M(L)$ has been identified by Kirby [74], in the sense that it is shown to be generated by certain moves on the link. (Craggs (unpublished) has also obtained results along these lines.) Kirby uses two moves; Fenn-Rourke [35] show that these can be incorporated into a single move. Also, Rolfsen (private communication) has provided the modification necessary to describe the equivalence relation appropriate to the more general process of Dehn surgery on a link.

Armed with these results, it is clear, in theory, how one might go about getting new invariants of 3-manifolds. For instance, with respect to any kind of complexity of a framed link, every 3-manifold will be obtained by surgery on some class of links of minimal complexity, so invariants of this class will be invariants of the 3-manifold. However, this point of view has not yet had much effect on the theory of 3-manifolds, mainly because, although the above-mentioned equivalence relation on links is easy to describe, it seems hard to decide in practice whether two given links are equivalent, or to find link-theoretic invariants of the equivalence relation. It is clear that more remains to be done in this direction.

The work described above also relates knot and link theory to 4-manifolds, and offers the prospect of obtaining, perhaps first, known results (Rohlin's $\Omega_4 \cong \mathbb{Z}$ theorem [127] is an obvious example), but ultimately, new results, about 4-manifolds via link theory. In this spirit, Kaplan has shown [68] that, given a framed link, it may be modified by Kirby's moves so as to make all the framings even. This, together with the Wallace-Lickorish theorem, implies the (known) result that every closed, orientable 3-manifold bounds a parallellizable 4-manifold.

We might also mention here <u>the</u> Rohlin Theorem, that the signature of a smooth closed, oriented, almost parallellizable 4-manifold is divisible by 16. Elementary proofs of this (assuming $\Omega_4 \cong \mathbb{Z}$) have been given by Casson and (independently) Matsumoto (both unpublished), and link-theoretic ideas are involved in these proofs.

Many questions concerning the existence of certain surfaces in 4-manifolds are equivalent, or closely related, to questions about knot and link concordance. Thus Tristram [148] used his p-signatures to show that a class $ax + by$ in $H_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ can be represented by a smoothly embedded 2-sphere only if a and b are coprime. (It is still unknown whether this condition is sufficient, except for the cases $|a| \leq 1$ or $|b| \leq 1$.) As we have seen in §12, signatures of knots (and links) are probably best studied from a 4-dimensional point of view anyway, so this kind of connection is not surprising.

Perhaps more surprising is the result of Casson (unpublished) that simply-connected surgery is possible in dimension 4 if each of a certain explicit set of infinite sequences of links contains a slice link. On the other hand, failure of the latter condition implies the existence of some kind of pathology in dimension 4. For example, if the sequence of (untwisted) doubles of the Whitehead link contains no slice link, then there is a 4-manifold proper homotopy equivalent to $S^2 \times S^2$-point whose end is not diffeomorphic to $S^3 \times R$, and a 4-dimensional counterexample to the McMillan cellularity criterion. (These results are also due to Casson.)

16. Knots and the 3-Sphere

All the abelian algebra discussed so far is valid for knots in homology 3-spheres. Similarly, all known knot concordance invariants are actually homology-cobordism invariants. The group of a knot in S^3, of course, has weight 1 (being generated by the conjugates of any meridian element), but again this is true of a knot in any homotopy 3-sphere. Still, it is clear that the theory of knots in the 3-sphere, having the concreteness and immediacy of the physical world, is of prime importance. Moreover, even properties which hold in more general settings might be more easily observed in the 3-sphere. This has certainly been the case historically. For example, the property $\Delta(1) = 1$ of the Alexander polynomial was first proved by means of knot projections [2]. (In fact the purely combinatorial view of knot theory, in which invariants are defined in terms of knot diagrams and then shown to be unchanged under the Reidemeister moves, dominated the subject for a long time [2], [3], [123], [125], [51].) Also, the symmetry property of link polynomials has been obtained [147] as a consequence of the existence of 'dual' Wirtinger presentations of the group. (See [95] for a 'co-ordinate-free' proof in the more general setting of a homology sphere.) Again, in trying to relate knot and link theory to 3- and 4-manifolds, presumably the hope is that one might be able to 'see' new information about the manifolds precisely because one is working with visualizable objects in ordinary space.

