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ON GRADIENT DYNAMICAL SYSTEMS

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We consider in this paper a C^∞ vector field X on a C^∞ compact manifold M^n (∂M, the boundary of M, may be empty or not) satisfying the following conditions:

(1) At each singular point β of X, there is a cell neighborhood N and a C^∞ function f on N such that X is the gradient of f on N in some riemannian structure on N. Furthermore β is a non-degenerate critical point of f. Let $\beta_1, \cdots, \beta_m$ denote these singularities.

(2) If $x \in \partial M$, X at x is transversal (not tangent) to ∂M. Hence X is not zero on ∂M.

(3) If $x \in M$ let $\varphi_t(x)$ denote the orbit of X (solution curve) through x satisfying $\varphi_0(x) = x$. Then for each $x \in M$, the limit set of $\varphi_t(x)$ as $t \to \pm\infty$ is contained in the union of the β_i.

(4) The stable and unstable manifolds of the β_i have normal intersection with each other.

This has the following meaning. The stable manifold W_i^* of β_i is the set of all $x \in M$ such that $\text{limit}_{t\to\infty}\varphi_t(x) = \beta_i$. The unstable manifold W_i of β_i is the set of all $x \in M$ such that $\text{limit}_{t\to-\infty}\,\varphi_t(x) = \beta_i$. It follows from conditions (1), (2) and a local theorem in [1, p. 330], that if β_i is a critical point of index λ, then W_i is the image of a 1–1, C^∞ map φ_i: $U \to M$, where $U \subset R^{n-\lambda}$ has the property if $x \in U$, $tx \in U$, $0 \leqq t \leqq 1$ and φ_i has rank $n - \lambda$ everywhere (see [4] for more details). A similar statement holds for W_i^* with the $U \subset R^\lambda$. Now for $x \in W_i$ (or W_i^*) let W_{ix} (or W_{ix}^*) be the tangent space of W_i (or W_i^*) at x. Then for each i, j, if $x \in W_i \cap W_j^*$, condition (4) means that

$$\dim W_i + \dim W_j^* - n = \dim (W_{ix} \cap W_{jx}^*) .$$

Here W_{ix} and W_{jx}^* are considered as subspaces of the tangent space to M at x.

For closed manifolds, these vector fields are a special case of those considered in [4].

THEOREM A. *Let f be a C^∞ function on a compact C^∞ manifold M^n with non-degenerate critical points. Suppose M is provided with a*

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riemannian metric and that grad f *is transversal to* ∂M. *Then* grad f *can be* C^1 *approximated by a vector field satisfying conditions* (1) *to* (4).

THEOREM B. *Let* X *be a* C^∞ *vector field on a compact* C^∞ *manifold* M^n *satisfying* (1)–(4). *Denote by* V_1 *those points of* ∂M *at which* X *is oriented in, and* V_2 *those points of* ∂M *at which* X *is oriented out. Then there is a* C^∞ *function* f *on* M *which has these properties:*

(a) *The critical points of* f *coincide with the singular points of* X *and* f *coincides with the function of condition* (1) *plus a constant in some neighborhood of each critical point.*

(b) *If* X *is not zero at* $x \in M$, *then it is transversal to the level hypersurface of* f *at* x.

(c) *If* $\beta \in M$ *is a critical point of* f, *then* $f(\beta)$ *is* $\lambda(\beta)$, *where* $\lambda(\beta)$ *is the index of* β.

(d) f *has value* $-\frac{1}{2}$ *on* V_1 *and* $n+\frac{1}{2}$ *on* V_2.

REMARK. It is easily proved from (a)–(d) that there is a riemannian metric on M such that grad $f = X$.

The next theorem follows easily from Theorems A and B.

THEOREM C. *Let* M^n *be a compact* C^∞ *manifold with* ∂M *equal to the disjoint union of* V_1 *and* V_2, *each* V_i *closed in* ∂M. *Then there exists a* C^∞ *function* f *on* M *with non-degenerate critical points, regular on* ∂M, $f(V_1) = -\frac{1}{2}$, $f(V_2) = n + \frac{1}{2}$ *and at a critical point* β *of* f, $f(\beta) =$ index β.

