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ON THE REGULAR AND SEMI-REGULAR FIGURES IN SPACE OF n DIMENSIONS.

By *Thorold Gosset.*

THE following is an attempt to give a complete list of all the regular and semi-regular figures in a Euclidean space of n dimensions where n is greater than 3. These results were obtained between two and three years ago, and as far as the regular figures are concerned agree with those given by Mr. Curjel in the April number of this year's *Messenger of Mathematics.** A list of the semi-regular hypersolids has not, it is believed, been previously given. For the sake of brevity a figure in a Euclidean space of n dimensions is called an n-ic figure, and an infinite n-ic fignre an $(n-1)$-ic check.

REGULAR FIGURES.

In space of n dimensions there are four regular figures.

1. n-ic *Pyramid.* Analogous to the tetrahedron. It is bounded by $n+1$ $(n-1)$-ic pyramids, n of which meet in each summit; and by $\frac{(n+1)!}{(r+1)!(n-r)!}$ r-ic pyramids, $\frac{n!}{r!(n-r)!}$ of which meet in each summit, where r may have any value from $(n-2)$ to 1. It has $n+1$ summits. $\frac{(n-t)!}{(r-t)!(n-r)!}$ r-ic pyramids meet in each t-ic pyramid.

The radius of its circumscribing sphere is $\frac{\sqrt{\{\frac{1}{2}n(n+1)\}}}{n+1}$ times the length of its edge. It is self-reciprocal.

2. n-ic *Double Pyramid.* Analogous to the octahedron. It is bounded by 2^n $(n-1)$-ic pyramids, 2^{n-1} of which meet in each summit; and by $\frac{2^{r+1}n!}{(r+1)!(n-r-1)!}$ r-ic pyramids, $\frac{2^r(n-1)!}{r!(n-r-1)!}$ of which meet in each summit, where r may have any value from $(n-2)$ to 1. It has $2n$ summits. $\frac{2^{r-t}(n-t-1)!}{(r-t)!(n-r-1)!}$ r-ic pyramids meet in each t-ic pyramid.

The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{2}$ times the length of its edge. Its reciprocal is the n-ic cube.

3. n-ic *Cube.* Analogous to the cube. It is bounded by $2n$ $(n-1)$-ic cubes, n of which meet in each summit; and by

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$\frac{2^{n-r}n!}{r!(n-r)!}$ r-ic cubes, $\frac{n!}{r!(n-r)!}$ of which meet in each summit, where r may have any value from $n-2$ to 1. It has 2^n summits. $\frac{(n-t)!}{(r-t)!(n-r)!}$ r-ic cubes meet in each t-ic cube. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{n}$ times the length of its edge. Its reciprocal is the n-ic double pyramid.

4. $(n-1)$-ic *Check*. Analogous to the pattern of a chess-board. This figure is infinite. It is bounded by ∞ $(n-1)$-ic cubes, 2^{n-1} of which meet in each summit; and by $\frac{(n-1)!}{(n-r-1)!r!}\infty$ r-ic cubes, $\frac{2^r(n-1)!}{r!(n-r-1)!}$ of which meet in each summit, where r may have any value from $n-2$ to 1. It has ∞ summits, $\frac{2^{r-t}(n-t-1)!}{(r-t)!(n-r-1)!}$ r-ic cubes meet in each t-ic cube. It is self-reciprocal.

In space of four dimensions there are three additional regular figures.

1. *Octahedric.* No analogy in three dimensions. It is bounded by 24 octahedra, 6 of which meet in each summit, and 3 of which meet in each edge; 96 triangles, 12 of which meet in each summit and 3 of which meet in each edge; and 96 edges, 8 of which meet in each summit. It has 24 summits. The radius of its circumscribing sphere is equal to the length of its edge. It is self-reciprocal.

2. *Tetrahedric.* Analogous to the icosahedron. It is bounded by 600 tetrahedra, 20 of which meet in each summit and 5 of which meet in each edge; 1200 triangles, 30 of which meet in each summit and 5 of which meet in each edge; and 720 edges, 12 of which meet in each summit. It has 120 summits. The radius of its circumscribing sphere is $\frac{1}{2}(\sqrt{5}+1)$ times the length of its edge. Its reciprocal is the dodecahedric.

3. *Dodecahedric.* Analogous to the dodecahedron. It is bounded by 120 dodecahedra, 4 of which meet in each summit and 3 of which meet in each edge; 720 pentagons, 6 of which meet in each summit and 3 of which meet in each edge; and 1200 edges, 4 of which meet in each summit. It has 600 summits. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{2}(3+\sqrt{5})$ times the length of its edge. Its reciprocal is the tetrahedric.

In space of five dimensions there are two additional regular figures, both of which are infinite.