In dealing with the 3-sphere, however, there is more involved than just convenience, for it is known that different 3-manifolds have different knot theories. More precisely, it is known that if $\mathcal{K}(M)$ denotes the set of isomorphism classes of groups of knots in the closed 3-manifold M, then $M \cong N$ if and only if $\mathcal{K}(M) = \mathcal{K}(N)$. The result in this generality is due to Jaco-Myers [63] (for orientable manifolds) and, independently, Row (unpublished). The fact that the 3-sphere is determined by its knot groups was apparently proved earlier by Connor (unpublished). The idea of trying to classify 3-manifolds by their knot theories goes back to Fox, who used it to recover the (known) classification of lens spaces.

This suggests the problem of trying to characterize the groups of knots in the 3-sphere. A characterization was given, several years ago, by Artin [5], but this is in terms of the existence of a particular kind of presentation, and whether this can be expressed more intrinsically is still unknown.

A good example of a problem which specifically concerns knots in the 3-sphere is the Smith conjecture that no non-trivial knot is the fixed-point set of a $\mathbb{Z}/p$-action on S^3 (clearly it is enough to consider p prime). This is false for knots in homology spheres. On the other hand, most of the partial results on the conjecture are essentially homological in nature. A notable exception is Waldhausen's proof [157] for the case $p = 2$, which uses (as it must) the geometry of the 3-sphere (in particular, the uniqueness of Heegaard splittings [156]) in an essential way.

17. Other Topics

Here we briefly mention one or two topics which we shall not be able to discuss in detail.

First, there is the whole question of symmetries of knots, For S^1-actions, the answer is known: only the unknot can be the fixed-point set of an S^1-action on S^3, and the only knots which are invariant under (effective) S^1-actions are the torus knots. (This follows from the theory of Seifert fibre spaces [134]; see [65].) For the case of $\mathbb{Z}/p$-actions on S^3 fixing a knot K, we of course have the Smith conjecture that K must be trivial. This is surely one of the major unsolved problems in knot theory. It is known to be true for $p = 2$ [157], and there exist various other partial results, including [16], [42], [45], [50], [56], [109]. Necessary conditions are given in [149] and [108], for a knot K to have a symmetry of order n in the sense that there is a homeomorphism h of S^3 of period n, with fixed-point set a circle disjoint from K, such that $h(K) = K$.

Given an unoriented knot K in oriented S^3, one can ask whether or not there exists an orientation-reversing homeomorphism of S^3 taking K to itself, (or equivalently, an orientation-preserving homeomorphism of S^3 taking K to its mirror-image). If there is, K is amphicheiral. If K is now oriented, one can ask whether there is an orientation-preserving homeomorphism of S^3 taking K onto K but reversing its orientation. If so, K is invertible. If K is amphicheiral, then, for example, all its branched covers will support orientation-reversing homeomorphisms. Because of this, amphicheirality is often relatively easy to detect [135]. Since many knot invariants are independent of the orientation of the knot, however, it is harder to establish non-invertibility. This was first done in [151], by analysing automorphisms of the group. See [71], [161] for further results. Two interesting conjectures relating these concepts to symmetries (see [75]) are: K is amphicheiral if and only if K is invariant under reflection through the origin (van Buskirk); and: K is invertible if and only if there is an orientation-preserving involution of S^3 taking K to itself, reversing its orientation (Montesinos). Apparently these are true for knots with small crossing number.

Alternating knots have always occupied a special place in the subject; for instance, their asphericity was proved [7] before the sphere theorem. Othere interesting results on alternating knots are contained in [21], [22], [101], [102], [103], [104], [105].

The important work of Schubert on unique factorization [129] and companionship [130], [131] should be mentioned.

For results on the genus of a knot see [136], [110], [59], [133], [22], [101].

The question of the uniqueness of Seifert surfaces of minimal genus has received considerable attention [4], [153], [162].

Finally, there is an extensive literature on the knots which arise as links of complex algebraic plane curve singularities. (These are certain iterated cables of the unknot.) See [97] and references therein.

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