All the Way with Gauss-Bonnet and the Sociology of Mathematics

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I was stimulated to write this story by the discussion in The American Mathematical Monthly between Peter Hilton and Jean Pederson on the one hand and Branko Grünbaum and G. C. Shephard on the other hand [HP] [GS]. The discussion as well as my story involves the Euler-Poincaré Number, alias the Euler Characteristic. The *discussion* centers on whether the Euler-Poincaré Number should be discussed in a historical way without mentioning the vast and dramatic generalization and depth of understanding that this most interesting invariant has acquired in this century.

My position in this discussion is that Topology should not be viewed as an advanced subject whose theorems and concepts should be avoided until graduate school. Rather it is the study of continuity, and thus underlies the most basic geometric results. In this paper I show how the basic concept of angle leads naturally to the basic topological ideas of *degree of mapping* and of the Euler-Poincaré Number.

My *story* spans the history of mathematics. It concerns what may be the most widely known non-obvious theorem of mathematics and it contains the same stunning generalization that characterizes the recent history of the Euler-Poincaré number. In fact, it concerns one of the most important and earliest of the applications of the Euler-Poincaré number. It shows the fickleness of mathematical fame, it shows the unreasonable power of unreasonable points of view, and it shows how easy it is for mathematicians to miss and forget beautiful and important theorems as well as simple and revealing points of view.

This is a history of the Gauss-Bonnet theorem as I see it. I am not a mathematical historian. I quote only secondary sources or first hand papers that I quickly scanned, and I did not conduct any thorough interviews. Nonetheless, I am writing this history because I have contributed the last sentence to it (for the moment).

I especially want to acknowledge the help of Hans Samelson. His scholarship greatly altered the thrust of earlier versions of this paper. He discovered Satz VI. He informed me of many points in this history; about Gauss' work, Descartes work, and Hopf's work. And he was a student of Hopf who generalized the Gauss-Bonnet theorem himself.

THE NORMAL MAP. What is the most widely known, not immediately obvious, mathematical theorem? I contend that is the following: *The sum of the interior angles of a triangle equals* π. The ordinary person might admit lightly that he doesn't quite remember the Pythagorean theorem, but if he does not know the sum of the angles equals 180 degrees, he brands himself as uneducated. I will call this theorem the *180 degree theorem.*

This 180 degree theorem was proved in the time of Thales. It has undergone a remarkable generalization through the ages, culminating in the Gauss-Bonnet Theorem as I give it here. The first generalization involves the concept of exterior angle. Exterior angles contain the same mathematical information as interior angles because (see Figure 1) they are related by a simple equation: $\alpha + \beta = \pi$, where α is an interior angle and β is the corresponding exterior angle. Now *the sum of the exterior angles of a polygon equals* 2π. This immediately implies the 180 degree theorem by the preceding equation.

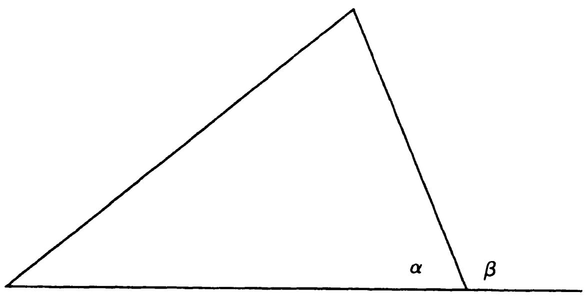

Figure 1

What is the angle between two straight lines intersecting at a point O? Let S^1 be the unit circle centered at O. Then the length of the arc of S^1 cut off by the lines (see Figure 2) is the angle between the lines. We regard angle as a property of a subset of the unit circle rather than as a number. This point of view is closer to the original Greek point of view. Regarding angle as a number is a more modern point of view.

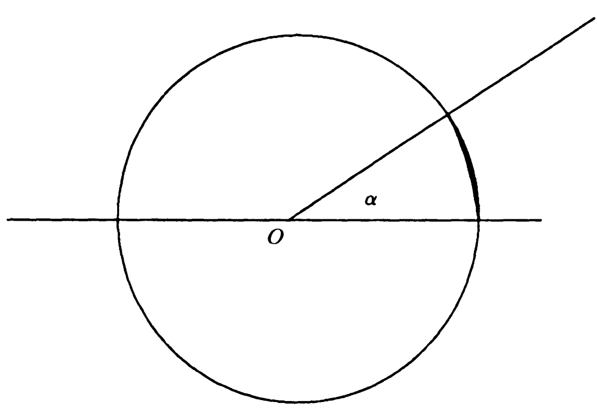

Figure 2

This Greek point of view is susceptible to immediate generalization. Just as angle is the length, or 1-volume, of a region of the unit circle in two space, we can think of the area, or 2-volume, of a region on the unit sphere in three space, denoted by S^2, as a representation of angle in three space. In general, angle in n-space can be thought of as the $(n-1)$-volume of a region on the unit sphere S^{n-1} in n-space.