For some motivation of these theorems see [4], [5], and [6]. In [4] Theorem A was announced for the case $\partial M = \emptyset$, while Theorem C was announced in [5] for the case $\partial M = \emptyset$. These theorems have implications in differential equations on one hand and topology on the other, both of which we will pursue in future papers.

As this paper was finished, an article by A. H. Wallace [7] appeared and seems to bear some relationship to this paper.

1. Proof of Theorem A.

First it is easily shown that there exist C^1 approximations f' of f such that f' is C^∞ and has distinct values at distinct critical points. Thus in proving Theorem A we can assume f has these properties.

LEMMA 1.1. *Let* f *be a* C^∞ *function on a compact riemannian manifold with non-degenerate critical points and* $X =$ grad f *is transversal to* ∂M. *Then a sufficiently close* C^1 *approximation* X' *of* X *with* $X' = X$ *in a neighborhood of the singular points, satisfies condition* (3) *above.*
(One does not need such strong hypotheses on X'.)

PROOF. One can assume that X and X' have the property that, except at singular points, dfX and dfX' are positive. Then an orbit $\varphi_t(x)$ of X or X' is either a singular point or has the property that $f\varphi_t(x)$ increases as t increases. Property (3) then follows. This fact that $f\varphi_t(x)$ increases as t increases is used in the rest of the paper without mentioning it again. It implies, for example, that there are no recurrent orbits of X and X'.

By 1.1 it is sufficient for the proof of Theorem A to show:

LEMMA 1.2. *If f is a C^∞ function on a compact C^∞ riemannian manifold M, with non-degenerate critical points, distinct critical points having distinct values and $X = \operatorname{grad} f$ transversal to ∂M, then there exist C^1 approximations Y of X satisfying condition* (4) *and $X = Y$ on some neighborhood of the critical points.*

Index the critical points β_i of f of 1.2 so that $f(\beta_i) > f(\beta_{i-1})$, $i = 1, \cdots, r$. Thus β_1 is the minimum of f. Denote by W_i and W_i^* respectively the unstable and stable manifolds associated to β_i. Let $\bar{\beta}_i = f(\beta_i)$, each i.

LEMMA 1.3. *Given sufficiently small $\varepsilon_1 > 0, j$, there is a C^1 approximation X' of X such that $X' = X$ outside of $f^{-1}(\bar{\beta}_j + \varepsilon_1, \bar{\beta}_j + 3\varepsilon_1)$ and in the X' system W_j and W_i^* have normal intersection, each i.*

("W_j in the X' system" has the obvious meaning.)

PROOF. Assume $f(\beta_j) + 3\varepsilon_1 < \bar{\beta}_{j+1}$. Let $\dim W_j = n - k$ and Q be the submanifold $f^{-1}(\bar{\beta}_j + 2\varepsilon_1) \cap W_j$ of M. Let $P = \{x = (x^1, \cdots, x^k) \mid \| x \| \leqq 1\}$ be the k-disk and $I_m = \{z \mid -m \leqq z \leqq m\}$, $m > 0$. Then for small enough m there is a diffeomorphism h of $I_m \times P \times Q$ onto a neighborhood U of Q sending identically $0 \times 0 \times Q$ onto Q and such that $X = \partial/\partial z'$ on U where $z' = h(z \times 0 \times 0)$ and $U \subset f^{-1}(\bar{\beta}_j + \varepsilon, \bar{\beta}_j + 3\varepsilon)$. We will identify points under h so that points of U will be represented by (z, x, y), $|z| \leqq m$, $\| x \| \leqq 1$ and $y \in Q$.

The proofs of the following two lemmas will be left to the reader.