1. *Octahedric Check.* It is bounded by ∞ octahedrics, 8 of which meet in each summit, 4 of which meet in each edge, and 3 of which meet in each triangle; 12 ∞ octahedra, 24 of which meet in each summit, 6 of which meet in each edge, and 3 of which meet in each triangle; 32 ∞ triangles, 32 of which meet in each summit, and 4 of which meet in each edge; and 24 ∞ edges, 16 of which meet in each summit. It has 3 ∞ summits. Its reciprocal is the double pyramidal check.

2. *Double Pyramidal Check.* It is bounded by 3 ∞ 4-ic double pyramids, 24 of which meet in each summit, 6 of which meet in each edge, and 3 of which meet in each triangle; 24 ∞ tetrahedra, 96 of which meet in each summit, 12 of which meet in each edge, and 3 of which meet in each triangle; 32 ∞ triangles, 96 of which meet in each summit, and 8 of which meet in each edge; and 12 ∞ edges, 24 of which meet in each summit. It has ∞ summits. Its reciprocal is the octahedric check.

Semi-regular Figures.

In space of n dimensions there is one infinite semi-regular figure.

$(n-1)$-ic *Semi-check.* Analogous to the pattern of a single file of a chess board. It is bounded by 2 $(n-2)$-ic semi-checks and ∞ $(n-1)$-ic cubes, one of the former meeting 2^{n-2} of the latter in each summit. It is also bounded by $\frac{(2n-r-2)(n-2)!}{r!(n-r-1)!}$ ∞ r-ic cubes, of which $\frac{2^{r-1}(2n-r-2)(n-2)!}{r!(n-r-1)!}$ meet in each summit, where r may have any value from $(r-2)$ to 1. It has 2 ∞ summits. Either

$$\frac{2^{r-t-1}(2n-r-t-2)(n-t-2)!}{(r-t)!(n-r-1)!} \text{ or } \frac{2^{r-t}(n-t-1)!}{(r-t)!(n-r-1)!}$$

r-ic cubes meet in each t-ic cube according to whether the t-ic cube is or is not a boundary of an $(n-2)$-ic semi-check.

In space of four dimensions there are six additional semi-regular figures.

1. *Tetroctahedric.* It is bounded by 5 octahedra and 5 tetrahedra, 3 of the former meeting 2 of the latter in each summit, and 2 of the former meeting 1 of the latter in each edge. It is also bounded by 30 triangles, 9 of which meet in each summit, and 3 of which meet in each edge; and by

30 edges, 6 of which meet in each summit. It has 10 summits. The radius of its circumscribing sphere is $\frac{1}{5}\sqrt{(15)}$ times the length of its edge.

2. *Tetricosahedric.* It is bounded by 24 icosahedra and 120 tetrahedra, 3 of the former meeting 5 of the latter in each summit; 480 triangles, 15 of which meet in each summit; and 432 edges, 9 of which meet in each summit. 2 icosahedra, 1 tetrahedron and 3 triangles meet in 6 out of the 9 edges meeting in each summit; and 1 icosahedron, 3 tetrahedra and 4 triangles meet in the remaining 3 out of the 9 edges meeting in each summit. It has 96 summits. The radius of its circumscribing sphere is $\frac{1}{2}(\sqrt{5}+1)$ times the length of its edge.

3. *Octicosahedric.* It is bounded by 120 icosahedra and 600 octahedra, 2 of the former meeting 5 of the latter in each summit, and 1 of the former meeting 2 of the latter in each edge. It is also bounded by 3600 triangles 15 of which meet n each summit, and 3 of which meet in each edge; and by 3600 edges, 10 of which meet in each summit. It has 720 summits. The radius of its circumscribing sphere is $\sqrt{(5+2\sqrt{5})}$ times the length of its edge.

4. *Simple Tetroctahedric Check.* This figure is infinite. It is bounded by ∞ octahedra and 2 ∞ tetrahedra, 6 of the former and 8 of the latter meeting in each summit; and 2 of the former and 2 of the latter meeting in each edge, in such a way that the 2 octahedra, and consequently the 2 tetrahedra are not adjacent. It is also bounded by 8 ∞ triangles, 24 of which meet in each summit, and 4 of which meet in each edge; and 6 ∞ edges, 12 of which meet in each summit. It has ∞ summits.

5. *Complex Tetroctahedric Check.* This figure is infinite. It is bounded by ∞ octahedra and 2 ∞ tetrahedra, 6 of the former and 8 of the latter meeting in each summit; and 2 of the former and 2 of the latter meeting in each edge. It is also bounded by 8 ∞ triangles, 24 of which meet in each summit and 4 of which meet in each edge; and 6 ∞ edges, 12 of which meet in each summit. It has ∞ summits. This figure differs from the simple tetroctahedric check in having the 2 octahedra and 2 tetrahedra, which meet in 6 out of the 12 edges meeting in each summit, placed so that the 2 octahedra and consequently the 2 tetrahedra are adjacent. The 2 octahedra and 2 tetrahedra, which meet in the remaining 6 out of the 12 edges meeting in each summit, are placed so that the 2 octahedra and consequently the 2 tetrahedra are not adjacent.