Now consider a plane curve σ connecting point A to point B (Figure 3). Consider the unit vectors tangent to σ at A and B. Translate these vectors to the origin, keeping the initial and translated vectors parallel. Then the arc on S^1 cut off by the two translated vectors represents the angle the curve has turned through.

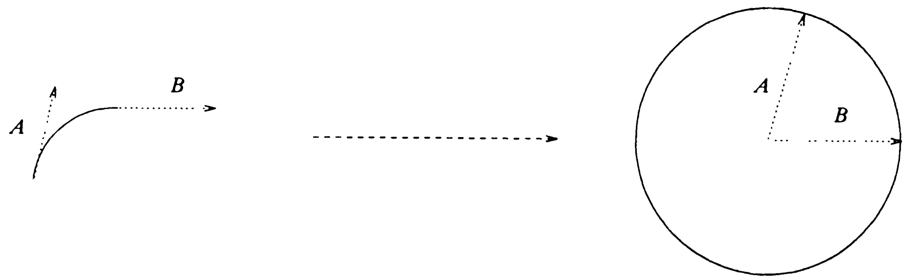

Figure 3

One thing that topologists have learned in developing Topology is that it almost always pays to convert things into functions or mappings. This procedure has spread throughout all of mathematics in the last half of this century. So in the case at hand, define a mapping from σ to the unit circle S^1 as follows: At each point P on σ, construct the unit tangent vector to σ at P, then parallel translate it to the origin; its end lies on the unit circle. Call this the *tangent map*.

Now let B approach A along σ. If we divide the angle between the tangent at B and the tangent at A by the length along σ from A to B, we have a quantity that approaches a limit if σ is smooth enough. This number is the *curvature* of σ at A. This is the same as saying that the curvature at A is the reciprocal of the ratio at A of the length of an infinitesimal arc on σ to the length of its image on S^1.

Now let us approximate a polygon by a smooth simple closed curve. Then the rate of change of the tangent (the *curvature* of the curve) corresponds to the exterior angle, and the total turning of the tangent (the *total curvature* of the closed curve) corresponds to the sum of the exterior angles. Now for simple closed curves, the tangent turns through 2π as it completes a tour of the closed non-self intersecting curve. That is, the total curvature is 2π. This then implies the exterior angles sum to 2π by continuity. This approximation of polygons by smooth curves is an argument known to the Greeks. So we have greatly generalized the original 180 degree theorem about the triangle by the theorem that the total curvature of a simple closed curve is 2π.

Instead of the tangents, we could consider the normals to σ. The normal varies exactly as the tangent does as a point moves along σ, so we could define the curvature of σ using normals instead of tangents. Thus we replace the tangent map with the *normal map* from σ to S^1. The advantage of using normals instead of tangents is that we can generalize curvature to surfaces in three space, for on surfaces in three-space, the normal direction is well-defined whereas there is no unique tangent direction.

We formalize this concept by introducing the idea of the *Gauss map*, also called the *normal map*. To each point on a smooth surface in three-space one can assign a unique unit normal vector pointing outside. This mapping maps the surface to the unit sphere. It is given by sending each point to its normal vector and then parallel transporting the unit vector through space so that the beginning of the

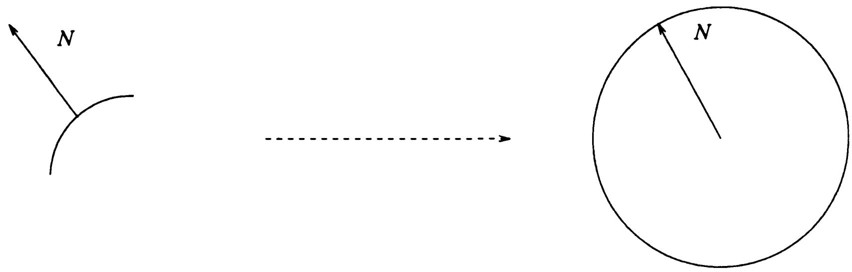

Figure 4

vector is at the center of the unit sphere and then taking the point on the unit sphere that corresponds to the tip of the transported unit vector (see Figure 4).

The same idea gives the normal map in dimension two from a closed curve to the unit circle, and from a smooth closed $(n-1)$ dimensional manifold M embedded in n-dimensional Euclidean space R^n to the unit sphere S^{n-1}. We let $\gamma : M \to S^{n-1}$ denote the normal map.

CURVATURE. Now we can define the concept of *normal curvature* at a point m of M in R^n. Let R be a small region around m in M. Let $\gamma(R)$ denote the image in S^{n-1} of R. Then the *normal curvature* at m, denoted $K(m)$, is the limit as R tends to m of the $(n-1)$ volume of $\gamma(R)$ divided by the $(n-1)$ volume of R. This is given a positive sign if γ preserves the orientation at m and a negative sign if γ reverses the orientation at m. In a suitable coordinate system, $K(m)$ is the Jacobian of γ at m.