LEMMA 1.4. *Let $I_m = [-m, m]$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that if $\bar{v} < \delta$, there is a C^∞ function $\beta(z)$ on I_m, zero in a neighborhood of ∂I_m, $0 \leqq \beta(z) \leqq \varepsilon$, $|\beta'(z)| \leqq \varepsilon$, and*

$$\int_0^{\pm m} \beta(z)dz = \pm \bar{v} .$$

LEMMA 1.5. *Let P be the k-disk as above. There is a C^∞ function γ on P which is zero in a neighborhood of ∂P, $0 \leqq \gamma \leqq 1$, $|(\partial\gamma/\partial x^i)| \leqq 2$ and $\gamma(x) = 1$ for $\| x \| \leqq 1/3$.*

With ε arbitrary, let δ_1 be the minimum of the δ in 1.4 and 1/100, and let g be the restriction of $\pi_P\colon I_m \times P \times Q \to 0 \times P \times 0 = P$ to $\bigcup_{i=1}^r (0 \times P \times Q) \cap W_i^*$. Now by Sard's theorem [3] choose $v \in P$ such that $\| v \| = \bar{v} < \delta_1$ and $+2v$ is a regular value of g. We can assume, using an orthogonal change of coordinates in P, that $v = (\bar{x}^1, \cdots, \bar{x}^k) = (\bar{v}, 0, \cdots, 0)$.

Let X' be the vector field on M which equals X outside U and on U is given by

$$X' = \frac{\partial}{\partial z} + \beta(z)\gamma(x)\frac{\partial}{\partial x^1},$$

where β and γ are chosen by 1.4 and 1.5. We claim that X' satisfies 1.3 if the ε of 1.4 has been chosen small enough.

To see that X' is well defined it is sufficient to note that the second term vanishes in a neighborhood of ∂U. It is easy to check that X' can be made arbitrarily close in the C^1 sense to X by choosing the ε of 1.4 small enough.

It remains to prove that W_i and W_j^* have normal intersection in the X' system for each i. So fix i in what follows.

Let ψ_t be the orbit in the X' system through x with $\psi_0(x) = x$ and denote by $W_i^{*\prime}$ and W_j' respectively W_i^* and W_j in the X' system. It is sufficient to prove $W_i^{*\prime}$ and W_j' have normal intersection in U since any point $q \in W_j' \cap W_i^{*\prime}$ is of the form $\psi_t(p)$, $p \in U$ and ψ_t preserves the property of normal intersection.

Let $V = \{(z, x, y) \in U \mid \|x\| \leqq 1/3\}$. On V,

$$X' = \frac{\partial}{\partial z} + \beta(z)\frac{\partial}{\partial x^1}$$

and integrating the corresponding system of differential equations, we get $z(t) = t + K_0$, $x^1(t) = \int_0^t \beta(t)dt + K_1$, with the other coordinates constant. Then as long as we are in V,

$$\psi_t(0, x, y) = \left(t, x^1 + \int_0^t \beta(t)dt, x^2, \cdots, x^k, y\right).$$

Using the main property of $\beta(z)$ in 1.4, $\psi_t(0, x, y)$ stays in V for $|t| \leqq m$, $\|x\| \leqq 1/6$, and $\psi_{\pm m}(0, x, y) = (\pm m, x \pm v, y)$ for $\|x\| \leqq 1/6$.

Let V_i and V_j denote respectively $W_i^{*\prime} \cap V'$ and $W_j' \cap V'$ where $V' = \{(0, x, y) \in U \mid \|x\| \leqq 1/6\}$. Then it is sufficient to show that V_i and V_j have normal intersection in $0 \times P \times Q$.

Since $W_j \cap 0 \times P \times Q = \{(0, 0, y) \in U \mid y \in Q\}$, and $W_j = W_j'$ when restricted to $\{(-m, x, y) \in U\}$ and also $\psi_{-m}^{-1}(-m, x, y) = (0, x + v, y)$ for $\|x\| \leqq 1/6$, we obtain $V_j = \{(0, +v, y) \in U\}$. Hence if $\pi_P \colon U \to P$ is the previously defined projection, $\pi_P(V_j) = +v$. If $\bar{g}$ is the restriction of π_P to V_i, then $\bar{g}^{-1}(+v) = V_j \cap V_i$.