6. *Tetroctahedric Semi-check.* This figure is infinite. It is bounded by ∞ octahedra, 2 ∞ tetrahedra, and 2 triangular checks (that is to say infinite 3-ic figures bounded by equilateral triangles meeting 6 at a point); 3 octahedra, 4 tetrahedra, and 1 triangular check meeting in each summit. It is also bounded by 10 ∞ triangles, 15 of which meet in each summit; and 9 ∞ edges, 9 of which meet in each summit. 1 octahedron, 1 tetrahedron, 1 triangular check, and 3 triangles meet in 6 out of the 9 edges meeting in each summit; and 2 octahedra, 2 tetrahedra, and 4 triangles meet in the remaining 3 out of the 9 edges meeting in each summit, in such a way that the 2 octahedra and consequently the 2 tetrahedra are not adjacent.

In space of five dimensions there is one additional semi-regular figure.

5-ic *Semi-regular.* It is bounded by 10 4-ic double pyramids, 16 4-ic pyramids, 120 tetrahedra, 160 triangles, 80 edges and 16 summits. 5 4-ic double pyramids, 5 4-ic pyramids, 30 tetrahedra, 30 triangles, and 10 edges meet in each summit; 3 4-ic double pyramids, 2 4-ic pyramids, 9 tetrahedra, and 6 triangles meet in each edge; 2 4-ic double pyramids, 1 4-ic pyramid, and 3 tetrahedra meet in each triangle. The radius of its circumscribing sphere is $\frac{1}{4}\sqrt{(10)}$ times the length of its edge.

In space of six dimensions there is one additional semi-regular figure.

6-ic *Semi-regular.* It is bounded by 27 5-ic double pyramids, 72 5-ic pyramids, 648 4-ic pyramids, 1080 tetrahedra, 720 triangles, 216 edges and 27 summits. 10 5-ic double pyramids, 16 5-ic pyramids, 120 4-ic pyramids, 160 tetrahedra, 80 triangles and 16 edges meet in each summit. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is $\frac{1}{3}\sqrt{6}$ times the length of its edge.

In space of seven dimensions there is one additional semi-regular figure.

7-ic *Semi-regular.* It is bounded by 126 6-ic double pyramids, 576 6-ic pyramids, 6048 5-ic pyramids, 12096 4-ic pyramids, 10080 tetrahedra, 4032 triangles, 756 edges and 56 summits. 27 6-ic double pyramids, 72 6-ic pyramids, 648 5-ic pyramids, 1080 4-ic pyramids, 720 tetrahedra, 216 triangles, and 27 edges meet in each summit. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 6-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{3}$ times the length of its edge.

In space of eight dimensions there is one additional semi-regular figure.

8-ic *Semi-regular*. It is bounded by 2160 7-ic double pyramids, 17280 7-ic pyramids, 207360 6-ic pyramids, 483840 5-ic pyramids, 4838404-ic pyramids, 241920 tetrahedra, 60480 triangles, 6720 edges and 240 summits. It has as many r-ic boundaries meeting in each summit as there are $(r-1)$-ic boundaries to the 7-ic semi-regular. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 7-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 6-ic semi-regular, or $(r-3)$-ic boundaries meeting in each $(t-3)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is equal to the length of its edge.

In space of nine dimensions there is one additional semi-regular figure, which is infinite.

9-ic *Semi-regular*. It is bounded by 135 ∞ 8-ic double pyramids, 1920 ∞ 8-ic pyramids, 25920 ∞ 7-ic pyramids, 61920 ∞ 6-ic pyramids, 80640 ∞ 5-ic pyramids, 48384 ∞ 4-ic pyramids, 15120 ∞ tetrahedra, 2240 ∞ triangles, 120 ∞ edges, and ∞ summits. It has as many r-ic boundaries meeting in each summit as there are $(r-1)$-ic boundaries to the 8-ic semi-regular; as many r-ic boundaries meeting in each edge as there are $(r-2)$-ic boundaries to the 7-ic semi-regular. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 8-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 7-ic semi-regular, or $(r-3)$-ic boundaries meeting in each $(t-3)$-ic boundary in the 6-ic semi-regular, or $(r-4)$-ic boundaries meeting in each $(t-4)$-ic boundary in the 5-ic semi-regular.

[The absolute (as distinguished from the relative) magnitude of the coefficients of ∞, which occur in giving the number of r-ic boundaries of the infinite figures are of course arbitrary, and are merely chosen so as to avoid fractions, and keep the integral coefficients as small as possible.]