Just as the reciprocal ratio of infinitesimal length at x on a curve to the length at the image $\gamma(x)$ is the definition of curvature of a curve in the plane at x, so is the reciprocal ratio of infinitesimal areas from x to $\gamma(x)$ the curvature of a surface at x in space. One would think that the same name would hold for the higher dimensional examples of ratios of infinitesimal volumes, but for historical reasons this did not happen. For the purposes of this paper I will call this number the *normal curvature* of M at x in R^n.

Let us pause and consider the reason that normal curvature, the natural generalization of angle, is not called curvature in dimensions higher than 2. It is because in dimension two, the normal curvature depends not on how the surface sits in R^3, but on the intrinsic geometry of the surface. That is, the curvature can be calculated by considering only the surface and not the ambient space. This is the famous Theorema Egregium of Gauss. So for higher dimension, curvature means the Riemann curvature tensor. This is based on the two dimensional curvature and does not agree at all with the normal curvature in higher dimensions and does not even make sense for dimension 1 curves. This curvature tensor plays an important role in differential geometry and physics, but it does not replace the normal curvature the way interior angles are replaced by exterior angles. Outside of dimension 2 they are very different concepts. This issue of intrinsic vs. extrinsic will play a key role in my story.

Now consider a compact $(n-1)$-dimensional manifold M in R^n, and assume that M has no boundary. Now M divides R^n into two pieces, the interior and the exterior. Let N denote the interior of M, which is a manifold with boundary M. Now if we integrate the normal curvature K over M, we get $\int K\,dM$, the analogue of the sum of the exterior angles. Call this the *Total Curvature* or the old fashioned *Curvatura Integra* of M in R^n. Now we can state our version of the Gauss-Bonnet theorem. Here the Euler-Poincaré number of N is $\chi(N)$.

Gauss-Bonnet Theorem. $\int K\,dM = \chi(N) \times$ (*the volume of* S^{n-1})

NORMAL DEGREE. The unit volume of the $(n - 1)$-sphere is 2π for the 1-sphere and 4π for the 2-sphere and it changes form for each dimension. Thus we define the *degree* of γ by the Curvatura Integra divided by the volume of the unit sphere corresponding to the dimension of M. The degree of γ is denoted by $\deg(\gamma)$ and is called the *normal degree*. The normal degree turns out to be an integer. In fact, this is a special case of the concept of *degree of a mapping*, an integer that plays a major role in Topology. In this notation we can write the Gauss-Bonnet theorem as the

Gauss-Bonnet-Hopf Theorem. $\deg(\gamma) = \chi(N)$.

The Euler-Poincaré number is the earliest invariant of Algebraic Topology. It is a vast generalization of a formula involving convex polyhedra due to Euler. There is evidence that Descartes knew about this formula a century before Euler, [S_2] or [St].

The degree of a map can be traced back to Kronecker and was well understood by L. E. J. Brouwer around 1913. The integral definition given here for the Gauss map can be generalized to maps between oriented closed manifolds of the same dimension. The most general definitions of the degree of a mapping and of the Euler-Poincaré number require Homology Theory. But both these concepts were discovered before homology was well understood and they can be used very effectively without knowledge of homology.

For a two dimensional surface N that can be divided up nicely by triangles that fit nicely together to form what we call a triangulation (as in Figure 5), the Euler-Poincaré number satisfies

$$\chi(N) = v - e + f$$

where v is the number of vertices, e is the number of edges, and f is the number of triangles in the triangulation.

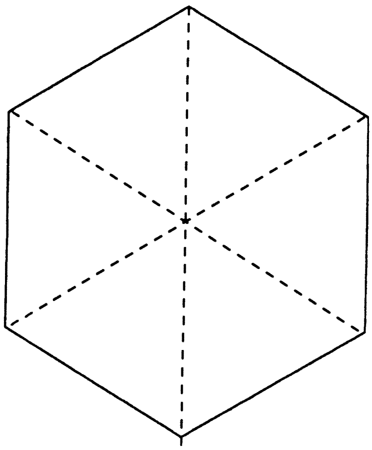

Figure 5

Given this, it is a simple matter to show that if N is bounded by a convex polygon, then $\chi(N) = 1$. Hence $\deg(\gamma) = 1$ by the Gauss-Bonnet-Hopf theorem so $\int K dM = 2\pi$, where K denotes the curvature of the curve in the plane. As we have said, this gives the 180 degree theorem.

Thus we have a tremendous generalization of the sum of angles concept valid for every dimension and given by a simple formula. We continue with the remarkable history of this result.

THE NINETEENTH CENTURY. The Gauss-Bonnet Theorem is so interesting that various authors could not resist including parts of its history in their textbooks. For example, Spivak [Sp] and Stillwell [St] give accounts of its early history.