Since the intersection of W_i^* and $W_i^{*\prime}$ with $\{(+m, x, y) \in U\}$ are the same and $\psi_{-m}(+m, x, y) = (0, x - v, y)$ we have

$$V_i = \{(0, x - v, y) \mid (0, x, y) \in W_i^* \cap V, \|x - v\| \leqq 1/6\} .$$

This implies that since g has a regular value at $+2v$, $\bar{g}$ has a regular value at $+v$. Hence $\dim V_i = \dim P + \dim (V_i \cap V_j)$ and since $\dim P = k$, V_i and V_j have normal intersection in $0 \times P \times Q$. This proves 1.3.

We show that 1.2 follows from 1.3 by induction on the following hypothesis:

$\mathscr{H}(q)$: There is a C^1 approximation X_q of X (of 1.2) such that $X_q = X$ in a neighborhood of the β_i, W_{r-p} and W_i^* have normal intersection in the X_q system for all $p \leqq q$ and all i.

Then $\mathscr{H}(0)$ is trivial and $\mathscr{H}(r)$ implies 1.2. We will now show that $\mathscr{H}(q-1)$ implies $\mathscr{H}(q)$. Given X_{q-1} by $\mathscr{H}(q-1)$ we will construct X_q. We can suppose that $df(X_{q-1}) = 0$ only on the β_i. Let $\varepsilon_1 = 1/4(\bar{\beta}_{q+1} - \bar{\beta}_q)$ and apply 1.3 to obtain an approximation X_q of X_{q-1} with $df(X_q) = 0$ only on the β_i, $X_q = X_{q-1}$ on a neighborhood of the β_i, and in the X_q system, W_i^* and W_{r-q} having normal intersection for all i. But also W_j and W_i^* will still have normal intersection in the X_q system for $j > r - q$ and all i since this is true in the X_{q-1} system, $X_q \equiv X_{q-1}$ on $f^{-1}([\bar{\beta}_{q+1}, \bar{\beta}_r])$ and $W_j \cap W_i^* \subset f^{-1}([\bar{\beta}_{q+1}, \bar{\beta}_r])$. This finishes the proof of 1.2.

2. Proof of Theorem B.

LEMMA 2.1. *Let X be a C^∞ vector field on a compact C^∞ manifold M^n satisfying* (1)–(4) *with V_1 and V_2 the subsets of ∂M described in Theorem* B. *Then there exists a set of disjoint closed $(n-1)$-dimensional submanifolds B_i of M, $i = -1, 0, 1, \cdots, n$ with the following properties:*

(i) $B_{-1} = V_1$, $B_n = V_2$.

(ii) *Each B_i is transversal everywhere to X.*

(iii) *Each B_k, $k \neq -1, n$, divides M into two regions whose closures we denote by G_k and H_k, with $G_k \supset G_{k-1}$, $H_k \supset H_{k+1}$ and G_k containing exactly those singular points of index $\leqq k$. For completeness we let $G_{-1} = B_{-1}$, $H_{-1} = M$, $G_n = M$ and $H_n = B_n$. Hence, for $k = -1, 0, \cdots, n$, $G_k \cap H_k = B_k$ and $G_k \cup H_k = M$.*

(iv) *On B_k, X is oriented into H_k.*

The proof goes by induction on k. Roughly having constructed B_{k-1}, we augment G_{k-1} by tubular neighborhoods of the stable manifolds corresponding to singular points of index k to obtain G_k (and hence B_k).

PROOF. Take $B_{-1} = V_1$ and assume we have constructed B_{k-1} with $M = G_{k-1} \cup H_{k-1}$, $G_{k-1} \cap H_{k-1} = B_{k-1}$, G_{k-1} containing those singular points of index $\leqq k-1$, and on B_{k-1}, X is oriented into H_{k-1}. We will now construct B_k.

Let $B_{k-1} \times [-1, 1]$ be a product neighborhood of B_{k-1} (in case $k = 0$, take $B_{k-1} \times [0,1]$) with $B_{k-1} = B_{k-1} \times 0$, $B_{k-1} \times [0,1] \subset H_{k-1}$ and $B_{k-1} \times t$ transversal to X for each t.

Denote by γ_i, $i = 1, \cdots, r$, the singularities of X of index k, and changing notation let $W_i^* = W_i^{*k}$ and $W_i = W_i^{n-k}$ denote the stable and unstable manifolds respectively of γ_i, $i = 1, \cdots, r$. Then if $x \in W_i^*$, the orbit of x passes through $\bar{V} = B_{k-1} \times 1$ by Lemma 3.1 of [4] at least once and hence exactly once (the proof of 3.1 in [5] is for closed manifolds but applies equally well to our case; this easy lemma is the only use we make of [4]).