Consider a geodesic triangle T on a surface in three space. The edges of the triangle are geodesics. Geodesics are what passes for straight lines on the surface; they are paths of shortest length on the surface. Let α, β, γ denote the interior angles of the triangle (Figure 6). Then if we integrate the curvature K over the triangle T, we get the Gauss-Bonnet Formula:

Gauss-Bonnet Formula for the geodesic triangle. $\int K dT = \alpha + \beta + \gamma - \pi$

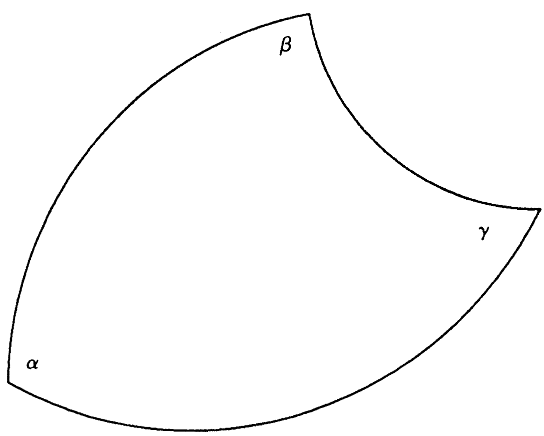

Figure 6

This formula immediately gives interesting corollaries:

If the triangle T is a plane triangle, then the geodesics are straight lines and K is identically equal to zero, so $\alpha + \beta + \gamma = \pi$. So the Gauss-Bonnet Formula implies the 180 degree theorem, but not at all in the same way that the Gauss-Bonnet-Hopf Theorem implies the 180 degree theorem.

If we divide the angular excess $\alpha + \beta + \gamma - \pi$ by the area of T, we get a number that is calculated intrinsically on the surface. As we let T shrink down to a point m, the ratio approaches the curvature $K(m)$ at m. Hence K is an intrinsic concept of the surface. This is Gauss' famous Theorema Egregium, but his published proof is not the argument just given. In an earlier unpublished manuscript, he gave this argument right after his proof of the Theorema Egregium.

If we triangulate a closed surface M with geodesic triangles, we get a Gauss-Bonnet formula for each triangle. If we add these equations up, we get on the left side the *total curvature* (also called the Curvatura Integra): $\int K dT$. On the right side we can rearrange the angles cleverly and end up with $4\pi \times \chi(M)/2$.

This agrees with what we named the Gauss-Bonnet Theorem, because for surfaces $\chi(M) = 2 \times \chi(N)$, where N is the part of space interior to the closed surface M. In fact, it is true that $\chi(M) = 2 \times \chi(N)$ for all even dimensional M. For odd dimensional closed manifolds M, however, $\chi(M) = 0$. These elementary topological facts along with 'intrinsic vs. non-intrinsic' play a key role in this story.

Gauss wrote down the preceding version of the 'Gauss-Bonnet Formula for the geodesic triangle' in an unpublished manuscript in 1825. In 1827, he published a book giving a differential formula, which if integrated would have given the generalization that Bonnet got of the Gauss-Bonnet formula; I was informed of this by Samelson.

In 1848, O. Bonnet extended the Gauss-Bonnet formula for a triangle to smooth closed curves on the surface. Here the sum of the angles is essentially replaced by the integral of the *geodesic curvature*. This generalized formula acquired the name Gauss-Bonnet sometime later. Probably Blaschke was the first to use the name in a textbook in the early 1920's.

If the geodesic triangles triangulate a closed surface S that is topologically a sphere, then Euler's Formula

$$v - e + f = 2$$

gives the first global Gauss-Bonnet theorem: $\int K dS = 4\pi$.

A lost manuscript of Descartes copied in Leibniz' hand was discovered some years later and published in the Comptes Rendus in 1860. A note by Bertrand immediately following Descartes' article points out its relationship to the global theorem. Bertrand notes that Descartes seems to get the polyhedral version of the global Gauss-Bonnet Theorem. He attributes the global theorem to Gauss. See [S_2] for an interesting account of this manuscript. However, we know that nobody understood the Euler-Poincaré Number at that time, and the result really held only for a surface diffeomorphic to a sphere. A good account of the difficulty involved with the development of the Euler-Poincaré Number is found in [La]. Indeed, the Hilton et al. discussion would fit right into the dialogues that Lakatos used to present his thesis.

Walter Dyck seems to be the first to realize that the Gauss-Bonnet Theorem should hold for more that just spherical surfaces. He did this in 1888. According to Hirsch [Hi], Dyck was the first to connect the degree with the Euler-Poincaré number and thus prove "what is wrongly called the Gauss-Bonnet Theorem".

An examination of Dyck's paper reveals pictures that are reminiscent of standard figures in Morse Theory, developed 50 years later. Dyck was a real pioneer, but he, like Descartes, was ahead of his time. Samelson tells me that he cannot find a statement of the global Gauss-Bonnet theorem in Gauss' works. So it appears that the global Gauss-Bonnet theorem should be called the Descartes-Dyck theorem.

Actually, part of this story shows that the name of a theorem is not really for an attribution. It is very convenient to have a name for important theorems, and the main point is that people should know approximately what theorem is meant by the name rather than who gets the credit. Still, one can reflect that Bonnet's name is famous and Dyck's is virtually unknown these days.

HOPF TO CHERN. Dyck worked at a time when two basic ideas—degree of a map and the Euler-Poincaré number—were not clearly understood. By 1925, these concepts were well-defined and were found to be useful. This was due in no small measure to Heinz Hopf.

Hopf made the biggest advance in [H_1]. He essentially proved that $\deg(\gamma) = \chi(M)/2$ for closed hypersurfaces of *even* dimension. The factor 1/2 is explained by the fact that $\chi(N) = \chi(M)/2$ whenever N is a compact odd-dimensional manifold with boundary M. Since $\chi(M) = 0$ for closed odd-dimensional manifolds, the theorem as stated by Hopf did not seem to generalize to the odd dimensional case, and in particular did not generalize the 180 degree theorem, which as we saw *is* generalized by the Gauss-Bonnet Formula.

Since the curvature of a surface is intrinsic in dimension 2, Hopf asked for intrinsic proofs and generalizations of his result [H_3]. He did this repeatedly and interested several mathematicians in the question. The story is told in [Gr].

Using Hermann Weyl's theory of tubes, two mathematicians independently answered Hopf's question in 1940. Allendoerfer [Al] and Fenchel [Fe] discovered that $\deg(\gamma)$ of the boundary of a tubular neighborhood of a closed $2n$ dimensional manifold embedded in a $2r$ dimensional Euclidean Space is equal to the integral of a $2n$ form constructed out of the components of the Riemannian curvature tensor and combined together as a Pfaffian. All this is too complicated to describe here. Since the tubular neighborhood has the same Euler-Poincaré Number as the embedded manifold, they got a formula for the Euler-Poincaré Number in terms of the Riemannian curvature of an embedded even dimensional manifold. This remarkable formula held for every Riemannian manifold because every Riemannian manifold can be isometrically embedded in some Euclidean space. However, this last result was not known until the 1950's, when it was proved by Nash.

Although the Allendoerfer-Fenchel Formula held only for an embedded manifold, it was obviously independent of the embedding and begged for an intrinsic proof. S. S. Chern provided one in 1944 [Ch]. This proof was so well received that the Allendoerfer-Fenchel Formula is frequently called the Gauss-Bonnet-Chern Formula or the Gauss-Bonnet-Chern Theorem. In fact, one of the goals of Gray's book [Gr] was to prevent the interesting methods of the Tube proof from being totally submerged by the powerful ideas of Chern's proof.

SATZ VI. Now we come to the most interesting part of the story. In 1956, Hopf gave lectures on global differential geometry at Stanford University. These lectures were honored by being published as volume number 1000 of Springer-Verlag's Lecture Notes In Mathematics in 1983 [H_4]. On pages 117–118, Hopf describes his version of the Gauss-Bonnet theorem for even dimensions. He does not mention the part that holds for odd dimensions. Because of this and various conversations, I wrote the following three paragraphs.

> It is clear that at that time Hopf did not know that the Gauss-Bonnet theorem held for all dimensions and thus was a generalization of the 180 degree theorem. Or else he knew it, but was embarassed to state it. Hopf certainly knew all the ingredients for the proof in all dimensions for many years, and the proof is of the same order of difficulty as his even dimensional proof. Had he known the version that held for all dimensions it seems likely he would not have asked for intrinsic proofs, since there are none in odd dimensions. So two very fruitful lines of research probably would not have been undertaken.
>
> Yet the Gauss-Bonnet-Hopf theorem was known to several topologists around the mid fifties, among them Milnor and Lashof. Nobody seems to

know who it was who first stated the theorem. At the time there were sophisticated generalizations and studies of deg(γ), for example [Ke] and [Mi]. Just recently Bredon, in his textbook [Br], stated and proved the result as "Theorem 12.11 (Lefschetz)". He proves it as a corollary of the Lefschetz fixed point theorem.

Finally, in 1960, the Gauss-Bonnet-Hopf theorem was stated in the literature, but in an even more generalized form by Samelson [S_1] and Haefliger [Ha]: Let N be a compact n-dimensional manifold with boundary M and let $f: N \to R^n$ be an immersion. Then the Gauss map $\gamma: M \to S^{n-1}$ can still be defined and $\deg(\gamma) = \chi(N)$.

After those words were written I received a letter from Hans Samelson. I had asked Samelson if he knew who had first discovered the Gauss-Bonnet-Hopf Theorem. After all, he had generalized it in [S_1]. In addition, he is a scholar about the Gauss-Bonnet Theorem, and he was a student of Heinz Hopf!

He thought it was Morse who first stated it. He could not find the reference, but on a hunch he looked at Hopf's 1927 paper [H_2]. There on page 248, Satz VI, the Gauss-Bonnet-Hopf theorem is clearly stated for all dimensions!