Let γ be one of the γ_i, $W = W_i$, $W^* = W_i^*$. One chooses from condition (1) an open neighborhood N of γ, f on N and $\delta > 0$ such that the $(n-k)$-disk bounded by $f^{-1}(\delta) \cap W = \bar{W}$ is in N. Let E_ε be the normal bundle of W in M restricted to $\bar{W}$ of vectors with magnitude $\leqq \varepsilon$. Denote by S_ε the image of E_ε under the exponential map. Assume $\varepsilon > 0$ is so small that S_ε is transversal to X.

If $\varepsilon > 0$ is sufficiently small one can define an imbedding $T\colon S_\varepsilon - \bar{W} \to V_1$ by sending $x \in S_\varepsilon - \bar{W}$ into the point of the orbit through x meeting V_1. Assume ε is this small and denote the image of T with $\gamma = \gamma_i$ by $K_{i\varepsilon}$ for each $i = 1, \cdots, r$. We assume that ε is small enough so that these $K_{i\varepsilon}$ are mutually disjoint.

Now define a C^∞ imbedding $F\colon \partial S_\varepsilon \times [-1,1] \to M$ by sending $(p, -1)$ into p, $(p, 1)$ into $T(p)$ and (p, t) into the orbit joining p and $T(p)$, the distance from p proportional to t. Then extend F to C^∞ imbedding of $\partial S_\varepsilon \times [-2, 2]$, which sends $p \times [-2, 2]$ into a single orbit, each p.

Next in the construction of G_k and B_k we modify F slightly to a new C^∞ imbedding. Fixing some riemannian metric on M, let $\nu(p, t)$ be the unit normal vector field on the image of F whose orientation is determined by the vectors on ∂S_ε oriented away from $\bar{W}$. For η, a small positive constant, let $F_\eta(p, t)$ be the point at distance ηt from $F(p, t)$ along the geodesic determined by $\nu(p, t)$.

Choose η so small that image $F_\eta = \operatorname{im} F_\eta$ is disjoint from the $K_{i\varepsilon}$, $\operatorname{im} F_\eta$ is transversal to X everywhere, and $\operatorname{im} F_\eta \cap S_\varepsilon$, $\operatorname{im} F_\eta \cap \bar{V}_1$ are diffeomorphic respectively to $\operatorname{im} F \cap S_\varepsilon$, $\operatorname{im} F \cap \bar{V}_1$.

Repeating this construction for each singular point γ_i we obtain a hypersurface (singular) B_k' in M made up of the following pieces:

(a) The part of S_ε bounded by $\operatorname{im} F_\eta \cap S_\varepsilon$, one corresponding to each γ_i;

(b) $\bar{V}_1$ minus pieces bounded by $\operatorname{im} F_\eta \cap \bar{V}_1$ and containing $W^* \cap \bar{V}_1$, one such piece corresponding to each γ_i; and

(c) the part of $\operatorname{im} F_\eta$ bounded by $\operatorname{im} F_\eta \cap S_\varepsilon$ and $\operatorname{im} F_\eta \cap \bar{V}_1$, one for each γ_i.

Then B_k' has the property that, on each piece, it is transversal to $X, M - B_k' = G_k' \cup H_k'$, with G_k' containing G_{k-1} and all the singular points of index k. In fact G_k' only fails to satisfy G_k of 2.1 in that $\partial G_k' = B_k'$ is not a differentiable submanifold, but has corners along im $F_\eta \cap \bar{V}_1$ and im $F_\eta \cap S_2$ for each singular point. This is easily modified however to obtain the desired G_k and B_k by the device of "straightening the angle" (see [2] for some discussion), the details of which we leave to the reader. This finishes the proof of 2.1.