It is a testament to Hopf's genius that even though he knew Satz VI for all dimensions, the fact that the even dimensional case was true for immersions, instead of merely embeddings (concepts not well understood then), must have led him to conjecture that there was an intrinsic proof in the even dimensional case.

A differentiable map between two manifolds of the same dimension is an *immersion* if the Jacobian of the map is not zero anywhere. It is an *embedding* if in addition the map is one-to-one. Thus, immersions are one-to-one in small neighborhoods of any point, whereas embeddings are globally one-to-one. This distinction generalizes to any mappings.

Now Satz VI, that is, the Gauss-Bonnet-Hopf theorem, was proved only for embeddings, whereas Hopf knew from [H_1], that for M an even dimensional manifold, $\deg(\gamma) = \chi(M)/2$ was true if M is immersed in Euclidean space of codimension one, i.e., the dimension of the Euclidean space is one higher than the dimension of M. By the way, since locally M is embedded in Euclidean space, there is a normal direction and so the Gauss map γ is still defined.

The distinction between the odd and even dimensional cases can be explained to anyone. A circle can be immersed in a plane with arbitrary normal degree, but a

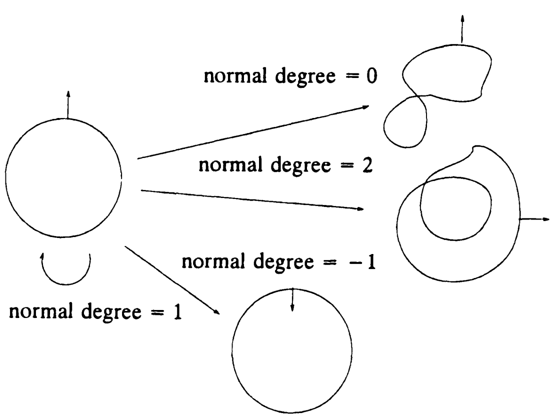

Figure 7

two-sphere can be immersed in three-space only with normal degree equal to one.

So Hopf's proof in [H_1] was not rendered superfluous by his proof of Satz VI, and he recognized the presence of intrinsicness in the difference. Thus he stimulated the Geometers with [H_3] to seek an intrinsic proof of the even dimensional Gauss-Bonnet-Hopf theorem. He also asked questions about the possible normal degrees of immersion for odd dimensional M. This stimulated Milnor's beautiful paper [Mi], and then [BK], wherein it is shown that the normal degree can take on the value of any odd integer.

THE UNASKED FOR ANSWER. In hindsight, we see that Hopf's question amounted to: Find a formula giving $\chi(M)$ in terms of the curvature tensor for even dimensional closed Riemannian manifolds. The more reasonable question should have been: Find a formula giving $\deg(\gamma)$ for all dimensions. Nobody asked this question. An answer has been found, however. It is what I will call the Topological Gauss-Bonnet Theorem to distinguish it from the Gauss-Bonnet-Chern Theorem.

This theorem immediately gives a proof of Satz VI as well as a proof for the immersion portion of the even dimensional part proved in [H_1]. The proof of this theorem requires nothing that was unknown in 1929. It is completely extrinsic. If Hopf had discovered this proof, it is unlikely he would have asked for an intrinsic proof of [H_1], and so some very important mathematics would not have been discovered so quickly. For the Allendoerfer-Fenchel Formula, now known as the Gauss-Bonnet-Chern Theorem, could not have been discovered by accident. It is too complicated. Very talented mathematicians were looking for it explicitly. On the other hand, the Topological Gauss-Bonnet Theorem is simple enough that it could have been discovered by accident. And it was!

Topological Gauss-Bonnet Theorem. *Let $f: N \to R^n$ be a map whose Jacobian is nonzero on the oriented boundary M of a compact n-manifold N. Then if x is the projection of R^n to some x-axis and $\nabla(x \circ f)$ is the gradient vector field of the composition of maps $(x \circ f)$ and Ind is its index, we have*

$$\deg(\gamma) = \chi(N) - \mathrm{Ind}(\nabla(x \circ f))$$

The fact that f has nonzero Jacobian on the boundary M of course means that f is an immersion on M. Since the composition $(x \circ f)$ is a map from N to the real line R, the gradient can be defined as in advanced calculus and gives a vector field on N. The *index* of a vector field, which is a new term in this paper, is another topological invariant that predates the start of Algebraic Topology. It was defined for vector fields in two dimensions by Poincaré in the late nineteenth century. Hopf generalized the index of a vector field for any dimensional manifold, and used the concept in his proofs of the Gauss-Bonnet-Hopf Theorem in [H_1] and [H_2].

The index of a vector field V is an integer. It is closely related to the degree of a map, yet it was defined earlier than that concept. In contrast to the degree of a map, the best definition of index does not necessarily need homology theory. In fact, it can be defined by means of a simple identity.

In 1929, Marston Morse [Mo] discovered a beautiful equation involving the index of a vector field V on a compact manifold N with boundary M; I call Morse's equation the Law of Vector Fields.

The Law of Vector Fields. *Let V be a vector field defined on N, and suppose V is not zero on the boundary M. Then* $\operatorname{Ind} V + \operatorname{Ind} \partial_- V = \chi(N)$, *where $\partial_- V$ is a vector field induced by V and defined on that part of the boundary M where V points inside.*

The vector field $\partial_- V$ is induced by V by considering the component vector field of V that is tangent to the boundary. Since $\partial_- V$ is defined on a one dimension lower space, part of the boundary M of N, an inductive scheme of calculating the index suggests itself. In fact, the Law of Vector fields is literally a self contained definition of the Index of vector fields by induction [gS]. This is elementary, but tricky, topology. Nonetheless, the whole theory of $\operatorname{Ind}(V)$ spins out from this simple 'A plus B equals C' equation. This equation is the key to the last part of the story.

Among the facts that follow easily from the Law of Vector Fields are two well known properties of the index, which combine with the Topological Gauss-Bonnet Theorem to give all the previous global results labled Gauss-Bonnet:

If V is a vector field with no zeros, then $\operatorname{Ind} V = 0$.

If V is a vector field on an odd dimensional manifold, then $\operatorname{Ind}(-V) = -\operatorname{Ind}(V)$ where $-V$ is the vector field in which every vector of V is reversed.

The Gauss-Bonnet-Hopf Theorem follows immediately from the first property, since if f is an embedding, the vector field $\nabla(x \circ f)$ is just ∇x, that is, a constant vector field parallel to the x-axis restricted to N. This has no zeros, so applying the Topological Gauss-Bonnet with the index zero yields the Gauss-Bonnet-Hopf Theorem. In fact, if f is an immersion, the vector field $\nabla(x \circ f)$ still has no zeros (because $x \circ f$ has no critical points). So we get Samelson and Haefliger's generalization of Gauss-Bonnet-Hopf from embeddings to immersions.

On the other hand, Hopf's first version in [H_1], that for even dimensional M *immersed* in R^{n+1} we have $\deg(\gamma) = \chi(M)/2$, follows from the second property. If we choose the x-axis in the Topological Gauss-Bonnet Theorem to run in the opposite direction, we reverse the direction of the gradient. The other two terms in the Topological Gauss-Bonnet Theorem certainly do not care which way the x-axis is going. So we must have $\operatorname{Ind}(\nabla(x \circ f)) = 0$. Thus $\deg(\gamma) = \chi(N) = \chi(M)/2$. The last equality follows because the Euler-Poincaré number for an even dimensional boundary is twice the Euler-Poincaré number of its bounded manifold.

There is one point that remains to be clarified. Does every orientable M that can be immersed in a codimension 1 Euclidean space bound an N so that the immersion can be extended to an f? The answer is yes. But I must admit that my way of proving this fact is immediate from a famous result of Thom's involving cobordism theory and Stiefel-Whitney numbers, which was not available until the 1950's.

THE ACCIDENTAL DISCOVERY. The Law of Vector Fields was discovered by Morse in 1929 [Mo]. In an interesting parallel with Satz VI, Morse rarely referred to the result or exploited its potential. Maybe it was because he was inventing Morse theory and may have thought unconsciously, as many topologist have, that all vector fields come from gradient vector fields. At any rate, this result was not used much and was virtually forgotten. When I rediscovered it in the 1980's, it took almost a year of questioning before someone told me about [Mo].

Ten years ago I shared the common misconception about how mathematics is created. I did not know the lessons of this story or of history. So I was shocked to find that most topologists were unaware of what I regarded as an elementary relationship satisfied by two classical topological concepts: index and Euler-Poincaré number. So I thought, perhaps, there might be some interesting unknown consequences of the Law of Vector Fields.

I thought of a simple scheme to try to exploit the Law of Vector Fields. I looked at interesting vector fields and plugged them into the equation. I had some success with various choices. When I plugged in what I called pullback vector fields, which generalize gradient vector fields, I got an equation involving the normal degree and the Euler-Poincaré number [G_1] [G_2]. It took a while before it occurred to me that I had generalized the Gauss-Bonnet Theorem. A simplified version of that result is the Topological Gauss-Bonnet Theorem as stated above. The only simplification is that I stated the result here for gradients since it is a concept familiar from advanced calculus. In fact, pullback vector fields may even be easier than gradients.

CONCLUSIONS. Mandelbrot, in proposing the name "fractals", complained that mathematicians do not give names to concepts and results. He was right. In the deepest sense, this story really revolves about the naming of theorems and of curvature.

But it also demonstrates that several of the bromides we have grown up with are seriously flawed: That great men do not overlook simple points. That there are no great results found in using old methods. That you can't discover something good unless you have asked the right questions. That mathematics progresses mostly by the work of a few great mathematicians; this particular misconception is called the Matthew Effect by historians of Science. It seems to me that what happened with the Gauss-Bonnet Theorem happened very frequently with the best of our mathematical ideas. Nobody seems to know who invented "Cartesian Coordinates", or who first thought about higher dimensional spaces. Great mathematicians are quoted denigrating ideas that blossomed and dominated mathematics. From our present vantage point, these ideas seem trivial, but our greatest predessesors had trouble grasping them. What seems to be trivial now was once the most difficult part of mathematics: infinity, velocity and acceleration, arbitrary axioms, abstract groups, functions.