LEMMA 2.2. *Let X be a C^∞ vector field on a manifold M^n satisfying conditions* (1), (2) *and* (3) *with only singular points of index k. Let V_1 and V_2 be as in Theorem* B. *Then there is a C^∞ function on M which satisfies conditions* (a), (b), *and* (c) *of Theorem* B *and has value $k - \frac{1}{2}$ on V_1, value $k + \frac{1}{2}$ on V_2.*

PROOF. Let $\gamma_1, \cdots, \gamma_r$ denote the singular points of X, W_i and W_i^*, their respective unstable and stable manifolds. We will first define the desired function in a neighborhood of $\bigcup_{i=1}^r (W_i \cup W_i^*)$. Let N_i and f_i be neighborhoods and functions of condition (1) but suppose also N_i is as in the proof of 2.1. Furthermore assume $f_i(\gamma_i) = k$ by adding appropriate constants.

Take $\gamma = \gamma_i$, some i, $f = f_i$, $W = W_i$, $W^* = W_i^*$, and $N = N_i$. Then let $f^{-1}(k + \delta) \cap N = R$, $f^{-1}(k - \delta) = R^-$, with δ chosen as in previous lemma, $R_\varepsilon = \{(x, y) \in R \mid \|y\| \leqq \varepsilon\}$, and $R_\varepsilon^- = \{(x, y) \in R^- \mid \|x\| \leqq \varepsilon\}$.

Fix a riemannian metric on M and take $\varepsilon = 1/10$. For $x \in R_\varepsilon$, re-define f on $\varphi_t(x)$, $t \geqq 0$, so that $f(\varphi_0(x)) = k + \delta$, $f(y) = k + \frac{1}{2}$ where y is the point of $\varphi_t(x)$ meeting V_2, and on the points between $\varphi_0(x)$ and y on $\varphi_t(x)$, f is defined proportionally to arc length. Thus we have obtained an f on a neighborhood of W satisfying the right boundary conditions, but is not differentiable on $f^{-1}(\delta)$. By a smoothing process similar to the one discussed by Milnor 8.1, 8.2 of [2], f can be made C^∞ on $f^{-1}(\delta)$.

In the same way using R_ε^-, one gets f defined on a neighborhood Q of W^* as well as a neighborhood of W which satisfies the condition $f(Q \cap V) = k - \frac{1}{2}$. This, by iteration, yields a function f defined on disjoint open neighborhoods P_i of $W_i \cup W_i^*$ which agrees with the f_i on some neighborhoods of the γ_i, of $f(P_i \cap V_1) = k - \frac{1}{2}$, $f(P_i \cap V_2) = k + \frac{1}{2}$, and f has only critical points at the γ_i. Furthermore f satisfies condition (b) of Theorem B. We can assume without loss of generality that the closures of the P_i are disjoint and if $x \in P_i$, all of $\varphi_t(x)$ lies in P_i. We will now extend f to all of M.

Choose $U_i \subset V_1 \cap P_i$, to be a compact neighborhood of $W_i^* \cap V_1$, $i = 1, \cdots, r$. Then let λ be a real C^∞ function on V_1 satisfying $0 \leqq \lambda \leqq 1$,

$\lambda = 1$ on each U_i, $\lambda = 0$ on $V_1 - \bigcup_{i=1}^{r} P_i \cap V_1$. For $x \in M - \bigcup_{i=1}^{r} (W_i \cup W_i^*)$ let $l(x)$ be the length of the orbit through x, $v(x)$ be the distance from $\{\varphi_t(x)\} \cap V_1$ to x along $\varphi_t(x)$ and $g(x) = k - \frac{1}{2} + (v(x))/(l(x))$. One can now show that the function $\bar{\lambda} f + (1 - \bar{\lambda}) g$ on M has the desired properties of the function of 2.2, where $\bar{\lambda}(x) = \lambda(\varphi_t(x) \cap V_1)$ or 1 if $\varphi_t(x)$ does not meet V_1.

Finally we prove Theorem B. Take f on the closure of $G_k - G_{k-1}$ of 2.1 to be the function of 2.2, $k = 0, 1, \cdots, n$. One obtains a well defined function and by smoothing this in a neighborhood of $B_0, \cdots, B_{n-1}$ as in the proof of 2.2, the desired function of Theorem B is obtained.

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