Finally, the story shows how mathematical challenges can have a great and good effect on the development of mathematics, even if the challenges were based on faulty points of view.

As an application of this last lesson I will issue a mathematical-historical challenge. Let us agree that a theorem generalizes a second theorem if the second has a short proof in which the first plays the predominate part. Then I propose the *Historical Fame Score* of any Theorem: The HFS is the product of three numbers H, F, and S.

H is the percent of the history of Mathematics covered between the time the first interesting special case was proved and its generalizing theorem was proved. The beginning of the History of Mathematics will be taken to be 300 BC in honor of Euclid and the unavailability of precise dates before that time.

F is the percent of mathematicians who know the most famous special case of the generalizing theorem.

S is the percent of results closely related to the subject matter of the generalizing theorem that receive new proofs, or new insights, or new corollaries from the generalizing theorem.

The maximum score is one million. I estimate that the Topological Gauss-Bonnet Theorem receives the maximum score. The challenge is to find generalizations with comparable scores.

REFERENCES

[Al] C. B. Allendoerfer, *The Euler Number of a Riemannian manifold*, Amer. J. Math. **62** (1940), 243–248.

[Br] Glenn E. Bredon, *Topology and Geometry*, Springer-Verlag: Graduate Text in Mathematics, New York (1993).

[BK] G. E. Bredon and A. Kosinski, *Vector fields on Π-manifolds*, Annals of Mathematics (2) **84** (1966), 85–90.

[Ch] S. S. Chern, *A simple intrinsic proof of the Gauss-Bonnet theorem for closed Riemannian manifolds*, Annals of Math. (2) **45** (1944), 747–752.

[Fe] W. Fenchel, *On the total curvature of Riemannian manifolds, I*, J. London Math. Soc. **15** (1940), 15–22.

[G_1] Daniel H. Gottlieb, *On the index of pullback vector fields*, Springer-Verlag, Lecture Notes in Mathematics **1350** (1987), 167–170.

[G_2] Daniel H. Gottlieb, *Zeroes of pullback vector fields and fixed point theory for bodies*, Contemporary Mathematics, Amer. Math. Soc. **96** (1989), 163–179.

[Gr] Alfred Gray, *Tubes*, Addison-Wesley, Redwood City, California (1990).

[gS] D. H. Gottlieb and G. Samaranayake, *Index of discontinuous vector fields*, New York Journal of Mathematics (to appear).

[GS] B. Grünbaum and G. C. Shephard, *A new look at Euler's Theorem for polyhedra*, Amer. Math. Monthly **101** (1994), 109–128.

[Ha] Andre Haefliger, *Quelques remarques sur les applications differentiables d'une surface dans le plan*, Ann. Inst. Fourier **10** (1960), 47–60.

[HP] P. J. Hilton and J. Pederson, *Euler's Theorem for polyhedra: A Topologist and a Geometer respond*, Amer. Math. Monthly **101** (1994), 959–962.

[Hi] Morris W. Hirsch, *Differential Topology*, Springer-Verlag, New York (1976).

[H_1] Heinz Hopf, *Über die Curvatura integra geschlossener Hyperflächen*, Math. Ann. **95** (1925), 340–376.

[H_2] Heinz Hopf, *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Annalen **96** (1927), 225–250.

[H_3] Heinz Hopf, *Differential Geometrie und topologische Gestalt*, Jahresbericht der Deutcher Math. Verein. **41** (1932), 209–229.

[H_4] Heinz Hopf, *Differential Geometry in the Large: Seminar Lectures NYU 1946 and Stanford 1956*, Lecture Notes in Mathematics, Springer Verlag **1000** (1983).

[Ke] M. E. Kervaire, *Courbure integrale generalisee et homotopie*, Math. Ann. **131** (1956), 219–252.

[La] Imre Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge University Press, Cambridge (1976).

[Mi] J. W. Milnor, *On the immersion of n-manifolds in (n + 1)-dimensional space*, Commentarii Math. Helvetici **30** (1956), 275–284.

[Mo] Marston Morse, *Singular points of vector fields under general boundary conditions*, Amer. J. Math **51** (1929), 165–178.

[S_1] Hans Samelson, *On immersion of manifolds*, Canadian J. Math. **12** (1960), 529–534.

[S_2] Hans Samelson, *Descartes and Differential Geometry*, Geometry, Topology, and Physics for Raoul Bott; ed. S. T. Yau; International Press, Boston (1995), 323–328.

[St] John Stillwell, *Mathematics and its History*, Springer-Verlag, New York (1974).

[Sp] Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, 2nd ed, vol. 1–5, Houston (1979).

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